Numerical Linear Algebra Homework Project 5: Constrained Optimization

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1 Test function (a)

We want to minimize the function

$$f_a(\mathbf{x}) = (x_1 - 4)^2 + x_2^2 \tag{1}$$

and the optimization is subject to the constraints

$$x_1 + x_2 \leq 2, \quad x_1 \geq 0, \quad x_2 \geq 0 \tag{2}$$

that identify the set $\mathcal{C}_a = \{\mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 \leq 2,\ x_1 \geq 0,\ x_2 \geq 0\}$. In other words, we want to find

$$f_a(\mathbf{x}^*) = \min_{\mathbf{x} \in \mathcal{C}_a} f_a(\mathbf{x}) \tag{3}$$

Since the function $f_a(\mathbf{x})$, reported in Eq. (1), is convex and continuous and the optimization set $\mathcal{C}_a$ is convex, there exists a unique minimum. In particular, it can be seen that, since there are no mixed terms in x_1x_2, the minimum can be found minimizing $(x_1 - 4)^2$ and x_2^2 separately. x_2^2 is minimized when $x_2 = 0$, that is compatible with the constraint set, and $(x_1 - 4)^2$ is minimized when $x_1 = 4$, that is incompatible with the constraint set. Since it has to be $x_1 + x_2 \leq 2$, the optimal choice of x_1 is $x_1 = 2$. Therefore we have:

$$\mathbf{x}^* = (2, 0), \qquad f_a(\mathbf{x}^*) = 4. \tag{4}$$

The Lagrangian of the problem is:

$$\mathcal{L}_a(\mathbf{x}, \boldsymbol{\lambda}) = f_a(\mathbf{x}) - \boldsymbol{\lambda}^T \mathbf{c}_a(\mathbf{x}), \quad \text{with } \mathbf{c}_a(\mathbf{x}) = \begin{bmatrix} 2 - x_1 - x_2 \\ x_1 \\ x_2 \end{bmatrix} \tag{5}$$

where the constraints of the problem are satisfied when $c_i(\mathbf{x}) \geq 0 \ \forall i = 1, 2, 3$. From this definition, it is possible to write the KKT conditions, namely: