Numerical Linear Algebra Homework Project 5: Constrained Optimization

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1 Function (a)

We want to minimize the function

$$f_a(\mathbf{x}) = (x_1 - 4)^2 + x_2^2 \tag{1}$$

subject to the constraints

$$x_1 + x_2 \leq 2, \quad x_1 \geq 0, \quad x_2 \geq 0, \tag{2}$$

which identify the set $\mathcal{C}_a = \{\mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 \leq 2,\ x_1 \geq 0,\ x_2 \geq 0\}$. In other words, we want to find

$$f_a(\mathbf{x}^*) = \min_{\mathbf{x} \in \mathcal{C}_a} f_a(\mathbf{x}). \tag{3}$$

Since the function $f_a(\mathbf{x})$, reported in Eq. (1), is convex and continuous and the constraint set $\mathcal{C}_a$ is convex, there exists a unique minimum. In particular, it can be seen that, since there are no mixed terms in x_1x_2, the minimum can be found minimizing $(x_1-4)^2$ and x_2^2 separately. x_2^2 is minimized when $x_2 = 0$, which is compatible with the constraint set, and $(x_1 - 4)^2$ is minimized when $x_1 = 4$, which is incompatible with the constraint set. Since it has to be $x_1 + x_2 \leq 2$, the optimal choice of x_1 is $x_1 = 2$. Therefore we have:

$$\mathbf{x}^* = (2, 0), \qquad f_a(\mathbf{x}^*) = 4. \tag{4}$$

Now we would like to find the minimum value both analytically and numerically.

The Lagrangian of the problem is:

$$\mathcal{L}_a(\mathbf{x}, \boldsymbol{\lambda}) = f_a(\mathbf{x}) - \boldsymbol{\lambda}^T \mathbf{c}_a(\mathbf{x}), \quad \text{with } \mathbf{c}_a(\mathbf{x}) = \begin{bmatrix} 2 - x_1 - x_2 \\ x_1 \\ x_2 \end{bmatrix}, \tag{5}$$

where the constraints of the problem are satisfied when $c_i(\mathbf{x}) \geq 0\ \forall i = 1, 2, 3$. From this definition, it is possible to write the KKT conditions, namely:

$$\begin{cases} \text{stationarity} & (2(x_1 - 4) + \lambda_1 - \lambda_2, 2x_2 + \lambda_1 - \lambda_3) = (0, 0) \\ \text{primal feasibility} & (2 - x_1 - x_2, x_1, x_2) \geq (0, 0, 0) \\ \text{dual feasibility} & (\lambda_1, \lambda_2, \lambda_3) \geq (0, 0, 0) \\ \text{complementarity} & (\lambda_1(2 - x_1 - x_2), \lambda_2 x_1, \lambda_3 x_2) = (0, 0, 0) \end{cases} \tag{6}$$

where the inequalities must be intended element-wise.

The complementarity conditions can be satisfied in eight different cases, depending on the fact that each constraint can be active or not. We recall that a constraint $c_i(\mathbf{x})$ is active if and only if $c_i(\mathbf{x}^*) = 0$. Note that, in terms of Lagrange multipliers, this definition requires that $\lambda_i^* > 0$, while for a non active constraint one has $\lambda_i = 0$. In the following, we identify each case with an array of m entries, where m is the number of constraints, and we assign to the i-th entry the label A or N depending on the fact that the i-th constraint is active or not. Associated to each case we report the possible solutions of the constrained minimization problem which are compatible with the corresponding complementarity condition.

	A/N constraints	Possible solutions $\mathbf{x}^*$
1	(A, N, N)	$\mathbf{x}^* = (x_1^*, -x_1^* + 2)$, with $x_1^* \in (0, 2)$
2	(A, A, A)	$\nexists\ \mathbf{x}^* \in \mathbb{R}^2$
3	(N, N, N)	$\mathbf{x}^* \in \mathring{\mathcal{C}}$
4	(N, A, A)	$\mathbf{x}^* = (0, 0)$
5	(A, A, N)	$\mathbf{x}^* = (0, 2)$
6	(A, N, A)	$\mathbf{x}^* = (2, 0)$
7	(N, A, N)	$\mathbf{x}^* = (0, x_2^*)$, with $x_2^* \in (0, 2)$
8	(N, N, A)	$\mathbf{x}^* = (x_1^*, 0)$ with $x_1^* \in (0, 2)$

Table 1: Table of the cases satisfying the complementarity condition. The i-th row of the table contains the array of the active/non-active states of the constraints (second column) and the possible solutions $\mathbf{x}^*$ of the constrained minimization problem which are compatible with that case (third column).

We want to find the solutions to the KKT system (6) by investigating if the conditions on Lagrange multipliers imposed by the constraint states and the range of possible solutions $\mathbf{x}^*$ reported in Tab. 1, which came from the complementarity condition, are compatible with the other equations of the system. The results that we find are reported below.

1. From the case (A, N, N) we have $\lambda_1^* > 0,\ \lambda_2^* = \lambda_3^* = 0$. This leads to $\lambda_1^* = 2$ and $\mathbf{x}^* = (3, -1)$, which is not compatible with the constraint set.

2. From the case (A, A, A) we have $\lambda_1^* > 0,\ \lambda_2^* > 0,\ \lambda_3^* > 0$. As can be seen from the second row of the Table 1, $\nexists\ \mathbf{x}^* \in \mathbb{R}^2$ which represents a solution to the constrained minimization problem.

3. From the case (N, N, N) we have $\lambda_1^* = \lambda_2^* = \lambda_3^* = 0$. In this case one finds $\mathbf{x}^* = (4, 0)$, which is outside the feasible set.

4. From the case (N, A, A) we have $\lambda_1^* = 0,\ \lambda_2^* > 0,\ \lambda_3^* > 0$. From the complementarity condition we already have and $\mathbf{x}^* = (0, 0)$. From the stationarity condition, we find $\lambda_2^* = -8$, which is not allowed, and $\lambda_3^* = 0$, which is not compatible with the state of the third constraint.

5. From the case (A, A, N) we have $\lambda_1^* > 0,\ \lambda_2^* > 0,\ \lambda_3^* = 0$. From the complementarity condition we already have and $\mathbf{x}^* = (0, 2)$. By inserting this value in the first equation of the KKT system we find $\lambda_1^* = -4$, $\lambda_2^* = -12$, which are both not allowed values.

6. From the case (A, N, A) we have $\lambda_1^* > 0,\ \lambda_2^* = 0,\ \lambda_3^* > 0$. In this case, one has $\mathbf{x}^* = (2, 0)$ from the complementarity condition and $\lambda_1^* = 4$, $\lambda_3^* = 4$, which are both acceptable solutions. Note that this is the solution that we have already found heuristically.

7. From the case (N, A, N) we have $\lambda_1^* = 0,\ \lambda_2^* > 0,\ \lambda_3^* = 0$. In this case, we find $\lambda_2 = -8$, which is not an acceptable solution. Note that one can also find $x_2^* = 0$, which is not compatible with the state of the constraint.

8. From the case (N, N, A) we have $\lambda_1^* = 0,\ \lambda_2^* = 0,\ \lambda_3^* > 0$. In this case, we find $\lambda_3 = 0$ and $\mathbf{x}^* = (4, 0)$, which is outside the feasible set.

As emerges from the study of the KKT system, the unique solution to the constrained minimization problem is $\mathbf{x}^* = (2, 0)$.

1.1 Function (b)