The Cauchy distribution characteristic function

We want to evaluate the characteristic function of the following distribution:

$$P_X(x) = \frac{1}{\pi} \frac{1}{1+x^2} \qquad x \in \mathbb{R}$$

In order to this, we have to evaluate the following integral:

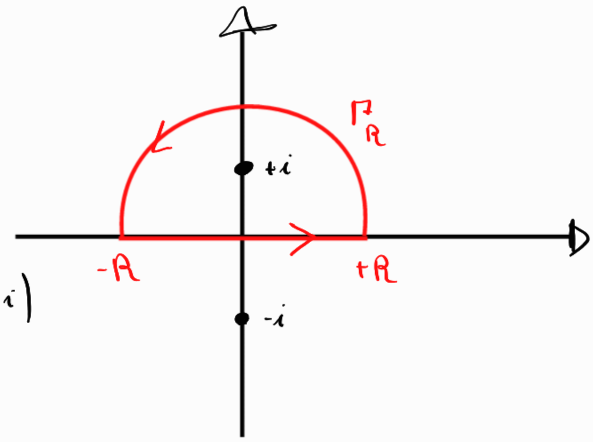
$$\varphi(k) = \langle e^{ikx} \rangle = \int_{\mathbb{R}} \frac{1}{\pi} \frac{e^{ikx}}{1+x^2} \, dx$$

For the evaluation we use the residue theory. We first evaluate the case in which k>0 (this hypothesis will be clarified later in the calculations)

• $(k>0)$

$$f(z) = \frac{e^{ikz}}{\pi(1+z^2)}$$

$$\oint_{[-R,+R] \cup \Gamma_R} f(z)\,dz = \int_{-R}^{+R} f(z)\,dz + \int_{\Gamma_R} f(z)\,dz = 2\pi i \,\mathrm{Res}(f(z), i)$$



Where in the last line we used the residue theorem. We evaluate now the residue by noticing that is a simple pole for f(z):

$$\mathrm{Res}(f(z), i) = \lim_{z \to i} (z-i) f(z) = \lim_{z \to i} \cancel{(z-i)} \frac{e^{ikz}}{\pi \cancel{(z-i)}(z+i)} = \frac{e^{-z}}{2i\pi}$$

We know focus our attention on the integral over the semicircle, by using Darboux's lemma.
Note: We can't imply immediatly that $|e^{ikz}|=1$ because z is complex, not real.
We briefly recall Darboux's lemma:

$$\left| \int_\gamma f(z)\,dz \right| \leq \mathrm{length}(\gamma) \max_{z \in \gamma} |f(z)|$$

So that in our case we have:

$$\left| \int_{\Gamma_R} \frac{e^{ikz}}{\pi(1+z^2)} \, dz \right| \leq \pi R \max_{z \in \Gamma_R} \left| \frac{e^{ikz}}{\pi(1+z^2)} \right|$$

The curve is parametrised by the following : $\Gamma_R : z = Re^{it} \quad t \in [0, \pi]$

So one has:

$$\left|\frac{e^{ikz}}{\pi(1+z^2)}\right| = \left|\frac{e^{ikRe^{it}}}{\pi(1+R^2e^{2it})}\right| = \frac{|e^{ikRe^{it}}|}{\pi|1+R^2e^{2it}|} = \frac{|e^{ikR\cos t}||e^{-kR\sin t}|}{\pi|1+R^2e^{2it}|} \le$$

Reverse triangle inequality

$$\le \quad |x-y| \ge ||x|-|y|| \quad \frac{1}{|x-y|} \le \frac{1}{||x|-|y||} \quad \le \frac{|e^{-kR\sin t}|}{\pi||R^2e^{2it}|-|1||} \le \frac{e^{-kR}}{\pi(R^2-1)}$$

(Where we are implying that $R^2>1$)

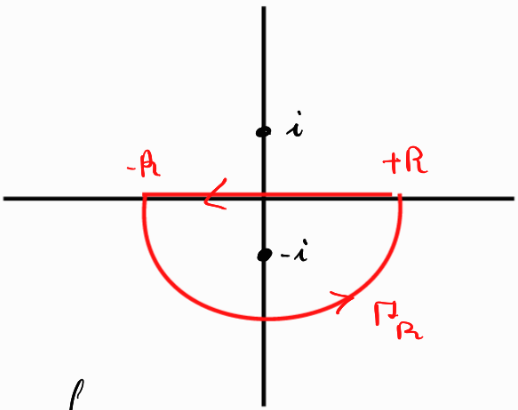
Notice that choosing k>0 is fundamental here, because this guarantees convergence in the limit of R to infinity.
So for the limit:

$$\lim_{R\to+\infty}\oint_{[-R,+R]\cup\Gamma_R} f(z)\,dz = \int_{\mathbb{R}} f(z)\,dz = e^{-k} \qquad k>0$$

Now we evaluate the result for k<0.

• $(k<0)$

Note that we must choose another contour to do the integration, for our consideration in the Darboux lemma in the previous point.
Going faster:



$$2\pi i\,\mathrm{Res}(f(z),-i) = -e^{z} = -\int_{-R}^{+R} f(z)\,dz + \int_{\Gamma_R} f(z)\,dz$$

We do now the majoration chain for the Darboux lemma: $(\Gamma_R : z = Re^{it} \quad t\in\pi,2\pi)$

$$\left|\frac{e^{ikz}}{\pi(1+z^2)}\right| = \left|\frac{e^{ikRe^{it}}}{\pi(1+R^2e^{2it})}\right| = \frac{|e^{ikRe^{it}}|}{\pi|1+R^2e^{2it}|} = \frac{|e^{ikR\cos t}||e^{-kR\sin t}|}{\pi|1+R^2e^{2it}|} \le$$

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(Where we are implying that $R^2>1$)

(The plus in the exponential is due to the fact that the sin is negative in the interval (pi, 2 pi))

So, in this case, for k<0 one has convergence, an obtains in the end:

$$\int_{\mathbb{R}} \frac{e^{ikx}}{\pi(1+x^2)}\,dx = e^{k} \qquad k<0$$

By outting the 2 results together, one obtains:

$$\int_{\mathbb{R}} \frac{e^{ikx}}{\pi(1+x^2)} dx = e^{-|k|} \qquad k \in \mathbb{R}$$

Notice that this function is not derivable in k=0, leading to the conclusion that the moments will be ill-defined.
Moreover from this result we can evaluate a more general one for the PDF:

$$P_x(x) = \frac{\gamma^2}{\pi(\gamma^2 + (x-x_0)^2)}$$

The characteristic function will be

$$\varphi(k) = \int_{\mathbb{R}} dx\, e^{ikx} \frac{\gamma^2}{\pi(\gamma^2+(x-x_0)^2)} = \int_{\mathbb{R}} \frac{e^{ikx}}{\pi\left(1+\left(\frac{x-x_0}{\gamma}\right)^2\right)} dx =$$

$$= \dots\ t = \frac{x-x_0}{\gamma} \dots = \int_{\mathbb{R}} \gamma\, dt \frac{e^{ik(\gamma t + x_0)}}{\pi(1+t^2)} =$$

$$= \gamma e^{ikx_0} \int_{\mathbb{R}} dt \frac{e^{ik\gamma t}}{\pi(1+t^2)} = \gamma e^{ikx_0} e^{-|k\gamma|} =$$

$$= \gamma e^{ikx_0} e^{-\gamma|k|} = \gamma e^{ikx_0 - \gamma|k|}$$