Contents

Partielle Integration (Produktintegration)

Idee aus der Produktregel (Ableitung)

$$f(x) = u(x)v(x)$$
$$f'(x) = u'(x)v(x) + u(x)v'(x)$$
$$f'(x) - u'(x)v(x) = u(x)v'(x) \mid \text{ integrieren}$$
$$\int_a^b f'(x) - u'(x)v(x) = \int_a^b u(x)v'(x)$$
$$\int_a^b f'(x)\mathrm{d}x - \int_a^b u'(x)v(x)\mathrm{d}x = \int_a^b u(x)v'(x)\mathrm{d}x$$
$$[f(x)]_a^b - \int_a^b u'(x)v(x)\mathrm{d}x = \int_a^b u(x)v'(x)\mathrm{d}x$$
$$[u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\mathrm{d}x = \int_a^b u(x)v'(x)\mathrm{d}x \text{ (Regel für die partielle Integration)}$$

Nebenrechnung

$$[f(x)]_{x=a}^{x=b} = f(x=b) - f(x=a)$$
$$[u(x)v(x)]_a^b = u(b)v(b) - u(a)v(a)$$

Anwendung

$$I = \int_a^b x\cos x\mathrm{d}x$$
$$x = u(x)$$
$$\cos x = v'(x)$$

$$u(x) = x$$
$$u'(x) = 1$$
$$v(x) = \sin x$$
$$v'(x) = \cos x$$

$$I = [x\sin x]_a^b - \int_a^b 1 \cdot \sin x\mathrm{d}x$$
$$I = [x\sin x]_a^b - [-\cos x]_a^b$$
$$I = [x\sin x + \cos x]_a^b$$

Nebenrechnung

$$f(x) = \sin x$$
$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$
$$f''''(x) = \sin x \text{ usw.}$$

Aufgabe 252/4

$$I = \int_0^\pi x \sin x \mathrm{d}x$$
$$u(x) = x$$
$$u'(x) = 1$$
$$v(x) = -\cos x$$
$$v'(x) = \sin x$$
$$I = [x \cdot (-\cos x)]_0^\pi - \int_a^\pi -\cos x \mathrm{d}x$$
$$I = [x \cdot (-\cos x)]_0^\pi - [-\sin x]_0^\pi$$
$$I = [x \cdot (-\cos x) + \sin x]_0^\pi$$

Aufgabe 252/5

(partielle Integration mehrfach anwenden)

$$I = \int_a^b x^2 \cos x \mathrm{d}x$$
$$a = -\frac{\pi}{2}$$
$$b = \frac{\pi}{2}$$
$$u(x) = x^2$$
$$u'(x) = 2x$$
$$v(x) = \sin x$$
$$v'(x) = \cos x$$
$$I = [x^2 \sin x]_a^b - \int_a^b 2x \sin x \mathrm{d}x$$
$$I = [x^2 \sin x]_a^b - ([2x(-\cos x)]_a^b - \int_a^b 2(-\cos x) \mathrm{d}x)$$
$$I = [x^2 \sin x]_a^b - ([2x(-\cos x)]_a^b - [2(-\sin x)]_a^b)$$
$$I = [x^2 \sin x]_a^b - [2x(-\cos x) - 2(-\sin x)]_a^b$$
$$I = [x^2 \sin x - 2x(-\cos x) - 2(-\sin x)]_a^b$$
$$I = [x^2 \sin x + 2x \cos x + 2 \sin x]_a^b$$

17.10.2014

Aufgabe 252/6 a)

(partielle Integration mehrfach anwenden)

$$a = -\frac{\pi}{2}$$
$$b = \frac{\pi}{2}$$
$$I = \int_a^b (\sin x)^2 \mathrm{d}x$$
$$I = \int_a^b \sin x \sin x \mathrm{d}x$$
$$u(x) = \sin x$$
$$u'(x) = \cos x$$
$$v(x) = -\cos x$$
$$v'(x) = \sin x$$
$$I = [\sin x \cdot (-\cos x)]_a^b - \int_a^b \cos x \cdot (-\cos x) \mathrm{d}x$$
$$I = [\cos x \cdot (-\sin x)]_a^b - \int_a^b -(\sin x) \cdot (-\sin x) \mathrm{d}x$$
$$I = [\cos x \cdot (-\sin x)]_a^b - [\cos x \cdot (-\sin x)]_a^b - [\cos x \cdot \cos x]_a^b$$
$$I = [\cos x \cdot (-\sin x) - \cos x \cdot (-\sin x) - \cos x \cdot \cos x]_a^b$$
$$I = [-\cos x \cdot \sin x + \cos x \cdot \sin x - \cos x \cdot \cos x]_a^b$$
$$I = [-(\cos x)^2 - \cos x \cdot \sin x + \cos x \cdot \sin x]_a^b$$

Aufgabe 252/7

(Stammfunktion ermitteln)

a)

$$f(x) = x \cdot (x-1)^4$$
$$u(x) = x$$
$$u'(x) = 1$$
$$v(x) = \frac{1}{5}(x-1)^5$$
$$v'(x) = (x-1)^4$$
$$F(x) = x \cdot \frac{1}{5}(x-1)^5 - \frac{1}{30}(x-1)^6$$

$$F(x) = \frac{1}{5}x(x-1)^5 - \frac{1}{30}(x-1)^6 \text{ (Musterlösung)}$$

b)

$$f(x) = (2x+5) \cdot (x+2)^8$$
$$u(x) = (2x+5)$$
$$u'(x) = 1$$
$$v(x) = \frac{1}{9}(x+2)^9$$
$$v'(x) = (x+2)^8$$
$$F(x) = (2x+5)\frac{1}{9}(x+2)^9 - \frac{1}{90}(x+2)^{10}$$

$$F(x) = (2x+5) \cdot \frac{1}{9}(x+2)^9 - \frac{2}{90}(x+2)^{10} \ \text{(Musterlösung)}$$

c)

$$f(x) = \cos x \cdot (x-1)$$
$$u(x) = \cos x$$
$$u'(x) = -\sin x$$
$$v(x) = \frac{1}{2}(x-1)^2$$
$$v'(x) = (x-1)$$
$$F(x) = \cos x \cdot \frac{1}{2}(x-1)^2 - \int (-\sin x) \cdot \frac{1}{2}(x-1)^2 \mathrm{d}x$$

Nebenrechnung

$$f(x) = (-\sin x) \cdot \frac{1}{2}(x-1)^2$$
$$u(x) = -\sin x$$
$$u'(x) = -\cos x$$
$$v(x) = \frac{1}{6}(x-1)^3$$
$$v'(x) = \frac{1}{2}(x-1)^2$$
$$F(x) = (-\sin x)\frac{1}{6}(x-1)^3 - \int (-\cos x)\frac{1}{6}(x-1)^3 \mathrm{d}x$$

$$F(x) = \cos x + (x-1) \cdot \sin x \ \text{(Musterlösung)}$$

Integration durch Substitution

Wiederholung

Ketternregel

Sei $F(x)$ die Stammfunktion von $f(x)$ und $g(x)$ differenzierbar

$$
\begin{gathered}
H(x) = F(g(x)) \\
H'(x) = f(g(x)) \cdot g'(x) \\
\int_a^b f(g(x)) \cdot g'(x) \mathrm{d}x \\
= [F(g(x))]_{x=a}^{x=b} \mid g(x) = z \\
= [F(z)]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(z) \mathrm{d}z
\end{gathered}
$$

$$
\begin{gathered}
c = g(a) \\
d = g(b) \\
\int_{g(c)}^{g(d)} f(g(x)) \cdot g'(x) \mathrm{d}x
\end{gathered}
$$

Beispiel 1

$$
\begin{gathered}
\int_0^2 \frac{4x}{\sqrt{1+2x^2}} \mathrm{d}x \\
= \int_0^2 f(g(x)) \cdot g'(x) \mathrm{d}x \\
= [F(z)]_{g(0)}^{g(2)} = 2 \cdot \sqrt{1 + 2 \cdot 2^2} - 2\sqrt{1} \\
= 6 - 2 = 4
\end{gathered}
$$

Nebenrechnung

$$
\begin{gathered}
g(x) = 1 + 2x^2 \\
g'(x) = 4x \\
f(g(x)) = f(z) = \frac{1}{\sqrt{z}} = z^{-\frac{1}{2}} \\
F(z) = 2z^{\frac{1}{2}}
\end{gathered}
$$

Beispiel 2

$$h(x) = \frac{x}{2(1+x^2)^2}$$

$$\int_a^b = \frac{x}{2(1+x^2)^2}\mathrm{d}x$$

$$= \int_a^b f(g(x)) \cdot g'(x)\mathrm{d}x$$

$$= \frac{1}{4}\int_a^b f(g(x)) \cdot g'(x)\mathrm{d}x$$

$$= \frac{1}{4}[F(z)]_{g(a)}^{g(b)}$$

$$= \frac{1}{4}[-\frac{1}{2(1+x^2)^2}]_a^b$$

$$H(x) = -\frac{1}{8 \cdot (1+x^2)^2}$$

Nebenrechnung

$$g(x) = 1 + x^2$$

$$g'(x) = 2x$$

$$f(g(x)) = f(z) = \frac{1}{z^3}$$

$$g'(x) \cdot f(z) = \frac{2x}{z^3}$$

Hausaufgabe 254/3a

$$f(x) = (3x-4)^4$$

$$z = g(x) = 3x - 4$$

$$g'(x) = 3$$

$$f(z) = z^4$$

$$F(z) = \frac{1}{5}z^5$$

$$\int_a^b f(x)\mathrm{d}x$$

$$= \int_a^b (3x-4)^4\mathrm{d}x$$

$$= \int_a^b f(g(x)\mathrm{d}x$$

$$= \frac{1}{3} \cdot 3\int_a^b f(g(x)\mathrm{d}x$$

$$= \frac{1}{3}\int_a^b f(g(x) \cdot 3\mathrm{d}x$$

$$= \frac{1}{3}\int_a^b f(g(x)) \cdot g'(x)\mathrm{d}x$$
$$= \frac{1}{3}[F(z)]_{g(a)}^{g(b)}$$
$$= \frac{1}{3}[\frac{1}{5}z^5]_{g(a)}^{g(b)}$$
$$= \frac{1}{3}[\frac{1}{5}(3x-4)^5]_a^b$$
$$= [\frac{1}{15}(3x-4)^5]_a^b$$
$$F(x) = \frac{1}{15}(3x-4)^5$$

Beispiel 3

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{1-(\sin t)^2}} \cos t\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{6}} \frac{1}{\cos t} \cdot \cos t\mathrm{d}t$$
$$= \int_0^{\frac{\pi}{6}} 1\mathrm{d}t = [t]_0^{\frac{\pi}{6}}$$
$$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

Nebenrechnung

$$x = g(t) = \sin t$$
$$g'(t) = \cos t$$
$$g(t_1) = \sin t_1 = 0 \rightarrow t_1 = 0$$
$$g(t_2) = \sin t_2 = \frac{1}{2} \rightarrow t_2 = \frac{\pi}{6}$$

Nebenrechnnung 2

$$(\sin x)^2 + (\cos x)^2 = 1$$
$$\cos x = \sqrt{1-(\sin x)^2}$$

30.09.2014

Wiederholung

Seite 254/3a

$$f(x) = (3x-4)^4$$
$$g(x) = 3x - 4$$
$$g'(x) = 3$$
$$f(g(x)) \cdot g'(x)$$
$$\int_{g(a)}^{g(b)} f(z)\mathrm{d}x = \int_a^b f(g(x))g'(x)\mathrm{d}x$$
$$f(x) = \frac{1}{3}\int_{g(a)}^{g(b)} (3x-4)^4 \cdot 3\mathrm{d}x$$
$$F(z) = \frac{1}{3}[\frac{1}{5}(3x-4)^5]_a^b$$
$$= [\frac{1}{15}(3x-4)^5]_a^b$$

Seite 254/3b

$$f(x) = \frac{(2-5x)^3}{3}$$

gesucht: Stammfunktion

$$\int_a^b f(x)\mathrm{d}x = \int_a^b h(g(x))\mathrm{d}x$$

Form der Substitution noch nicht erreicht... ($g'(x)$ fehlt)

$$= -5 \cdot \frac{1}{-5}\int_a^b h(g(x))\mathrm{d}x$$
$$= \frac{1}{-5}\int_a^b h(g(x)) \cdot (-5)\mathrm{d}x$$
$$= \frac{1}{-5}\int_a^b h(g(x)) \cdot g'(x)\mathrm{d}x$$
$$= \frac{1}{-5}\int_{g(a)}^{g(b)} h(z)\mathrm{d}z$$
$$= \frac{1}{-5}\int_{g(a)}^{g(b)} \frac{z^3}{3}\mathrm{d}z$$
$$= \frac{1}{-5}[\frac{z^4}{12}]_{g(a)}^{g(b)}$$
$$= \frac{1}{-5}[\frac{(2-5x)^4}{12}]_a^b$$
$$= [-\frac{(2-5x)^4}{60}]_a^b$$
$$F(x) = -\frac{1}{60} \cdot (2-5x)^4$$

Nebenrechnung

$$g(x) = 2 - 5x$$
$$g'(x) = -5$$
$$h(x) = \frac{x}{3}$$
$$f(x) = h(g(x))$$

Seite 254/3c

Ergebnis:

$$F(x) = \frac{2}{5(5x+1)} = \frac{2}{25x+5}$$

Seite 255/10

Bestimmen Sie eine Stammfunktion des Integranden und berechnen Sie damit das Integral. Überprüfen Sie ihr Ergebnis mit dem GTR.

Vorraussetzung

$$h(z) = \frac{1}{z} = z^{-1}$$
$$H(z) = \ln z + c$$

a)

$$\int_0^5 \frac{2x}{1+x^2} \mathrm{d}x$$

$$\int_0^5 f(g(t))g'(t)\mathrm{d}t$$
$$= \frac{1}{2x} \int_{g(0)}^{g(5)} f(z)\mathrm{d}z$$
$$= \frac{1}{2x} \int_{g(0)}^{g(5)} \frac{2x}{z}\mathrm{d}z$$
$$= \frac{1}{2x} \int_{g(0)}^{g(5)} 2x \cdot \frac{1}{z}\mathrm{d}z$$
$$= \frac{1}{2x}[2x \cdot ln(z)]_0^5$$
$$= [\frac{1}{2x} \cdot 2x \cdot ln(z)]_0^5$$
$$= [ln(z)]_0^5$$

Lösung

$$F(x) = \ln x^2 + 1$$

$$\int_0^5 f(x)\mathrm{d}x = \ln 26 \approx 3,26$$

Nebenrechnung

$$g(t) = 1 + x^2$$

$$g'(t) = 2x$$

$$f(t) = \frac{2x}{t}$$

b)

$$\int_1^3 \frac{x^2}{1+x^3}\mathrm{d}x$$

c)

$$\int_1^4 \frac{x-3}{x^2-6x}\mathrm{d}x$$

d)

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x}\mathrm{d}x$$