Spring mass walking model

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0.1 Walking model

In [2], a model of two compliant legs is proposed. In this model, when one leg is on the ground, the system is equivalent to a inverted spring pendulum, this is called the single support phase. When the model is not in a single support phase, it is in a double support phase, where the two springs in this phase influence the movement of the CoM at the same time, reproducing a bipedal spring-mass walking. The springs accumulate and deliver energy so that the system remains convervative with no energy losses.
In single phase support, the dynamics of the center of mass (CoM) is as follows,

$$\ddot{x} = \frac{F_1}{m}\frac{x - x_{t1}}{l_1}, \tag{1}$$

$$\ddot{y} = \frac{F_1}{m}\frac{y - y_{t1}}{l_1} - g, \tag{2}$$

where (x, y) are the coordinates of the center of mass (CoM and (x_{ti}, y_{ti}) are the coordenates of the respective toes for the leg 1 and leg 2. With double support, the dynamics are similar to the previous case with the difference that we are in the presence of two springs, this is,

$$\ddot{x} = \frac{F_1}{m}\frac{x - x_{t1}}{l_1} + \frac{F_2}{m}\frac{x - x_{t2}}{l_2}, \tag{3}$$

$$\ddot{y} = \frac{F_1}{m}\frac{y - y_{t1}}{l_1} + \frac{F_2}{m}\frac{y - y_{t2}}{l_2} - g, \tag{4}$$

with F_i being the force applied on the mass by the respective leg,

$$F_i = k(l_0 - l_i) \geq 0 \quad i = 1, 2\,, \tag{5}$$

l_0 is the natural length of the spring, l_i is the respective length,

$$l_i = \sqrt{(x - x_{ti})^2 + (y - y_{ti})^2} \quad i = 1, 2\,. \tag{6}$$

Fig. (1) illustrates the model, single support to double support transitions occur when the center of mass drops to a height of $y \sin \alpha$ where α is the angle of attack and the vertical velocity is negative. Double support to single support transitions occur when the spring deflection $l_0 - l_i$ of one of the legs return to 0.

Let the Poincaré section be defined as the vector $\psi_n = (y_n, \beta_n)^T$ at the Vertical Leg Orientation (VLO). Everytime the system is in the single support phase and the leg is in it's vertical position we record the height of the CoM, y_n and the angle that the velocity makes with the ground, β_n. This state of the system defines the **stride** n of the simulation. Alternatively a **step** is defined when the system passes from single support to double support.

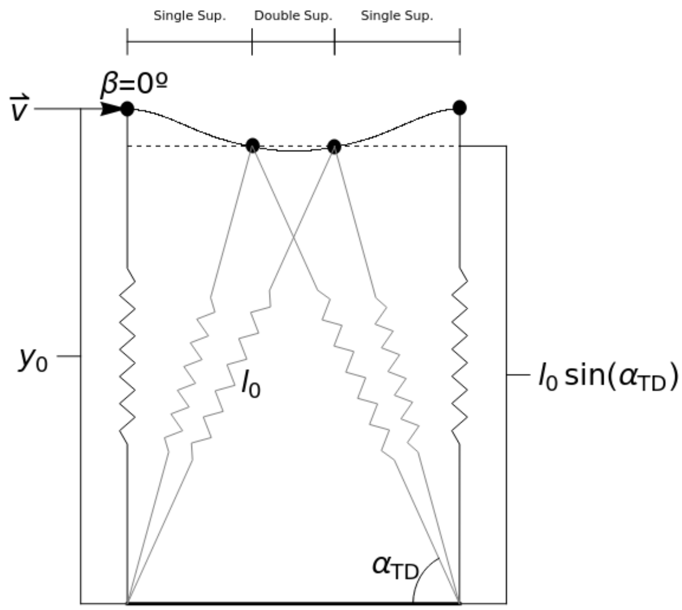

Figure 1: Bipedal spring-mass walking model described in [2]. The initial velocity is given by $\vec{v}$, in this case, $\beta = 0$, this is the velocity is initially parallel to the ground, α_{TD} is the angle of attack for each leg, this is, the aperture that the leg has when it starts the double support phase. In this example, a step occurs when the system passes from single support to double support and a stride is defined when the leg becomes aligned with the toe, this is , the leg returns to it's vertical leg orientation

This model is energetically conservative, therefore, the energy E is a constant with

$$E = \frac{k(l_0 - y_n)^2}{2} + mgy_n + m\frac{v_n^2}{2}. \tag{7}$$

We can express the absolute value of the velocity in terms of the energy. In this model, the only parameters in the initial conditions that can alter the stability of the system for a certain angle of attack α, and energy E are β_0, the angle of the initial velocity with the ground and y_0, the initial vertical position of the CoM.

By allowing the simulation to run over one stride/step, we can associate a map, which is called the Poincaré map, by defining a function A which iterates the Poincaré section, $\psi_n = (y_n, \beta_n)$, this way,

$$\psi_{n+1} = A\psi_n. \quad (8)$$

By recurrence we can apply the A function $n+1$ times from the initial state ψ_0 to get to the final step $n+1$. Not all solutions are admited, if in any instance, the CoM ends up falling this is, $y < 0$, or starts walking backwards, $v_x < 0$, or the leg leaves the ground, $y > l_0$, the set of initial parameters is rejected as a possible stable combination.

1 Simulation Results (walking model)

In this section a method to analyze the stability of the system is displayed regarding the model in section (0.1). The fourth order Runge-Kutta method [1] was applied in order to simulate the Eqs. (1)-(4). A different time steps were used for different situations.

1.1 Fixed points

We are interested in determining the fixed points of the system, this is, the points which repeat after an iteration for a grid of associated parameters. This way, we determine the Poincaré section which we will define by, instead of $\psi_n = [y_n, \theta_n]$, as $\psi_n = [y_n, \Delta x_n]$ [1] and the subsequent Poincaré section $\psi_{n+1} = [y_{n+1}, \Delta x_{n+1}]$. Some of the parameters were kept fixed such as the spring stiffness $k = 14000$ N/m, the spring rest length $l_0 = 1$ m, the mass, $m = 80$ Kg and the angle of the initial velocity at VLO $\beta = 0$.

The grid in which the model was studied consists on the variation of 3 parameters, the energy E which changes the initial velocity of the leg and has a variation in the interval $[800, 840]$ J with $\Delta E = 1$J, the angle of attack α which changes the aperture of the leg and it was studied by dividing the interval $[\pi/2 - \pi/5, \pi/2]$ rad into 30 subintervals to access each angle in this range. Also, the initial position y_0 was studied by dividing the interval $[l_0 \sin\theta, l_0]$ into 25 subintervals. This way the parameter space contains $40 \times 30 \times 25 = 30000$ possible configurations in which the system can be stable. Out of all of this possibilities a filtering was applied based on which configurations had successfully completed 2 strides with a time step on the Runge-Kutta of $\Delta t = 0.001$ s.

The criteria for failling to complete a simulation is the same as in the previous section (0.1),this is, configurations in which the center of mass

[1] The definition of the Poincaré section is done this way because the difference in length of the stride provides a more accurate measurement on the fixed points

falls ($y < 0$) or goes backwards ($v_x < 0$) or leaves the ground ($y > l_0$), were not suitable as valid candidates for stable configurations of the system. The results of this filtering were that 8195 out of 30000 configurations had successfully completed 2 strides and were selected as candidates for fixed points.

Having this subset of fixed point candidates, configurations in which the differences $|\Delta x_2 - \Delta x_1| < 0.001$, $|(y_1 - y_0)| < 0.001$ and $(y_2 - y_1)| < 0.001$ were elligible as fixed points. After determining the fixed points, it is important to know if they are associated with a stable or unstable region. We can do this by determining the eigenvalues of the Jacobian [2],

$$J = \begin{bmatrix} \frac{\partial \Delta x_{n+1}}{\partial \Delta x_n} & \frac{\partial \Delta x_{n+1}}{\partial y_n} \\ \frac{\partial y_{n+1}}{\partial \Delta x_n} & \frac{\partial y_{n+1}}{\partial y_n} \end{bmatrix}, \tag{9}$$

where each derivative can be computed by the following expression,

$$\frac{\partial \Delta x_{n+1}}{\partial \Delta x_n} = \frac{\frac{\partial \Delta x_{n+1}}{\partial t}}{\frac{\partial \Delta x_n}{\partial t}}. \tag{10}$$

If the eigenvalues of the fixed point lie in the circle of radius 1, this is, if $|\lambda_{1,2}| < 1$, then the fixed point is considered stable, otherwise, is considered a unstable point [3]. With this, in the subset of 8195 configurations which completed 2 strides, 11 were selected as fixed points, in Table (1) the parameters for this points are displayed aswell as the respective eigenvalues of matrices (9).

Energy[J]	α_0	y_0[m]	λ_1	λ_2	Stability
800	1.215	0.950	0.629	0	Stable
801	1.152	0.962	1.711	0	Unstable
809	1.131	0.970	1.388	0	Unstable
812	1.236	0.953	1.860	0	Unstable
814	1.131	0.973	1.144	0	Unstable
821	1.089	0.982	0.536	0	Stable
825	1.110	0.983	1.044	0	Unstable
827	1.236	0.947	-1.063	0	Unstable
828	1.089	0.986	0.734	0	Stable
834	1.089	0.991	0.739	0	Stable
839	1.068	0.995	0.467	0	Stable

Table 1: Fixed points table regarding the walking model. In this table we can see that if the energy is higher, the points with higher stability will be those with smaller angles, otherwise the case is reversed.

1.2 Step incrementation

Having determined the fixed points and their respective type, an estimation of the stable and unstable zones of the parameter space was determined. This was done by incrementing the step of the configurations that completed 2 strides (1.1), so that the maximum number of steps,*maxsteps*, increases by 1 from [3,10] with a Runge-Kutta time step of $\Delta t = 0.00075$ s. If the simulation registered a success when the step was incremented, we increment another step, otherwise, we associate to that configuration a maximum number of steps of $maxsteps - 1$. If a configuration point reached 10 steps, we registered the number of steps to that configuration to be *maxsteps*. Fig. (2) shows the final result obtained aswell with the points that succeeded 10 steps as the fixed points associated.

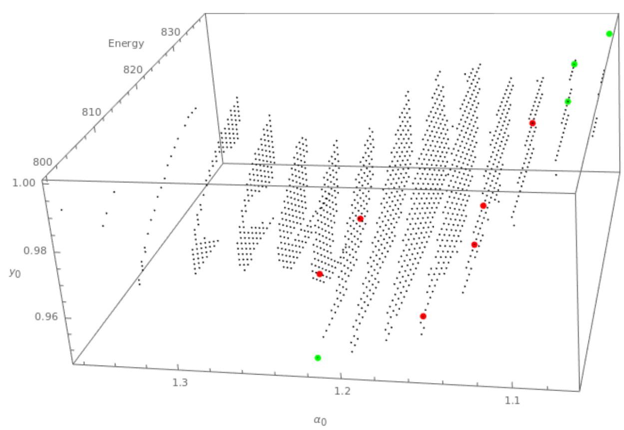

Figure 2: Successful 10 step configurations with stable fixed points in green and unstable fixed points in red. In this 3D plot the Energy was varied from 800 and 840, y_0 from 0.8 to 1 and α_0 from 0.94 to 1.57 and only this configurations remained. The fixed points displayed are associated to Table (1). There is a preference for the stable configurations to have angle of attack of $\approx [1.05,1.25]$

Figures (3) and (4) simulate the behaviour of the Center of Mass in the walking model for respectively a stable and an unstable fixed point a constant angle of attack policy.

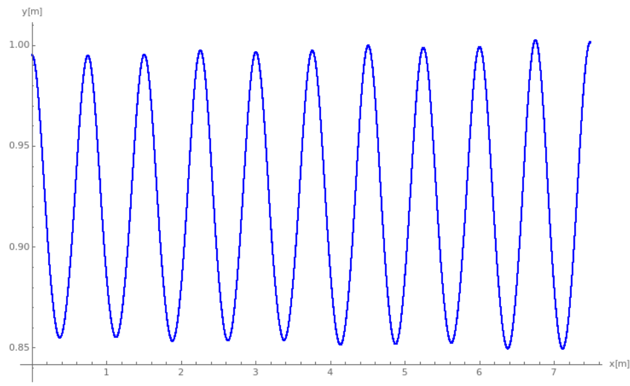

Figure 3: Position of the CoM of the stable fixed point E=839J, α_0= 1.068,y_0=0.995, Runge-Kutta timestep $\Delta t = 0.0005$s. In this example, the fixed point successfully covers all 10 strides with no instability.

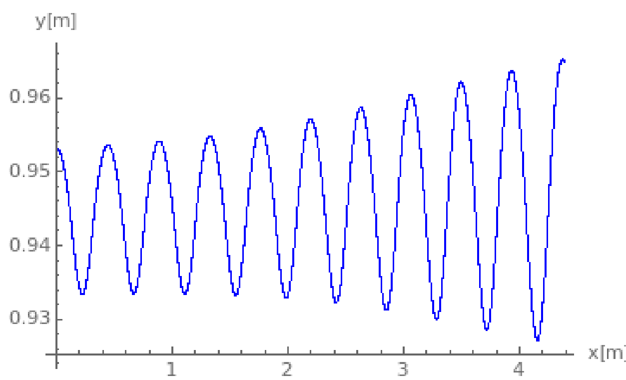

Figure 4: Position of the CoM of the unstable fixed point E=812J, α_0= 1.236,y_0=0.953, Runge-Kutta timestep $\Delta t = 0.0005$s. In this example, the fixed point successfully covers all 10 strides, but the amplitude increases in each stride bringing the system to an unstable configuration.

Appendix

In this appendix I explain how the programs that I developed work. All the notebooks are designed to at least work using the option "Evaluate Notebook" from the menu Evaluation.

2legs.nb

In this mathematica code, I pick a set of parameters and initial conditions to apply into the Runge-Kutta method to evaluate a simulation of the center of mass. Using the Manipulate method, it is possible to view an animation of the movement as well as to access the trajectory using the plot associated.

2legs_pontosfixos.nb

In this program I obtain a table which is the result of a scan of 3 parameters: energy, angle of atack and initial position regarding a successful 2 stride simulation. In this table, besides this 3 parameters information, other important measures are required to obtain the fixed points such as y_1, y_2, x_1, x_2 which are the positions of the CoM at Vertical Leg Orientation at the first and second stride as well it's respective velocities to calculate the Jacobian matrix according to Eq. (9). To know the precise location and velocity of the Center of Mass at Vertical Leg Orientation an interpolation of third order is applied from each Runge-Kutta point so that we can precisely access each value at the horizontal position of the toe.

leitura.nb

In this program I filter all the successful 2 stride configurations by requiring that Δx_1, Δx_2, Δy_1 and $\Delta y_2 < 0.001$. This gives me access to the index of the fixed point. From this list i calculate the associated eigenvalues of the jacobian matrix and determine if they are unstable or stable

2legs_avail.nb

In this program I build a table with the maximum number of steps regarding the successful 2 stride configuration points. If that configuration succeeds, I continue to iterate the maximum step until 10, otherwise I associate to that configuration a maximum step number of $maxsteps - 1$.

References

[1] George B Arfken and Hans J Weber. *Mathematical methods for physicists*. 1999.

[2] Hartmut Geyer, Andre Seyfarth, and Reinhard Blickhan. "Compliant leg behaviour explains basic dynamics of walking and running". In: *Proceedings of the Royal Society of London B: Biological Sciences* 273.1603 (2006), pp. 2861–2867.

[3] Steven H Strogatz. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering (Boulder, CO*. 2001.