# Numerical Linear Algebra Homework Project 3: Eigenvalues and Eigenvectors

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## Problem 2

Here we consider approximations to the eigenvalues and eigenfunctions of the one-dimensional Laplace operator $L[u] := -\frac{d^2u}{dx^2}$ on the unit interval $[0,1]$ with boundary conditions $u(0) = u(1) = 0$. A scalar $\lambda$ is said to be an eigenvalue of $L$ (with homogeneous Dirichlet boundary conditions) if there exists a twice-differentiable function $u : [0,1] \to \mathbb{R}$, not identically zero in $[0,1]$, such that

$$-u''(x) = \lambda u(x) \text{ on } [0,1] \text{ with } u(0) = u(1) = 0. \tag{1}$$

In this case $u$ is said to be an *eigenfunction* of $L$ corresponding to the eigenvalue $\lambda$. Obviously, eigenfunctions are defined up to a nonzero scalar multiple. The eigenvalues and eigenfunctions of $L$ are easily found to be $\lambda_j = j^2\pi^2$ and $u_j(x) = \alpha \sin(j\pi x)$ for any nonzero constant $\alpha$, which we can take to be 1. Here $j$ is a positive integer; hence, the operator $L$ has an infinite set of (mutually orthogonal) eigenfunctions $\{u_j\}_{j=1}^{\infty}$ corresponding to the discrete spectrum of eigenvalues $\lambda_{j}{}_{j=1}^{\infty}$. Note that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j \to \infty$ as $j \to \infty$. Also, each eigenvalue is simple in the sense that (up to a scalar multiple) there is a unique eigenfunction corresponding to it. Approximations to the eigenvalues and eigenfunctions can be obtained by discretizing the interval $[0,1]$ bymeansof $N+2$ evenly spaced points: $x_i = ih$ where $i = 0, 1, ..., N+1$ and $h = 1/(N+1)$. The second derivative operator can then be approximated by centered finite differences:

$$-\frac{d^2u}{dx^2}(x_i) \approx \frac{-u(x_{i-1} + 2u(x_i) - 2u(x_{i+1}))}{h^2} \tag{2}$$

and therefore the continuous (differential) eigenproblem (1) can be approximated by the discrete (algebraic) eigenvalue problem

$$h^{-2}T_N\mathbf{u} = \lambda\mathbf{u} \tag{3}$$

where we have set

$$T_N = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}, \text{ and } \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \tag{4}$$

with $u_i := u(x_i)$. It can be shown that the $N \times N$ matrix $T_N$ has eigenvalues $\mu_j = 2(1 - \cos\frac{\pi j}{N+1})$ for $j = 1, \ldots, N$, corresponding to the eigenvectors $\mathbf{u}_j$, where $\mathbf{u}_j(k) = \sqrt{\frac{2}{N+1}}\sin(\frac{jk\pi}{N+1})$ is the $kth$ entry in $\mathbf{u}_j$. Notice that the eigenvectors $\mathbf{u}_j$ are normalized with respect to the 2-norm: $\mathbf{u}_j^T\mathbf{u}_j = 1$. Also notice that the eigenvalues of $T_N$ lie in the interval $(0,4)$. Hence, the eigenvalues of $h^{-2}T_N$ lie in the interval $(0, 4(N+1)2)$.

**(1)** Since we are considering $j \ll N$ and $N \gg 1$ we can identify the Taylor expansion of $\cos x$ with $x = \frac{\pi j}{N+1}$:

$$\cos \frac{\pi j}{N+1} = \cos \pi j h = 1 - \frac{1}{2}\pi^2 j^2 h^2 + O(h^4), \tag{5}$$

that leads us to approximate the smallest eigenvalues of $2h^{-2}T_N$ as follows:

$$2h^{-2}(1 - \cos \frac{\pi j}{N+1}) = 2h^{-2}(1 - 1 + \frac{1}{2}\pi^2 j^2 h^2 + O(h^4)) = \pi^2 j^2 + O(h^2) \simeq \pi^2 j^2, \tag{6}$$

where we used that $h = 1/N+1$.
For the largest eigenvalue of $T_N$, we have that $j = N$, therefore we can not truncate anymore the taylor expansion of the cosine if we want a good approximation. We can compute the $N - th$ eigenvalue of $T_N$ in the limit of $N \gg 1$:

$$\mu_N = 2(1 - \cos \pi \frac{N}{N+1}) = 2(1 - \cos(\pi - \pi h))) = 2(1 + cos\pi h) = 4 - \pi^2 h^2 + O(h^4) \tag{7}$$

Therefore, we have

$$h^{-2}\mu_N = 4(N+1)^2 - \pi^2 + O(h^2) \tag{8}$$

that is not a good approximation of $\lambda_N = \pi^2 N^2$
**(2)** We want to compare the eigenvectors $\mathbf{u}_j$ of $T_N$ with the eigenfunctions of $L$, up to the normalization constant, that we will set to 1 for both. If we recall that $x_k = kh \; \forall k = 1, \dots, N$ we can observe that the $k - th$ component of the eigenvector $\mathbf{u}_j$ is equal to the $j - th$ eigenfunction $u_j(x)$ computed in corrispondence of the value $x = x_k$:

$$u_j(x_k) = \sin(j\pi x_k) = \sin(j\pi kh) = \sin\left(\frac{j\pi k}{N+1}\right) = \mathbf{u}_j(k). \tag{9}$$

**(3)** Now we compute the spectral condition number of $T_N$ in the limit of $N \gg 1$. We recall that the eigenvalues of $T_N$ are

$$\mu_j = 2\left(1 - \cos \frac{\pi j}{N+1}\right) = 2\left(1 - \cos \pi j h\right) \tag{10}$$

$$\begin{aligned} h^{-2}\mu_1 &= h^{-2}2\left(1 - 1 + \frac{1}{2}\pi^2 h^2 - \frac{1}{4!}\pi^4 h^4 + O(h^6)\right) \\ &= \pi^2 - \frac{1}{12}\pi^4 h^2 + O(h^4) \end{aligned} \tag{11}$$

$$\begin{aligned} h^{-}2\mu_N &= h^{-2}2(1 + cos(\pi h)) \\ &= 2h^{-2}\left(1 + 1 - \frac{1}{2}\pi^2 h^2 + \frac{1}{4!}\pi^4 h^4 + O(h^6)\right) \\ &= 4h^{-2} - \pi^2 + \frac{1}{12}\pi^4 h^2 + O(h^4) \end{aligned} \tag{12}$$

$$\begin{aligned} k_2(T_N) = \frac{h^{-2}\mu_N}{h^{-2}\mu_1} &= \frac{4h^{-2} - \pi^2 + \frac{1}{12}\pi^4 h^2 + O(h^4)}{\pi^2(1 - \frac{1}{12}\pi^2 h^2 + O(h^4))} \\ &= \frac{4h^{-2}}{\pi^2} - 1 + \frac{4h^{-2}}{\pi^2}\frac{\pi^2 h^2}{12} + O(h^2) \\ &= \frac{4}{\pi^2}(N+1)^2 - \frac{2}{3} + O(N^{-2}) \end{aligned} \tag{13}$$