Evaluation of the multidimensional Gaussian integral

We want to evaluate

$$Z[A] = \int_{\mathbb{R}^n} d^n x \; e^{-\frac{1}{2}\bar{x}^t A \bar{x}}$$

A is a square matrix of order n, symmetric and with strictly positive eigenvalues

We make a brief comment on the conditions over A. Must be square in order to the matrix dot product to make sense. Simmetry is important in order to introduce a meaningful change of variable that permits the integral resolution.
Note that a symmetric matrix has real eigenvalues, but not necessarely positive. This condition must be considered in order to guarantee the integral convergence. I.e. If I have eigenvalue 0, I'm integrating 1 over the real numbers, which is divergent.

Let's now go to the integral resolution.
Every symmetric matrix can be diagonalized by the introduction of an invertible matrix P, which is also orthogonal:

$$D = P^{-1} A P$$

$$P P^{-1} = P^{-1} P = \mathbb{1}_n$$ invertibility

$$P^t = P^{-1}$$ orthogonality

Where D is a diagonal matrix that contains in its diagonal the eigenvalues of A. Note that this condition can also be stated as D is similar to A.
So we propose in the integral the following change of variables:

$$\bar{y} = P^{-1}\bar{x} \qquad \bar{x} = P\,\bar{y}$$

So that, upon evaluation of the Jacobian, the integral becomes:

$$Z[A] = \int_{\mathbb{R}^n} d^n y \; \det J \; e^{-\frac{1}{2}(P\bar{y})^t A (P\bar{y})} = \int_{\mathbb{R}^n} d^n y \; |\det J| \; e^{-\frac{1}{2}\bar{y}^t P^t A P \bar{y}} =$$

$$= \int_{\mathbb{R}^n} d^n y \; |\det J| \; e^{-\frac{1}{2}\bar{y}^t D \bar{y}} = \int_{\mathbb{R}^n} d^n y \; |\det J| \; e^{\sum_i -\frac{1}{2}\lambda_i y_i^2}$$

Note that we implicitly use the P matrix orthogonality .
Now we briefly prove that det(J) =1

$$\left|\det\left(\frac{\partial \bar{x}}{\partial \bar{y}}\right)\right| = \left|\det\left(P\,\mathbb{1}_n\right)\right| = 1$$

$$\det(P^t P) = \det(\mathbb{1}_n)$$

$$(\det P)^2 = 1$$

$$\det P = \pm 1$$

So we can finally solve our integral as a product of 1 dimensional integrals.

$$\mathcal{Z}[A] = \int_{\mathbb{R}^n} d^n y \; e^{\sum_i -\frac{1}{2}\lambda_i y_i^2} = \sqrt{\frac{(2\pi)^n}{\lambda_1 \lambda_2 \cdots \lambda_n}}$$

We now find a compact form for the denominator

$$D = P^{-1} A P$$

$$\det D = \det(P^{-1} A P)$$

$$\lambda_1 \cdots \lambda_n = (\det P)^2 \cdot \det A$$

$$\lambda_1 \cdots \lambda_n = \det A.$$

So we can finally write

$$\mathcal{Z}[A] = \int_{\mathbb{R}^n} d^n x \; e^{-\frac{1}{2}\bar{x}^t A \bar{x}} = \sqrt{\frac{(2\pi)^n}{\det A}}$$

$\square$