FEM

Nuomizong

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Chapter 1

Strain

The deformation of a thin straight rod into a closed loop. The length of the rod remains almost unchanged during the deformation, which indicates that the strain is small. In this particular case of bending, displacements associated with rigid translations and rotations of material elements in the rod are much greater than displacements associated with straining.

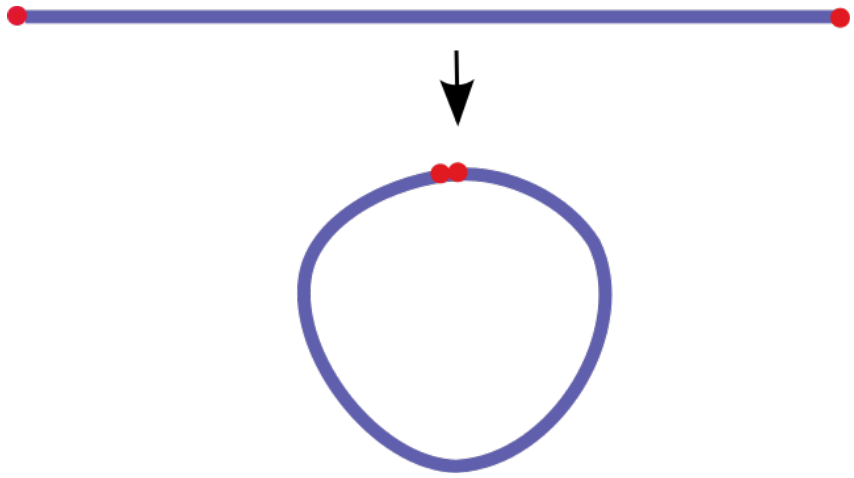


Figure 1.1: deformation of rod

1.1 Preliminary Definitions

- displacement - the total movement of a point with respect to a fixed reference coordinates is called *displacement*
- Deformation - the relative movement of a point with respect to another point on the body is called *deformation*
- Lagrangian Strain - *Lagrangian Strain* is computed from deformation by using the original undeformed geometry as the reference geometry
- Eulerian Strain - *Eulerian Strain* is computed from deformation by using the final deformed geometry as the reference geometry.
- Deflection - a term to describe the magnitude to which a structural element is displaced when subject to an applied load.

- Elongations ($L_f > L_o$) result in *positive* normal strains. Contractions ($L_f < L_o$) result in *negative* normal strains.

$$L_0 = x_B - x_A$$

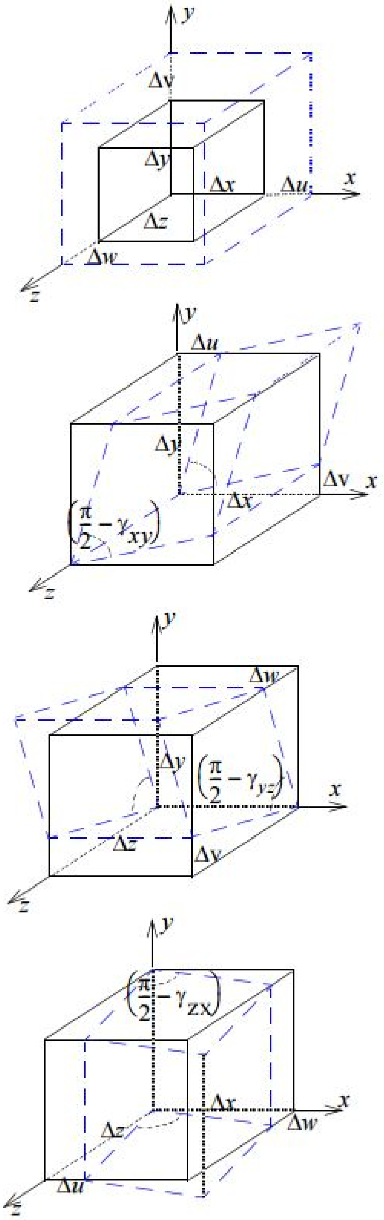
$$L_f = (x_B + u_B) - (x_A + u_A) = L_o + (u_B -$$

$$\varepsilon_{av} = \frac{u_B - u_A}{x_B - x_A}$$

Figure 1.2: Average normal strain

1.2 Average Normal Strain

1.3 Strain components



$$\varepsilon_{xx} = \frac{\Delta u}{\Delta x}$$

$$\varepsilon_{yy} = \frac{\Delta v}{\Delta y}$$

$$\varepsilon_{zz} = \frac{\Delta w}{\Delta z}$$

$$\gamma_{xy} = \frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x}$$

$$\gamma_{yx} = \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y} = \gamma_{xy}$$

$$\gamma_{yz} = \frac{\Delta v}{\Delta z} + \frac{\Delta w}{\Delta y}$$

$$\gamma_{zy} = \frac{\Delta w}{\Delta y} + \frac{\Delta v}{\Delta z} = \gamma_{yz}$$

$$\gamma_{zx} = \frac{\Delta w}{\Delta x} + \frac{\Delta u}{\Delta z}$$

$$\gamma_{xz} = \frac{\Delta u}{\Delta z} + \frac{\Delta w}{\Delta x} = \gamma_{zx}$$

Figure 1.3: strain components

1.4 Engineering Strain

- Engineering strain matrix:

$$\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{bmatrix}$$

- Plane strain matrix:

$$\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Strain at a point

$$
\begin{aligned}
\varepsilon_{xx} &= \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x} \right) \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} \\
\gamma_{xy} = \gamma_{yx} &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x} \right) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\gamma_{yz} = \gamma_{zy} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\
\gamma_{zx} = \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}
\end{aligned}
$$

1.5 linearized Cauchy strain

Our technique uses linearized Cauchy strain and stiffness warping to avoid linearization artifacts for large deformations.
The elastic energy density of a deformable body is defined in terms of **stress** and **strain** within the object. For the latter, we employ the linear Cauchy strain ϵ, which depends on the Jacobian $\Delta \mathbf{u}$ of the deformation field $\mathbf{u}$:

$$\epsilon = \frac{1}{2}(\Delta \mathbf{u} + \Delta \mathbf{u}^T) \tag{1.5.1}$$

The strain of the material in turn causes internal forces, represented by the 3×3 stress matrix σ. We assume a **Hookean material**, i.e., a linear stress-strain relation. Note that ϵ and σ both are 3×3 matrices.
Representing the stress by a 6D vector as well reduces the constitutive relation 1.5.1 to a simple 6×6 matrix product:

$$\sigma = \mathbf{C}\epsilon \tag{1.5.2}$$

where the **constitutive matrix C** only depends on the material' elasticity modulus and Poisson ratio, controlling stiffness and volume preservation.
With stress and strain defined at any material point $\mathbf{x}$, the total elastic energy $U(\mathbf{u})$ can finally be computed as the integral of stress times strain over the object's volume:

$$U(\mathbf{u}) = \frac{1}{2} \int_v \sigma^T \epsilon = \frac{1}{2} \int_V \epsilon^T \mathbf{C} \epsilon \tag{1.5.3}$$

1.6 Discretize energy function

In order to discretize the energy function 1.5.3 the continuum object is decomposed into a finite number of elements, and each node $i \in 1, \dots, n$ of this decomposition is associated wit a material position $\mathbf{x}_i$, a displacement value $\mathbf{u}_i = \mathbf{u}(\mathbf{x}_i)$, and a scalar shape function $\Phi_i(\mathbf{x})$. With this, the continuous function $\mathbf{u}(\mathbf{x})$ can be approximated by:

$$\mathbf{u}(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{u}_i \Phi_i(\mathbf{x}) \tag{1.6.1}$$

While graphics applications typically employ **tetrahedron** or **hexahedral** decomposition, our goal is to find an FEM formulation for **convex polyhedron**. This requires interpolation function Φ_i for convex polyhedra that are suitable for FEM computation.

Chapter 2

Stress

2.1 Hooke's Law

Hooke's law is a principle of physics that states that the **force** F needed to extend or compress a **spring** by some distance X is proportional to that distance. That is: $F = kX$, where k is a constant factor characteristic of the spring: its **stiffness**, and X is small compared to the total possible deformation of the spring.

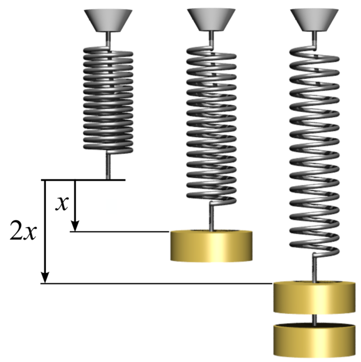


Figure 2.1: Hooke's law of spring

Hooke's law is only a **first-order linear approximation** to the real response of springs and other elastic bodies to applied forces. It must eventually fail once the forces exceed some limit, since no material can be compressed beyond a certain minimum size, or stretched beyond a maximum size, without some permanent deformation or change of state. Many materials will noticeably deviate from Hooke's law well before those **elastic limits** are reached.

On the other hand, Hooke's law is an accurate approximation for most solid bodies, as long as the forces and deformations are small enough. For this reason, Hooke's law is extensively used in all branches of science and engineering, and is the foundation of many disciplines such as seismology, molecular mechanics and acoustics. It is also the fundamental principle behind the spring scale, the manometer, and the balance wheel of the mechanical clock.

The modern theory of elasticity generalizes Hooke's law to say that the **strain (deformation)** of an elastic object or material is proportional to the **stress** applied to it. However, since general stresses and strains may have multiple independent components, the "proportionality factor" may no longer be just a single real number, but rather a linear map (a tensor) that can be represented by a matrix of real numbers.

In this general form, Hooke's law makes it possible to deduce the relation between strain and stress for complex objects in terms of intrinsic properties of the materials it is made of. For example, one can deduce that a homogeneous rod with uniform cross section will behave like a simple spring when stretched, with a stiffness k directly proportional to its cross-section area and inversely proportional to its length.

2.2 Stiffness Tensor for continuous media

The stresses and strains of the material inside a continuous elastic material (such as a block of rubber, the wall of a boiler, or a steel bar) are connected by a linear relationship that is mathematically similar to Hooke's spring law, and is often referred to by that name.
However, the strain state in a solid medium around some point cannot be described by a single vector. The same parcel of material, no matter how small, can be compressed, stretched, and sheared at the time, along different directions. Likewise, the stresses in that parcel can be at once pushing, pulling and shearing.
In order to capture this complexity, the relevant state of the medium around a point must be represented by two second-order tensors, the **strain tensor** ε (in lieu of the displacement X) and the **stress tensor** σ (replacing the restoring force F). The analogous of Hooke's spring law for continuous media is then

$$\sigma = -\mathbf{C}\varepsilon \tag{2.2.1}$$

The symmetry of the **Cauchy stress tensor** ($\sigma_{ij} = \sigma_{ji}$) and the generalized Hooke's laws:

$$\sigma_{ij} = c_{ijkl}\varepsilon_k l \tag{2.2.2}$$

implies that $c_{ijkl} = c_{jikl}$. Similarly, the symmetry of the **infinitesimal strain tensor** implies that $c_{ijkl} = c_{ijlk}$. These symmetries are called the **minor symmetries** of the **stiffness tensor** (**C**). This reduces the number of elastic constants from 81 to 36.
If in addition, since the displacement gradient and the Cauchy stress are work conjugate, the stress-strain relation can be deprived from a strain energy density function (U), then

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} \implies c_{ijkl} = \frac{\partial^2 U}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}} \tag{2.2.3}$$

The arbitrariness of the order of differentiation implies that $c_{ijkl} = c_{klij}$. These are called the **major symmetries** of the stiffness tensor. This reduces the number of elastic constants to 21 from 36. The major and minor symmetries indicate that the stiffness tensor has only 21 independent components.
where **C** is a forth-order tensor (that is, a linear map between second-order tensors) called the **stiffness tensor** or **elasticity tensor**. One may also write as

$$\varepsilon = -\mathbf{S}\sigma \tag{2.2.4}$$

where the tensor **S**, called the compliance tensor, represents the inverse of said linear map.
In a Cartesian coordinate system, the stress and strain tensors can be represented by 3×3 matrices

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \qquad \sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{2.2.5}$$

Being a linear mapping between the nine numbers σ_{ij} and the nine numbers ε_{kl}, the stiffness tensor **C** is represented by a matrix of $3 \times 3 \times 3 \times 3 = 81$ real numbers C_{ijkl}. Hooke's law then says that

$$\sigma_{ij} = -\sum_{k=1}^{3}\sum_{l=1}^{3} C_{ijkl}\varepsilon_{kl} \tag{2.2.6}$$

where i and j are 1, 2, or 3.
All three tensors generally vary from point to point inside the medium, and may vary with time as well. The strain tensor ϵ merely specifies the displacement of the medium particles in the neighborhood of the point, while the stress tensor σ specifies the forces that neighboring parcels of the medium are exerting on each other. Therefore, they are independent of the composition and physical state of the material. The stiffness tensor **C** , on the other hand, is a property of the material, and often depends on physical state variables such as temperature, pressure, and microstructure.
Due to the inherent symmetries of σ, ϵ , and **C** , only 21 elastic coefficients of the latter are independent. For isotropic media (which have the same physical properties in any direction), **C** can be reduced to only two independent numbers, the bulk modulus K and the shear modulus G , that quantify the material's resistance to changes in volume and to shearing deformations, respectively.

2.3 Principal stresses

The majority of material strength data is based on uniaxial tensile test results. Usually, allthat you have to work with is the yield strength S_y and/or the ultimate tensile strength S_u.

This is fine if you only have the one normal stress component present: this is true for simple tension or compression members and for parts loaded only in bending.

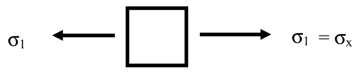


Figure 2.2: uniaxial stress

In this case, failure (defined as the onset of plastic deformation) occurs when

$$\sigma_x = \sigma_1 = \frac{S_y}{n} \tag{2.3.1}$$

n is the factor of safety.
In many loading cases, we have more than just one normal stress component.E.g. in torsion, we have a single shear stress component:

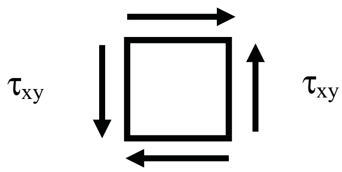


Figure 2.3: shear stress

Or, combined bending and torsion in a shaft:

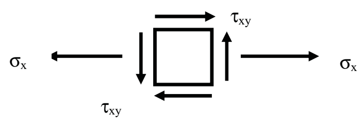


Figure 2.4: combined stress

These cases can all be reduced to a simple biaxial case by finding the principal stresses, σ_1 and σ_2:

2.4 Mises stress

$$\sigma_v^2 = \frac{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6\left(\sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2\right)}{2} \tag{2.4.1}$$

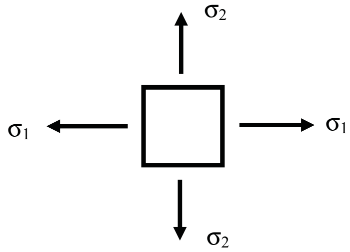


Figure 2.5: principal stress

Chapter 3

Plate

3.1 Constant strain triangle

When a flat plate is subjected to both inplane and transverse or normal loads as shown in Figure 3.1 any point inside the plate can have displacement components u, v, and w parallel to x, y, and z axes, respectively. In the small deflection (or linear) theory of thin plates, the transverse deflection w is uncoupled from the inplane deflections u and v. Consequently, the stiffness matrices for the inplane and transverse deflections are also uncoupled and they can be calculated independently. Thus, if a plate is subjected to inplane loads only, it will undergo deformation in its plane only. In this case, the plate is said to be under the action of "membrane" forces. Similarly, if the plate is subjected to transverse loads (and/or bending moments), any point inside the plate experiences essentially a lateral displacement w (inplane displacements u and v, are also experienced because of the rotation of the plate element). In this case. the plate is said to be under the action of bending forces. The inplane and bending analysis of plates is considered in this chapter. If the plate elements are used for the analysis of three-dimensional structures. such as folded plate structures, both inplane and bending actions have to be considered in the development of element properties. This aspect of coupling the membrane and bending actions of a plate element is also considered in this chapter.

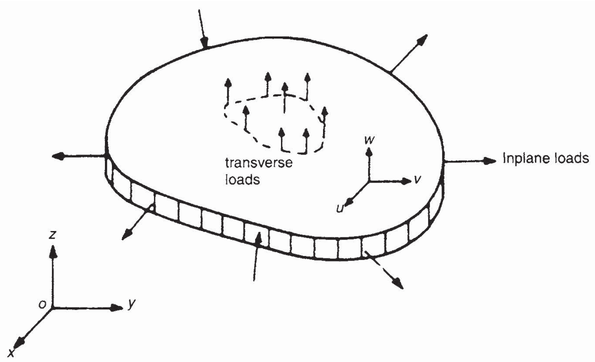


Figure 3.1: Inplane and transverse loads in a plane

3.1.1 Displacement and shape function

The triangular membrane element is considered to lie in the xy plane of a local xy coordinate system as shown in Figure 3.2. The nodes are arranged as i, j and$m$ in anti-clockwise. Each node includes 2 degrees of freedom . The displacement components can be expressed as:

$$\mathbf{q}^{(e)} = \begin{bmatrix} u_i & v_i & u_j & v_j & u_m & v_m \end{bmatrix} \tag{3.1.1}$$

By assuming a linear displacement variation inside the element and introducing a set of area coordinates, the displacement model can be expressed as:

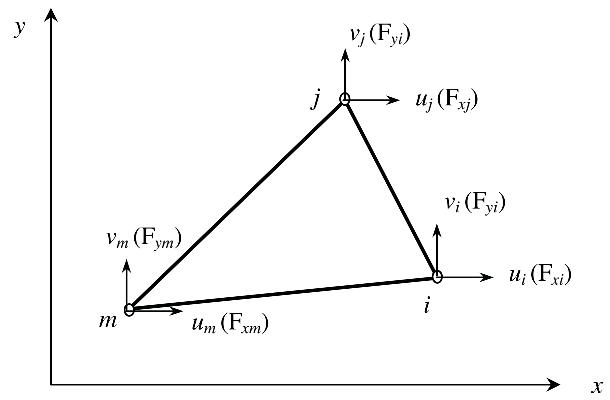


Figure 3.2: Triangular membrane element

$$
\begin{aligned}
u &= a_1 + a_2x + a_3y \\
v &= a_4 + a_5x + a_6y
\end{aligned}
\tag{3.1.2}
$$

By considering the displacements u_i and v_i as the local degrees of freedom of node i, j, m, the constants $a_1, \cdots, a_6$ can be evaluated. Thus, by using the conditions

$$
\begin{aligned}
u_i &= a_1 + a_2x_i + a_3y_i & \qquad v_i &= a_4 + a_5x_i + a_6y_i \\
u_j &= a_1 + a_2x_j + a_3y_j & \qquad v_j &= a_4 + a_5x_j + a_6y_j \\
u_m &= a_1 + a_2x_m + a_3y_m & \qquad v_m &= a_4 + a_5x_m + a_6y_m
\end{aligned}
$$

We can express the constants $a_1, \cdots, a_6$ in terms of the nodal degrees of freedom:

$$
a_1 = \frac{1}{2\Delta}\begin{vmatrix} u_i & x_i & y_i \\ u_j & x_j & y_j \\ u_m & x_m & y_m \end{vmatrix}
\qquad
a_2 = \frac{1}{2\Delta}\begin{vmatrix} 1 & u_i & y_i \\ 1 & u_j & y_j \\ 1 & u_m & y_m \end{vmatrix}
\qquad
a_3 = \frac{1}{2\Delta}\begin{vmatrix} 1 & x_i & u_i \\ 1 & x_j & u_j \\ 1 & x_m & u_m \end{vmatrix}
$$

Where

$$
2\Delta = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}
$$

This leads to the displacement model:

$$
\begin{aligned}
\vec{U} &= \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} N \end{bmatrix} \vec{q}^{(e)} \\
&= \begin{bmatrix} N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) & 0 \\ 0 & N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= x_jy_m - x_my_j & \qquad a_2 &= x_my_i - x_iy_m & \qquad a_3 &= x_iy_j - x_jy_i \\
b_1 &= y_j - y_m & \qquad b_2 &= y_m - y_i & \qquad b_3 &= y_i - y_j \\
c_1 &= x_m - x_j & \qquad c_2 &= x_i - x_m & \qquad c_3 &= x_j - x_i \\
N_i &= \frac{(a_i + b_ix + c_iy)}{2\Delta} & \qquad N_j &= \frac{(a_j + b_jx + c_jy)}{2\Delta} & \qquad N_m &= \frac{(a_m + b_mx + c_my)}{2\Delta}
\end{aligned}
$$

3.1.2 element Strain

By using the relations:

$$\epsilon = \left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{array} \right\} = \left\{ \begin{array}{c} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{array} \right\} \tag{3.1.3}$$

the components of strain can be expressed in terms of nodal displacements as:

$$\epsilon = \mathbf{B}\mathbf{q}^{(e)} \tag{3.1.4}$$

where

$$\mathbf{B} = \frac{1}{2\Delta} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \tag{3.1.5}$$

3.1.3 element stress

The stress-strain relations are given by:

$$\sigma = \mathbf{D}\epsilon \tag{3.1.6}$$

where

$$\sigma = \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} \tag{3.1.7}$$

and

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{3.1.8}$$

3.1.4 element stiffness

the stiffness matrix of the element $\mathbf{K}_e$ can be found by using:

$$\mathbf{K}_e = \iiint_\Omega \mathbf{B}^T\mathbf{D}\mathbf{B}d\Omega = \iint_\Delta \mathbf{B}^T\mathbf{D}\mathbf{B}hd\Delta \tag{3.1.9}$$

where Ω denotes the volume of the element. If the plate thickness is taken as a constant h, the evaluation of the integral in eq 3.1.9 presents no difficult since the elements of the matrices **B** and **D** are all constants (not functions of x and y). Hence, Eq 3.1.9 can be rewritten as:

$$\mathbf{K}_e = \mathbf{B}^T\mathbf{D}\mathbf{B}h \iint_\Delta d\Delta = h\Delta\mathbf{B}^T\mathbf{D}\mathbf{B} \tag{3.1.10}$$

3.2 Discrete Kirchhoff triangle

A large number of plate t)ending elements tmve beell developed and reported in the literature. In the classical theory of thin plates discussed in this section, certain simplifying approximations are made. One of the important assumptions made is that shear deformation is negligible. Some elements have been developed by including the effect of transverse shear deformation also.

references

According to thin plate theory, the deformation is completely described by the transverse deflection of the middle surface of the plate (w) only. Thus. if a displacement model is assumed for w. the continuity of not only w tut also its derivatives has to be maintained between adjacent elements.

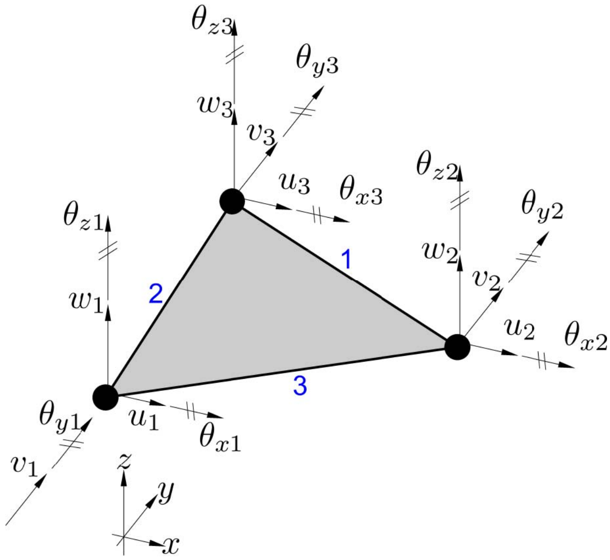


Figure 3.3: Degrees of freedom of triangle element

3.2.1 Displacement model

At each node of triangular plate element shown in figure 3.3. the transverse displacement w and rotations about the x and y axes are taken as the degrees of freedom. Since there are nine displacement degrees of freedom in the element, the assumed polynomial for $w(x, y)$ must also contain nine constant terms. To maintain geometric isotropy, the displacement model is taken as:

$$w = a_1L_1+a_2L_2+a_3L_3+a_4L_2L_3+a_5L_3L_1+a_6L_1L_2+a_7(L_2L_3^2-L_3L_2^2)+a_8(L_3L_1^2-L_1L_3^2)+a_9(L_1L_2^2-L_2L_1^2) \tag{3.2.1}$$

The first three terms represent rigid displacements and terms 3 ˜6 correspond to constant strain. The constants $(a_1, a_2, \cdots, a_9)$ have to be determined from the nodal conditions:
substitute area coordinates of nodes into eq 3.2.1, we have:

$$a_1 = w_1 \quad a_2 = w_2 \quad a_3 = w_3 \tag{3.2.2}$$

By computing derivatives of deflection w to area coordinates:

$$\begin{aligned}
\frac{\partial w}{\partial L_1} &= w_1 - w_3 - a_4L_2 + a_5(L_3 - L_1) + a_6L_2 + a_7(L_2^2 - 2L_2L_3) + a_8(4L_1L_3 - L_1^2 - L_3^2) + a_9(L_2^2 - 2L_1L_2) \\
\frac{\partial w}{\partial L_2} &= w_2 - w_3 + a_4(L_3 - L_2) - a_5L_1 + a_6L_1 + a_7(L_3^2 + L_2^2 - 4L_2L_3) + a_8(2L_1L_3 - L_1^2) + a_9(2L_1L_2 - L_1^2)
\end{aligned} \tag{3.2.3}$$

In the equation, $w_{,L_{ij}}$ means derivative of deflection w to area coordinate L_i at node j. The constants can be solved:

$$\begin{aligned}
a_4 &= \frac{w_{,L_{23}} - w_{,L_{22}}}{2} \\
a_5 &= \frac{w_{,L_{13}} - w_{,L_{11}}}{2} \\
a_6 &= \frac{w_{,L_{12}} + w_{,L_{21}} - w_{,L_{11}} - w_{,L_{22}}}{2} \\
a_7 &= w_3 - w_2 + \frac{w_{,L_{23}} + w_{,L_{22}}}{2} \\
a_8 &= w_1 - w_3 - \frac{w_{,L_{11}} + w_{,L_{13}}}{2} \\
a_9 &= w_2 - w_1 + \frac{w_{,L_{11}} + w_{,L_{12}} - w_{,L_{21}} - w_{,L_{22}}}{2}
\end{aligned} \tag{3.2.4}$$

Note the L_3 is not a dependent variable, it should be regarded as $1 - L_1 - L_2$. substitue area coordinates of three nodes into equation 3.2.3, we get:

$$
\begin{aligned}
w_{,L_{11}} &= w_1 - w_3 - a_5 - a_8 & w_{,L_{21}} &= w_2 - w_3 - a_5 + a_6 - a_8 - a_9 \\
w_{,L_{12}} &= w_1 - w_3 - a_4 + a_6 + a_7 + a_9 & w_{,L_{22}} &= w_2 - w_3 - a_4 + a_7 \\
w_{,L_{13}} &= w_1 - w_3 + a_5 - a_8 & w_{,L_{23}} &= w_2 - w_3 + a_4 + a_7
\end{aligned}
$$

substitute eq 3.2.2 and 3.2.4 into eq 3.2.1, it gives:

$$
w = \begin{bmatrix} \bar{\mathbf{N}}_i & \bar{\mathbf{N}}_j & \bar{\mathbf{N}}_m \end{bmatrix} \begin{bmatrix} \bar{\delta}_i \\ \bar{\delta}_j \\ \bar{\delta}_m \end{bmatrix} \tag{3.2.5}
$$

where:

$$
\begin{aligned}
\bar{\delta}_1 &= \begin{bmatrix} w_1 & w_{,L_{11}} & w_{,L_{21}} \end{bmatrix} & \bar{\delta}_2 &= \begin{bmatrix} w_2 & w_{,L_{12}} & w_{,L_{21}} \end{bmatrix} & \bar{\delta}_3 &= \begin{bmatrix} w_3 & w_{,L_{13}} & w_{,L_{23}} \end{bmatrix} \\
\bar{\mathbf{N}}_1 &= \begin{bmatrix} N_1 & N_{,L_{11}} & N_{,L_{21}} \end{bmatrix} & \bar{\mathbf{N}}_2 &= \begin{bmatrix} N_2 & N_{,L_{12}} & N_{,L_{21}} \end{bmatrix} & \bar{\mathbf{N}}_3 &= \begin{bmatrix} N_3 & N_{,L_{13}} & N_{,L_{23}} \end{bmatrix}
\end{aligned}
$$

the components of the shape function are:

$$
\begin{aligned}
N_1 &= L_1 - (L_1 L_2^2 - L_2 L_1^2) + (L_3 L_1^2 - L_1 L_3^2) \\
N_{L_{11}} &= \frac{-L_1 L_2 - L_3 L_1 + (L_1 L_2^2 - L_2 L_1^2) - (L_3 L_1^2 - L_1 L_3^2)}{2} \\
N_{L_{21}} &= \frac{L_1 L_2 + (L_2 L_1^2 - L_1 L_2^2)}{2} \\
N_2 &= L_2 - (L_2 L_3^2 - L_3 L_2^2) + (L_1 L_2^2 - L_2 L_1^2) \\
N_{L_{12}} &= \frac{L_1 L_2 + (L_1 L_2^2 - L_2 L_1^2)}{2} \\
N_{L_{22}} &= \frac{-L_2 L_3 - L_1 L_2 + (L_2 L_3^2 - L_3 L_2^2) - (L_1 L_2^2 - L_2 L_1^2)}{2} \\
N_3 &= L_3 - (L_3 L_1^2 - L_1 L_3^2) + (L_2 L_3^2 - L_3 L_2^2) \\
N_{L_{13}} &= \frac{L_1 L_3 + (L_1 L_3^2 - L_3 L_i^2)}{2} \\
N_{L_{23}} &= \frac{L_2 L_3 + (L_2 L_3^2 - L_3 L_2^2)}{2}
\end{aligned}
$$

as we have:

$$
\begin{aligned}
\frac{\partial x}{\partial L_1} &= x_1 - x_3 = c_1 \\
\frac{\partial y}{\partial L_1} &= y_1 - y_3 = -b_1 \\
\frac{\partial x}{\partial L_2} &= x_2 - x_3 = -c_1 \\
\frac{\partial y}{\partial L_2} &= x_2 - x_3 = b_1
\end{aligned} \tag{3.2.6}
$$

this leads to

$$
\begin{aligned}
\frac{\partial w}{\partial L_1} &= c_2 \frac{\partial w}{\partial x} - b_2 \frac{\partial w}{\partial y} = -b_2 \theta_x - c_2 \theta_y \\
\frac{\partial w}{\partial L_2} &= -c_2 \frac{\partial w}{\partial x} + b_2 \frac{\partial w}{\partial y} = b_2 \theta_x + c_2 \theta_y
\end{aligned} \tag{3.2.7}
$$

The relation of nodal displacements between cardisian system and area system is:

$$\bar{\delta}_i = \begin{bmatrix} w_i \\ w_{,L_{11}} \\ w_{,L_{21}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -b_2 & -c_2 \\ 0 & b_1 & c_1 \end{bmatrix} \begin{bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{bmatrix} = \mathbf{P}\delta_i \tag{3.2.8}$$

Use eq 3.2.8, the eq 3.2.5 can be re-written as:

$$w = \begin{bmatrix} \bar{\mathbf{N}}_1 & \bar{\mathbf{N}}_2 & \bar{\mathbf{N}}_3 \end{bmatrix} \begin{bmatrix} \mathbf{P} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} \tag{3.2.9}$$

where

$$\begin{aligned}
\mathbf{N}_i &= \bar{\mathbf{N}}_i \mathbf{P} = \begin{bmatrix} N_i & N_{xi} & N_{yi} \end{bmatrix} \\
N_i &= L_i + L_i^2 L_j + L_i^2 L_m - L_i L_j^2 - L_i L_m^2 \quad (i = 1, 2, 3) \\
N_{xi} &= b_j L_i^2 L_m - b_m L_i^2 L_j + \frac{(b_j - b_m)}{2} L_i L_j L_m \\
N_{yi} &= c_j L_i^2 L_m - c_m L_i^2 L_j + \frac{c_j - c_m}{2} L_i L_j L_m
\end{aligned}$$

3.2.2 Stiffness matrix

In eq **??**, the shape function is using L_1 and L_2 as independent variables. To obtain stiffness matrix in global space, transformation matrix is introduced:

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} \partial/\partial L_1 \\ \partial/\partial L_2 \end{bmatrix} \tag{3.2.10}$$

and

$$\begin{bmatrix} \partial^2/\partial x^2 \\ \partial^2/\partial y^2 \\ 2\partial^2/\partial x \partial y \end{bmatrix} = \frac{1}{4\Delta^2} \mathbf{T} \begin{bmatrix} \partial^2/\partial L_1^2 \\ \partial^2/\partial L_2^2 \\ \partial^2/\partial L_1 \partial L_2 \end{bmatrix} \tag{3.2.11}$$

where

$$\mathbf{T} = \begin{bmatrix} b_1^2 & b_2^2 & 2b_1 b_2 \\ c_1^2 & c_2^2 & 2c_1 c_2 \\ 2b_1 c_1 & 2b_2 c_2 & 2(b_1 c_2 + b_2 c_1) \end{bmatrix} \tag{3.2.12}$$

the element strain matrix is now:

$$\epsilon = \mathbf{B}\delta^{\mathbf{e}} = z \begin{bmatrix} \mathbf{B}_i & \mathbf{B}_j & \mathbf{B}_m \end{bmatrix} \begin{bmatrix} \delta_i \\ \delta_j \\ \delta_m \end{bmatrix} \tag{3.2.13}$$

where

$$\mathbf{B}_k = -\begin{bmatrix} \mathbf{N}_{k,xx} \\ \mathbf{N}_{k,yy} \\ \mathbf{N}_{k,xy} \end{bmatrix} = -\frac{1}{4\Delta^2} \mathbf{T} \begin{bmatrix} \mathbf{N}_{k,11} \\ \mathbf{N}_{k,22} \\ \mathbf{N}_{k,12} \end{bmatrix} \quad (k = 1, 2, 3) \tag{3.2.14}$$

and the element stiffness matrix in global space is:

$$\mathbf{K}^e = \frac{h^3}{12} \iint_\Omega \mathbf{B}^T \mathbf{D} \mathbf{B} dxdy \tag{3.2.15}$$

In eq 3.2.15, n is number of gauss points, W_i is weight, (L_{i1}, L_{i2}, L_{i3}) is integration point.

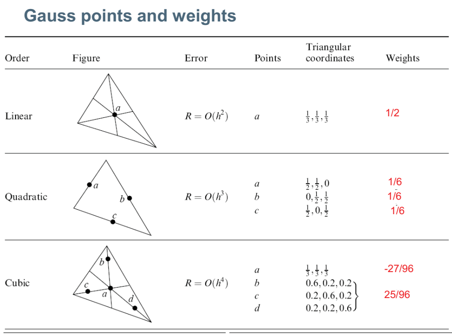
3.2.3 Gauss quadrature

To simplify computation, we use quadrature for solving stiffness integral:
Firstly, the integral over triangular domain Ω is converted to:

$$\iint_{\Omega} f(L_1, L_2, L_3)dxdy = 2\Delta \int_0^1 \int_0^{1-L_1} f(L_1, L_2, L_3)dL_1 dL_2 \tag{3.2.16}$$

the right term can be calcultated using:

$$\int_0^1 \int_0^{1-L_1} f(L_1, L_2, L_3)dL_1 dL_2 = \sum_{i=1}^{n} W_i f(L_{i1}, L_{i2}, L_{i3}) \tag{3.2.17}$$



Gauss points and weights

Order	Figure	Error	Points	Triangular coordinates	Weights
Linear		$R = O(h^2)$	a	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$	1/2
Quadratic		$R = O(h^3)$	a b c	$\frac{1}{2}, \frac{1}{2}, 0$ $0, \frac{1}{2}, \frac{1}{2}$ $\frac{1}{2}, 0, \frac{1}{2}$	1/6 1/6 1/6
Cubic		$R = O(h^4)$	a b c d	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ 0.6, 0.2, 0.2 0.2, 0.6, 0.2 0.2, 0.2, 0.6	-27/96 25/96

Figure 3.4: gauss points and weights

3.3 Drilling degree. to be added

3.4 Transformation Matrix

The shape functions are defined in the plane of the triangle, i.e. z-coordinates for the nodes are equal to zero

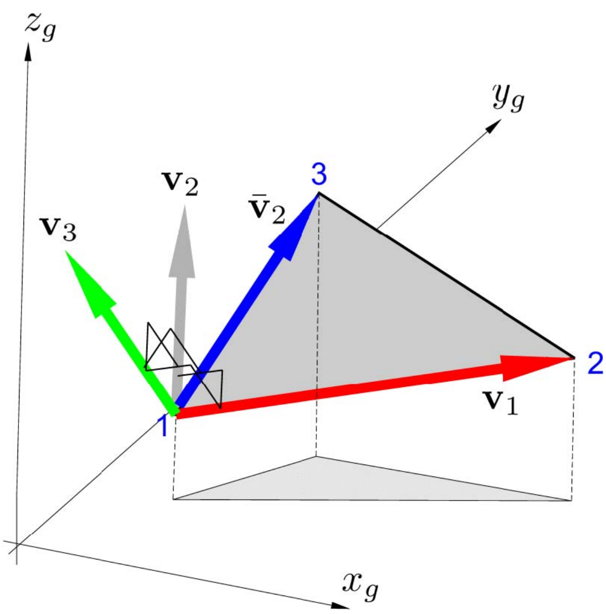


Figure 3.5: transformation matrix in 3d

Assuming that the triangular element under consideration is an interior element of a large structure. let the node numbers 1, 2, and 3 of the element correspond to the node numbers i, j. and k, respectively, of the global system.

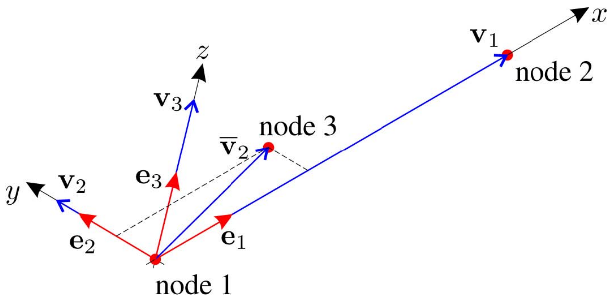


Figure 3.6: unit vectors describing xyz sysem.

Then place the origin of the local xg system at node 1 (node i), and take the y axis along the edge 1 2 (edge ij) and the x axis perpendicular to the y axis directed toward node 3 (node k) as shown in Figure 3.6.
We have:

$$\mathbf{v}_1 = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$$

$$\bar{\mathbf{v}}_2 = \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{v}_1 \times \bar{\mathbf{v}}_2$$

$$\mathbf{v}_2 = \mathbf{v}_3 \times \mathbf{v}_1$$

the unit vectors can be expressed:

$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|}$$

$$\mathbf{e}_2 = \frac{\mathbf{v}_2}{|\mathbf{v}_2|}$$

$$\mathbf{e}_3 = \frac{\mathbf{v}_3}{|\mathbf{v}_3|}$$

So the transformation matrix is:

$$\mathbf{T} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \tag{3.4.1}$$

$$\begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = \mathbf{T}^T \begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix} \tag{3.4.2}$$

3.5 global element stiffness matrix

The full transformation matrix (12x12) can be made from $T(3x3)$ and multiply the local element stiffness matrix with the transformation to obtain the global element stiffness matrix:

$$\mathbf{T}_g = \begin{bmatrix} \mathbf{T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{T} \end{bmatrix} \tag{3.5.1}$$

$$\mathbf{K}_{eg} = \mathbf{T}_g \mathbf{K}_e \mathbf{T}_g^T \tag{3.5.2}$$

Chapter 4

Space beam

4.1 Introduction

short introducti

4.2 Displacement model

Each node of beam element has 6 degrees of freedom, corresponding to 6 nodal forces. For a beam of structure with nodes i and j, it isshown in figure 4.1. In right hand coordinate system, take x axis as element axis then y and z axes become main moments axes of cross-section.

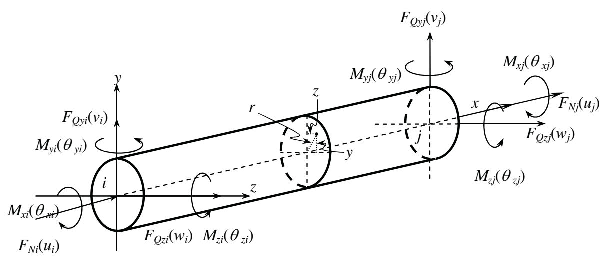


Figure 4.1: Beam element

The nodal displacements are:

$$\delta_i = \begin{bmatrix} u_i & v_i & w_i & \theta_{xi} & \theta_{yi} & \theta_{zi} \end{bmatrix}^T \quad \delta_j = \begin{bmatrix} u_j & v_j & w_j & \theta_{xj} & \theta_{yj} & \theta_{zj} \end{bmatrix}^T \tag{4.2.1}$$

or simplier:

$$\delta^e = \begin{bmatrix} \delta_i^T & \delta_j^T \end{bmatrix}^T \tag{4.2.2}$$

By assuming a linear interpolation for axial transplacement u and angular rotation θ_x, and deflection v and w are represented in cubic terms, so we have:

$$\begin{aligned} u &= a_0 + a_1 x \\ v &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 \\ w &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 \\ \theta_x &= e_0 + e_1 x \end{aligned} \tag{4.2.3}$$

Meanwhile, we can write axial nodal displacements, deflections and rotations in a vector form:

$$\begin{aligned}\delta_u &= \begin{bmatrix} u_i & u_j \end{bmatrix}^T \\ \delta_v &= \begin{bmatrix} v_i & \theta_{zj} & v_j & \theta_{zj} \end{bmatrix}^T \\ \delta_w &= \begin{bmatrix} w_i & \theta_{yj} & w_j & \theta_{wj} \end{bmatrix}^T \\ \delta_\theta &= \begin{bmatrix} \theta_{xi} & \theta_{xj} \end{bmatrix}^T \end{aligned} \tag{4.2.4}$$

Now eq 4.2.3 becoms:

$$u = \mathbf{N}_u \delta_u \quad \theta_x = \mathbf{N}_\theta \delta_\theta \quad v = \mathbf{N}_v \delta_v \quad w = \mathbf{N}_w \delta_w \tag{4.2.5}$$

Any displacement inside the element is expressed as:

$$\mathbf{u} = \begin{bmatrix} u \\ v \\ w \\ \theta_x \end{bmatrix} = \begin{bmatrix} \mathbf{H}_u \\ \mathbf{H}_v \\ \mathbf{H}_w \\ \mathbf{H}_\theta \end{bmatrix} \mathbf{A}^{-1} \delta^e = \mathbf{N} \delta^e \tag{4.2.6}$$

where:

$$\mathbf{H}_u(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.2.7}$$

$$\mathbf{H}_v(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & x & 0 & x^2 & 0 & 0 & 0 & x^3 \end{bmatrix} \tag{4.2.8}$$

$$\mathbf{H}_w(x) = \begin{bmatrix} 0 & 0 & 1 & 0 & x & 0 & 0 & 0 & x^2 & 0 & x^3 & 0 \end{bmatrix} \tag{4.2.9}$$

$$\mathbf{H}_\theta(x) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \end{bmatrix} \tag{4.2.10}$$

and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & l & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & l & 0 & l^2 & 0 & 0 & 0 & l^3 \\ 0 & 0 & 1 & 0 & l & 0 & 0 & 0 & l^2 & 0 & l^3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & l & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2l & 0 & 3l^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2l & 0 & 0 & 0 & 3l^2 \end{bmatrix} \tag{4.2.11}$$

4.3 strain and stress

When beam undergoes tensile, compressional, bending and torsional deformation, it's axial strain is composed of three parts: tensile or compressional strain ϵ_0, bending strain ϵ_{by} and ϵ_{bz}. The shear strain caused by torsion is γ. Hence,

$$\epsilon = \begin{bmatrix} \epsilon_0 \\ \epsilon_{by} \\ \epsilon_{bz} \\ \gamma \end{bmatrix} = \begin{bmatrix} u' \\ -yv'' \\ -zw'' \\ r\theta'_x \end{bmatrix} = \begin{bmatrix} \mathbf{H}'_u \\ -y\mathbf{H}''_v \\ -z\mathbf{H}''_w \\ r\mathbf{H}'_\theta \end{bmatrix} \mathbf{A}^{-1} \delta^e = \mathbf{B} \delta^e \tag{4.3.1}$$

where y and z are point in the cross-section, and r is the distance to x axis.
By using Hooke's law, the stress can be expressed:

$$\sigma = \begin{bmatrix} \sigma_0 \\ \sigma_{by} \\ \sigma_{bz} \\ \tau \end{bmatrix} = \begin{bmatrix} E\mathbf{H}'_u \\ -Ey\mathbf{H}''_v \\ -Ez\mathbf{H}''_w \\ Gr\mathbf{H}'_\theta \end{bmatrix} \mathbf{A}^{-1} \delta^e = \mathbf{D}\mathbf{B} \delta^e \tag{4.3.2}$$

where

$$\mathbf{D} = \mathrm{diag}(E, E, EG) \tag{4.3.3}$$

4.4 stiffness matrix

The stiffness matrix now can be integrated expressively:

$$\mathbf{K}^e = \iint dA \int_0^l \mathbf{B}^T \mathbf{D} \mathbf{B} dx$$
$$= \begin{bmatrix}
\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{l^3} & 0 & 0 & 0 & \frac{6EI_z}{l^2} & 0 & -\frac{12EI_z}{l^3} & 0 & 0 & 0 & \frac{6EI_z}{l^2} \\
0 & 0 & \frac{12EI_y}{l^3} & 0 & \frac{6EI_y}{l^2} & 0 & 0 & 0 & -\frac{12EI_y}{l^3} & 0 & \frac{6EI_y}{l^2} & 0 \\
0 & 0 & 0 & \frac{GJ_k}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ_k}{l} & 0 & 0 \\
0 & 0 & \frac{6EI_y}{l^2} & 0 & \frac{4EI_y}{l} & 0 & 0 & 0 & -\frac{6EI_y}{l^2} & 0 & \frac{2EI_y}{l} & 0 \\
0 & \frac{6EI_z}{l^2} & 0 & 0 & 0 & \frac{4EI_z}{l} & 0 & -\frac{6EI_z}{l^2} & 0 & 0 & 0 & \frac{2EI_z}{l} \\
-\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{l} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{l^3} & 0 & 0 & 0 & -\frac{6EI_z}{l^2} & 0 & \frac{12EI_z}{l^3} & 0 & 0 & 0 & -\frac{6EI_z}{l^2} \\
0 & 0 & -\frac{12EI_y}{l^3} & 0 & -\frac{6EI_y}{l^2} & 0 & 0 & 0 & \frac{12EI_y}{l^3} & 0 & -\frac{6EI_y}{l^2} & 0 \\
0 & 0 & 0 & -\frac{GJ_k}{l} & 0 & 0 & 0 & 0 & 0 & \frac{GJ_k}{l} & 0 & 0 \\
0 & 0 & \frac{6EI_y}{l^2} & 0 & \frac{2EI_y}{l} & 0 & 0 & 0 & -\frac{6EI_y}{l^2} & 0 & \frac{4EI_y}{l} & 0 \\
0 & \frac{6EI_z}{l^2} & 0 & 0 & 0 & \frac{2EI_z}{l} & 0 & -\frac{6EI_z}{l^2} & 0 & 0 & 0 & \frac{4EI_z}{l}
\end{bmatrix} \tag{4.4.1}$$

In eq 4.4.1, $I_y = \iint z^2 dA$ and $I_z = \iint y^2 dA$ are main moments of beam cross-section with respect to y and z axes respectively, J_k is second area moments about x axis.

4.5 Transformation matrix

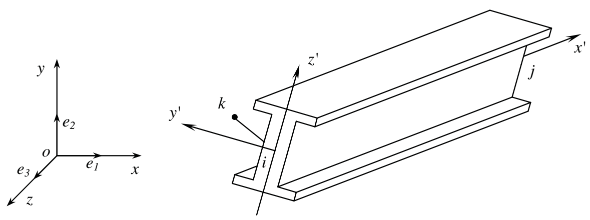


Figure 4.2: transformation matrix

4.6 variable beam cross-section... to be added

Chapter 5

Failure theories

In materials science, the strength of a material is its ability to withstand an applied load without failure or plastic deformation. The field of strength of materials deals with forces and deformations that result from their acting on a material. A load applied to a mechanical member will induce internal forces within the member called stresses when those forces are expressed on a unit basis. The stresses acting on the material cause deformation of the material in various manners. Deformation of the material is called strain when those deformations too are placed on a unit basis. The applied loads may be axial (tensile or compressive), or rotational (strength shear). The stresses and strains that develop within a mechanical member must be calculated in order to assess the load capacity of that member. This requires a complete description of the geometry of the member, its constraints, the loads applied to the member and the properties of the material of which the member is composed. With a complete description of the loading and the geometry of the member, the state of stress and of state of strain at any point within the member can be calculated. Once the state of stress and strain within the member is known, the strength (load carrying capacity) of that member, its deformations (stiffness qualities), and its stability (ability to maintain its original configuration) can be calculated. The calculated stresses may then be compared to some measure of the strength of the member such as its material yield or ultimate strength. The calculated deflection of the member may be compared to a deflection criteria that is based on the member's use. The calculated buckling load of the member may be compared to the applied load. The calculated stiffness and mass distribution of the member may be used to calculate the member's dynamic response and then compared to the acoustic environment in which it will be used.

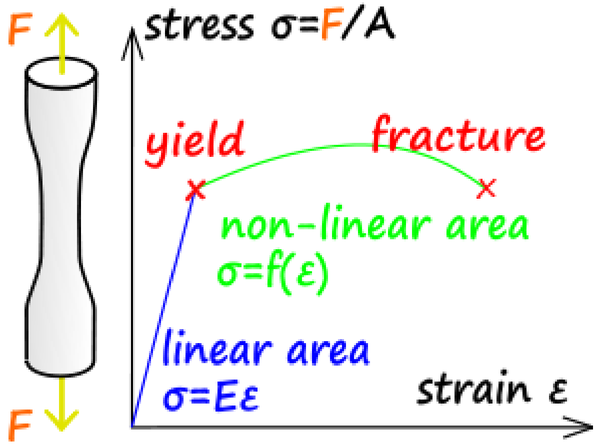


Figure 5.1: Basic static response of a specimen under tension

Material strength refers to the point on the engineering stress–strain curve (yield stress) beyond which the material experiences deformations that will not be completely reversed upon removal of the loading and as a result the member will have a permanent deflection. The ultimate strength refers to the point on the engineering stress–strain curve corresponding to the stress that produces fracture.

5.1 Strength terms

Uniaxial stress is expressed by

$$\sigma = \frac{F}{A} \tag{5.1.1}$$

where F is the $force[N]$ acting on an area $A[m^2]$. The area can be the undeformed area or the deformed area, depending on whether engineering stress or true stress is of interest.

Mechanical properties of materials include the yield strength, tensile strength, fatigue strength, crack resistance, and other characteristics.

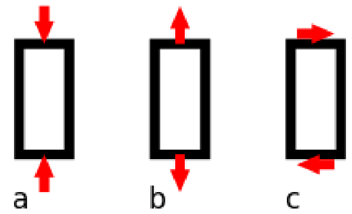


Figure 5.2: A material being loaded in a) compression, b) tension, c) shear

- Yield strength is the lowest stress that produces a permanent deformation in a material. In some materials, like aluminium alloys, the point of yielding is difficult to identify, thus it is usually defined as the stress required to cause 0.2 % plastic strain. This is called a 0.2 % proof stress.
- Compressive strength is a limit state of compressive stress that leads to failure in a material in the manner of ductile failure (infinite theoretical yield) or brittle failure (rupture as the result of crack propagation, or sliding along a weak plane - see shear strength).
- Tensile strength or ultimate tensile strength is a limit state of tensile stress that leads to tensile failure in the manner of ductile failure (yield as the first stage of that failure, some hardening in the second stage and breakage after a possible "neck" formation) or brittle failure (sudden breaking in two or more pieces at a low stress state). Tensile strength can be quoted as either true stress or engineering stress, but engineering stress is the most commonly used.
- Fatigue strength is a measure of the strength of a material or a component under cyclic loading,[7] and is usually more difficult to assess than the static strength measures. Fatigue strength is quoted as stress amplitude or stress range ($\Delta\sigma = \sigma_{\max} - \sigma_{\min}$), usually at zero mean stress, along with the number of cycles to failure under that condition of stress.
- Impact strength, is the capability of the material to withstand a suddenly applied load and is expressed in terms of energy. Often measured with the Izod impact strength test or Charpy impact test, both of which measure the impact energy required to fracture a sample. Volume, modulus of elasticity, distribution of forces, and yield strength affect the impact strength of a material. In order for a material or object to have a high impact strength the stresses must be distributed evenly throughout the object. It also must have a large volume with a low modulus of elasticity and a high material yield strength.

5.2 Failure theories

There are four failure theories: maximum shear stress theory, maximum normal stress theory, maximum strain energy theory, and maximum distortion energy theory. Out of these four theories of failure, the maximum normal stress theory is only applicable for brittle materials, and the remaining three theories are applicable for ductile materials. Of the latter three, the distortion energy theory provides most accurate results in majority of the stress conditions. The strain energy theory needs the value of Poisson's ratio of the part material, which is often not readily available. The maximum shear stress theory is conservative. For simple unidirectional normal stresses all theories are equivalent, which means all theories will give the same result.

- Maximum Shear stress Theory- This theory postulates that failure will occur if the magnitude of the maximum shear stress in the part exceeds the shear strength of the material determined from uniaxial testing.
- Maximum normal stress theory - This theory postulates that failure will occur if the maximum normal stress in the part exceeds the ultimate tensile stress of the material as determined from uniaxial testing. This theory deals with brittle materials only. The maximum tensile stress should be less than or equal to ultimate tensile stress divided by factor of safety. The magnitude of the maximum compressive stress should be less than ultimate compressive stress divided by factor of safety.
- Maximum strain energy theory - This theory postulates that failure will occur when the strain energy per unit volume due to the applied stresses in a part equals the strain energy per unit volume at the yield point in uniaxial testing.
- Maximum distortion energy theory - This theory is also known as shear energy theory or von Mises-Hencky theory. This theory postulates that failure will occur when the distortion energy per unit volume due to the applied stresses in a part equals the distortion energy per unit volume at the yield point in uniaxial testing. The total elastic energy due to strain can be divided into two parts: one part causes change in volume, and the other part causes change in shape. Distortion energy is the amount of energy that is needed to change the shape.

For ductile materials there are two commonly used strength theories - the Maximum Shear Stress (MSS) or Tresca theory and the von Mises or Distortion Energy theory.

5.2.1 Maximum Shear Stress:

This states that failure occurs when the maximum shear stress in the component being designed equals the maximum shear stress in a uniaxial tensile test at the yield stress:

This gives $\tau_{\max} = \frac{S_y}{2n}$ or $|\sigma_1 – \sigma_2| = \frac{S_y}{n}$ or $|\sigma_2 – \sigma_3| = \frac{S_y}{n}$ or $|\sigma_3 – \sigma_1| = \frac{S_y}{n}$

whichever of the last three leads to the safest result. The latter usually involves σ_3 being zero, i.e. plane stress, and both σ_1 and σ_2 having the same sign. Note that the yield strength is reduced by the factor of safety n.

5.2.2 von Mises or Distortion Energy Theory:

distortion energy

For ductile metals and alloys, according to the Maximum Shear Stress failure theory (aka “Tresca”) the only factor that affects dislocation slip is the maximum shear stress in the material. This is really a 1-dimensional explanation; a single parameter (maximum shear stress) is the only thing that causes yielding. However, the Tresca theory does work well in a 3-dimensional world. None the less, a slight improvement upon Tresca’s theory is warranted. Yielding (dislocation slip) is somewhat better explained (i.e. it is better supported by empirical data) by considering strain energy.

If we apply a load to a material it will deform. The units of energy are force*distance, so when a load is applied and the material deforms, we are putting energy into the material. This energy introduced into the material due to the loading is referred to as “strain energy.” We prefer to normalize strain energy by unit volume, and when we do so, this is referred to as strain energy density. The area under a stress-strain curve is the energy per unit volume (stress*strain has units of force per area such as N/mm^2, which is the same as energy per unit volume $N - mm/mm^3$. We will be assuming linear elastic material only. Most metals and alloys are linear elastic prior to the onset of plastic deformation, so this is a valid assumption.

The strain energy is composed of two distinct forms – volume changes and distortion (angular change). Normal strains cause a change in volume, shear strain cause distortions. The total stain energy is the sum of distortion energy and volume energy:

$$U_{\text{total}} = U_{\text{distortion}} + U_{\text{volume}} \tag{5.2.1}$$

Where U_{total} = total strain energy,
$U_{distortioin}$ = strain energy due to distortioni,

U_{volume} = strain energy due to volume change (aka hydrostatic strain energy)

We will develop equations for total strain energy (U_{total}) and volume energy (U_{volume}), anddetermine the distortion energy (which is really what we are interested in) from:

$$U_{\text{distortion}} = U_{\text{total}} - U_{\text{volume}} \tag{5.2.2}$$

For general (3D) loading, the total strain energy is given in terms of principal stresses and strains:

$$U_{\text{total}} = \frac{1}{2}\left(\epsilon_1\sigma_1 + \epsilon_2\sigma_2 + \epsilon_3\sigma_3\right) \tag{5.2.3}$$

Using Hooke's law $\epsilon_1 = [\sigma_1 - \nu(\sigma_2 + \sigma_3)]/E$, etc. the total strain energy equation 5.2.3 can be written in terms of stress only:

$$U_{\text{total}} = \frac{1}{2}E\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_2\sigma_3 + \sigma_1\sigma_3 + \sigma_1\sigma_2\right)\right) \tag{5.2.4}$$

Remember that hydrostatic stress causes volume change and that it is invariant (hydrostatic stress is a scalar – it is not directionally dependent – therefore it does not vary depending upon axis orientation). "Invariant" means "does not vary." The hydrostatic stress can be determined from the average magnitudes of the three principal stresses:

$$\sigma_{\text{hydrostatic}} = \sigma_{\text{ave}} = \left(\sigma_1 + \sigma_2 + \sigma_3\right)/3 \tag{5.2.5}$$

$\sigma_{\text{hydrostatic}}$ is the stress condition that causes volume change. It is invariant. For a moment, let's consider it alone. Let's consider a loading condition that was purely hydrostatic with magnitude of shydrostatic as calculated in equation 5.2.5. If the only stress in this material is $\sigma_{\text{hydrostatic}}$ then for this special loading condition the 3 principal stresses would be equal to $\sigma_{\text{hydrostatic}}(\sigma_1 = \sigma_2 = \sigma_3 = \sigma_{\text{hydrostatic}})$. Equation 5.2.4 would become:

$$U = \frac{1-2\nu}{6E}\left(\sigma_{\text{hyd}}^2 + \sigma_{\text{hyd}}^2 + \sigma_{\text{hyd}}^2 + 2\left(\sigma_{\text{hyd}}^2\sigma_{\text{hyd}}^2 + \sigma_{\text{hyd}}^2\sigma_{\text{hyd}}^2 + \sigma_{\text{hyd}}^2\sigma_{\text{hyd}}^2\right)\right) \tag{5.2.6}$$

For purely hydrostatic loading condition that we assumed in equation 5.2.6, there is no distortion energy ($U_{\text{distortion}} = 0$) so $U_{\text{total}} = U_{\text{volume}}$. But what about our part which may have distortion energy? Regardless of the existence of distortion energy or not, equation 5.2.6 – being based on the invariant hydrostatic stress – is the energy due to volume change, U_{volume}:

$$U = \frac{3(-2\nu)}{2E}\sigma_{\text{hyd}}^2 \tag{5.2.7}$$

Substituting equation 5.2.5 into 5.2.7 gives:

$$U_{\text{volume}} = \frac{1-2\nu}{6E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_2\sigma_3 + 2\sigma_1\sigma_3 + 2\sigma_1\sigma_2\right) \tag{5.2.8}$$

To determine the strain energy due to distortion only (not volume change) we subtract equation 5.2.8 from equation 5.2.4:

$$\begin{aligned}U_{\text{distortion}} &= U_{\text{total}} - U_{\text{volume}} \\ &= \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_2\sigma_3 + \sigma_1\sigma_3 + \sigma_1\sigma_2\right)\right) - \frac{1-2\nu}{6E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_2\sigma_3 + 2\sigma_1\sigma_3 + 2\sigma_1\sigma_2\right) \\ &= \frac{1+\nu}{3E}\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}\end{aligned} \tag{5.2.9}$$

Remember, the Maximum Shear Stress theory works pretty well in predicting yielding of ductile metals and alloys, but we are trying to improve upon it a bit. Why does Maximum Shear Stress theory work well? Because indeed it is shear stress that causes dislocation slip (aka plastic deformation). What sort of strain do shear stresses produce? They produce shear strains — in other words, distortion. What is the equation for maximum shear stress? It is: $\tau_{max} = (\sigma_1 - \sigma_3)/2$. It is the difference between principal stress divided by 2. What do we see in equation 5.2.9? The differences between all principal stress divided by 2. Equation 5.2.9 combines the maximum shear stress in each of the 3 principal planes into a single equation. It should not be surprising that "distortion strain energy" is related to maximum shear stress. Shear stress cause shear strain, which is distortion.

The Distortion Energy failure theory (which we will discuss next) is a bit more mathematically sophisticated than the Maximum Shear Stress failure theory, but is really very similar. Rather than considering only the maximum shear stress at a point, it combines the maximum shear stress at a point in the 3 principal planes. These two theories give very similar results, but Distortion Energy does match empirical data better.

distortion energy failure theory

For uniaxial tensile loading (as is used to create a stress-strain curve), $\sigma_2 = \sigma_3 = 0$, and at the onset of yielding, $F/A = \sigma_1 = S_{\text{ys}}$ (at onset of yielding).

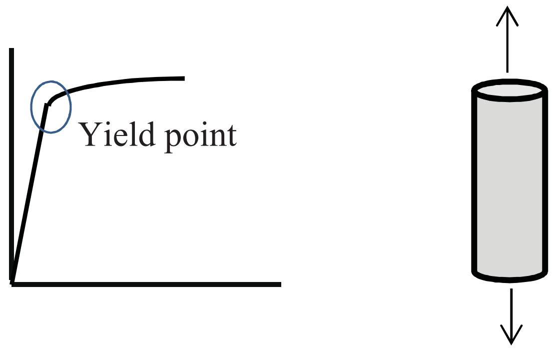


Figure 5.3: yield point in uniaxial test

Therefore, for uniaxial loading at the onset of yielding (the stress shown on the stress-strain curve that we call "yield strength") we substituting S_{ys} for σ_1 and $\sigma_2 = \sigma_3 = 0$ into equation 5.2.9

$$U_{\text{distortion}} = \frac{1+\nu}{3E} S_{\text{ys}}^2 \tag{5.2.10}$$

The Distortion Energy Theory states that when the distortion energy in a material equals or exceeds the distortion energy present at the onset of yielding in uniaxial loading tensile test for that material, the part will experience plastic deformation (i.e. it will yield):

$$U_{distortion,part} \leq U_{distortion,uniaxialtest} yieldingoccurs \tag{5.2.11}$$

Equating the distortion energy in general 3-dimensional stress condition (equation 5.2.9) and distortion energy in simple uniaxial loading (equation 5.2.10); from equations 5.2.9 and 5.2.10 into equation 5.2.11:

$$\begin{gathered}\frac{1+\nu}{3E}\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2} \leq \frac{1+\nu}{3E} S_{\text{ys}}^2 \\ \sigma_{\text{eff}} = \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}} \leq S_{\text{ys}}\end{gathered} \tag{5.2.12}$$