Electron DOS evaluation in the free electron model

We show here different methods to evaluate the electronic DOS according to the system dimensionality.

- 3 D

Method 1: We consider

$$D(E) = g \frac{V}{(2\pi)^3} \int \delta(E - E(\bar{k}))\, d^3k = g \frac{V}{(2\pi)^3} \int \delta\left(E - \frac{\hbar^2 k^2}{2m}\right) d^3k =$$

$$= g \frac{V}{(2\pi)^3} \int_{\mathbb{R}^+} \delta\left(E - \frac{\hbar^2 k^2}{2m}\right) 4\pi k^2\, dk =$$

Change of variables

$$\frac{\hbar^2 k^2}{2m} = \alpha \rightarrow \frac{\hbar^2}{m} k\, dk = d\alpha$$

$$= g \frac{V}{(2\pi)^3} 4\pi \int_{\mathbb{R}^+} \delta(E - \alpha) \sqrt{\alpha} \left(\frac{2m}{\hbar^2}\right)^{3/2} d\alpha = \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{E}$$

$$g = 2$$

Method 2:

$$D(E) = g \frac{V}{(2\pi)^3} \int_{shell} dS_E \frac{1}{|\bar{\nabla}_k (E(\bar{k}))|} =$$

$$E(\bar{k}) = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2)$$

$$|\bar{\nabla}_{\bar{k}}(E(\bar{k}))| = \frac{\hbar^2}{m} k$$

Notice that we are integrating over a surface that is the boundary of a sphere of fixed radius

$$k^2 = \frac{2mE}{\hbar^2}$$

Over this sphere the value of 1/k is constant

$$= g \frac{V}{(2\pi)^3} \int_{shell} \frac{m}{\hbar^2} \frac{1}{k}\, dS_E =$$

$$= g \frac{V}{(2\pi)^3} \frac{m}{\hbar^2} \frac{1}{k} \int_{shell} dS_E =$$

$$= g \frac{V}{(2\pi)^3} \frac{m}{\hbar^2} \frac{1}{k} 4\pi k^2 = g \frac{V}{2\pi^2} \frac{m}{\hbar^2} \sqrt{\frac{2m}{\hbar^2}} \sqrt{E} = g \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{E} =$$

$$= \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{E}$$

2D

$$D(E) = g \frac{S}{(2\pi)^2} \int \delta(E - E(\bar{k}))\, d^2k = g \frac{S}{(2\pi)^2} \int_{\mathbb{R}^+} \delta\left(E - \frac{\hbar^2 k^2}{2m}\right) 2\pi k\, dk =$$

$$\frac{\hbar^2 k^2}{2m} = \beta \rightarrow \frac{\hbar^2}{m} k\, dk = d\beta$$

$$= g \frac{S}{2\pi} \frac{m}{\hbar^2} \int_{\mathbb{R}^+} \delta\left(E - \beta\right) d\beta = g \frac{S}{2\pi} \frac{m}{\hbar^2} = \frac{S}{\pi} \frac{m}{\hbar^2}$$

1D

$$D(E) = g \frac{L}{2\pi} \int \delta(E - E(\bar{k}))\, dk = g \frac{L}{2\pi} \int_{\mathbb{R}^+} \delta\left(E - \frac{\hbar^2 k^2}{2m}\right) dk =$$

$$\gamma = \frac{\hbar^2 k^2}{2m}$$

$$d\gamma = \frac{\hbar^2}{m} k\, dk$$

$$g \frac{L}{2\pi} \frac{m}{\hbar^2} \sqrt{\frac{\hbar^2}{2m}} \int_{\mathbb{R}^+} \delta\left(E - \gamma\right) \frac{1}{\sqrt{\gamma}}\, d\gamma = g \frac{L}{2\pi} \sqrt{\frac{2m}{\hbar^2}} \frac{1}{\sqrt{E}} = \frac{L}{\pi} \sqrt{\frac{2m}{\hbar^2}} \frac{1}{\sqrt{E}} =$$

0D

$$D(E) = g\, \delta(E - E_0)$$