Evaluation of the translated multidimensional Gaussian integral

$$\mathcal{I}[A,\bar{b}] = \int_{\mathbb{R}^n} d^n x \; e^{-\frac{1}{2}\bar{x}^t A x + \bar{x}^t \cdot \bar{b}}$$

We want to try the same strategy employed in the 1 dimensional case, so we begin by finding the stationary point of the exponent:

$$\bar{\nabla}_{\bar{x}}\left(-\frac{1}{2}\bar{x}^t A \bar{x} + \bar{x}^t \cdot \bar{b}\right) = 0$$

Considering a generic component of the gradient:

$$\partial_{x_k}\left(-\frac{1}{2}\sum_i \sum_j A_{ij} x_i x_j + \sum_i x_i b_i\right) =$$

$$= -\frac{1}{2}\sum_i \sum_j A_{ij}\left(x_i \delta_{jk} + x_j \delta_{ik}\right) + \sum_i \delta_{ik} b_i = -\frac{1}{2}\left[\sum_i A_{ik} x_i + \sum_j A_{kj} x_j\right] + b_k =$$

$$= \underset{A_{kj} = A_{jk}}{\text{Due to } A \text{ simmetry}} = -\frac{1}{2}\left[2\sum_i A_{ik} x_i\right] + b_k = -\sum_i A_{ik} x_k + b_k$$

So that one has:

$$\bar{\nabla}_{\bar{x}}\left(-\frac{1}{2}\bar{x}^t A \bar{x} + \bar{x}^t \cdot \bar{b}\right) = -A\bar{x} + \bar{b} = 0 \longrightarrow \bar{x} = A^{-1}\bar{b}$$

A is invertible, given the fact that $\det A \neq 0$

So, we propose the following change of variable:

$$\bar{y} = \bar{x} - A^{-1}\bar{b} \longrightarrow \bar{x} = \bar{y} + A^{-1}\bar{b}$$

And that will give us:

$$\mathcal{I}[A,\bar{b}] = \int_{\mathbb{R}^n} d^n x \; e^{-\frac{1}{2}\bar{x}^t A x + \bar{x}^t \cdot \bar{b}} = \int_{\mathbb{R}^n} d^n y \left|\frac{\partial \bar{x}}{\partial \bar{y}}\right| e^{-\frac{1}{2}(\bar{y}+A^{-1}\bar{b})^t A (\bar{y}+A^{-1}\bar{b}) + (\bar{y}+A^{-1}\bar{b})^t \cdot \bar{b}}$$

One can see that the determinant of the Jacobian is 1. This is also obvious is we consider that the cange we made is a translation.
Continuing we have:

$$\mathcal{I}[A,\bar{b}] = \int_{\mathbb{R}^n} d^n y \; e^{-\frac{1}{2}(\bar{y}+A^{-1}\bar{b})^t A (\bar{y}+A^{-1}\bar{b}) + (\bar{y}+A^{-1}\bar{b})^t \cdot \bar{b}} =$$

$$= \int_{\mathbb{R}^n} d^n y \; e^{-\frac{1}{2}\left[\bar{y}^t + \bar{b}^t {A^{-1}}^t\right] A\left(\bar{y} + A^{-1}\bar{b}\right) + \left[\bar{y}^t + \bar{b}^t {A^{-1}}^t\right] \cdot \bar{b}} =$$

We used
$[A+B]^t = A^t + B^t$
$[AB]^t = B^t A^t$

$$= \int_{\mathbb{R}^n} d^n y \; e^{-\frac{1}{2}\left[\bar{y}^t + \bar{b}^t {A^{-1}}^t\right]\left(A\bar{y} + \bar{b}\right) + \bar{y}^t \bar{b} + \bar{b}^t {A^{-1}}^t \bar{b}} =$$

$$= \int_{\mathbb{R}^n} d^n y \; e^{-\frac{1}{2}\left[\bar{y}^t A \bar{y} + \bar{y}^t \bar{b} + \bar{b}^t {A^{-1}}^t A \bar{y} + \bar{b}^t {A^{-1}}^t \bar{b}\right] + \bar{y}^t \bar{b} + \bar{b}^t {A^{-1}}^t \bar{b}} =$$

Before continuing we make the following observation: the inverse of a symmetric matrix is itself symmetric, so in our calculation we can simplify greatly by considering that $(A^{-1})^t = A^{-1}$
So we apply this semplification and obtain:

$$= \int_{\mathbb{R}^n} d^n y \; e^{-\frac{1}{2}\left[\bar{y}^t A \bar{y} + \bar{y}^t \bar{b} + \bar{b}^t A^{-1} A \bar{y} + \bar{b}^t A^{-1} \bar{b}\right] + \bar{y}^t \bar{b} + \bar{b}^t A^{-1} \bar{b}} =$$

$$= \int_{\mathbb{R}^n} d^n y \; e^{-\frac{1}{2}\left[\bar{y}^t A \bar{y} + \bar{y}^t \bar{b} + \bar{b}^t \bar{y} + \bar{b}^t A^{-1} \bar{b}\right] + \bar{y}^t \bar{b} + \bar{b}^t A^{-1} \bar{b}} =$$

Note that $\bar{b}^t \bar{y} = \bar{y}^t \bar{b}$ by the dot product commutativity

$$= \int_{\mathbb{R}^n} d^n y \; e^{-\frac{1}{2}\left[\bar{y}^t A \bar{y} + 2\bar{y}^t \bar{b} + \bar{b}^t A^{-1} \bar{b}\right] + \bar{y}^t \bar{b} + \bar{b}^t A^{-1} \bar{b}} =$$

$$= \int_{\mathbb{R}^n} d^n y \; e^{-\frac{1}{2}\bar{y}^t A \bar{y} + \frac{1}{2}\bar{b}^t A^{-1} \bar{b}} = e^{\frac{1}{2}\bar{b}^t A^{-1} \bar{b}} \int_{\mathbb{R}^n} d^n y \; e^{-\frac{1}{2}\bar{y}^t A \bar{y}} = e^{\frac{1}{2}\bar{b}^t A^{-1} \bar{b}} \mathcal{Z}[A, \bar{0}]$$

$\square$