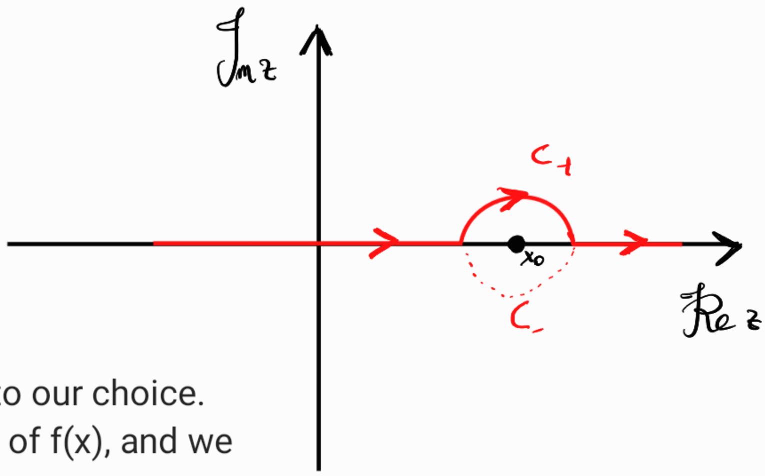
The Sokhotski-Plemelj formula

The general idea is to build a formula to evaluate the following integral:

$$\int_{\mathbb{R}} \frac{f(x)}{x-x_0}\,dx$$

Where f(x) is a complex function of real variable which is integral over the real line, is there holomorphic (or analytic), and moreover $f(x_0)\neq 0$

The problem with this integral is that it doesn't converge in the usual sense.
We must then consider the integrand as a distribution and evaluate the integral in the sense of the distribution.
In our proof this fact will be masked quite artificially, but a more rigorous approach must consider this fact.



We begin by circling our singularity with a circle of radius epsilon, which upon integration will be be put to the limit of epsilon to 0.
We can circle the singularity in 2 ways, either from above with C_+ or from below with C_-.
We will need different hypothesis according to our choice.
In any case we consider the complex version of f(x), and we call it f(z), then:

C_+ choice : f(z) is holomorphic (or analytic) in Im(z)>0 and f(z) tends uniformly to 0 as |z| tends to infinity with respect to 0<arg(z)<pi

C_- choice : f(z) is holomorphic (or analytic) in Im(z)<0 and f(z) tends uniformly to 0 as |z| tends to infinity with respect to pi<arg(z)<2pi (or -pi<arg(z)<0)

Now we evaluate specifically the countour integral over C_+ or C_- with the respective hypothesis:

$C_+ : z = x_0 + \varepsilon e^{it} \quad t \in [\pi, 0]$

$$\int_{C_+} \frac{f(z)}{z-x_0}\,dz = \int_{\pi}^{0} \frac{f(x_0+\varepsilon e^{it})}{\cancel{\varepsilon e^{it}}}\, i\cancel{\varepsilon e^{it}}\,dt = i\int_{\pi}^{0} f(x_0+\varepsilon e^{it})\,dt$$

$$\lim_{\varepsilon\to 0^+} i\int_{\pi}^{0} f(x_0+\varepsilon e^{it})\,dt = \text{uniformly convergence} = i\int_{\pi}^{0} \lim_{\varepsilon\to 0^+} f(x_0+\varepsilon e^{it})\,dt =$$

$$i f(x_0)\int_{\pi}^{0} dt = -i\pi f(x_0)$$

$C_-: z = x_0 + \varepsilon e^{it} \quad t \in [\pi, 2\pi]$

$$\int_{C_-} \frac{f(z)}{z - x_0}\, dz = \int_{\pi}^{2\pi} \frac{f(x_0 + \varepsilon e^{it})}{\cancel{\varepsilon e^{it}}}\, i\cancel{\varepsilon e^{it}}\, dt = i \int_{\pi}^{2\pi} f(x_0 + \varepsilon e^{it})\, dt$$

$$\lim_{\varepsilon \to 0^+} i \int_{\pi}^{2\pi} f(x_0 + \varepsilon e^{it})\, dt \underset{\text{convergence}}{\overset{\text{uniformly}}{=}} = i \int_{\pi}^{2\pi} \lim_{\varepsilon \to 0^+} f(x_0 + \varepsilon e^{it})\, dt =$$

$$i f(x_0) \int_{\pi}^{2\pi} dt = i\pi f(x_0)$$

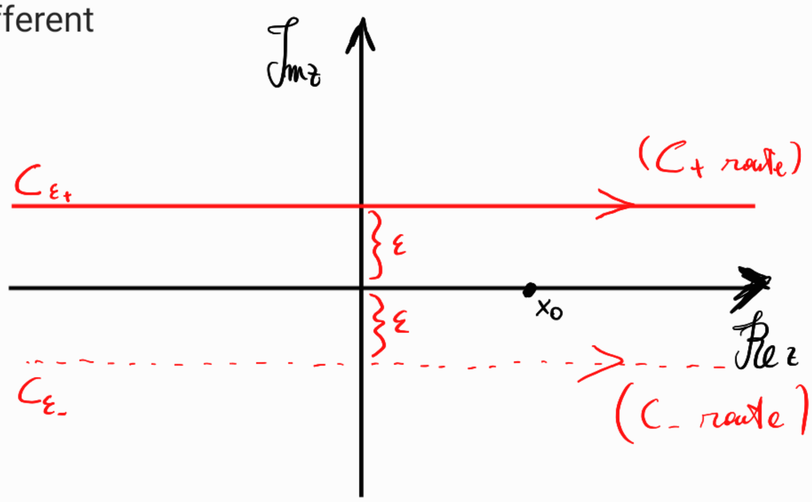
So that one can write in the end:

$$\int_{C_{\delta_+}} \frac{f(z)}{z - x_0}\, dz = \text{P.V.} \int_{\mathbb{R}} \frac{f(x)}{x - x_0}\, dx - i\pi f(x_0) \qquad (\text{over } C_+)$$

$$\int_{C_{\delta_-}} \frac{f(z)}{z - x_0}\, dz = \text{P.V.} \int_{\mathbb{R}} \frac{f(x)}{x - x_0}\, dx + i\pi f(x_0) \qquad (\text{over } C_-)$$

Now we rewrite the LHS with the following manipulation, according to the circling choice made above.

We write the parametrization for the different routes:



C_+ route)

$$\begin{cases} x = t \\ y = \varepsilon \end{cases} \quad t \in \mathbb{R} \qquad C_{\varepsilon_+}: \begin{array}{l} z = t + i\varepsilon \\ t \in \mathbb{R} \end{array}$$

C_- route)

$$\begin{cases} x = t \\ y = -\varepsilon \end{cases} \quad t \in \mathbb{R} \qquad C_{\varepsilon_-}: \begin{array}{l} z = t - i\varepsilon \\ t \in \mathbb{R} \end{array}$$

Notice that, in order to have $C_{\varepsilon_\pm} \simeq C_{\delta_\pm} \iff \varepsilon \simeq \delta$

So that one has:

$$\int_{C_{\varepsilon_+}} \frac{f(z)}{z - x_0}\, dz = \int_{\mathbb{R}} \frac{f(t + i\varepsilon)}{t + i\varepsilon - x_0}\, dt \qquad (C_+ \text{ route})$$

$$\int_{C_{\varepsilon_-}} \frac{f(z)}{z - x_0}\, dz = \int_{\mathbb{R}} \frac{f(t - i\varepsilon)}{t - i\varepsilon - x_0}\, dt \qquad (C_- \text{ route})$$

Sometimes, with notation abuse, one writes, in a disgusting way:

$$\int_{C_{\varepsilon_+}} \frac{f(z)}{z - x_0}\, dz \equiv \int_{-\infty + i\varepsilon}^{+\infty + i\varepsilon} \frac{f(z)}{z - x_0}\, dz \quad ; \quad \int_{C_{\varepsilon_-}} \frac{f(z)}{z - x_0}\, dz \equiv \int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} \frac{f(z)}{z - x_0}\, dz$$

Going now to the limit of epsilon to zero, one obtains the final expression (one usually renames t with x)

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \frac{f(x)}{x - x_0 + i\varepsilon}\, dx = \text{P.V.} \int_{\mathbb{R}} \frac{f(x)}{x - x_0}\, dx - i\pi f(x_0) \qquad (\text{Upper analicity})$$

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \frac{f(x)}{x - x_0 - i\varepsilon}\, dx = \text{P.V.} \int_{\mathbb{R}} \frac{f(x)}{x - x_0}\, dx + i\pi f(x_0) \qquad (\text{Lower analicity})$$

A clarification over the LHS. The limit has not been taken yet, so the idea is the following: we replace:

$$f(x + i\varepsilon) \simeq f(x)$$

Given that f is analytic over the real line, we can expand in any point:

$$f(x + i\varepsilon) = f(x) + i\varepsilon f'(x_0) + \frac{(i\varepsilon)^2}{2!} f''(x_0) + o(\varepsilon^3) \simeq f(x) \qquad (\text{for small } \varepsilon)$$

Important note: If we consider that:

$$f(x_0) = \int_{\mathbb{R}} \delta(x - x_0) f(x)\, dx$$

We can rewrite the 2 expressions as:

$$\lim_{\varepsilon \to 0^+} \frac{1}{x - x_0 + i\varepsilon} = \text{P.V.} \frac{1}{x - x_0} - i\pi \delta(x - x_0) \qquad (\text{Upper analicity})$$

$$\lim_{\varepsilon \to 0^+} \frac{1}{x - x_0 - i\varepsilon} = \text{P.V.} \frac{1}{x - x_0} + i\pi \delta(x - x_0) \qquad (\text{Lower analicity})$$