Report

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Problem 3

Suppose that $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix and the LU factorization of A exists and has been computed. Consider two given vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we can define the matrix $\tilde{A} = A + \mathbf{u}\mathbf{v}^T$
(1a) Prove that $\tilde{A}$ is nonsingular if and only if $\mathbf{v}^T A^{-1}\mathbf{u} \neq 1$.
Proof: We start proving that $\det(\tilde{A}) \neq 0$ implies that $\mathbf{v}^T A^{-1}\mathbf{u} \neq 1$. We can choose an orthonormal basis $\mathcal{B} = \{\mathbf{e}_i\}_{i=1,\dots,n}$ of $\mathbb{R}^n$ such that $\mathbf{u} = \alpha_1\mathbf{e}_1$ and $\mathbf{v} = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2$ with $\alpha_1, \beta_1, \beta_2 \in \mathbb{R}$. We can represent the matrix A with respect to the basis $\mathcal{B}$, denoting with $a_{ij} = \mathbf{e}_i^T A\mathbf{e}_j$ the element of the $i-th$ row and $j-th$ column of the matrix A written in the basis $\mathcal{B}$. We can represent the matrix $\mathbf{u}\mathbf{v}^T$ with respect to the basis $\mathcal{B}$, obtaining $\mathbf{u}\mathbf{v}^T = \alpha_1\beta_1\mathbf{e}_1\mathbf{e}_1^T + \alpha_1\beta_2\mathbf{e}_1\mathbf{e}_2^T$. Knowing that we can compute the determinant of a matrix M $n \times n$ using the formula:

$$\det(M) = \sum_{j=1}^{n} m_{ij} C_{ij}(M), \tag{0.1}$$

where C_{ij} is the cofactor of the element (i, j) of the matrix M, the determinant of $\tilde{A}$ is:

$$\det(\tilde{A}) = \det(A) + \alpha_1\beta_1 C_{11}(A) + \alpha_1\beta_2 C_{12}(A). \tag{0.2}$$

Since we know that $\det(\tilde{A}) \neq 0$, we can write:

$$\det(A) + \alpha_1\beta_1 C_{11}(A) + \alpha_1\beta_2 C_{12}(A) \neq 0, \tag{0.3}$$

and, therefore:

$$\alpha_1\beta_1 \frac{C_{11}(A)}{\det(A)} + \alpha_1\beta_2 \frac{C_{12}(A)}{\det(A)} \neq -1, \tag{0.4}$$

where we divided for $\det(A)$ both sides of the equation, knowing that $\det(A) \neq 0$. At this point, it is straightforward to verify that

$$\mathbf{v}^T A^{-1}\mathbf{u} = \alpha_1\beta_1 \frac{C_{11}(A)}{\det(A)} + \alpha_1\beta_2 \frac{C_{12}(A)}{\det(A)} \tag{0.5}$$

writing $\mathbf{u}$ and $\mathbf{v}$ in terms of $\mathbf{e}_1$ and $\mathbf{e}_2$ and $A^{-1} = \frac{1}{\det(A)}(cof(A))^T$, where $cof(A)$ is the matrix of cofactors of A. This proves that:

$$\mathbf{v}^T A^{-1}\mathbf{u} \neq -1. \tag{0.6}$$

In order to prove the converse implication, we consider again the expression for the determinant of $\tilde{A}$:

$$\det(\tilde{A}) = \det(A) + \alpha_1\beta_1 C_{11}(A) + \alpha_1\beta_2 C_{12}(A) = \det(A)\left(1 + \alpha_1\beta_1 \frac{C_{11}(A)}{\det(A)} + \alpha_1\beta_2 \frac{C_{12}(A)}{\det(A)}\right). \tag{0.7}$$

Here we can recognize the expression of $\mathbf{v}^T A^{-1}\mathbf{u}$, obtaining:

$$\det(\tilde{A}) = \det(A)(1 + \mathbf{v}^T A^{-1}\mathbf{u}). \tag{0.8}$$

Now, knowing that $\det(A) \neq 0$ and $\mathbf{v}^T A^{-1}\mathbf{u} \neq -1$, we obtain

$$\det(\tilde{A}) \neq 0 \tag{0.9}$$

and this concludes the proof.

(1b) Show that:

$$\tilde{A}^{-1} = A^{-1} - \alpha A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}, \quad \text{where } \alpha = \frac{1}{\mathbf{v}^T A^{-1}\mathbf{u} + 1}. \tag{0.10}$$

Proof: We start noticing that the last expression is well defined since $\tilde{A}$ invertible implies $\mathbf{v}^T A^{-1}\mathbf{u} + 1 \neq 0$. Now we can manipulate the (0.10) multiplying both sides to the left for A and to the right for $\tilde{A}$, obtaining:

$$\begin{aligned} A &= \tilde{A} - \alpha\mathbf{u}\mathbf{v}^T A^{-1}\tilde{A} \\ &= A + \mathbf{u}\mathbf{v}^T - \alpha\mathbf{u}\mathbf{v}^T A^{-1}(A + \mathbf{u}\mathbf{v}^T) \\ &= A + (1-\alpha)\mathbf{u}\mathbf{v}^T - \alpha\mathbf{u}\mathbf{v}^T A^{-1}\mathbf{u}\mathbf{v}^T. \end{aligned} \tag{0.11}$$

Subtracting A from each side and dividing both sides for α (that is nonzero $\forall \mathbf{v}^T A^{-1}\mathbf{u} \in \mathbb{R}$) we obtain:

$$\mathbf{u}\mathbf{v}^T A^{-1}\mathbf{u}\mathbf{v}^T = \mathbf{u}\mathbf{v}^T(\alpha^{-1} - 1). \tag{0.12}$$

Finally, since $\mathbf{v}^T A^{-1}\mathbf{u} = \alpha^{-1} - 1$, the identity (0.12) is verified and this concludes the proof.

(1c) Assuming that LU factorization of A is already available, describe an $\mathcal{O}(n^2)$ algorithm to solve $\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$ for any right-hand side $\tilde{\mathbf{b}}$.
Supposing that $\tilde{A}$ is invertible, we can write the solution $\tilde{\mathbf{x}}$ using the Sherman-Morrison formula for $\tilde{A}$:

$$\tilde{\mathbf{x}} = \tilde{A}^{-1}\tilde{\mathbf{b}} = A^{-1}\tilde{\mathbf{b}} - \frac{A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}\tilde{\mathbf{b}}}{\mathbf{v}^T A^{-1}\mathbf{u} + 1} \tag{0.13}$$

Algorithm:

- Compute $\mathbf{x}$ s.t. $A\mathbf{x} = \tilde{\mathbf{b}}$ and $\mathbf{y}$ s.t. $A\mathbf{y} = \mathbf{u}$ using backward and forward substitutions. This requires $\mathcal{O}(n^2)$ operations.
- Compute $\gamma = \frac{\mathbf{v}^T\mathbf{x}}{\mathbf{v}^T\mathbf{y}+1}$. This requires $\mathcal{O}(n)$ operations.
- Compute $\tilde{\mathbf{x}} = \mathbf{x} - \gamma\mathbf{y}$. This requires $\mathcal{O}(n)$ operations.

(2) Assuming again that the LU factorization of A existes and has been computed, describe an efficient algorithm for solving the *bordered system*

$$\begin{bmatrix} A & \mathbf{u} \\ \mathbf{v}^T & \beta \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ c \end{bmatrix}, \tag{0.14}$$

where z is unknown and β and c are given scalars. When does this system have a unique solution?
Solution:
Putting

$$A' = \begin{bmatrix} A & \mathbf{u} \\ \mathbf{v}^T & \beta \end{bmatrix}, \mathbf{x}' = \begin{bmatrix} \mathbf{x} \\ z \end{bmatrix}, \mathbf{b}' = \begin{bmatrix} \mathbf{b} \\ c \end{bmatrix}, \tag{0.15}$$

the system above rewrites as:

$$A'\mathbf{x}' = \mathbf{b}'. \tag{0.16}$$

The LU factorization of the matrix $A' = L'U'$ exists and the matrices L' and U' take the following form:

$$L' = \begin{bmatrix} L & \mathbf{0} \\ \mathbf{f}^T & 1 \end{bmatrix}, U' = \begin{bmatrix} U & \mathbf{g} \\ \mathbf{0} & \gamma \end{bmatrix}. \tag{0.17}$$

In order to get the values of $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, we impose $A' = L'U'$, obtaining the following system:

$$\begin{cases} L\mathbf{g} = \mathbf{u} \\ U^T\mathbf{f} = \mathbf{v} \\ \mathbf{f}^T\mathbf{g} + \gamma = \beta \end{cases}. \tag{0.18}$$

Here we can find $\mathbf{f}$ and $\mathbf{g}$ with forward substitutions with $\mathcal{O}(n^2)$ operations. Therefore, we can rewrite the last equation as:

$$\gamma = \beta - \mathbf{v}^T A^{-1}\mathbf{u}. \tag{0.19}$$

In order to impose that the bordered system has a unique solution, we have to require that $\det(A') \neq 0$, that is true if and only if all the diagonal elements of U' are nonzero and, given that $\det(A) \neq 0$, this means requiring that $\gamma \neq 0$. Therefore, the condition for ne uniquenes of the solution becomes:

$$\gamma = \beta - \mathbf{v}^T A^{-1}\mathbf{u} \neq 0 \Rightarrow \quad \mathbf{v}^T A^{-1}\mathbf{u} \neq \beta. \tag{0.20}$$