Addendum to the calculation of the mean field alpha exponent

We want to show that in the neighborhood of the critical temperature, we can consider for the bulk free energy (free energy density)

$$f_b = f_0 + \frac{A}{2} m^2 + \frac{B}{4} m^4 \qquad \text{If } T \to T_c \qquad \begin{array}{l} A \sim a_0 (T - T_c) \\ B \sim b_0 \end{array}$$

This evaluation needs some care, so we detail it here.
First we recall for the beta exponent, i.e. for the magnetization at the critical temperature:

$$m_0 \sim (T - T_c)^{1/2}$$

Then if we consider the free energy density in the neighborhood of the critical temperature :

$$f_b = f_0 + \frac{A}{2} m_0^2 + \frac{B}{4} m_0^4 \sim f_0 + A\,(T - T_c) + B\,(T - T_c)^2$$

Now we continue by expanding the coefficients A and B:

$$A = \beta z J (1 - \beta z J) \qquad B = \frac{1}{3\beta} (\beta z J)^4$$

Now we consider

$$\beta z J = \frac{T}{T_c} \equiv x \qquad \text{If } T \to T_c \iff x \to 1$$

$$A = x(1 - x) = x - x^2 \qquad\qquad f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

$$B = \frac{1}{3\beta} x^4$$

Then

$$A = 0 - (x - 1) + \mathcal{O}(x - 1)^2$$

$$B = \frac{1}{\alpha} \left(1 + 4(x - 1) + \mathcal{O}(x - 1)^2 \right)$$

Notice that the second term in the B expansion is a subleasing term. In fact notice what happend if we put it in the free energy expression:

$$f_b \sim f_0 + a_0 (T - T_c)^2 + b_0 (T - T_c)^2 + b_1 (T - T_c)^4$$

Notice

$x - 1 \sim T - T_c$

$b_1 (T - T_c)^4 \ll b_0 (T - T_c)^2$

So in the end we have

$$f_b \sim f_0 + (a_0 + b_0)(T - T_c)^2$$