Linear monoatomic chain with mass defect - calculations

We start by the equations of motion

$$\begin{cases} m\ddot{u}_n = \tilde{k}\left[u_{n+1} + u_{n-1} - 2u_n\right] & n \neq 0 \\ M\ddot{u}_0 = \tilde{k}\left[u_1 + u_{-1} - 2u_0\right] & n = 0 \end{cases}$$

We want to show that the solution is:

$$u_n = A e^{-\kappa|n|a - i\omega t} \qquad \kappa \in \mathbb{C},\ \forall n.$$

We obtain by substitution

$$\begin{cases} -\omega^2 m A e^{-\kappa|n|a - i\omega t} = \tilde{k} A e^{i\omega t}\left[e^{-\kappa|n+1|a} + e^{-\kappa|n-1|a} - 2e^{-\kappa|n|a}\right] & n \neq 0 \\ -\omega^2 M A e^{-i\omega t} = \tilde{k} A e^{i\omega t}\left[e^{-\kappa a} + e^{-\kappa a} - 2\right] & n = 0 \end{cases}$$

$$\begin{cases} \omega^2 m e^{-\kappa|n|a} = \tilde{k}\left[2e^{-\kappa|n|a} - e^{-\kappa|n+1|a} - e^{-\kappa|n-1|a}\right] & n \neq 0 \quad \mathrm{I} \\ \omega^2 M = 2\tilde{k}\left[1 - e^{-\kappa a}\right] & n = 0 \end{cases}$$

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- $n \geq 1$

$$\omega^2 m e^{-\kappa n a} = \tilde{k}\left[2e^{-\kappa n a} - e^{-\kappa(n+1)a} - e^{-\kappa(n-1)a}\right]$$

$$\omega^2 m = \tilde{k}\left[2 - e^{-\kappa a} - e^{\kappa a}\right]$$

- $n \geq -1$

$$\omega^2 m e^{\kappa n a} = \tilde{k}\left[2e^{\kappa n a} - e^{\kappa(n+1)a} - e^{\kappa(n-1)a}\right]$$

$$\omega^2 m = \tilde{k}\left[2 - e^{\kappa a} - e^{-\kappa a}\right]$$

So, in the end

$$\begin{cases} \omega^2 m = \tilde{k}\left[2 - e^{\kappa a} - e^{-\kappa a}\right] & n \neq 0 \\ \omega^2 M = 2\tilde{k}\left[1 - e^{-\kappa a}\right] & n = 0 \end{cases}$$