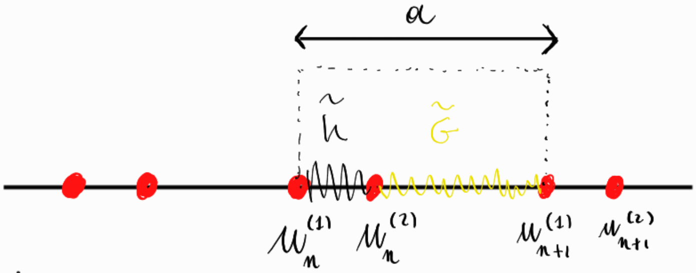
Biatomic linear chain - Force calculation

We begin by building the total potential energy in the framework of the harmonic approximation



$$U = U_0 + \frac{\tilde{K}}{2}\sum_{n=1}^{N}\left(u_n^{(2)} - u_n^{(1)}\right)^2 + \frac{\tilde{G}}{2}\sum_{n=1}^{N}\left(u_{n+1}^{(1)} - u_n^{(2)}\right)^2$$

Now we evaluate directly the forces by the expressions:

$$M\ddot{u}_m^{(1)} = -\frac{\partial U}{\partial u_m^{(1)}} \qquad \mathrm{I}$$

$$M\ddot{u}_m^{(2)} = -\frac{\partial U}{\partial u_m^{(2)}} \qquad \mathrm{II}$$

$$\mathrm{I} \quad M\ddot{u}_m^{(1)} = -\frac{\tilde{K}}{2}\sum_{n=1}^{N} 2\left(u_n^{(2)} - u_n^{(1)}\right)\left(-\delta_{m,n}\right) - \frac{\tilde{G}}{2}\sum_{n=1}^{N} 2\left(u_{n+1}^{(1)} - u_n^{(2)}\right)\delta_{n+1,m} \qquad \frac{\partial u_m^{(2)}}{\partial u_n^{(1)}} = 0 \ \forall n,m$$

$$M\ddot{u}_m^{(1)} = -\frac{\tilde{K}}{2}\sum_{n=1}^{N} 2\left(u_n^{(2)} - u_n^{(1)}\right)\left(-\delta_{m,n}\right) - \frac{\tilde{G}}{2}\sum_{n=1}^{N} 2\left(u_{n+1}^{(1)} - u_n^{(2)}\right)\delta_{n,m-1}$$

$$M\ddot{u}_m^{(1)} = \tilde{K}\left(u_m^{(2)} - u_m^{(1)}\right) - \tilde{G}\left(u_m^{(1)} - u_{m-1}^{(2)}\right)$$

$$M\ddot{u}_m^{(1)} = -\tilde{K}\left(u_m^{(1)} - u_m^{(2)}\right) - \tilde{G}\left(u_m^{(1)} - u_{m-1}^{(2)}\right)$$

$$\mathrm{II} \quad M\ddot{u}_m^{(2)} = -\frac{\tilde{K}}{2}\sum_{n=1}^{N} 2\left(u_n^{(2)} - u_n^{(1)}\right)\left(\delta_{m,n}\right) - \frac{\tilde{G}}{2}\sum_{n=1}^{N} 2\left(u_{n+1}^{(1)} - u_n^{(2)}\right)\left(-\delta_{n,m}\right) \qquad \frac{\partial u_m^{(1)}}{\partial u_n^{(2)}} = 0 \ \forall n,m$$

$$M\ddot{u}_m^{(2)} = -\tilde{K}\left(u_m^{(2)} - u_m^{(1)}\right) + \tilde{G}\left(u_{m+1}^{(1)} - u_m^{(2)}\right)$$

$$M\ddot{u}_m^{(2)} = -\tilde{K}\left(u_m^{(2)} - u_m^{(1)}\right) - \tilde{G}\left(u_m^{(2)} - u_{m+1}^{(1)}\right)$$

So that, in the end:

$$\begin{cases} M\ddot{u}_m^{(1)} = -\tilde{K}\left(u_m^{(1)} - u_m^{(2)}\right) - \tilde{G}\left(u_m^{(1)} - u_{m-1}^{(2)}\right) \\ M\ddot{u}_m^{(2)} = -\tilde{K}\left(u_m^{(2)} - u_m^{(1)}\right) - \tilde{G}\left(u_m^{(2)} - u_{m+1}^{(1)}\right) \end{cases}$$