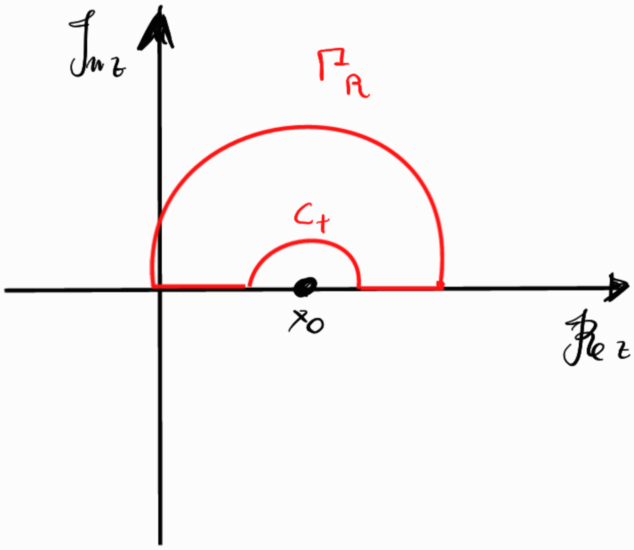
Proof of the Kramers-Kronig relations

We consider a generic complex function of real variable, and consider the following integral over the contour C in the picture aside, for its complex version: (note we are implicitly considering the hypothesis of Sokhotski-Plemelj formula in the upper analicity version)



$$\oint_C \frac{f(z)}{z-x_0}\,dz =$$

$$= \int_{[x_0-R,\,x_0-\delta]\cup C^+\cup[x_0+\delta,\,x_0+R]} \frac{f(z)}{z-x_0}\,dz + \int_{\Gamma_R} \frac{f(z)}{z-x_0}\,dz =$$

$= \underset{\text{(or Cauchy - Goursat)}}{\text{By the residue Theorem}} = 0$

Now the idea is to take the limit $R \to +\infty$.

We show now by Darboux lemma that the second integral in the LHS tends to 0 in the aforementioned limit.

$$\Gamma_R : \; z = x_0 + R e^{it} \quad t \in [0,\pi]$$

$$\left| \int_{\Gamma_R} \frac{f(z)}{z-x_0}\,dz \right| \leq \pi R \max_{z\in\Gamma_R} \left| \frac{f(z)}{z-x_0} \right| = \pi R \left| \frac{f(x_0+Re^{it})}{Re^{it}} \right| = \pi f(x_0+Re^{it})$$

We know that by that the hypothesis of the Sokhotski-Plemelj formula that:

$$f(x_0+Re^{it}) \to 0 \qquad \text{As } R \to +\infty \quad \text{While } t\in[0,\pi]$$

So we can imply that, in the limit $R \to +\infty$:

$$\int_{[x_0-\infty,\,x_0-\delta]\cup C^+\cup[x_0+\delta,\,x_0+\infty]} \frac{f(z)}{z-x_0}\,dz = 0$$

Now the LHS is the Sokhotskj-Plemely integral, so we can write directly:

$$\text{P.V.}\int_{\mathbb{R}} \frac{f(x)}{x-x_0}\,dx - i\pi f(x_0) = 0$$

Now we use the fact that the function is a complex function of real variable, i.e. it can be written as:

$$f(x) = u(x) + i\,v(x)$$

So we obtain upon substitution:

$$\text{P.V.} \int_{\mathbb{R}} \frac{u(x) + i v(x)}{x - x_0} \, dx - i\pi \left(u(x_0) + i v(x_0) \right) = 0$$

$$\frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{u(x)}{x - x_0} \, dx + i \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{v(x)}{x - x_0} \, dx = - v(x_0) + i \, u(x_0)$$

Finally we obtain by comparing the real parts between them and the imaginary parts between them:

$$\boxed{\begin{aligned} v(x_0) &= -\frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{u(x)}{x - x_0} \, dx \\ u(x_0) &= \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{u(x)}{x - x_0} \, dx \end{aligned}}$$

Kramers - Kronig

Upper plane version