N-order moment of a Gaussian pdf

Given a Gaussian p.d.f. we want to prove

$$\langle x^n \rangle = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (n-1)!!\,\sigma^2 & \text{if } n \text{ is even} \end{cases}$$

The first part is trivial, because we evaluate an odd function over a symmetric integral, so the integral is 0.
For the second part we start by the following result:

$$\int_{\mathbb{R}} e^{-\frac{a}{2}x^2}\,dx = \sqrt{\frac{2\pi}{a}}$$

And we differentiate both side respect to a

$$\int_{\mathbb{R}} -\frac{x^2}{2} e^{-\frac{a}{2}x^2}\,dx = -\frac{1}{2}\sqrt{2\pi}\,\frac{1}{a^{\frac{3}{2}}}$$

$$\int_{\mathbb{R}} x^2 e^{-\frac{a}{2}x^2}\,dx = \frac{\sqrt{2\pi}}{a^{\frac{3}{2}}}$$

We continue to differentiate.

$$\int_{\mathbb{R}} -\frac{x^4}{2} e^{-\frac{a}{2}x^2}\,dx = -\frac{3}{2}\sqrt{2\pi}\,\frac{1}{a^{\frac{5}{2}}}$$

$$\int_{\mathbb{R}} x^4 e^{-\frac{a}{2}x^2}\,dx = 3\cdot\frac{\sqrt{2\pi}}{a^{\frac{5}{2}}}$$

We continue:

$$\int_{\mathbb{R}} x^6 e^{-\frac{a}{2}x^2}\,dx = 5\cdot 3\,\frac{\sqrt{2\pi}}{a^{\frac{7}{2}}}$$

$$\int_{\mathbb{R}} x^8 e^{-\frac{a}{2}x^2}\,dx = 7\cdot 5\cdot 3\,\frac{\sqrt{2\pi}}{a^{\frac{7}{2}}}$$

Then we can prove by induction that:

$$\int_{\mathbb{R}} x^n e^{-\frac{a}{2}x^2}\,dx = \frac{(n-1)(n-3)\cdots 3\cdot 1\,\sqrt{2\pi}}{a^{\frac{n-1}{2}}} = (n-1)!!\sqrt{\frac{2\pi}{a^{n-1}}}$$

Now, in order to obtain $\langle x^n \rangle$ we consider $\alpha = \frac{1}{\sigma^2}$ and multiply both sides by $\frac{1}{\sqrt{2\pi\sigma^2}}$

We will obtain

$$\int_{\mathbb{R}} x^n e^{-\frac{\alpha}{2}x^2} dx = (n-1)!! \sqrt{\frac{2\pi}{\alpha^{n-1}}}$$

$$\int_{\mathbb{R}} x^n e^{-\frac{x^2}{2\sigma^2}} dx = (n-1)!! \sqrt{2\pi\sigma^{2n-2}}$$

$$\langle x^n \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} x^n e^{-\frac{x^2}{2\sigma^2}} dx = (n-1)!! \sqrt{2\pi\sigma^{2n-2}} \frac{1}{\sqrt{2\pi\sigma^2}} = (n-1)!! \sigma^n$$

$\square$