Report

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Problem 2

Consider the $n \times n$ Wilkinson matrix

$$W_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ -1 & 1 & 0 & \cdots & 1 \\ -1 & -1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & 1 \end{bmatrix} \tag{0.1}$$

(1) Compute the LU factorization of W_5.

We are interested to compute two $n \times n$ matrices, L_5 and U_5, that are, respectively, a unit lower triangular matrix and an upper triangular matrix that satisfy the identity $W_5 = L_5 U_5$. We start writing the expression of W_5:

$$W_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}, \tag{0.2}$$

now we compute $\mathbf{m}_1$:

$$\mathbf{m}_1 = \begin{bmatrix} 0 \\ W_{21}/W_{11} \\ W_{31}/W_{11} \\ W_{41}/W_{11} \\ W_{51}/W_{11} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}. \tag{0.3}$$

Defining $\mathbf{e}_i$ as the vectors with 1 in the $i-th$ element and 0 otherwise, we can compute:

$$M_1 = \mathcal{I}_5 - \mathbf{m}_1 \mathbf{e}_1^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{0.4}$$

Using the expression of M_1, we can compute $W_5^{(1)}$:

$$W_5^{(1)} = M_1 W_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & -1 & -1 & 1 & 2 \\ 0 & -1 & -1 & -1 & 2 \end{bmatrix}. \tag{0.5}$$

The second iteration proceeds in a similar way:

$$\mathbf{m}_2 = \begin{bmatrix} 0 \\ 0 \\ W_{32}^{(1)}/W_{22}^{(1)} \\ W_{42}^{(1)}/W_{22}^{(1)} \\ W_{52}^{(1)}/W_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \tag{0.6}$$

$$M_2 = \mathcal{I}_5 - \mathbf{m}_2 \mathbf{e}_2^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}. \tag{0.7}$$

Using the expression of M_2, we can compute $W_5^{(2)}$:

$$W_5^{(2)} = M_2 W_5^{(1)} = M_2 M_1 W_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -1 & 4 \end{bmatrix}. \tag{0.8}$$

Now we start the third iteration:

$$
\mathbf{m}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ W_{43}^{(2)}/W_{33}^{(2)} \\ W_{53}^{(2)}/W_{33}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \tag{0.9}
$$

$$
M_3 = \mathcal{I}_5 - \mathbf{m}_3 \mathbf{e}_3^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}. \tag{0.10}
$$

Using the expression of M_3, we can compute $W_5^{(3)}$:

$$
W_5^{(3)} = M_3 W_5^{(2)} = M_3 M_2 M_1 W_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & -1 & 8 \end{bmatrix}. \tag{0.11}
$$

Similarly, we can perform the last iteration:

$$
\mathbf{m}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ W_{54}^{(2)}/W_{44}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \tag{0.12}
$$

$$
M_4 = \mathcal{I}_5 - \mathbf{m}_4 \mathbf{e}_4^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \tag{0.13}
$$

Using the expression of M_4, we can compute $W_5^{(4)}$:

$$
W_5^{(4)} = M_4 W_5^{(3)} = M_4 M_3 M_2 M_1 W_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 16 \end{bmatrix} = U_5. \tag{0.14}
$$

We can get L_5 from $L_5 = M_1^{-1} M_2^{-1} M_3^{-1} M_4^{-1}$ and knowing that $M_i^{-1} = \mathcal{I}_5 + \mathbf{m}_i \mathbf{e}_i^T$:

$$
L_5 = M_1^{-1} M_2^{-1} M_3^{-1} M_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}. \tag{0.15}
$$

(2) It is possible to guess the LU factorization of $W_n = L_n U_n$:

$$
L_n = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ -1 & \cdots & \cdots & -1 & 1 \end{bmatrix}, \quad U_n = \begin{bmatrix} 1 & 0 & \cdots & 0 & 2^0 \\ 0 & 1 & \ddots & \vdots & 2^1 \\ \vdots & \ddots & \ddots & 0 & 2^2 \\ \vdots & & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & 2^{n-1} \end{bmatrix} \tag{0.16}
$$

(3) Here is the function that generates the $n \times n$ Wilkinson matrix:
*** INSERIRE CODICE (funzione wilkin(n)) ***
(4-5) Here is the code that performs the numerical experiment for each $n = 2, \ldots, 60$:
*** INSERIRE CODICE (funzione check_when_lufact_W_fails) ***

The largest value of n for which $W_n \mathbf{x} = \mathbf{b}$ can be solved accurately is 54, that means that for $n = 55$ the program returns an inaccurate value for the solution $\mathbf{x}$. In this case, instead of computing the value $\mathbf{x} = \mathbf{e} = [1, \ldots, 1]$, it computes $\tilde{\mathbf{x}} = [1, \ldots, 1, 0, 1]$. In other words we have:

$$\mathbf{x}_{54} = 0 \neq \mathbf{e}_{54} = 1. \tag{0.17}$$

In order to understand the motivation uder this behavior, we verified that the matrices L_{55} and U_{55} were computed accurately. In the following is reported the code that verifies this computation:
*** INSERIRE CODICE (funzioni expected_LU_wilkin, compute_error_lufact_W) ***

Once verified that the matrices L_{55} and U_{55} are correct, we know that the problems is in the calculation of the forward and backward substitution. We recall that having done the LU factorization of the matrix W_{55} allows us to solve the system $W_{55}\mathbf{x} = \mathbf{b}_{55}$, where $\mathbf{b}_{55} = W_{55}\mathbf{e}$, by solving (in order) the linear systems with forward and backward substitutions:

$$\begin{cases} L\mathbf{y} = \mathbf{b} \\ U\mathbf{x} = \mathbf{y} \end{cases}, \tag{0.18}$$

where we named $L_{55} = L$, $U_{55} = U$ and $\mathbf{b}_{55} = \mathbf{b}$ for clarity. The solution to the first system is $\mathbf{y} = [2^0 + 1, 2^1 + 1, \ldots, 2^{n-2} + 1, 2^{n-1}]$. After that, we compute the solution of the second system starting from the bottom:

$$\mathbf{x}_{55} = \mathbf{y}_{55}/U_{55,55} = \frac{2^{54}}{2^{54}} = 1, \tag{0.19}$$

so far so goood. Now we update the vector $\mathbf{y}$ as follows:

$$\mathbf{y}^{(1)} = \mathbf{y} - \mathbf{x}_{55}\mathbf{u}_{55}, \tag{0.20}$$

where $\mathbf{u}_{55}$ is the $55-th$ and last column of U. Here we have, considering the $54-th$ component of $\mathbf{y}^{(1)}$:

$$[\mathbf{y}^{(1)}]_{54} = [\mathbf{y}]_{54} - \mathbf{x}_{55}U_{54,55} = 2^{53} + 1 - 2^{53}. \tag{0.21}$$

Problem 3

Suppose that $A \in \mathbb{R}^{n \times n}$ is a nonsingular matrix and the LU factorization of A exists and has been computed. Consider two given vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we can define the matrix $\tilde{A} = A + \mathbf{u}\mathbf{v}^T$
(1a) Prove that $\tilde{A}$ is nonsingular if and only if $\mathbf{v}^T A^{-1}\mathbf{u} \neq 1$.
Proof: We start proving that $\det(\tilde{A}) \neq 0$ implies that $\mathbf{v}^T A^{-1}\mathbf{u} \neq 1$. We can choose an orthonormal basis $\mathcal{B} = \{\mathbf{e}_i\}_{i=1,\ldots,n}$ of $\mathbb{R}^n$ such that $\mathbf{u} = \alpha_1\mathbf{e}_1$ and $\mathbf{v} = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2$ with $\alpha_1, \beta_1, \beta_2 \in \mathbb{R}$. We can represent the matrix A with respect to the basis $\mathcal{B}$, denoting with $a_{ij} = \mathbf{e}_i^T A \mathbf{e}_j$ the element of the $i-th$ row and $j-th$ column of the matrix A written in the basis $\mathcal{B}$. We can represent the matrix $\mathbf{u}\mathbf{v}^T$ with respect to the basis $\mathcal{B}$, obtaining $\mathbf{u}\mathbf{v}^T = \alpha_1\beta_1\mathbf{e}_1\mathbf{e}_1^T + \alpha_1\beta_2\mathbf{e}_1\mathbf{e}_2^T$. Knowing that we can compute the determinant of a matrix M $n \times n$ using the formula:

$$\det(M) = \sum_{j=1}^{n} m_{ij}C_{ij}(M), \tag{0.22}$$

where C_{ij} is the cofactor of the element (i, j) of the matrix M, the determinant of $\tilde{A}$ is:

$$\det(\tilde{A}) = \det(A) + \alpha_1\beta_1 C_{11}(A) + \alpha_1\beta_2 C_{12}(A). \tag{0.23}$$

Since we know that $\det(\tilde{A}) \neq 0$, we can write:

$$\det(A) + \alpha_1\beta_1 C_{11}(A) + \alpha_1\beta_2 C_{12}(A) \neq 0, \tag{0.24}$$

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and, therefore:

$$\alpha_1\beta_1\frac{C_{11}(A)}{\det(A)} + \alpha_1\beta_2\frac{C_{12}(A)}{\det(A)} \neq -1, \tag{0.25}$$

where we divided for $\det(A)$ both sides of the equation, knowing that $\det(A) \neq 0$. At this point, it is straightforward to verify that

$$\mathbf{v}^T A^{-1}\mathbf{u} = \alpha_1\beta_1\frac{C_{11}(A)}{\det(A)} + \alpha_1\beta_2\frac{C_{12}(A)}{\det(A)} \tag{0.26}$$

writing $\mathbf{u}$ and $\mathbf{v}$ in terms of $\mathbf{e}_1$ and $\mathbf{e}_2$ and $A^{-1} = \frac{1}{\det(A)}(cof(A))^T$, where $cof(A)$ is the matrix of cofactors of A. This proves that:

$$\mathbf{v}^T A^{-1}\mathbf{u} \neq -1. \tag{0.27}$$

In order to prove the converse implication, we consider again the expression for the determinant of $\tilde{A}$:

$$\det(\tilde{A}) = \det(A) + \alpha_1\beta_1 C_{11}(A) + \alpha_1\beta_2 C_{12}(A) = \det(A)\left(1 + \alpha_1\beta_1\frac{C_{11}(A)}{\det(A)} + \alpha_1\beta_2\frac{C_{12}(A)}{\det(A)}\right). \tag{0.28}$$

Here we can recognize the expression of $\mathbf{v}^T A^{-1}\mathbf{u}$, obtaining:

$$\det(\tilde{A}) = \det(A)(1 + \mathbf{v}^T A^{-1}\mathbf{u}). \tag{0.29}$$

Now, knowing that $\det(A) \neq 0$ and $\mathbf{v}^T A^{-1}\mathbf{u} \neq -1$, we obtain

$$\det(\tilde{A}) \neq 0 \tag{0.30}$$

and this concludes the proof.

(1b) Show that:

$$\tilde{A}^{-1} = A^{-1} - \alpha A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}, \quad \text{where } \alpha = \frac{1}{\mathbf{v}^T A^{-1}\mathbf{u} + 1}. \tag{0.31}$$

Proof: We start noticing that the last expression is well defined since $\tilde{A}$ invertible implies $\mathbf{v}^T A^{-1}\mathbf{u} + 1 \neq 0$. Now we can manipulate the (0.31) multiplying both sides to the left for A and to the right for $\tilde{A}$, obtaining:

$$\begin{aligned}
A &= \tilde{A} - \alpha\mathbf{u}\mathbf{v}^T A^{-1}\tilde{A} \\
&= A + \mathbf{u}\mathbf{v}^T - \alpha\mathbf{u}\mathbf{v}^T A^{-1}(A + \mathbf{u}\mathbf{v}^T) \\
&= A + (1-\alpha)\mathbf{u}\mathbf{v}^T - \alpha\mathbf{u}\mathbf{v}^T A^{-1}\mathbf{u}\mathbf{v}^T.
\end{aligned} \tag{0.32}$$

Subtracting A from each side and dividing both sides for α (that is nonzero $\forall \mathbf{v}^T A^{-1}\mathbf{u} \in \mathbb{R}$) we obtain:

$$\mathbf{u}\mathbf{v}^T A^{-1}\mathbf{u}\mathbf{v}^T = \mathbf{u}\mathbf{v}^T(\alpha^{-1} - 1). \tag{0.33}$$

Finally, since $\mathbf{v}^T A^{-1}\mathbf{u} = \alpha^{-1} - 1$, the identity (0.33) is verified and this concludes the proof.

(1c) Assuming that LU factorization of A is already available, describe an $\mathcal{O}(n^2)$ algorithm to solve $\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$ for any right-hand side $\tilde{\mathbf{b}}$.
Supposing that $\tilde{A}$ is invertible, we can write the solution $\tilde{\mathbf{x}}$ using the Sherman-Morrison formula for $\tilde{A}$:

$$\tilde{\mathbf{x}} = \tilde{A}^{-1}\tilde{\mathbf{b}} = A^{-1}\tilde{\mathbf{b}} - \frac{A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}\tilde{\mathbf{b}}}{\mathbf{v}^T A^{-1}\mathbf{u} + 1} \tag{0.34}$$

Algorithm:

- Compute $\mathbf{x}$ s.t. $A\mathbf{x} = \tilde{\mathbf{b}}$ and $\mathbf{y}$ s.t. $A\mathbf{y} = \mathbf{u}$ using backward and forward substitutions. This requires $\mathcal{O}(n^2)$ operations.

- Compute $\gamma = \frac{\mathbf{v}^T\mathbf{x}}{\mathbf{v}^T\mathbf{y}+1}$. This requires $\mathcal{O}(n)$ operations.

- Compute $\tilde{\mathbf{x}} = \mathbf{x} - \gamma\mathbf{y}$. This requires $\mathcal{O}(n)$ operations.

(2) Assuming again that the LU factorization of A existes and has been computed, describe an efficient algorithm for solving the *bordered system*

$$\begin{bmatrix} A & \mathbf{u} \\ \mathbf{v}^T & \beta \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ c \end{bmatrix}, \tag{0.35}$$

where z is unknown and β and c are given scalars. When does this system have a unique solution?
Solution:
Putting

$$A' = \begin{bmatrix} A & \mathbf{u} \\ \mathbf{v}^T & \beta \end{bmatrix}, \mathbf{x}' = \begin{bmatrix} \mathbf{x} \\ z \end{bmatrix}, \mathbf{b}' = \begin{bmatrix} \mathbf{b} \\ c \end{bmatrix}, \tag{0.36}$$

the system above rewrites as:

$$A'\mathbf{x}' = \mathbf{b}'. \tag{0.37}$$

The LU factorization of the matrix $A' = L'U'$ exists and the matrices L' and U' take the following form:

$$L' = \begin{bmatrix} L & \mathbf{0} \\ \mathbf{f}^T & 1 \end{bmatrix}, U' = \begin{bmatrix} U & \mathbf{g} \\ \mathbf{0} & \gamma \end{bmatrix}. \tag{0.38}$$

In order to get the values of $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, we impose $A' = L'U'$, obtaining the following system:

$$\begin{cases} L\mathbf{g} = \mathbf{u} \\ U^T\mathbf{f} = \mathbf{v} \\ \mathbf{f}^T\mathbf{g} + \gamma = \beta \end{cases}. \tag{0.39}$$

Here we can find $\mathbf{f}$ and $\mathbf{g}$ with forward substitutions with $\mathcal{O}(n^2)$ operations. Therefore, we can rewrite the last equation as:

$$\gamma = \beta - \mathbf{v}^T A^{-1}\mathbf{u}. \tag{0.40}$$

In order to impose that the bordered system has a unique solution, we have to require that $\det(A') \neq 0$, that is true if and only if all the diagonal elements of U' are nonzero and, given that $\det(A) \neq 0$, this means requiring that $\gamma \neq 0$. Therefore, the condition for the uniquenes of the solution becomes:

$$\gamma = \beta - \mathbf{v}^T A^{-1}\mathbf{u} \neq 0 \Rightarrow \quad \mathbf{v}^T A^{-1}\mathbf{u} \neq \beta. \tag{0.41}$$