Lecture 4: Markov chains

Exercise Draw the digraphs and classify the states of the Markov chains defined by the following transition matrices:

$$(A) \quad \boldsymbol{\mathcal{P}} = \begin{pmatrix} 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad (B) \quad \boldsymbol{\mathcal{P}} = \begin{pmatrix} 0.3 & 0.4 & 0 & 0 & 0.3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{4}$$

Exercise Consider the Markov chain wth transition matrix

$$\boldsymbol{\mathcal{P}} = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{pmatrix} \tag{5}$$

(a) Show that the MC is irreducible and aperiodic

(b) Suppose to start the process in state 1; find the probability that it is in state 3 after two steps.

(c) Find the matrix which is the limit of $\mathcal{P}^n$ as $n \to \infty$.

Exercise Two boxes, A and B, contain 2 white and 3 red balls respectively. At each time one extracts one ball from each box, and puts it back in the opposite box (1 event). Let us call by a_1 the state corresponding to the number i of the read balls in box A. Determine the occurrence probability of state a_2 after 3 events and in the limit of many events.

Lecture 5: MC simulation of a 2D Ising model by the Metropolis algorithm

Program Write a program that simulates a 2D Ising model with periodic boundary conditions by using the Metropolis acceptance matric a_{ij} and the matrix Γ_{ij} based on the local spin-flip (Glauber) proposed move.

Simulation By assuming $k_B = 1$ and $J = 1$ simulate the 2D Ising model for different values of temperatures (at least 3, one below, one above, and one close to the critical temperature $T_c = \frac{2}{\ln(1+\sqrt{2})}$ and 3 different values of L (for instance 25,50 and 100).

Equilibration time, averages and fluctuations After having determined the equilibrium time and disregarding the samples for $t < \tau_{eq}$ estimate the ensemble averages of the magnetisation per spin, the energy per spin, and the corresponding fluctuations (specific heat and magnetic susceptibility).

Integrated correlation time and critical slowing-down Estimate the autocorrelation time of the magnetisation and the energy for the MC simulations proposed above and estimate the errors accordingly.