Electron velocità lemma

We want to prove that:

$$\langle \Psi | \hat{p}^2 | \Psi \rangle = \langle u | (\hat{\bar{p}} + \hbar \bar{k})^2 | u \rangle \qquad \text{Where } \Psi(\bar{r}, \bar{k}) = e^{i\bar{k}\cdot\bar{r}} u(\bar{k}, \bar{r})$$

Proof:

$$\langle \Psi | \hat{p}^2 | \Psi \rangle = \int d^3\bar{r}\; u^\dagger e^{-i\bar{k}\cdot\bar{r}} (\hat{p}^2) u e^{i\bar{k}\cdot\bar{r}} = \int d^3\bar{r}\; u^\dagger e^{-i\bar{k}\cdot\bar{r}} \left[-\hbar^2 \nabla_{\bar{r}}^2 \left(u e^{i\bar{k}\cdot\bar{r}} \right) \right]$$

$$* = \int d^3\bar{r}\; u^\dagger e^{-i\bar{k}\cdot\bar{r}} \left[-\hbar^2 \left(u \nabla_{\bar{r}}^2 e^{i\bar{k}\cdot\bar{r}} + e^{i\bar{k}\cdot\bar{r}} \nabla_{\bar{r}}^2 u + 2 \bar{\nabla}_r u \cdot \bar{\nabla}_r e^{i\bar{k}\cdot\bar{r}} \right) \right] =$$

$$= \int d^3\bar{r}\; u^\dagger e^{-i\bar{k}\cdot\bar{r}} \left[-\hbar^2 \left(-k^2 u\, e^{i\bar{k}\cdot\bar{r}} + e^{i\bar{k}\cdot\bar{r}} \nabla_{\bar{r}}^2 u - 2i\bar{k}\, e^{i\bar{k}\cdot\bar{r}} \bar{\nabla}_r u \right) \right] =$$

$$= \int d^3\bar{r}\; u^\dagger \left[-\hbar^2 \left(-k^2 u + \nabla_{\bar{r}}^2 u - 2i\bar{k} \bar{\nabla}_r u \right) \right] =$$

$$= \int d^3\bar{r}\; u^\dagger \left[\hbar^2 k^2 + \hat{p}^2 + 2\hbar \bar{k}\cdot\hat{\bar{p}} \right] u = \int d^3 r\; u^\dagger (\hat{\bar{p}} + \hbar\bar{k})^2 u = \langle u | (\hat{\bar{p}} + \hbar\bar{k})^2 | u \rangle$$

$\square$

$$* \; \nabla^2 (fg) = f \nabla^2 g + g \nabla^2 f + 2 \bar{\nabla} f \cdot \bar{\nabla} g$$