Gradient approximation on an hypercubic lattice

Suppose to have an hypercubic lattice with N spins, for each of which we then consider the anglo that it forms with the first axis.
We have then the collection:

$$\{\vartheta_1, \vartheta_2, \dots, \vartheta_N\}$$

Suppose now that we have N>>1, i.e a large collection of spins.
Then we can, instead of the discretization, consider a scalar field that for each position gives back the spin angle:

$$\{\vartheta_1, \vartheta_2, \dots, \vartheta_N\} \longrightarrow \vartheta(\bar{r}_i) \equiv \vartheta_i$$

Suppose now to be on a d-dimensional hypercubic lattice of lattice constant a.
Then clearly, for the nearest neighbors:

$$\bar{r}_J = \bar{r}_i + a\,\hat{e}_{ij} \qquad \hat{e}_{ij} = \{\pm\hat{e}_1, \pm\hat{e}_2, \dots, \hat{e}_d\}$$

If we now consider the following Taylor expansion:

$$\vartheta(\bar{r}_J) = \vartheta(\bar{r}_i + a\,\hat{e}_{ij}) =$$
$$= \vartheta(\bar{r}_i) + a\,\hat{e}_{ij}\cdot\bar{\nabla}\vartheta(\bar{r}_i) + \mathcal{O}(a^2)$$

Recall: If $\|u\| \ll 1$

$$f(\bar{R}+\bar{u}) = f(\bar{R}) + \bar{u}\cdot\bar{\nabla} f(\bar{R}) + \mathcal{O}(\|\bar{u}\|^2)$$

So we obtain in the end:

$$\vartheta_J = \vartheta_i + a\,\hat{e}_{ij}\cdot\bar{\nabla}\vartheta(\bar{r}_i) + \mathcal{O}(a^2) \longleftrightarrow \vartheta_i - \vartheta_J \simeq -a\,\hat{e}_{ij}\cdot\bar{\nabla}\vartheta(\bar{r}_i)$$

Let's now consider the sum:

$$\sum_{\langle i,J\rangle} \frac{1}{2}(\vartheta_i - \vartheta_J)^2 = \sum_i \sum_{J\in N(i)} \frac{1}{4}(\vartheta_i - \vartheta_J)^2$$

$N(i)$ set of the n.n. of element i.

Using the obtained approximation we have:

$$\sum_i \sum_{J\in N(i)} \frac{1}{4}(\vartheta_i - \vartheta_J)^2 \simeq \frac{a^2}{4}\sum_i \sum_{J\in N(i)} \left[\hat{e}_{ij}\cdot\bar{\nabla}\vartheta(\bar{r}_i)\right]^2$$

Now we divide the set unit vectors oriented along the cardinal axis and not, i.e.

$$N(i) = N_+(i) \cup N_-(i)$$

e.g. 2D cubic lattice

$$N(i) = \{\pm\hat{i}, \pm\hat{j}\}$$
$$N_+(i) = \{+\hat{i}, +\hat{j}\}$$
$$N_-(i) = \{-\hat{i}, -\hat{j}\}$$

In this way clearly:

$$\frac{a^2}{4}\sum_i \sum_{J\in N(i)} \left[\hat{e}_{ij}\cdot\bar{\nabla}\vartheta(\bar{r}_i)\right]^2 = \frac{a^2}{4}\sum_i \left[\left(\sum_{J\in N_-(i)} \left(-\hat{e}_j\cdot\bar{\nabla}\vartheta(\bar{r}_i)\right)^2\right) + \left(\sum_{J\in N_+(i)} \left(-\hat{e}_j\cdot\bar{\nabla}\vartheta(\bar{r}_i)\right)^2\right)\right] =$$

$$= \frac{a^2}{4}\sum_i \sum_{k=1}^{d} 2\left(\frac{\partial\vartheta}{\partial e_k}\right)^2 = \frac{a^2}{2}\sum_i |\bar{\nabla}\vartheta|^2$$

Now finally we want to make the sum an integral.
The idea is the same behind the series-integral exchange in Solid State Physics:

$$\sum_{p \in S} \longleftrightarrow \frac{1}{V_n^{\varphi}} \int d^d p_c \quad \text{For a generic space}$$

$$\sum_{i} \longleftrightarrow \frac{1}{a^d} \int d^d \bar{r} \quad \text{In our case}$$

In the end one has

$$\sum_{\langle i,j \rangle} \frac{1}{2} (\vartheta_i - \vartheta_j)^2 \sim \frac{a^{2-d}}{2} \int d^d r \, |\bar{\nabla} \vartheta|^2$$

Note that for $d=2$ $\frac{a^{2-d}}{2} = \frac{1}{2}$