Study of the function <k> in the single-ended zipper model

$$\langle k \rangle = x \frac{d}{dx}(\ln Z_N) =$$

$$= \frac{N x^N}{x^N - 1} - \frac{x}{x-1} \qquad x \equiv g e^{-\beta \varepsilon_0} > 0$$

1) Mathematical way

We begin by studying the limiting behaviour

$$\lim_{x \to 0^+} \langle k \rangle = 0 \qquad , \quad \lim_{x \to +\infty} \langle k \rangle = N - 1$$

As is the function is defined in $\mathbb{R}^+ / \{1\}$, so we study the limit in this neighborhood:

$$\lim_{x \to 1} \langle k \rangle = \lim_{x \to 1} \frac{N x^N}{x^N - 1} - \frac{x}{x-1} =$$

$$= \lim_{x \to 1} \frac{N x^N (x-1) - x(x^N - 1)}{(x^N - 1)(x-1)} =$$

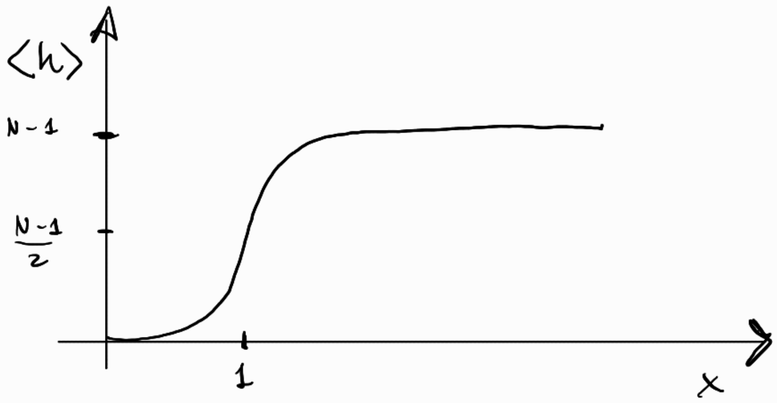
$$= \lim_{x \to 1} \frac{N x^{N+1} - N x^N - x^{N+1} + x}{x^{N+1} - x^N - x + 1} = \frac{0}{0} =$$

$$= \text{De L'Hopital} = \lim_{x \to 1} \frac{N(N+1) x^N - N^2 x^{N-1} - (N+1) x^N + 1}{(N+1) x^N - N x^{N-1} - 1} = \frac{0}{0} =$$

$$= \text{De L'Hopital} = \lim_{x \to 1} \frac{N^2 (N+1) x^{N-1} - N^2 (N-1) x^{N-2} - N(N+1) x^{N-1}}{N(N+1) x^{N-1} - N(N-1) x^{N-2}} =$$

$$= \frac{N^2(N+1) - N^2(N-1) - N(N+1)}{N(N+1) - N(N-1)} = \frac{2N^2 - N(N+1)}{2N} = \frac{2N - N - 1}{2} = \frac{N-1}{2}$$

This is a rough sketch of the function



2) Mumbo-jumbo way (Physics way)

Given that we want to study the function in the neighborhood of $x=1$, we consider the function $\langle n \rangle$ with parameter $x = 1+\varepsilon$

$$\langle n \rangle = x \frac{d}{dx}\left[\ln\left(\frac{1-x^n}{1-x} \right) \right] \sim (1+\varepsilon) \frac{d}{d\varepsilon}\left[\ln\left(\frac{1-(1+\varepsilon)^n}{-\varepsilon} \right) \right] \quad \forall x \in B(1,\delta)$$

We now consider the function in parenthesis

$$\ln\left(\frac{1-(1+\varepsilon)^n}{-\varepsilon} \right) = \ln\left(-\frac{1}{\varepsilon}\left(1 - \sum_{k=1}^{n} \binom{n}{k} \varepsilon^k \right) \right) =$$

$$= \ln\left[-\frac{1}{\varepsilon}\left(1 - 1 - \binom{n}{1}\varepsilon - \binom{n}{2}\varepsilon^2 - \binom{n}{3}\varepsilon^3 - \mathcal{O}(\varepsilon^4) \right) \right] =$$

$$= \ln\left[\binom{n}{1} + \binom{n}{2}\varepsilon + \binom{n}{3}\varepsilon^2 + \mathcal{O}(\varepsilon^3) \right] \equiv \ln(P(\varepsilon))$$

So we have

$$\langle n \rangle \doteq (1+\varepsilon) \frac{d}{d\varepsilon} \ln(P(\varepsilon)) = \frac{1+\varepsilon}{P(\varepsilon)} \frac{dP(\varepsilon)}{d\varepsilon}$$

And

$$\lim_{x \to 1} \langle n \rangle = \lim_{\varepsilon \to 0} \frac{1+\varepsilon}{P(\varepsilon)} P'(\varepsilon) = \frac{P'(0)}{P(0)} = \frac{\binom{n}{2}}{\binom{n}{1}} = \frac{\frac{n!}{2!(n-2)!}}{n} =$$

$$= \frac{n-1!}{2\,(n-2)!} = \frac{n-1}{2}.$$

this method, while cumbersome, is better if we are interested in next order term, say in the response function @ the critical point:

$$\chi_{b_n} \equiv \frac{1}{n} \frac{d\langle k \rangle}{dx} \simeq \frac{1}{n} \frac{d\langle k \rangle}{d\varepsilon}$$

So

$$\lim_{x \to 0} \chi_{b_n} = \lim_{\varepsilon \to 0} \frac{1}{n} \frac{d\langle k \rangle}{d\varepsilon} = \frac{1}{n} \frac{P''(0)}{P(0)} = \frac{1}{n} \frac{\binom{n}{3}}{\binom{n}{1}} = \frac{\frac{n!}{3!(n-3)!}}{n^2} =$$

$$= \frac{(n-1)(n-2)}{6n} \propto n$$