FEM

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Contents

Chapter 1

Plate

1.1 Dimension of plate

thick	thin	very thin	
Length / thickness	~ 5 to ~ 10	~ 10 to ~ 100	> 100
physical characteristics	transverse shear deformations $\varepsilon_{13} \neq 0$	negligible transverse shear deformations $\varepsilon_{13} \approx 0$	geometrically non-linear

1.2 positive directions

The sign convention used in the plate section is defined in this section. Notice that unlike some publications on this subject, the loading term $p(x, y)$ points to the same direction as the z-axis. Please refer to the classical plate theory for more details.

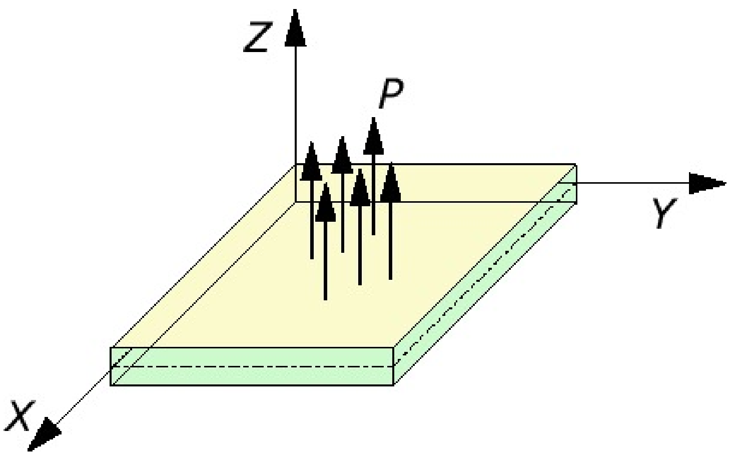


Figure 1.1: The loading p and deflection w in the z direction indicated are positive.

For the sign of moment resultants, a moment caused by a positive stress with a positive arm is considered positive and the subscript of the stress is used to denote this moment.

Therefore, positive M_x points toward the y direction under the right-hand rule.

When a flat plate is subjected to both inplane and transverse or normal loads as shown in Figure 1.5 any point inside the plate can have displacement components u, v, and w parallel to x, y, and z axes, respectively. In the small deflection (or linear) theory of thin plates, the transverse deflection w is uncoupled from the inplane deflections u and v. Consequently, the stiffness matrices for the inplane and transverse deflections are also uncoupled and they can be calculated independently. Thus, if a plate is subjected to inplane loads only, it will undergo deformation in its plane only. In this case, the plate is said to be under the action of ”membrane” forces. Similarly, if the plate is subjected to transverse loads (and/or bending moments), any point inside the plate experiences essentially a lateral

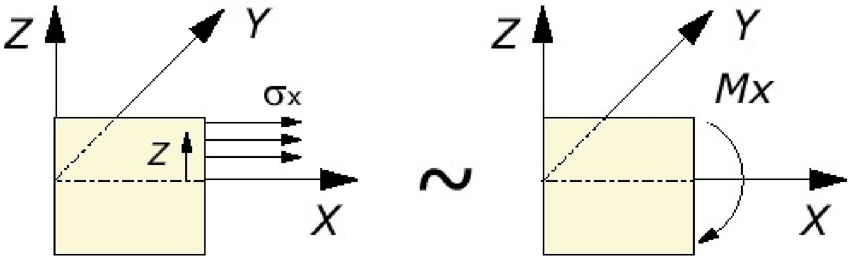


Figure 1.2

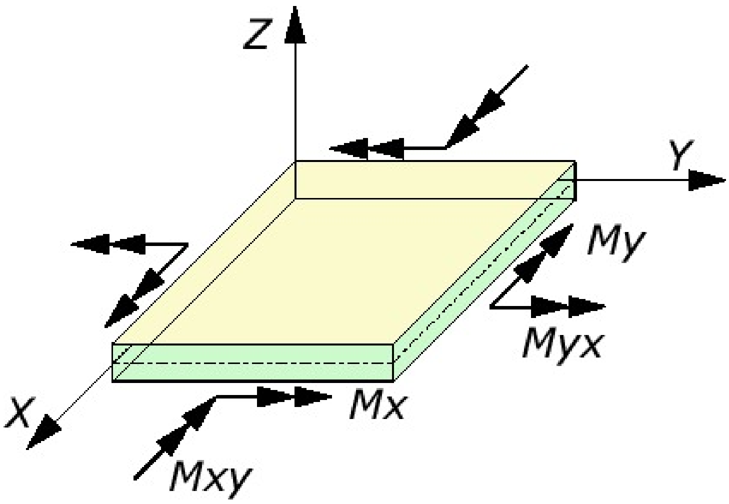


Figure 1.3: Positive directions for bending moments.

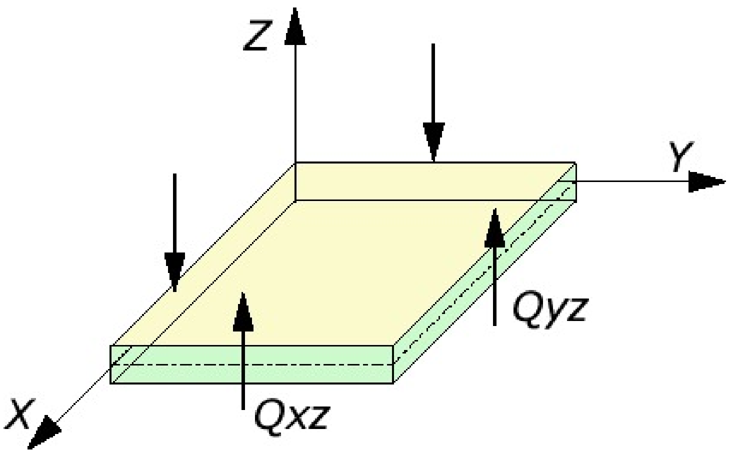


Figure 1.4: Positive directions for shear forces.

displacement w (inplane displacements u and v, are also experienced because of the rotation of the plate element). In this case. the plate is said to be under the action of bending forces. The inplane and bending analysis of plates is considered in this chapter. If the plate elements are used for the analysis of three-dimensional structures. such as folded plate structures, both inplane and bending actions have to be considered in the development of element properties. This aspect of coupling the membrane and bending actions of a plate element is also considered in this chapter.

The CST and LST triangles are variations of the Pascal triangles as shown bellow:

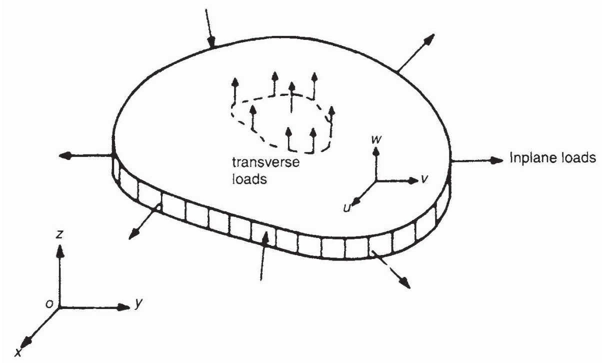


Figure 1.5: Inplane and transverse loads in a plane

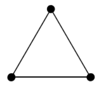
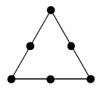
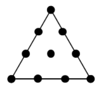
Terms in Pascal Triangle	Polynomial Degree	Number of Terms	Triangle
1	0(constant)	1	
$x \quad y$	1(linear)	3	 CST
$x^2 \quad xy \quad y^2$	2(quadratic)	6	 LST
$x^3 \quad x^2y \quad xy^2 \quad y^3$	3(cubic)	10	 QST

Figure 1.6: variations of the Pascal triangles

1.3 Constant Strain Triangle

1.3.1 Nodal displacements

The triangular membrane element is considered to lie in the xy plane of a local xy coordinate system as shown in Figure 1.7. The nodes are arranged as $1, 2$ and 3 in anti-clockwise. Each node includes 2 degrees of freedom . The displacement components can be expressed as:

$$\mathbf{q}^e = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \tag{1.3.1}$$

1.3.2 Shape functions

By assuming a linear displacement variation inside the element and introducing a set of area coordinates, the displacement model can be expressed as:

$$\begin{aligned} u &= a_1 + a_2 x + a_3 y \\ v &= a_4 + a_5 x + a_6 y \end{aligned} \tag{1.3.2}$$

By considering the displacements u_i and v_i as the local degrees of freedom of node $1, 2, 3$, the constants $a_1, \cdots, a_6$ can be evaluated. Thus, by using the conditions

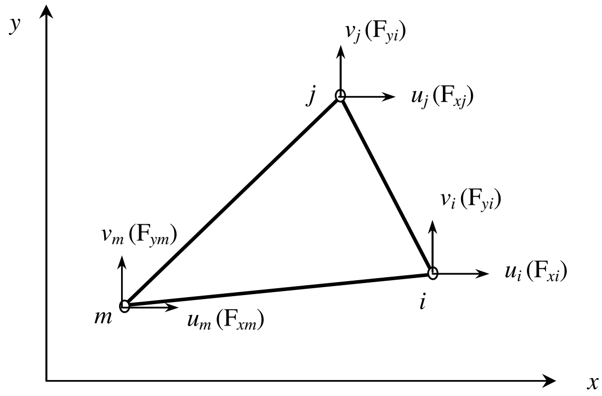


Figure 1.7: Triangular membrane element

$$
\begin{aligned}
u_1 &= a_1 + a_2 x_1 + a_3 y_1 & \qquad v_1 &= a_4 + a_5 x_1 + a_6 y_1 \\
u_2 &= a_1 + a_2 x_2 + a_3 y_2 & \qquad v_2 &= a_4 + a_5 x_2 + a_6 y_2 \\
u_3 &= a_1 + a_2 x_3 + a_3 y_3 & \qquad v_3 &= a_4 + a_5 x_3 + a_6 y_3
\end{aligned}
$$

We can express the constants $a_1, \cdots, a_6$ in terms of the nodal degrees of freedom:

$$
a_1 = \frac{1}{2\Delta}\begin{vmatrix} u_1 & x_1 & y_1 \\ u_2 & x_2 & y_2 \\ u_3 & x_3 & y_3 \end{vmatrix} \qquad
a_2 = \frac{1}{2\Delta}\begin{vmatrix} 1 & u_1 & y_1 \\ 1 & u_2 & y_2 \\ 1 & u_3 & y_3 \end{vmatrix} \qquad
a_3 = \frac{1}{2\Delta}\begin{vmatrix} 1 & x_1 & u_1 \\ 1 & x_2 & u_2 \\ 1 & x_3 & u_3 \end{vmatrix}
$$

Where

$$
2\Delta = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}
$$

This leads to the displacement model:

$$
\mathbf{N}\mathbf{q}^e = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}
$$

$$
\begin{aligned}
a_1 &= x_2 y_3 - x_3 y_2 & \qquad a_2 &= x_3 y_1 - x_1 y_3 & \qquad a_3 &= x_1 y_2 - x_2 y_1 \\
b_1 &= y_2 - y_3 & \qquad b_2 &= y_3 - y_1 & \qquad b_3 &= y_1 - y_2 \\
c_1 &= x_3 - x_2 & \qquad c_2 &= x_1 - x_3 & \qquad c_3 &= x_2 - x_1 \\
N_1 &= \frac{(a_1 + b_1 x + c_1 y)}{2\Delta} & \qquad N_2 &= \frac{(a_2 + b_2 x + c_2 y)}{2\Delta} & \qquad N_3 &= \frac{(a_3 + b_3 x + c_3 y)}{2\Delta}
\end{aligned}
$$

1.3.3 Element Strain

By using the relations:

$$
\varepsilon = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \tag{1.3.3}
$$

the components of strain can be expressed in terms of nodal displacements as:

$$\varepsilon = \mathbf{B}\mathbf{q}^e \tag{1.3.4}$$

where

$$\mathbf{B} = \frac{1}{2\Delta}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \tag{1.3.5}$$

1.3.4 Element stress

The stress-strain relations are given by:

$$\sigma = \mathbf{D}\varepsilon \tag{1.3.6}$$

where

$$\sigma = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \tag{1.3.7}$$

and

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{1.3.8}$$

1.3.5 Element stiffness

the stiffness matrix of the element $\mathbf{K}_e$ can be found by using:

$$\mathbf{K}_e = \iiint_\Omega \mathbf{B}^T\mathbf{D}\mathbf{B}d\Omega = \iint_\Delta \mathbf{B}^T\mathbf{D}\mathbf{B}hd\Delta \tag{1.3.9}$$

where Ω denotes the volume of the element. If the plate thickness is taken as a constant h, the evaluation of the integral in eq 1.3.9 presents no difficult since the elements of the matrices $\mathbf{B}$ and $\mathbf{D}$ are all constants (not functions of x and y). Hence, Eq 1.3.9 can be rewritten as:

$$\mathbf{K}_e = \mathbf{B}^T\mathbf{D}\mathbf{B}h\iint_\Delta d\Delta = h\Delta\mathbf{B}^T\mathbf{D}\mathbf{B} \tag{1.3.10}$$

1.4 Linear Strain Triangle

1.4.1 Nodal displacements

Each node of Linear Strain Triangular has two degrees of freedom: displacements in the x and y directions.

Let u_i and v_i represent the node i displacement components in the x and y directions, respectively.

$$\mathbf{q}^e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ q_{12} \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{Bmatrix} \tag{1.4.1}$$

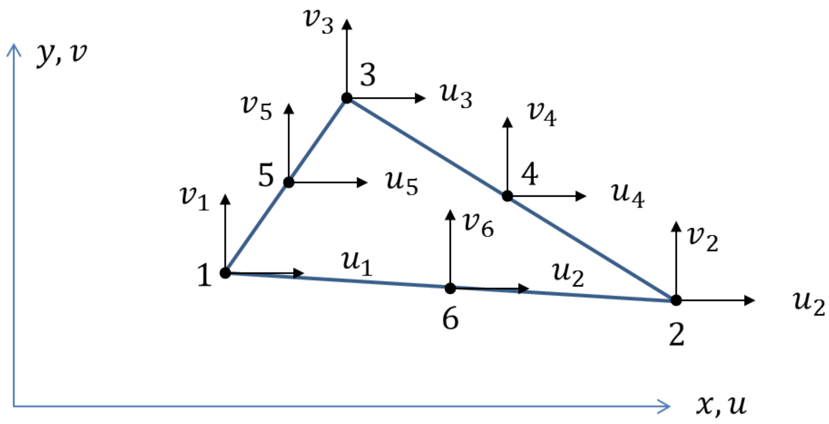


Figure 1.8: Linear strain triangle element

1.4.2 Shape functions

The variation of the displacements over the element may be expressed as:

$$\begin{aligned} u(x,y) &= a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 \\ v(x,y) &= a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2 \end{aligned} \tag{1.4.2}$$

The coefficients a_1 through a_6 can be calculated by evaluating the displacement u at each node.

$$\delta = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \mathbf{H}_\delta \mathbf{a} \tag{1.4.3}$$

To obtain the values for the **a**'s substitute the coordinates of the nodal points into the above equations:

$$\begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \\ v_1 \\ \vdots \\ v_5 \\ v_6 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_6 & y_6 & x_6^2 & x_6 y_6 & y_6^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_5 & y_5 & x_5^2 & x_5 y_5 & y_5^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_6 & y_6 & x_6^2 & x_6 y_6 & y_6^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \\ a_7 \\ \vdots \\ a_{11} \\ a_{12} \end{Bmatrix} = \mathbf{A}_\delta \mathbf{a} \tag{1.4.4}$$

Solving for the **a**'s:

$$\begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \\ a_7 \\ \vdots \\ a_{11} \\ a_{12} \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_6 & y_6 & x_6^2 & x_6 y_6 & y_6^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_5 & y_5 & x_5^2 & x_5 y_5 & y_5^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_6 & y_6 & x_6^2 & x_6 y_6 & y_6^2 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \\ v_1 \\ \vdots \\ v_5 \\ v_6 \end{Bmatrix} = \mathbf{A}_\delta^{-1} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \tag{1.4.5}$$

The general displacement expressions in terms of interpolation functions and the nodal degrees of freedom are:

$$\{\delta\} = [N]\{u\}, \quad \text{where } \mathbf{N} = \mathbf{H}_\delta \mathbf{A}_\delta^{-1} \tag{1.4.6}$$

1.4.3 Element strains

The strains over two dimensional element are:

$$\{\varepsilon\} = \left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{array} \right\} = \left\{ \begin{array}{c} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{array} \right\} = \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix} \left\{ \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_{11} \\ a_{12} \end{array} \right\} \tag{1.4.7}$$

Therefore, we have

$$\{\varepsilon\} = [B]\{q\} = \begin{bmatrix} 0 & 1 & 0 & 2x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x & 2y \\ 0 & 0 & 1 & 0 & x & 2y & 0 & 1 & 0 & 2x & y & 0 \end{bmatrix} \mathbf{A}_\delta^{-1} \left\{ \begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_6 \\ v_1 \\ \vdots \\ v_5 \\ v_6 \end{array} \right\} \tag{1.4.8}$$

1.4.4 Elemental stresses

The stresses are given as:

$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\} = [D] \left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{array} \right\} \tag{1.4.9}$$

For plane stress, $[D]$ is:

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{1.4.10}$$

For plane strain, $[D]$ is:

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \tag{1.4.11}$$

1.4.5 Element stiffness matrix

The stiffness matrix can be defined as:

$$[k] = \int_V [B]^T [D][B] dV \tag{1.4.12}$$

1.5 Discrete Kirchhoff Triangle

the stiffness matrix of the element $\mathbf{K}_e$ can be found by using:

$$\mathbf{K}_e = \iiint_\Omega \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega = \iint_\Delta \mathbf{B}^T \mathbf{D} \mathbf{B} h d\Delta \tag{1.5.1}$$

where Ω denotes the volume of the element. If the plate thickness is taken as a constant h, the evaluation of the integral in eq 1.5.1 presents no difficult since the elements of the matrices $\mathbf{B}$ and $\mathbf{D}$ are all constants (not functions of x and y). Hence, Eq 1.5.1 can be rewritten as:

$$\mathbf{K}_e = \mathbf{B}^T \mathbf{D} \mathbf{B} h \iint_\Delta d\Delta = h\Delta \mathbf{B}^T \mathbf{D} \mathbf{B} \tag{1.5.2}$$

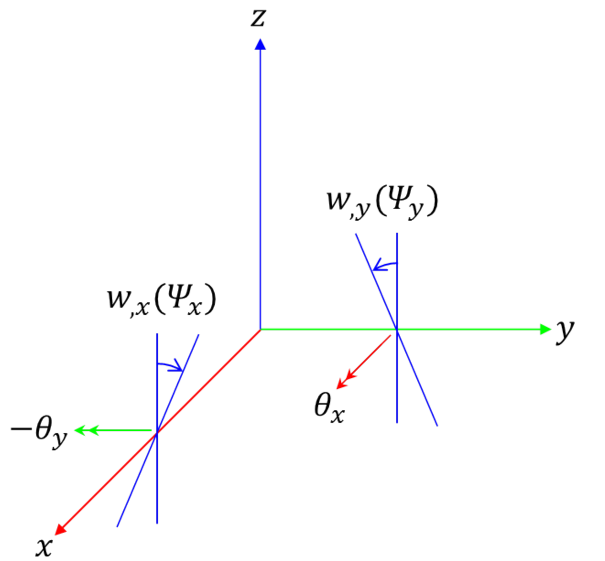


Figure 1.9: Plate transverse displacements and angular rotations

1.6 Discrete Kirchhoff triangle

A large number of plate bending elements have been developed and reported in the literature. In the classical theory of thin plates discussed in this section, certain simplifying approximations are made. One of the important assumptions made is that shear deformation is negligible. Some elements have been developed by including the effect of transverse shear deformation also.

According to thin plate theory, the deformation is completely described by the transverse deflection of the middle surface of the plate (w) only. Thus. if a displacement model is assumed for w. the continuity of not only w tut also its derivatives has to be maintained between adjacent elements.

1.6.1 Displacement model

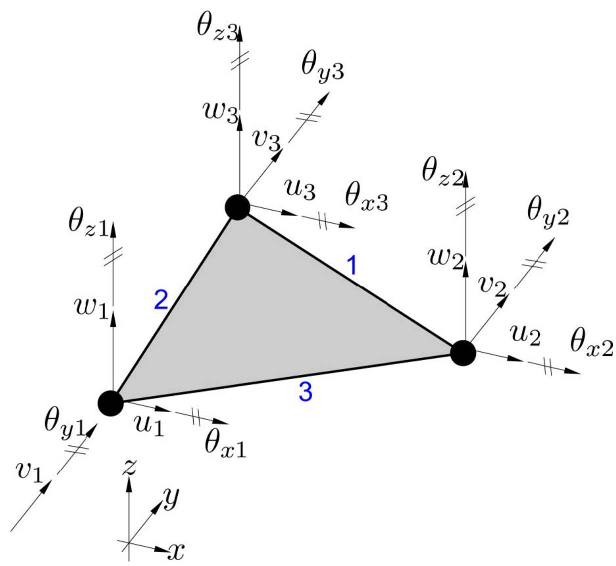


Figure 1.10: Degrees of freedom of triangle element

At each node of triangular plate element shown in figure 1.10. the transverse displacement w and rotations about the x and y axes are taken as the degrees of freedom. Since there are nine displacement degrees of freedom in the element, the assumed polynomial for $w(x, y)$ must also contain nine constant terms. To maintain geometric isotropy, the displacement model is taken as:

$$w = a_1L_1 + a_2L_2 + a_3L_3 + a_4L_2L_3 + a_5L_3L_1 + a_6L_1L_2 + a_7(L_2L_3^2 - L_3L_2^2) + a_8(L_3L_1^2 - L_1L_3^2) + a_9(L_1L_2^2 - L_2L_1^2) \tag{1.6.1}$$

The first three terms represent rigid displacements and terms 3 ˜6 correspond to constant strain. The constants $(a_1, a_2, \cdots, a_9)$ have to be determined from the nodal conditions:

substitute area coordinates of nodes into eq 1.6.1, we have:

$$a_1 = w_1 \quad a_2 = w_2 \quad a_3 = w_3 \tag{1.6.2}$$

By computing derivatives of deflection w to area coordinates:

$$\begin{aligned}
\frac{\partial w}{\partial L_1} &= w_1 - w_3 - a_4L_2 + a_5(L_3 - L_1) + a_6L_2 + a_7(L_2^2 - 2L_2L_3) + a_8(4L_1L_3 - L_1^2 - L_3^2) + a_9(L_2^2 - 2L_1L_2) \\
\frac{\partial w}{\partial L_2} &= w_2 - w_3 + a_4(L_3 - L_2) - a_5L_1 + a_6L_1 + a_7(L_3^2 + L_2^2 - 4L_2L_3) + a_8(2L_1L_3 - L_1^2) + a_9(2L_1L_2 - L_1^2)
\end{aligned} \tag{1.6.3}$$

In the equation, $w_{,L_{ij}}$ means derivative of deflection w to area coordinate L_i at node j. The constants can be solved:

$$\begin{aligned}
a_4 &= \frac{w_{,L_{23}} - w_{,L_{22}}}{2} \\
a_5 &= \frac{w_{,L_{13}} - w_{,L_{11}}}{2} \\
a_6 &= \frac{w_{,L_{12}} + w_{,L_{21}} - w_{,L_{11}} - w_{,L_{22}}}{2} \\
a_7 &= w_3 - w_2 + \frac{w_{,L_{23}} + w_{,L_{22}}}{2} \\
a_8 &= w_1 - w_3 - \frac{w_{,L_{11}} + w_{,L_{13}}}{2} \\
a_9 &= w_2 - w_1 + \frac{w_{,L_{11}} + w_{,L_{12}} - w_{,L_{21}} - w_{,L_{22}}}{2}
\end{aligned} \tag{1.6.4}$$

Note the L_3 is not a dependent variable, it should be regarded as $1 - L_1 - L_2$. substitue area coordinates of three nodes into equation 1.6.3, we get:

$$\begin{aligned}
w_{,L_{11}} &= w_1 - w_3 - a_5 - a_8 & w_{,L_{21}} &= w_2 - w_3 - a_5 + a_6 - a_8 - a_9 \\
w_{,L_{12}} &= w_1 - w_3 - a_4 + a_6 + a_7 + a_9 & w_{,L_{22}} &= w_2 - w_3 - a_4 + a_7 \\
w_{,L_{13}} &= w_1 - w_3 + a_5 - a_8 & w_{,L_{23}} &= w_2 - w_3 + a_4 + a_7
\end{aligned}$$

substitute eq 1.6.2 and 1.6.4 into eq 1.6.1, it gives:

$$w = \begin{bmatrix} \bar{\mathbf{N}}_i & \bar{\mathbf{N}}_j & \bar{\mathbf{N}}_m \end{bmatrix} \begin{bmatrix} \bar{\delta}_i \\ \bar{\delta}_j \\ \bar{\delta}_m \end{bmatrix} \tag{1.6.5}$$

where:

$$\begin{aligned}
\bar{\delta}_1 &= \begin{bmatrix} w_1 & w_{,L_{11}} & w_{,L_{21}} \end{bmatrix} \quad \bar{\delta}_2 = \begin{bmatrix} w_2 & w_{,L_{12}} & w_{,L_{21}} \end{bmatrix} \quad \bar{\delta}_3 = \begin{bmatrix} w_3 & w_{,L_{13}} & w_{,L_{23}} \end{bmatrix} \\
\bar{\mathbf{N}}_1 &= \begin{bmatrix} N_1 & N_{,L_{11}} & N_{,L_{21}} \end{bmatrix} \quad \bar{\mathbf{N}}_2 = \begin{bmatrix} N_2 & N_{,L_{12}} & N_{,L_{21}} \end{bmatrix} \quad \bar{\mathbf{N}}_3 = \begin{bmatrix} N_3 & N_{,L_{13}} & N_{,L_{23}} \end{bmatrix}
\end{aligned}$$

the components of the shape function are:

$$
\begin{aligned}
N_1 &= L_1 - (L_1L_2^2 - L_2L_1^2) + (L_3L_1^2 - L_1L_3^2)\\
N_{L_{11}} &= \frac{-L_1L_2 - L_3L_1 + (L_1L_2^2 - L_2L_1^2) - (L_3L_1^2 - L_1L_3^2)}{2}\\
N_{L_{21}} &= \frac{L_1L_2 + (L_2L_1^2 - L_1L_2^2)}{2}\\
N_2 &= L_2 - (L_2L_3^2 - L_3L_2^2) + (L_1L_2^2 - L_2L_1^2)\\
N_{L_{12}} &= \frac{L_1L_2 + (L_1L_2^2 - L_2L_1^2)}{2}\\
N_{L_{22}} &= \frac{-L_2L_3 - L_1L_2 + (L_2L_3^2 - L_3L_2^2) - (L_1L_2^2 - L_2L_1^2)}{2}\\
N_3 &= L_3 - (L_3L_1^2 - L_1L_3^2) + (L_2L_3^2 - L_3L_2^2)\\
N_{L_{13}} &= \frac{L_1L_3 + (L_1L_3^2 - L_3L_i^2)}{2}\\
N_{L_{23}} &= \frac{L_2L_3 + (L_2L_3^2 - L_3L_2^2)}{2}
\end{aligned}
$$

as we have:

$$
\begin{aligned}
\frac{\partial x}{\partial L_1} &= x_1 - x_3 = c_1\\
\frac{\partial y}{\partial L_1} &= y_1 - y_3 = -b_1\\
\frac{\partial x}{\partial L_2} &= x_2 - x_3 = -c_1\\
\frac{\partial y}{\partial L_2} &= x_2 - x_3 = b_1
\end{aligned}
\tag{1.6.6}
$$

this leads to

$$
\begin{aligned}
\frac{\partial w}{\partial L_1} &= c_2\frac{\partial w}{\partial x} - b_2\frac{\partial w}{\partial y} = -b_2\theta_x - c_2\theta_y\\
\frac{\partial w}{\partial L_2} &= -c_2\frac{\partial w}{\partial x} + b_2\frac{\partial w}{\partial y} = b_2\theta_x + c_2\theta_y
\end{aligned}
\tag{1.6.7}
$$

The relation of nodal displacements between cardisian system and area system is:

$$
\bar{\delta}_i = \begin{bmatrix} w_i \\ w_{,L_{11}} \\ w_{,L_{21}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -b_2 & -c_2 \\ 0 & b_1 & c_1 \end{bmatrix} \begin{bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{bmatrix} = \mathbf{P}\delta_i \tag{1.6.8}
$$

Use eq 1.6.8, the eq 1.6.5 can be re-written as:

$$
w = \begin{bmatrix} \bar{\mathbf{N}}_1 & \bar{\mathbf{N}}_2 & \bar{\mathbf{N}}_3 \end{bmatrix} \begin{bmatrix} \mathbf{P} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} \tag{1.6.9}
$$

where

$$
\begin{aligned}
\mathbf{N}_i &= \bar{\mathbf{N}}_i\mathbf{P} = \begin{bmatrix} N_i & N_{xi} & N_{yi} \end{bmatrix}\\
N_i &= L_i + L_i^2L_j + L_i^2L_m - L_iL_j^2 - L_iL_m^2 \quad (i = 1, 2, 3)\\
N_{xi} &= b_jL_i^2L_m - b_mL_i^2L_j + \frac{(b_j - b_m)}{2}L_iL_jL_m\\
N_{yi} &= c_jL_i^2L_m - c_mL_i^2L_j + \frac{c_j - c_m}{2}L_iL_jL_m
\end{aligned}
$$

1.6.2 Stiffness matrix

In eq **??**, the shape function is using L_1 and L_2 as independent variables. To obtain stiffness matrix in global space, transformation matrix is introduced:

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \frac{1}{2\Delta}\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}\begin{bmatrix} \partial/\partial L_1 \\ \partial/\partial L_2 \end{bmatrix} \tag{1.6.10}$$

and

$$\begin{bmatrix} \partial^2/\partial x^2 \\ \partial^2/\partial y^2 \\ 2\partial^2/\partial x\partial y \end{bmatrix} = \frac{1}{4\Delta^2}\mathbf{T}\begin{bmatrix} \partial^2/\partial L_1^2 \\ \partial^2/\partial L_2^2 \\ \partial^2/\partial L_1\partial L_2 \end{bmatrix} \tag{1.6.11}$$

where

$$\mathbf{T} = \begin{bmatrix} b_1^2 & b_2^2 & 2b_1b_2 \\ c_1^2 & c_2^2 & 2c_1c_2 \\ 2b_1c_1 & 2b_2c_2 & 2(b_1c_2 + b_2c_1) \end{bmatrix} \tag{1.6.12}$$

the element strain matrix is now:

$$\epsilon = \mathbf{B}\delta^{\mathbf{e}} = z\begin{bmatrix} \mathbf{B}_i & \mathbf{B}_j & \mathbf{B}_m \end{bmatrix}\begin{bmatrix} \delta_i \\ \delta_j \\ \delta_m \end{bmatrix} \tag{1.6.13}$$

where

$$\mathbf{B}_k = -\begin{bmatrix} \mathbf{N}_{k,xx} \\ \mathbf{N}_{k,yy} \\ \mathbf{N}_{k,xy} \end{bmatrix} = -\frac{1}{4\Delta^2}\mathbf{T}\begin{bmatrix} \mathbf{N}_{k,11} \\ \mathbf{N}_{k,22} \\ \mathbf{N}_{k,12} \end{bmatrix} \quad (k = 1, 2, 3) \tag{1.6.14}$$

and the element stiffness matrix in global space is: h3 / 12

$$\mathbf{K}^e = \frac{h^3}{12}\iint_\Omega \mathbf{B}^T\mathbf{D}\mathbf{B}dxdy \tag{1.6.15}$$

In eq 1.6.15, n is number of gauss points, W_i is weight, (L_{i1}, L_{i2}, L_{i3}) is integration point.

1.6.3 Gauss quadrature

To simplify computation, we use quadrature for solving stiffness integral:

Firstly, the integral over triangular domain Ω is converted to:

$$\iint_\Omega f(L_1, L_2, L_3)dxdy = 2\Delta\int_0^1\int_0^{1-L_1} f(L_1, L_2, L_3)dL_1dL_2 \tag{1.6.16}$$

the right term can be calcultated using:

$$\int_0^1\int_0^{1-L_1} f(L_1, L_2, L_3)dL_1dL_2 = \sum_{i=1}^{n} W_i f(L_{i1}, L_{i2}, L_{i3}) \tag{1.6.17}$$

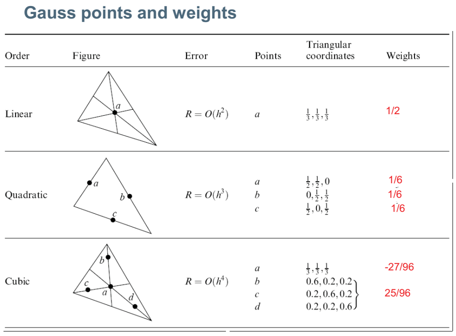
1.7 Drilling degree. to be added

1.8 Transformation Matrix

The shape functions are defined in the plane of the triangle, i.e. z-coordinates for the nodes are equal to zero

Assuming that the triangular element under consideration is an interior element of a large structure. let the node numbers 1, 2, and 3 of the element correspond to the node numbers i, j. and k, respectively, of the global system. Then place the origin of the local xg system at node 1 (node i), and take the y axis along the edge 1 2 (edge ij) and the x axis perpendicular to the y axis directed toward node 3 (node k) as shown in Figure 1.13.

We have:



Gauss points and weights

Order	Figure	Error	Points	Triangular coordinates	Weights
Linear		$R = O(h^2)$	a	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	1/2
Quadratic		$R = O(h^3)$	a	$\frac{1}{2}, \frac{1}{2}, 0$	1/6
			b	$0, \frac{1}{2}, \frac{1}{2}$	1/6
			c	$\frac{1}{2}, 0, \frac{1}{2}$	1/6
Cubic		$R = O(h^4)$	a	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	-27/96
			b	0.6, 0.2, 0.2	25/96
			c	0.2, 0.6, 0.2	
			d	0.2, 0.2, 0.6	

Figure 1.11: gauss points and weights

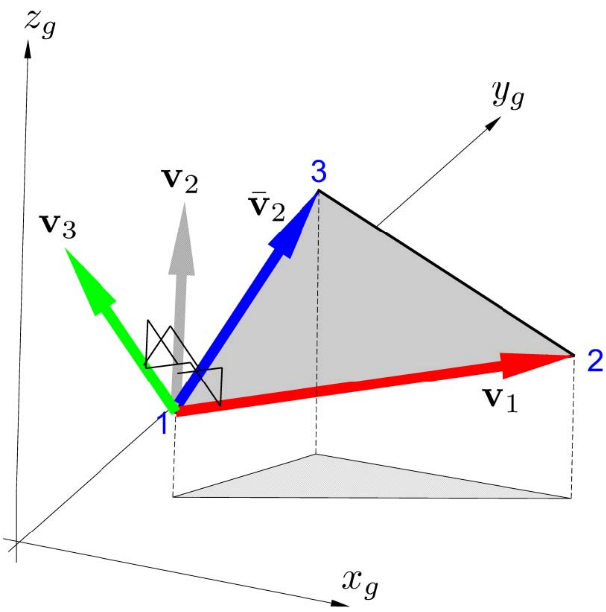


Figure 1.12: transformation matrix in 3d

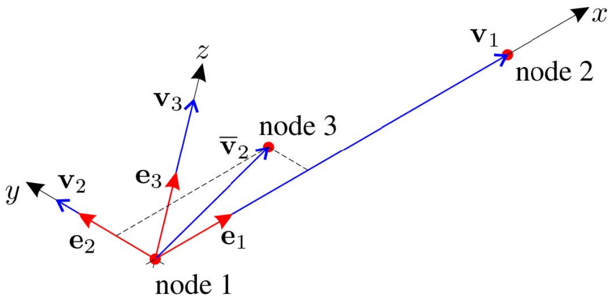


Figure 1.13: unit vectors describing xyz sysem.

$$
\begin{aligned}
\mathbf{v}_1 &= \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \\
\bar{\mathbf{v}}_2 &= \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix} \\
\mathbf{v}_3 &= \mathbf{v}_1 \times \bar{\mathbf{v}}_2 \\
\mathbf{v}_2 &= \mathbf{v}_3 \times \mathbf{v}_1
\end{aligned}
$$

the unit vectors can be expressed:

$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|}$$
$$\mathbf{e}_2 = \frac{\mathbf{v}_2}{|\mathbf{v}_2|}$$
$$\mathbf{e}_3 = \frac{\mathbf{v}_3}{|\mathbf{v}_3|}$$

So the transformation matrix is:

$$\mathbf{T} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \tag{1.8.1}$$

$$\begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = \mathbf{T}^T \begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix} \tag{1.8.2}$$

1.9 global element stiffness matrix

The full transformation matrix ($12x12$) can be made from $T(3x3)$ and multiply the local element stiffness matrix with the transformation to obtain the global element stiffness matrix:

$$\mathbf{T}_g = \begin{bmatrix} \mathbf{T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{T} \end{bmatrix} \tag{1.9.1}$$

$$\mathbf{K}_{eg} = \mathbf{T}_g \mathbf{K}_e \mathbf{T}_g^T \tag{1.9.2}$$

1.10 Transform matrix