Numerical Linear Algebra Homework Project 4: Unconstrained Optimization

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In this project we want to perform unconstrained optimization using the Newton method both in its standard form and by considering its variants which make use of the backtracking and the trust region approach. We will apply these methods to four test functions and we will compare these approaches to see how convergence changes.

1 Algorithms

bjdkf bjkfd b

2 Test Functions

dfbfd

(a)

The first function that we want to consider is the following:

$$f(x_1, x_2) = (x_1 - 2)^4 + (x_1 - 2)^2 x_2^2 + (x_2 + 1)^2. \tag{1}$$

This function assumes a minimum value, equal to 0, in the point $(x_1^*, x_2^*) = (2, -1)$. We would like to obtain this minimum value by using the standard Newton algorithm implemented by the function ****Newton**** in the library ***Project_4.py ****. We report below the gradient and the Hessian of f, which must be passed to the function ****Newton****.

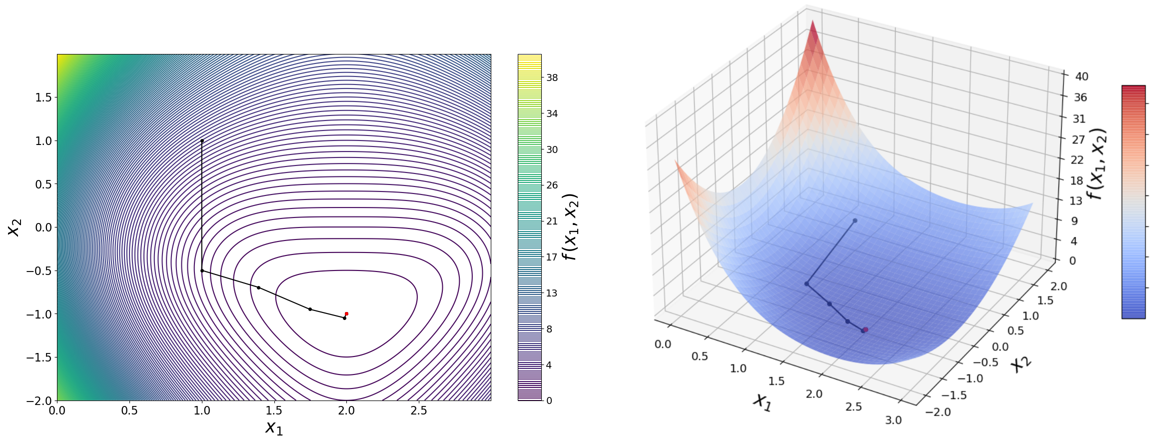
$$\nabla f(x_1, x_2) = \begin{bmatrix} 4(x_1 - 2)^3 + 2x_2^2(x_1 - 2) \\ 2x_2(x_1 - 2)^2 + 2x_2(x_2 - 2) \end{bmatrix}, \tag{2}$$

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 12(x_1 - 2)^2 + 2x_2^2 & 4x_2(x_1 - 2) \\ 4x_2(x_1 - 2) & 2(x_1 - 2)^2 + 2 \end{bmatrix}. \tag{3}$$

k	$\|\mathbf{x}_k - \mathbf{x}^*\|_2$	$f_1(\mathbf{x}_k) - f_1(\mathbf{x}^*)$	$-\nabla f_1(\mathbf{x}_k)^T[\nabla^2 f_1(\mathbf{x}_k)]^{-1}\nabla f_1(\mathbf{x}_k)$
0	2.236	6.000	-9.000
1	1.118	1.500	-1.761
2	6.805×10^{-1}	4.092×10^{-1}	-5.552×10^{-1}
3	2.592×10^{-1}	6.489×10^{-2}	-1.237×10^{-1}
4	5.012×10^{-2}	2.531×10^{-3}	-5.026×10^{-3}
5	1.277×10^{-3}	1.632×10^{-6}	-3.262×10^{-6}
6	1.659×10^{-6}	2.754×10^{-12}	-5.508×10^{-12}
7	1.404×10^{-12}	1.971×10^{-24}	-3.943×10^{-24}

Table 1: Table

k	$\|\mathbf{x}_k - \mathbf{x}^*\|_2$	$f_1(\mathbf{x}_k) - f_1(\mathbf{x}^*)$	$-\nabla f_1(\mathbf{x}_k)^T[\nabla^2 f_1(\mathbf{x}_k)]^{-1}\nabla f_1(\mathbf{x}_k)$
0	5.745	5.223×10^{2}	-1.045×10^{3}
1	9.769×10^{-13}	2.842×10^{-14}	-9.948×10^{-27}
2	1.688×10^{-13}	0.000	-2.005×10^{-26}



(a) (b)

Figure 1: Contour plot for the function $f^{(a)}(x_1, x_2)$ where the intermediate points are obtained by using the standard Newton method.

2.1 (b)

$f : \mathbb{R}^4 \to \mathbb{R},\ f(\mathbf{x}) = \mathbf{b}^T + \frac{1}{2}\mathbf{x}^T H \mathbf{x}$

$$\mathbf{b} = (5.04, -59.4, 146.4, -96.6)^T \tag{4}$$

$$H = \begin{bmatrix} 0.16 & -1.2 & 2.4 & -1.4 \\ -1.2 & 12.0 & -27.0 & 16.8 \\ 2.4 & -27.0 & 64.8 & -42.0 \\ -1.4 & 16.8 & -42.0 & 28.0 \end{bmatrix} \tag{5}$$

2.2 (c)

$f : \mathbb{R}^2 \to \mathbb{R},\ f(x_1, x_2) = (1.5 - x_1(1 - x_2))^2 + (2.25 - x_1(1 - x_2^2))^2 + (2.625 - x_1(1 - x_2^3))^2$

k	$\|\mathbf{x}_k - \mathbf{x}^*\|_2$	$f_1(\mathbf{x}_k) - f_1(\mathbf{x}^*)$	$-\nabla f_1(\mathbf{x}_k)^T[\nabla^2 f_1(\mathbf{x}_k)]^{-1}\nabla f_1(\mathbf{x}_k)$
0	5.009	8.17×10^{1}	-1.455×10^{2}
1	8.66×10^{-1}	2.423	-4.527
2	6.494×10^{-2}	2.407×10^{-2}	-4.643×10^{-2}
3	1.393×10^{-1}	3.45×10^{-3}	-6.32×10^{-3}
4	2.103×10^{-2}	1.383×10^{-4}	-2.704×10^{-4}
5	1.377×10^{-3}	2.863×10^{-7}	-5.717×10^{-7}
6	3.033×10^{-6}	2.186×10^{-12}	-4.372×10^{-12}
7	2.836×10^{-11}	1.233×10^{-22}	-2.466×10^{-22}
8	4.441×10^{-16}	4.437×10^{-31}	-8.79×10^{-31}

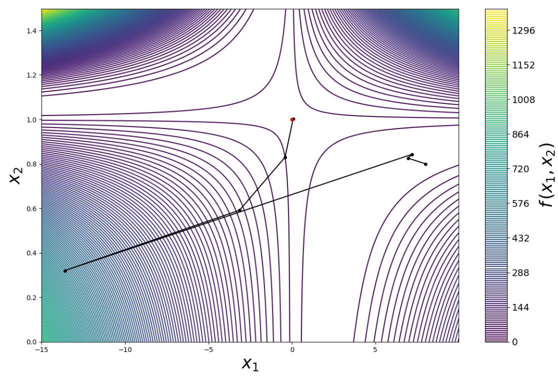
Table 2: $x_0 = (8, 0.2)^T$, backtracking = False

k	$\|\mathbf{x}_k - \mathbf{x}^*\|_2$	$f_1(\mathbf{x}_k) - f_1(\mathbf{x}^*)$	$-\nabla f_1(\mathbf{x}_k)^T[\nabla^2 f_1(\mathbf{x}_k)]^{-1}\nabla f_1(\mathbf{x}_k)$
0	5.009	2.043	-3.291
1	3.963	2.553×10^{-1}	-5.885×10^{-2}
2	4.218	2.328×10^{-1}	-6.421×10^{-1}
3	1.661×10^{1}	5.743×10^{2}	-9.291×10^{2}
4	6.137	5.226×10^{1}	-6.327×10^{1}
5	3.426	1.596×10^{1}	-3.709
6	2.974	1.421×10^{1}	-8.445×10^{-3}
7	3.041	1.42×10^{1}	-5.688×10^{-8}
8	3.041	1.42×10^{1}	-1.083×10^{-18}
9	3.041	1.42×10^{1}	0.000

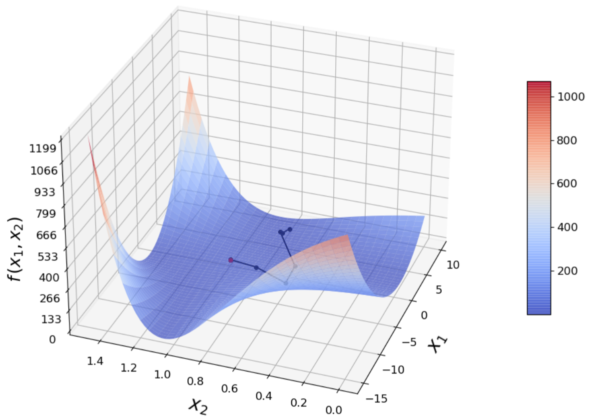
Table 3: $x_0 = (8, 0.2)^T$, backtracking = False *** PROBLEMA? ***

k	$\|\mathbf{x}_k - \mathbf{x}^*\|_2$	$f_1(\mathbf{x}_k) - f_1(\mathbf{x}^*)$	$-\nabla f_1(\mathbf{x}_k)^T[\nabla^2 f_1(\mathbf{x}_k)]^{-1}\nabla f_1(\mathbf{x}_k)$
0	5.009	2.043	-3.291
1	3.963	2.553×10^{-1}	-5.885×10^{-2}
2	4.218	2.328×10^{-1}	-6.421×10^{-1}
3	2.920	2.131×10^{-1}	-7.337×10^{-2}
4	2.471	1.649×10^{-1}	-1.224×10^{-1}
5	1.340	1.502×10^{-1}	-1.22×10^{-1}
6	1.192	7.989×10^{-2}	-1.697×10^{-1}
7	6.453×10^{-1}	5.012×10^{-2}	-5.027×10^{-2}
8	3.335×10^{-1}	1.73×10^{-2}	-2.418×10^{-2}
9	8.357×10^{-2}	3.009×10^{-3}	-5.269×10^{-3}
10	2.128×10^{-2}	1.004×10^{-4}	-1.967×10^{-4}
11	2.767×10^{-4}	1.503×10^{-7}	-3.005×10^{-7}
12	1.103×10^{-6}	2.293×10^{-13}	-4.585×10^{-13}
13	2.404×10^{-13}	8.166×10^{-25}	-1.633×10^{-24}
14	0.000	0.000	0.000

Table 4: $x_0 = (8, 0.2)^T$, backtracking = True



(a)



(b)

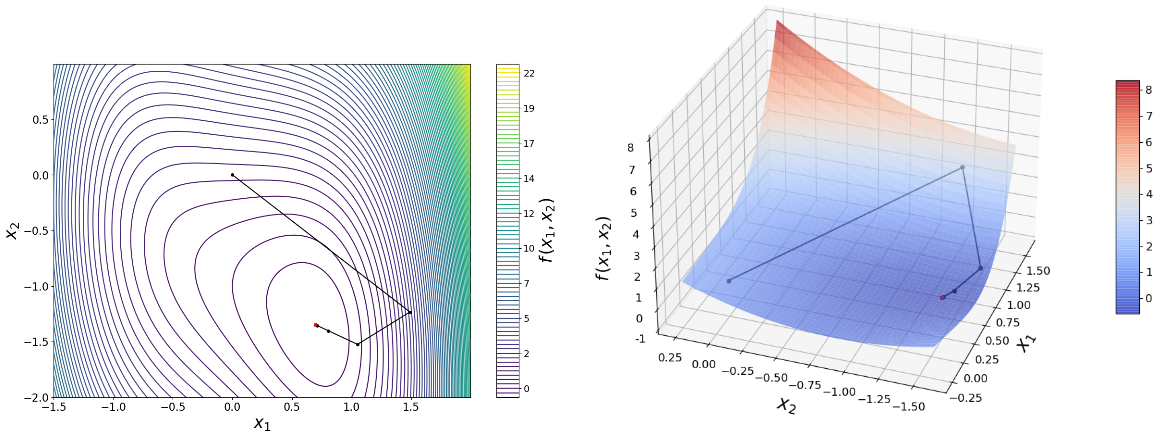
Figure 2: Contour plot for the function $f^{(c)}(x_1, x_2)$ where the intermediate points are obtained by using the standard Newton method.

2.3 (d)

$f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = x_1^4 + x_1 x_2 + (1 + x_2)^2$

k	$\|\mathbf{x}_k - \mathbf{x}^*\|_2$	$f_1(\mathbf{x}_k) - f_1(\mathbf{x}^*)$	$-\nabla f_1(\mathbf{x}_k)^T [\nabla^2 f_1(\mathbf{x}_k)]^{-1} \nabla f_1(\mathbf{x}_k)$
0	1.119×10^{-1}	2.385×10^{-2}	-4.688×10^{-2}
1	1.209×10^{-2}	3.79×10^{-4}	-7.539×10^{-4}
2	1.264×10^{-3}	4.365×10^{-6}	-8.725×10^{-6}
3	1.272×10^{-4}	4.4×10^{-8}	-8.889×10^{-8}
4	1.272×10^{-5}	1.649×10^{-12}	-8.906×10^{-10}
5	1.271×10^{-6}	-4.392×10^{-10}	-8.907×10^{-12}
6	1.256×10^{-7}	-4.436×10^{-10}	-8.907×10^{-14}

Table 5: $x_0 = (0.75, -1.25)^T$, backtracking = False



(a) (b)

Figure 3: Contour plot for the function $f^{(d)}(x_1, x_2)$ where the intermediate points are obtained by using the standard Newton method.