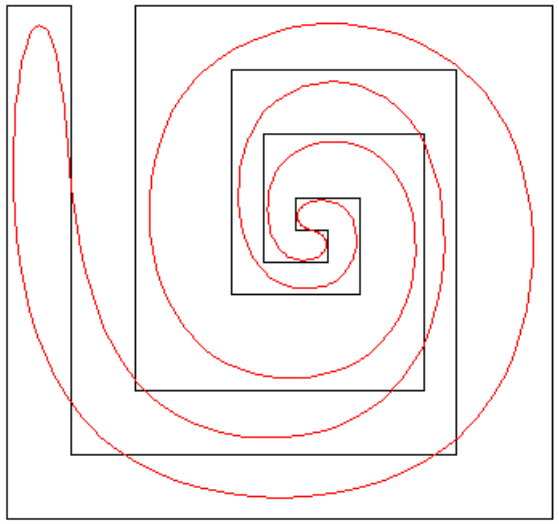
Curve subdivision

Georgia Tech College of Computing
Interactive & Intelligent Computing

- Smoothing polyloops
- B-spline, Bezier and 4-points subdivisions
- The Jarek subdivision schemes for polyloops
- Surfaces

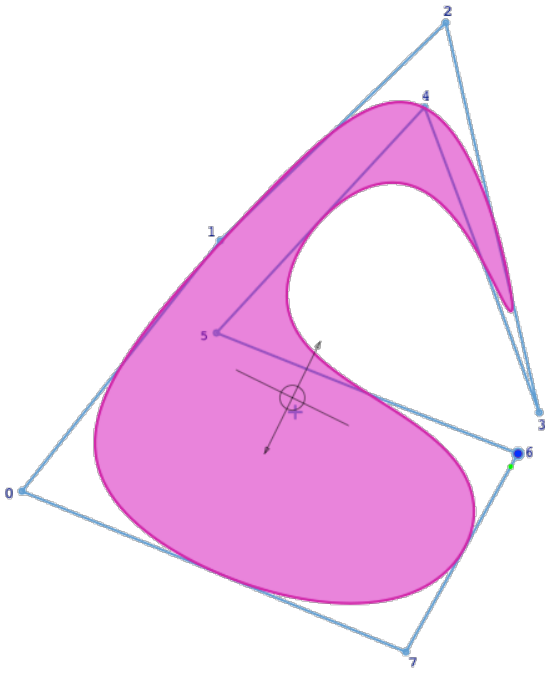
Updated November 9, 2012

Motivation

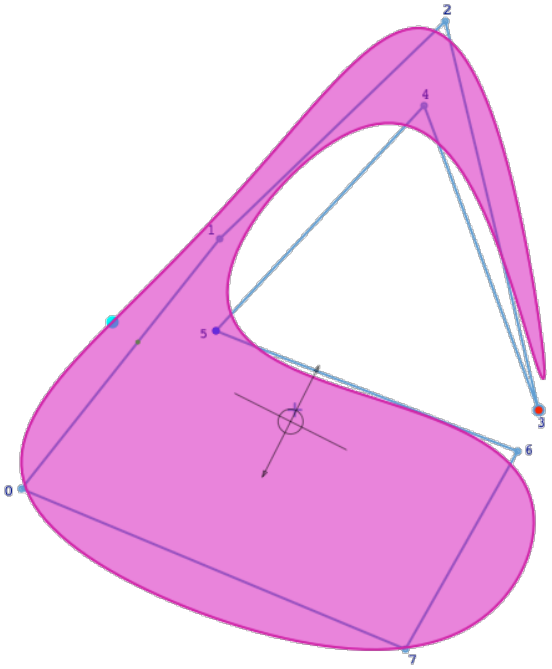


- **Smooth** curves and surfaces are used for aesthetic, manufacturing, and analysis applications where discontinuities due to piecewise linear approximations would create misleading artifacts.
- Designers like to specify such curves by adjusting a **control** network with only a few vertices
- How to go from a crude 2D control polyloop (black) to a smooth curve (red)?
- **Suggestions**?

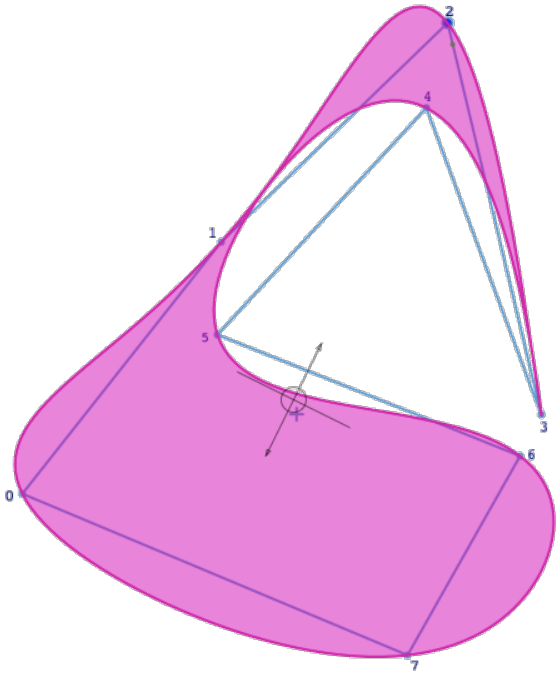
Different requirements



smooth

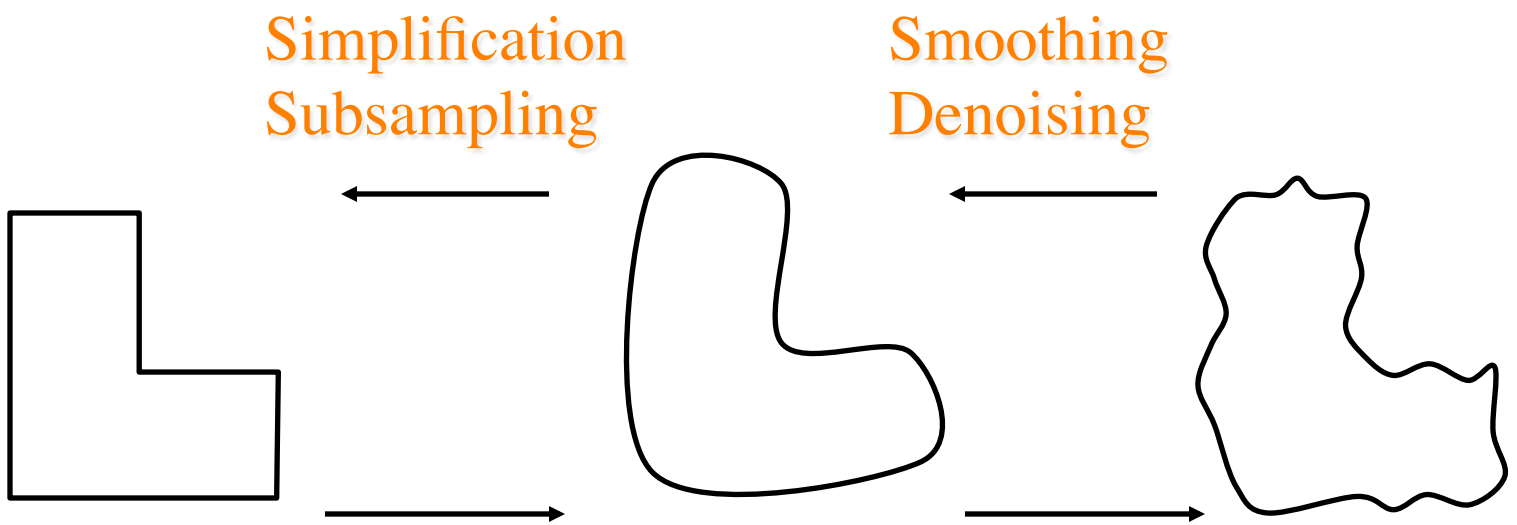


compromise



interpolating

Operators on polyloops

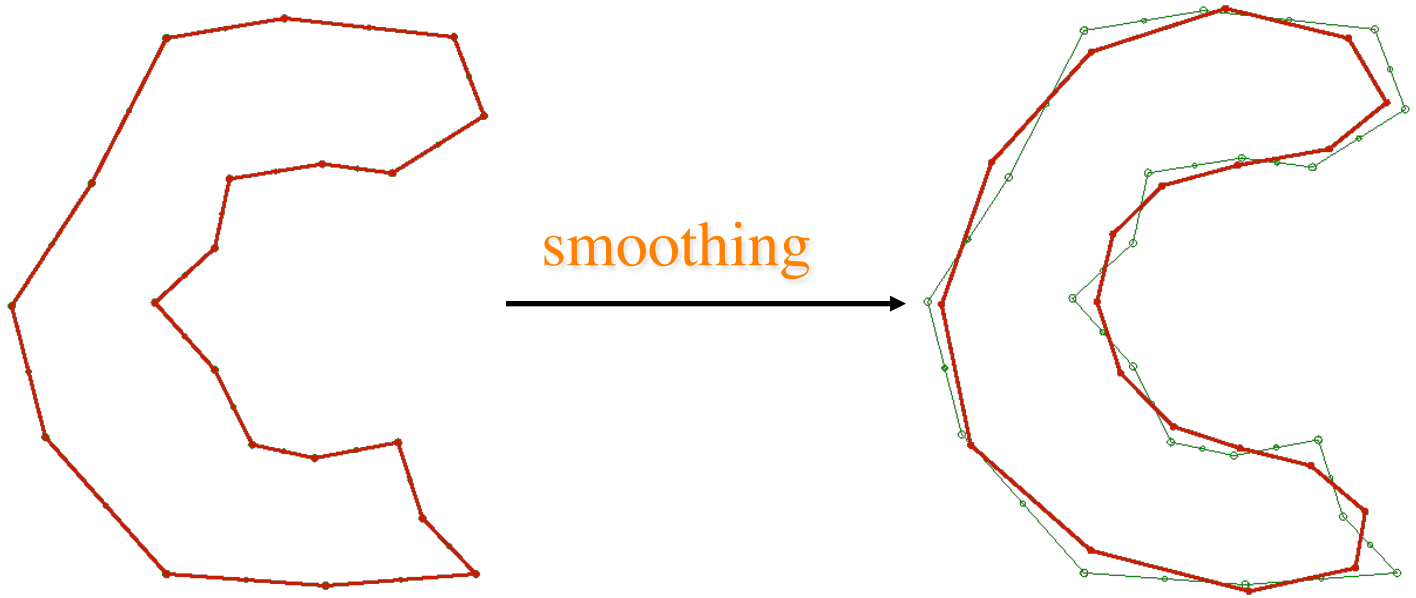


Smoothing a polyloop

Given a polyloop L, we wish to move its vertices to produce a curve J that resembles L, but is "smoother" (less jerk?)

Smoothing does not change the number of vertices,

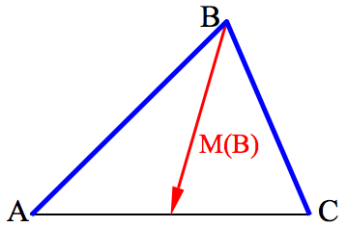
it only moves them



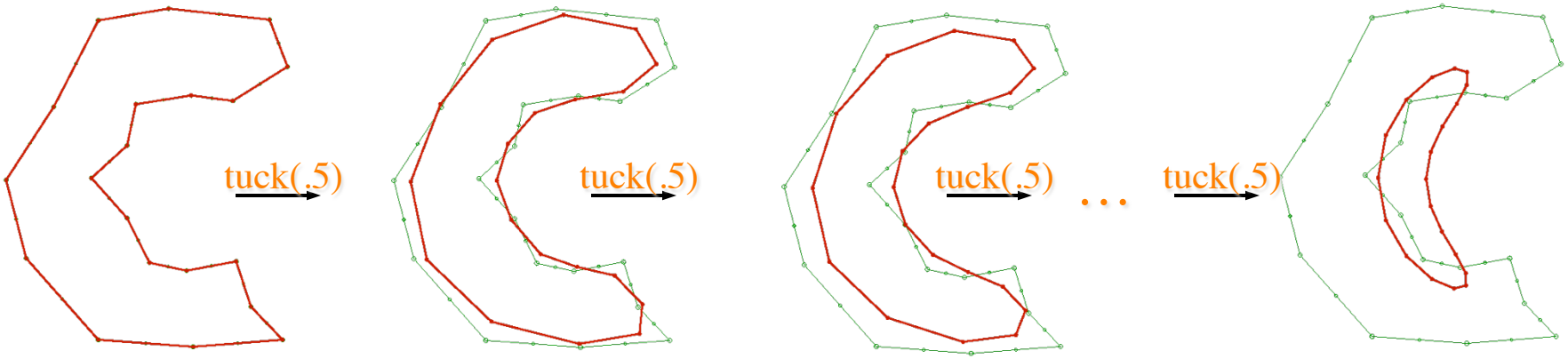
Tuck smoothing

tuck(s) moves the vertices by executing the following two steps:

- For each vertex B, compute M(B)=(BA+BC)/2
  - A (r. C) is the previous (r. next) vertex in the cyclic order
  - Moving B to B+M(B) brings it to the average of its neighbors
- For each vertex B, move B to B+sM(B)

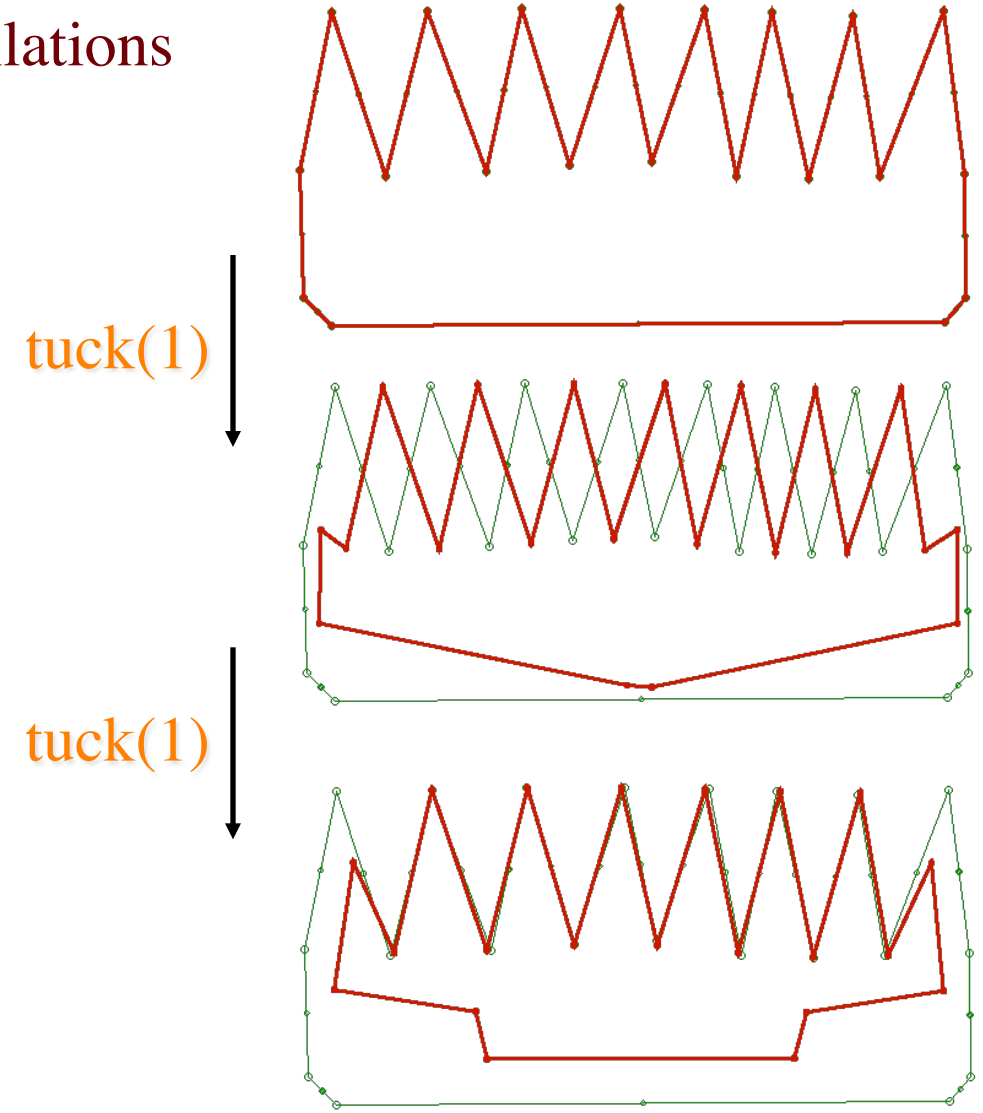


Repeating tuck(s) tends to eliminate sharp features, but shrinks the curve



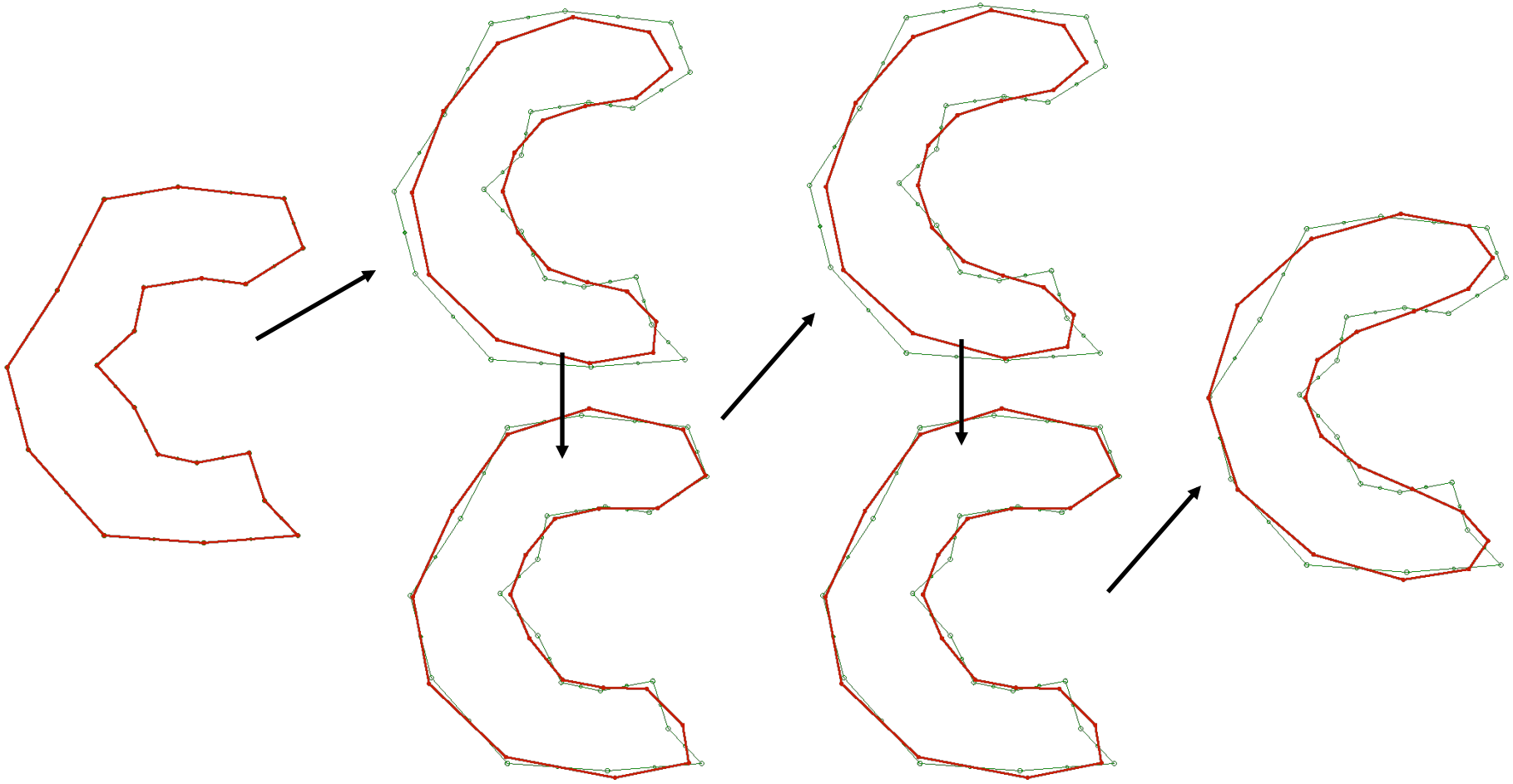
Avoid oscillations

Keep s in [0,2/3] to avoid oscillations



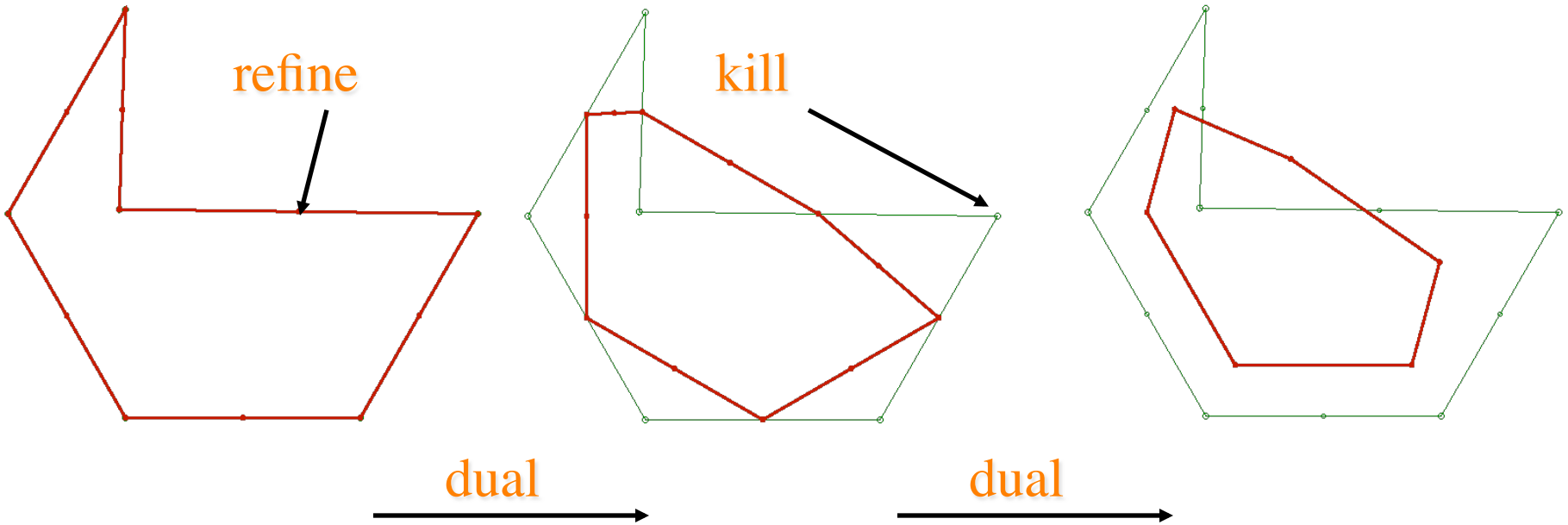
Avoid shrinking: **Tuck-tuck** smoothing

- Alternate **tuck(s)**, which shrinks, and **tuck(–s)**, which expands
  - untuck cannot recover the sharp features lost during the tuck



Tuck(1/2) = dual of dual

- Tuck(.5) is the dual of the dual
  - refine introduces a new vertex in the middle of each edge
  - kill deletes the old vertices
  - dual = (refine, kill)
  - tuck(.5) = (dual, dual)



Cubic fit tuck-tuck: (–A+4B+4D–E)/6

Compute a cubic polynomial curve, $\mathbf{C}(t)=at^3+bt^2+ct+d$, through vertices A, B, D, E satisfying: $\mathbf{C}(-2)=A$, $\mathbf{C}(2)=E$, $\mathbf{C}(-1)=B$, and $\mathbf{C}(1)=D$

We want to insert a point $C=\mathbf{C}(0)=d$ between B and E. Solve for d

$\mathbf{C}(-2)=-8a+4b-2c+d=A$ and $\mathbf{C}(2)=8a+4b+2c+d=E$ yields $A+E=8b+2d$

$\mathbf{C}(-1)=-a+b-c+d=B$, and $\mathbf{C}(1)=a+b+c+d=D$ yields $A+E=8b+2d$

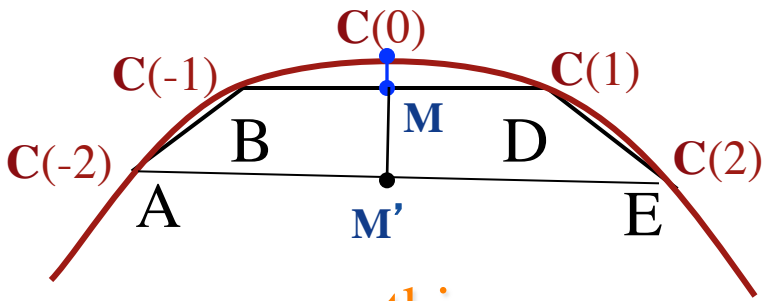
$A+E=8b+2d$ and $A+E=8b+2d$ yields $6d=4(B+D)-(A+E)$

$d = (-A+4B+4D-E)/6 = \mathbf{M+M'M/3}$, with $M=(B+D)/2$, $M'=(A+E)/2$

To insert such a cubic fit for each edge: tuck($2/\sqrt{6}$), tuck($-2/\sqrt{6}$)

Note that $2/\sqrt{6}=0.82$, which may be unstable

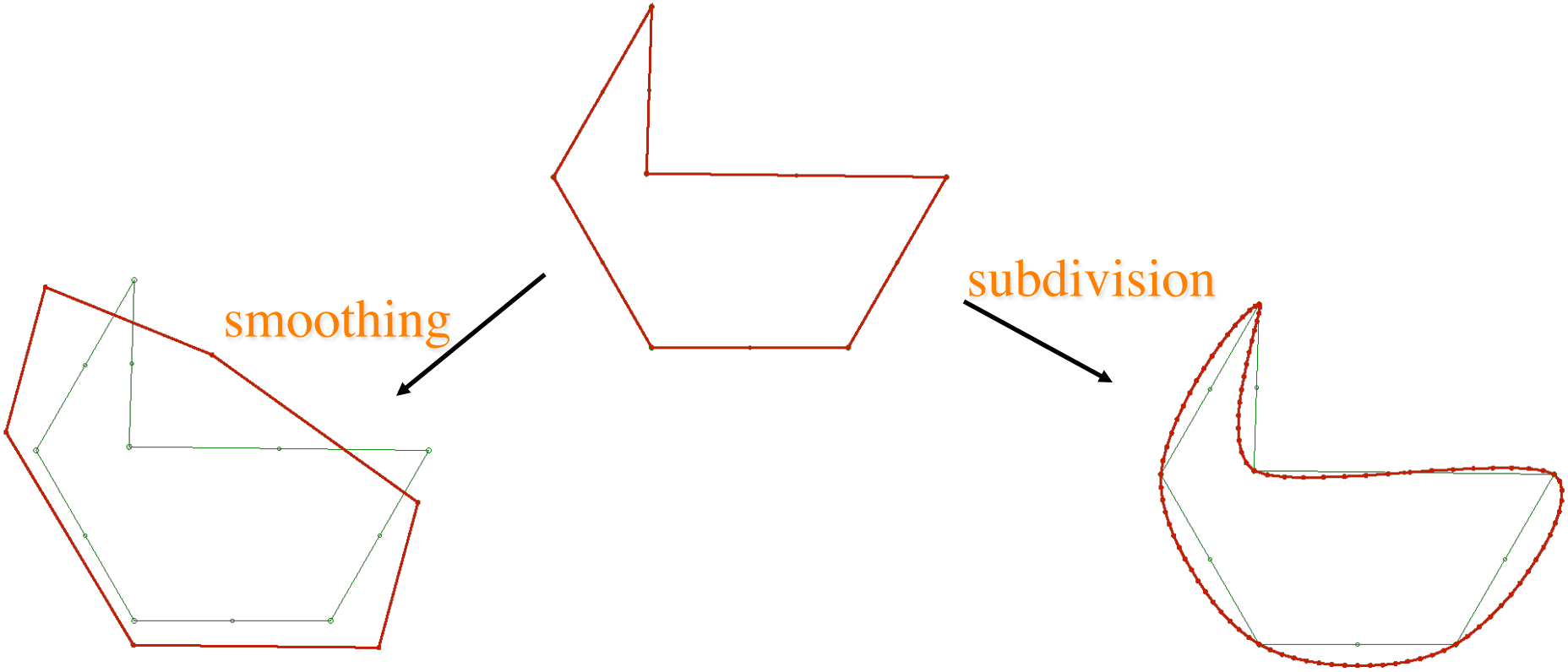
*Move **towards** C*



smoothing
bulge M'M/3

Smoothing vs subdivision

- **Smoothing** should be performed when the desired curve has been densely **sampled** but the resulting polyloop has **undesired sharp features** (acquisition noise, too many details, sharp changes in direction or curvature)
- **Subdivision** refines the polyloop (**adds more vertices**). It should be performed when the polyloop is a **crude** outline of the desired curve.



Polyloop refine/smooth operators

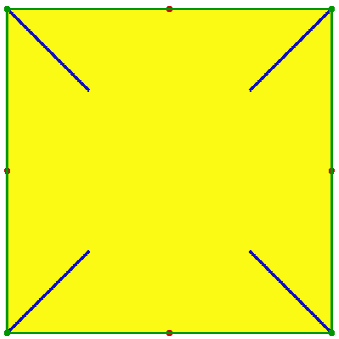
We will use the following operators:

- r (*refine*) introduces a new vertex in the middle of each edge
- k (*kill*) deletes the old vertices (those present before r)
- d (*dual*) performs (r, k)
- t (*tuck*) performs tuck(1/2), which is (d, d)
- u_s (*parameterized untuck*) performs tuck(–s)
- u (*untuck*) performs tuck(–1/2), same as $u_{.5}$

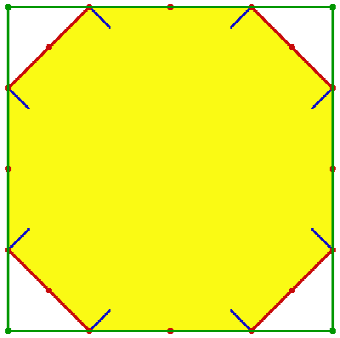
Write a polyloops **smoothing** scheme using these

Quadratic uniform B-spline (r,d)

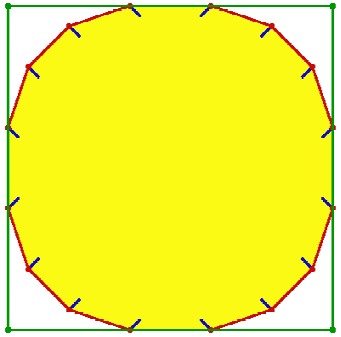
Repeating (r,d) converges to a piecewise quadratic B-spline curve that interpolates the mid-edge points



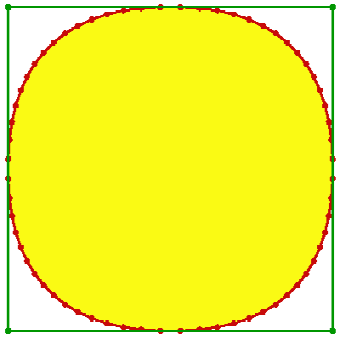
r



rd



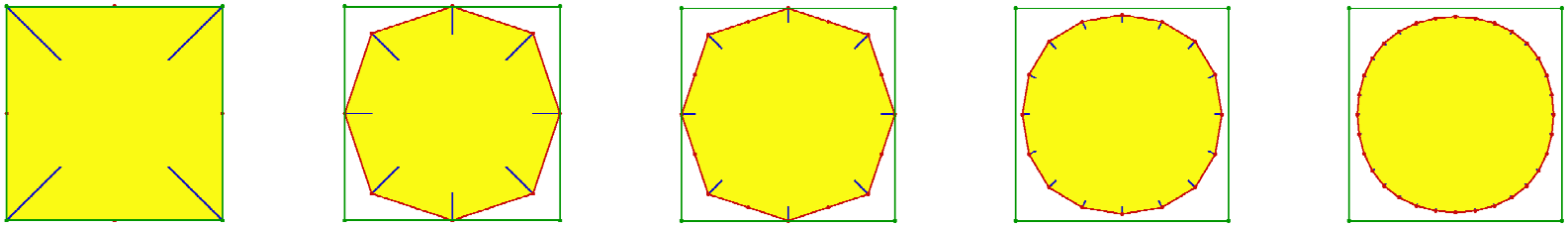
rdrd



rdrdrdrd

Cubic uniform B-spline (r,t)

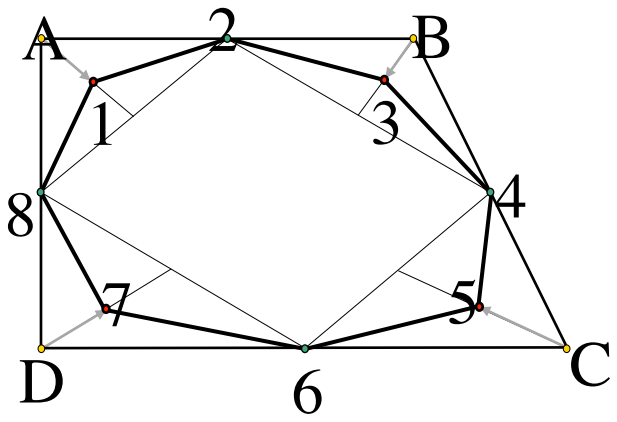
Repeating (r,t), which is (r,d,d) converges to a piecewise cubic B-spline curve that has second degree continuity (no jerk)

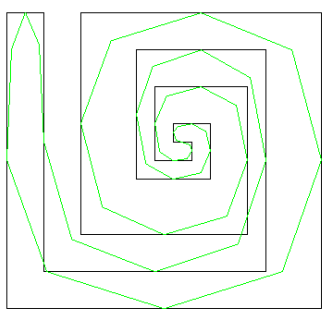


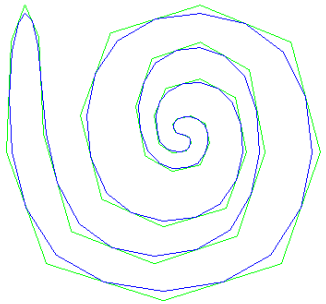
rt rtr rtrt rtrtrt

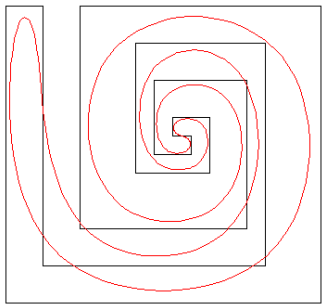
Split&tweak B-spline subdivision

- Given a control polygon, for example (A,B,C,D), repeat the following sequence of two steps, until all consecutive 4-tuples of control points are nearly co-linear.
  1. "Split": insert a new control point in the middle of each edge (2,4,6,8)
  2. "Tweak": move the old control points half-way towards the average of their new neighbors (1,3,5,7)
- The polyloop converges rapidly to a smooth curve, which happens to be a cubic B-spline curve.



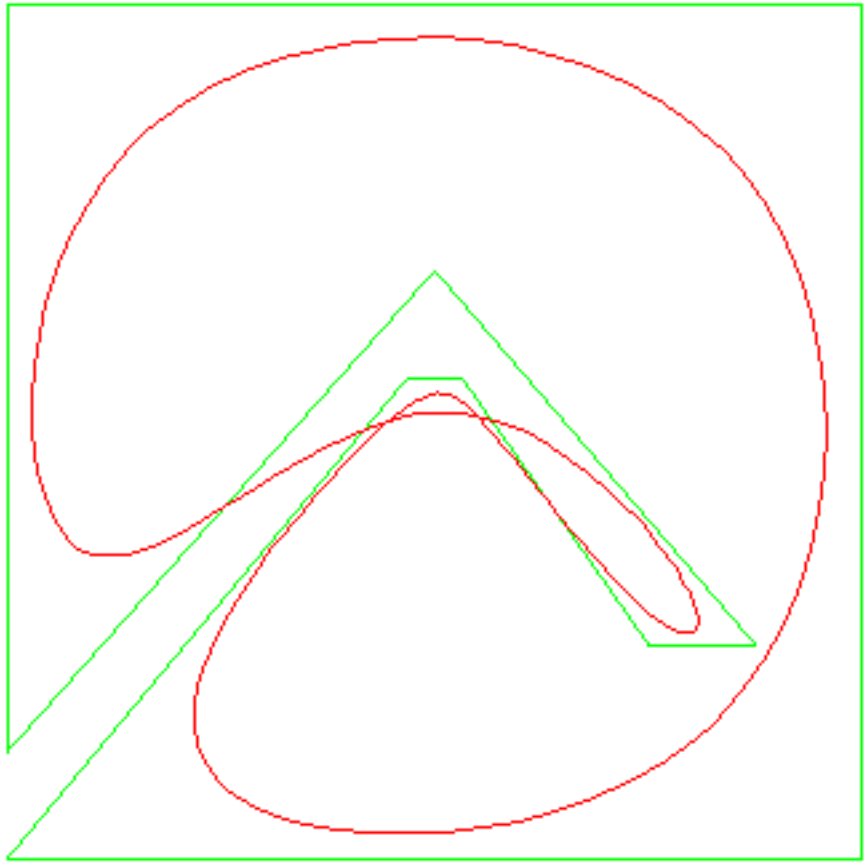






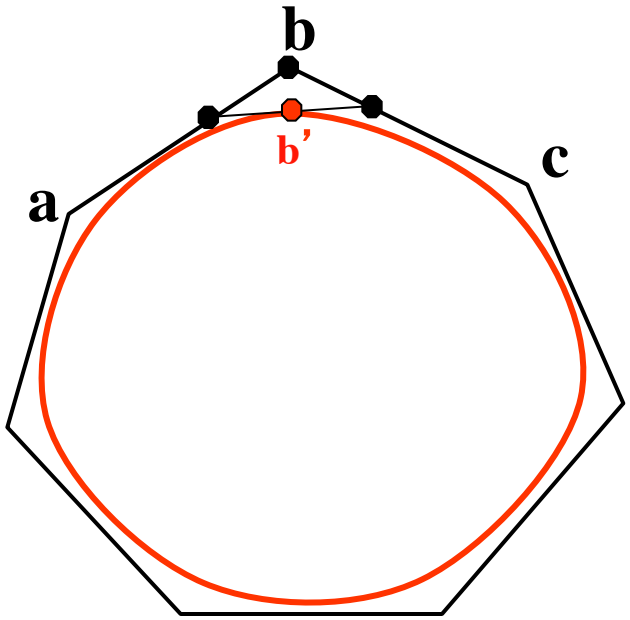
Can the limit curve self-intersect?

- Can it self-intersect when P does not?



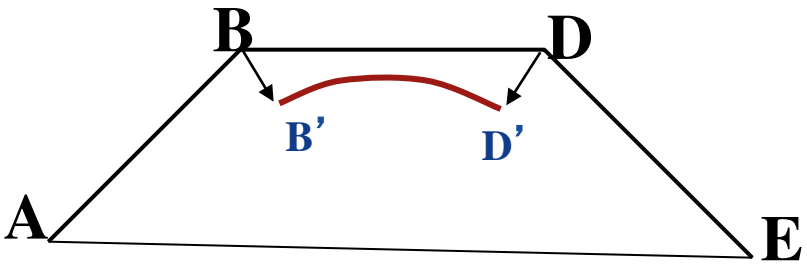
Where do the control vertices converge to?

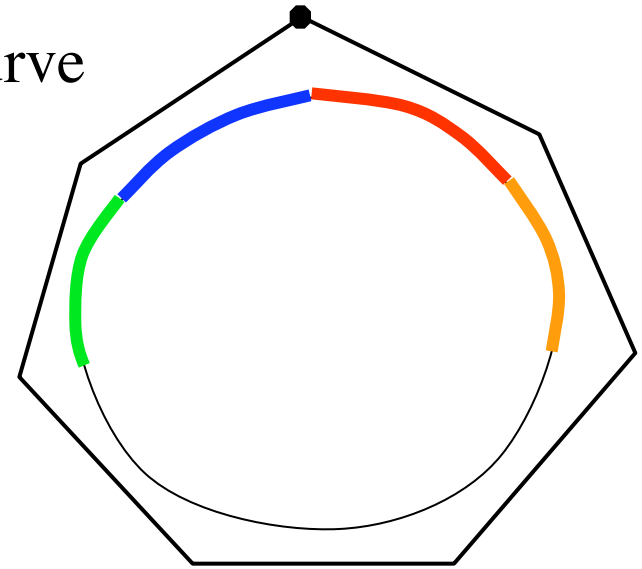
- The original control vertex B **converges** to B’=(A+4B+C)/6
- The **tangent** at B’ to the B-spline curve is parallel to AC



Local control for B-spline curves

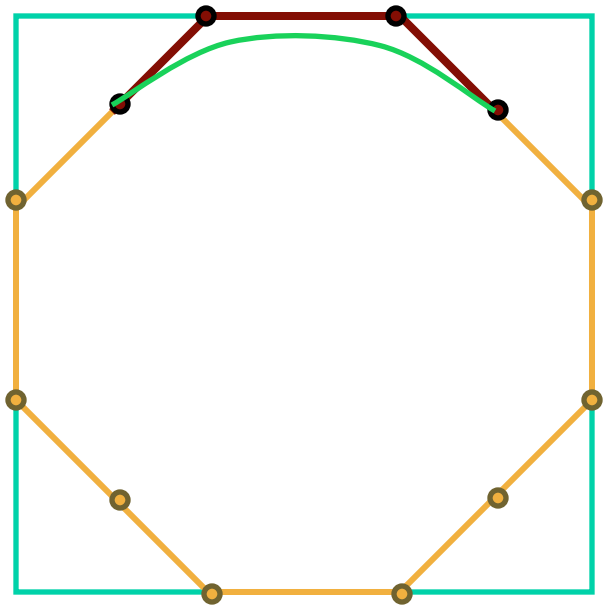
- Trace the original vertices **during subdivision**
  - Do they move along straight lines?
- They subdivide the final curve into **spans**
- **Each span is influenced by 4 control vertices**
- **Each control vertex influences 4 spans**
- Each span converges to a cubic Bezier curve

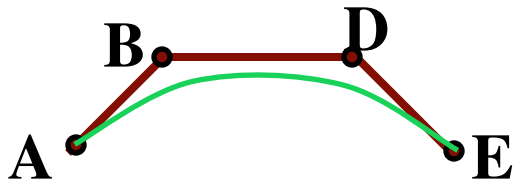




Cubic B-spline–to–Bezier conversion

- Introduce 2 new vertices in each edge to split it in 3 equal parts
- Move the old vertices to the average of their new neighbors
- Make groups of 4 consecutive vertices (each sharing its old vertices with consecutive groups)
- Each group is the control polygon (A,B,D,E) of a Bezier curve





How to evaluate the Bezier control polygon

Let t be the parameter in [0,1].

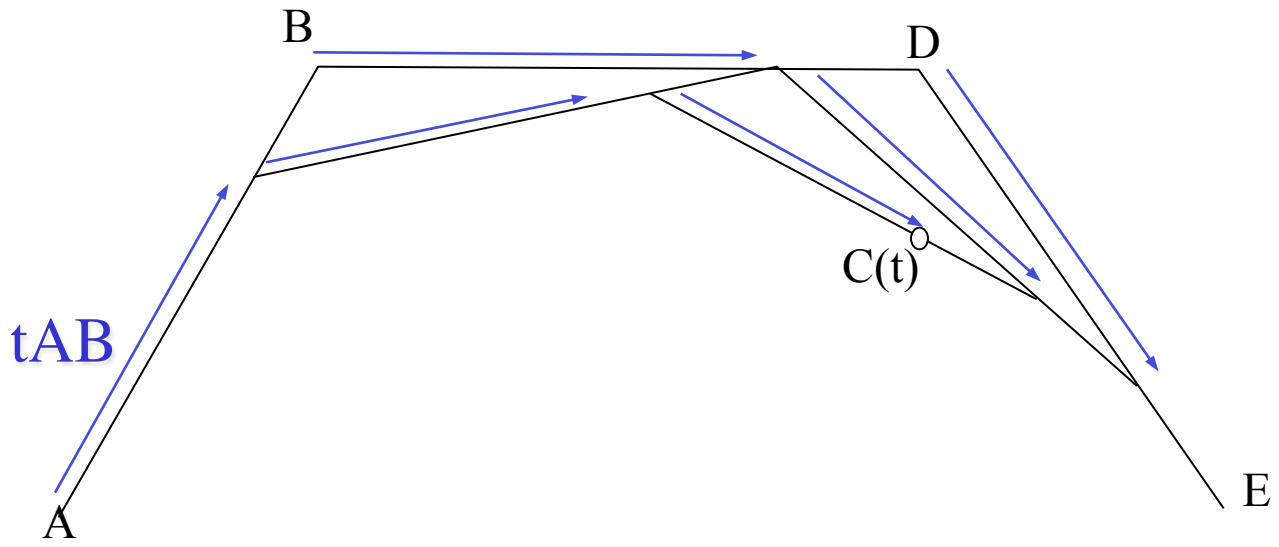
Let C(t) be the point on the Bezier curve corresponding to t

You can compute t as

C(t)=s(s(s(A,t,B),t,s(B,t,C)) , t , s(s(B,t,C),t,s(C,t,D))

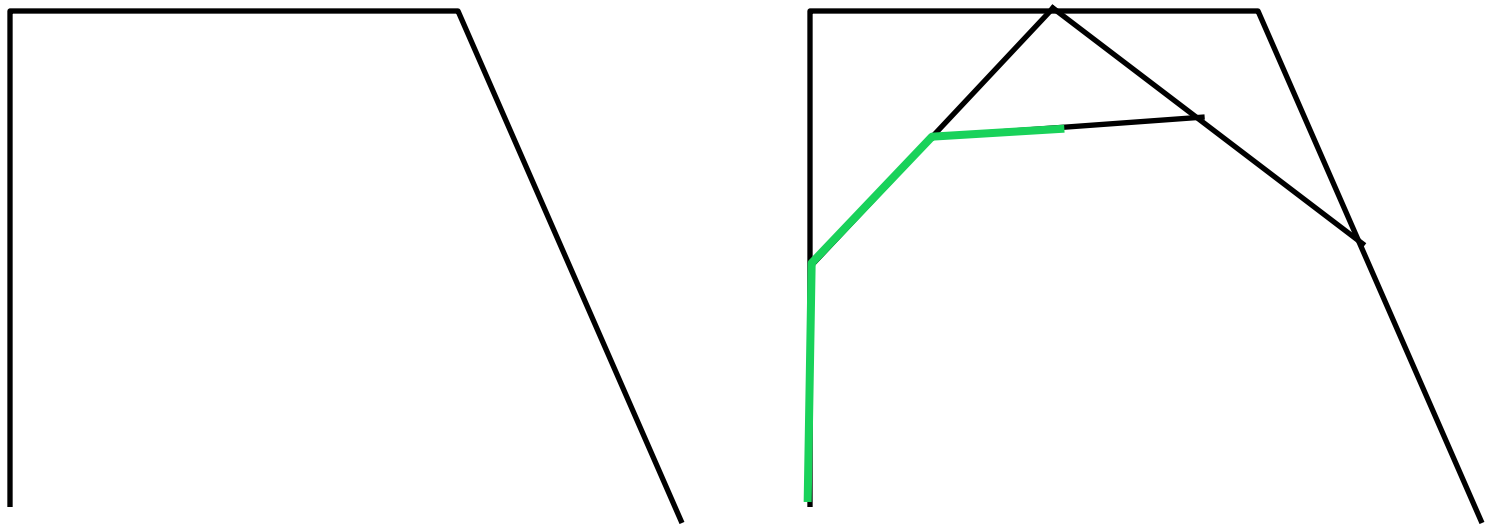
where s(A,t,B) returns A+tAB

Note that this is a cubic polynomial in t



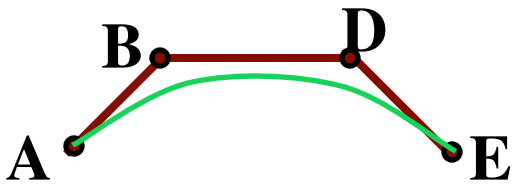
How to subdivide the Bezier control polygon

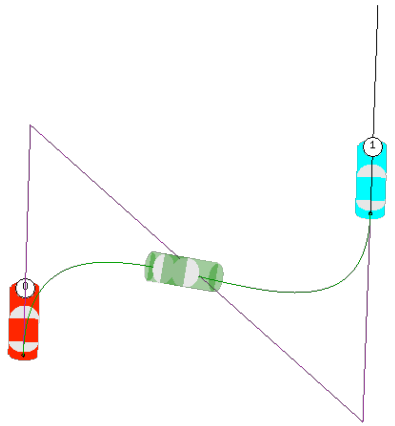
- Perform the evaluation of C(0.5) and keep the construction points



Convex hull

- Each Bezier curve lies in the convex hull of 4 control vertices
- The convex hull of 4 points in 3D?
  - A tetrahedron
- The convex hull of 4 points in 2D?
  - Contained in the union of 3 triangles: ABD, ABE, BDE



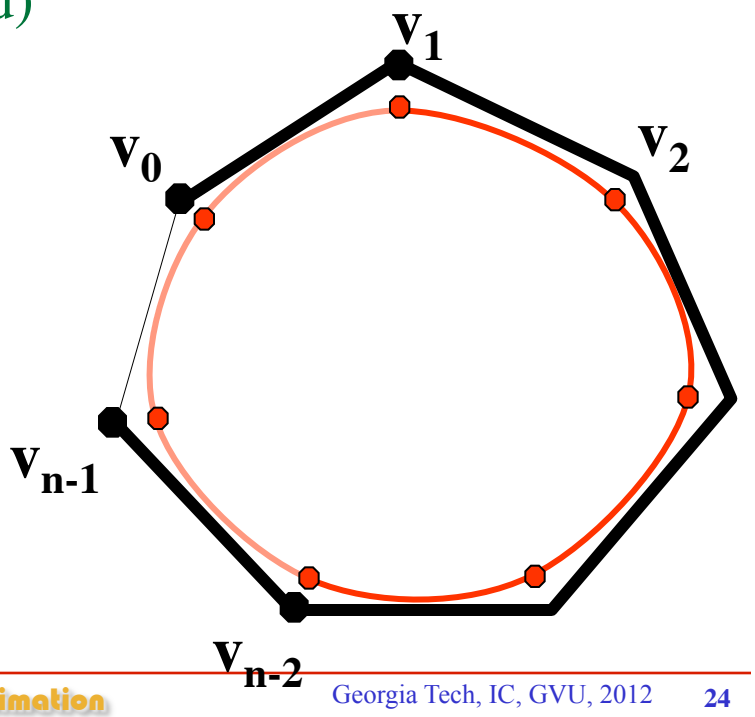


How can we use the convex hull property?

- Display uncertainty (region guaranteed to contain the limit curve)
  - Shade triangles (in 2D) or tetrahedra (in 3D)
- Decide when to stop the subdivision
  - All triangles or tetrahedra are nearly linear
- Quickly decide when two B-spline curves in 2D are not intersecting
  - The convex hulls do not overlap
  - Useful for accelerating collision detection and for computing the exact collision time

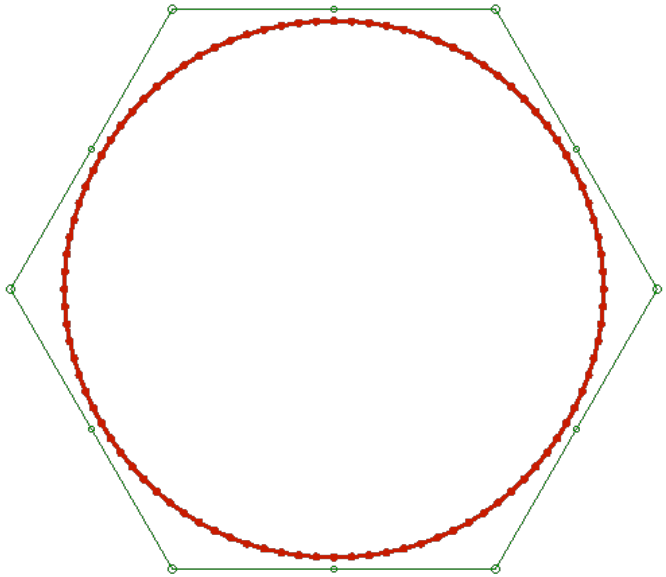
Open curves

- If you want to subdivide an open polygon
  - Trace the original vertices through the subdivision process
  - They split the curve into spans (one per original edge)
- Delete the span (v_{n-1},v_n) and the two adjacent spans
  - v_0 should not influence the head (beginning of the curve)
  - v_{n-1} should not influence the tail (end)

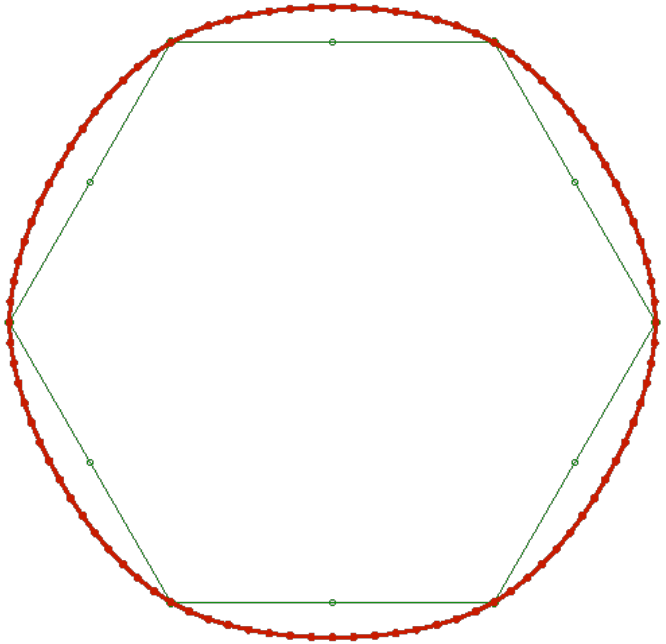


B-spline curves are not interpolating

- The cubic B-spline is **approximating**
  - It does not go through the control vertices
- You may prefer the 4-point, which is **interpolating**



B-spline (approximating)



4-point (interpolating)

4-point subdivision

- We want to leave the original vertices of L where they are.
- Hence, we displace (bulge-out) each inserted mid-edge point C.
- By how much? Fit a cubit C=**C**(t) through A, B, D, and E
  - **C**(t)=at^3+bt^2+ct+d, and thus **C**(0)=d

We use different constraints than the cubic fit tuck-tuck!

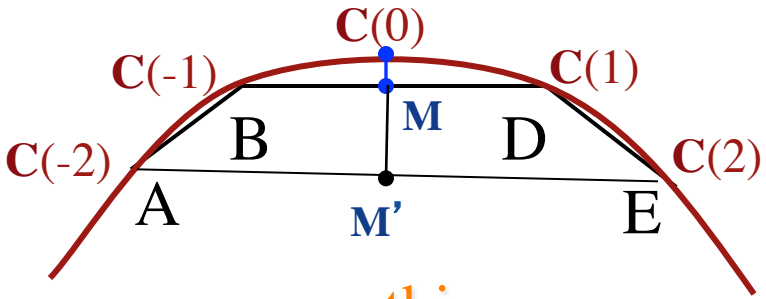
- B=**C**(–1)= –a+b–c+d
- D=**C**(1)= a+b+c+d
- A=**C**(–3)= –27a+9b–3c+d
- E=**C**(3)= 27a+9b+3c+d
- C=**C**(0)=d=M+M' M/8

M=(B+D)/2=b+d

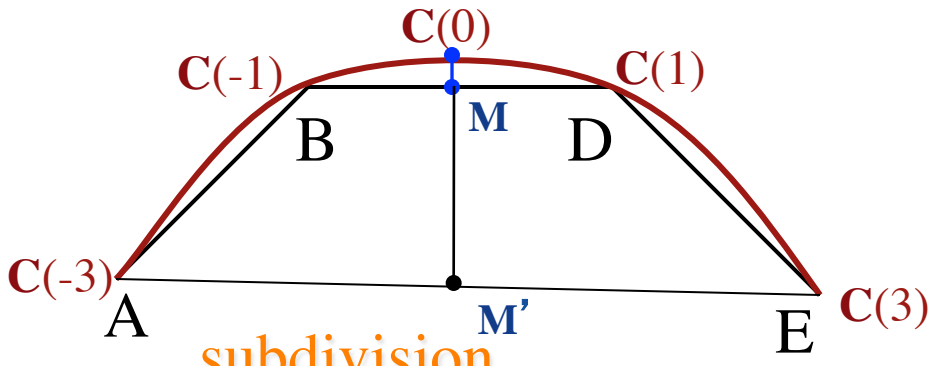
M' =(A+E)/2=9b+d

M' –9M=–8d

8M+M–M' =8d



smoothing
bulge M' M/3

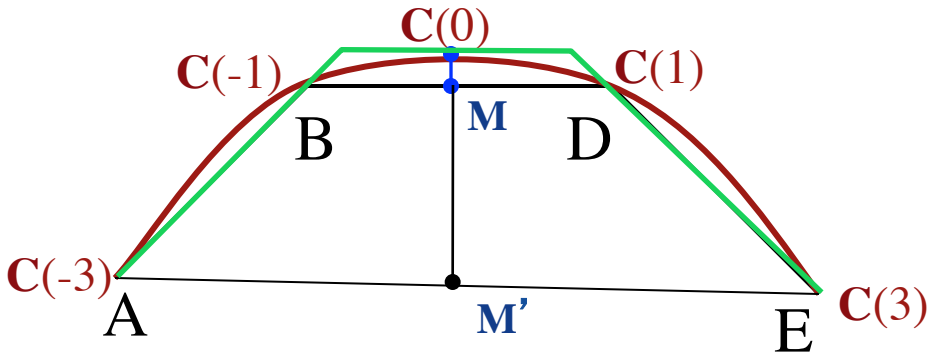


subdivision
bulge M' M/8

Implementing 4-point subdivision

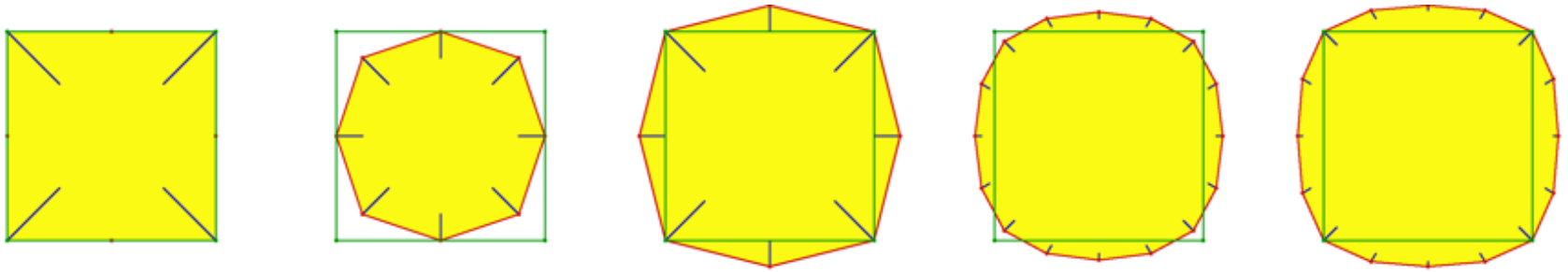
- Design the construction of **C=M+M' M/8** using s(A,t,B)
- Avoid computing M'
  - instead extrapolate AB and ED and take the mid-point
- We will use this construction for smoothing key-frames motions

Compare with your neighbor



s(s(A,9/8,B) ,.5, s(E,9/8,D))

4-point = (r,t,u_1)



r rt rtu_1 rtu_1rt rtu_1rtu_1

Refine (r) puts a mid-edge point C at M.

Surprise! (t,u_1) produces the cubic-fit bulge M'M/8

Verify by inspection…

Compare cubic B-spline & 4-point

- Which one interpolates the original vertices?
- Which one produces the smoothest curve?
  - When is this important?
- Can both have cusps (non smooth points)?
- How can you force the B-spline curve to go through a point?
- Which one has a convex-hull property?
  - When is this important?

Cubic B-spline and four-points

- From James Koch

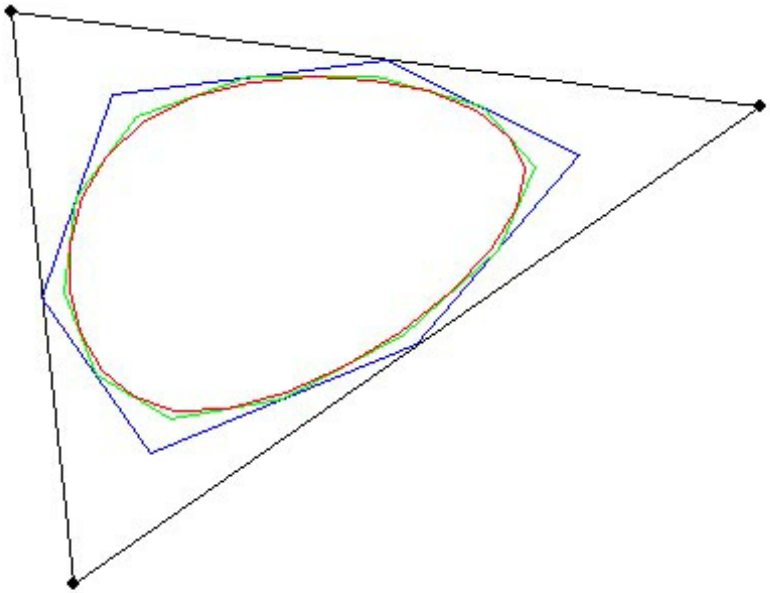


Figure 1. A triangle with several iterations of B-spline subdivision

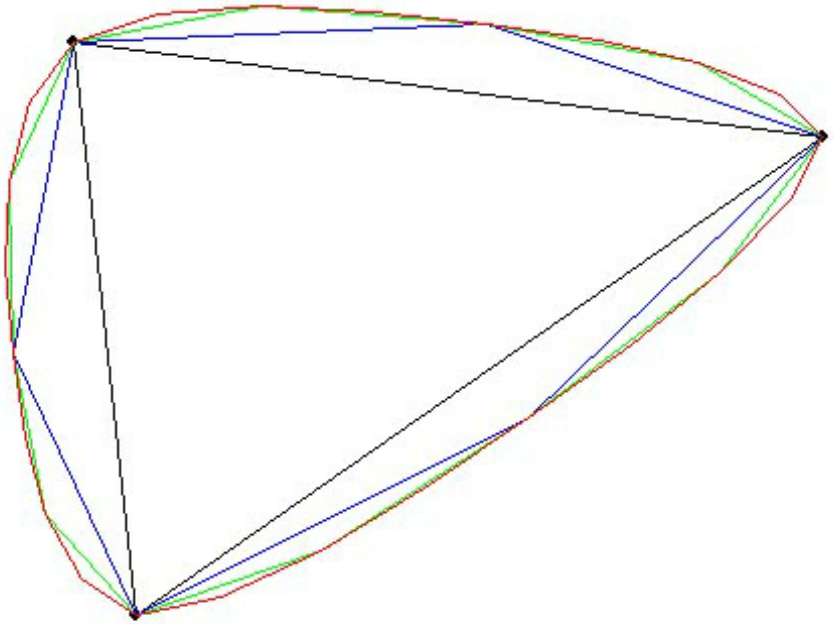
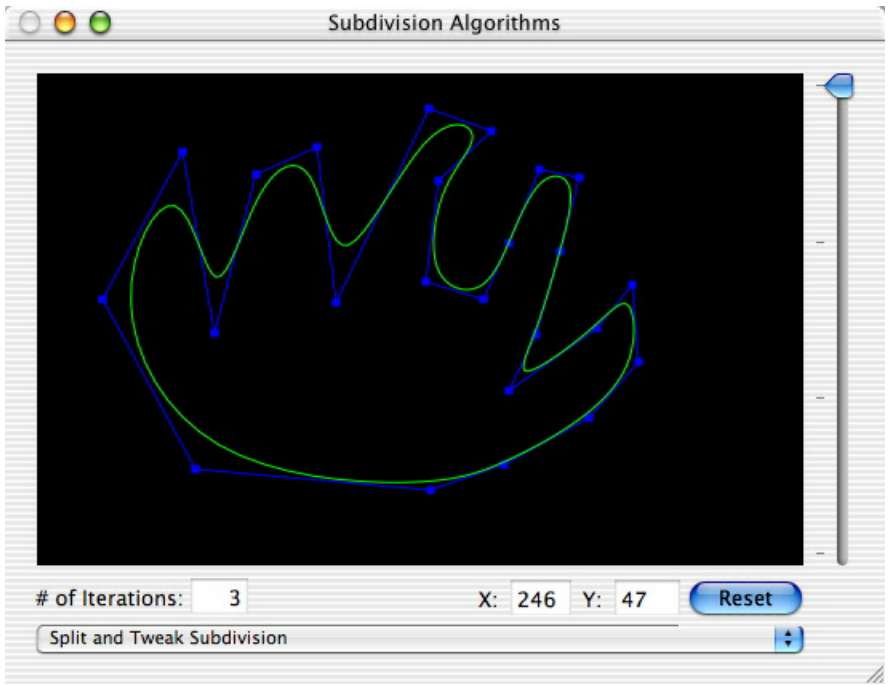
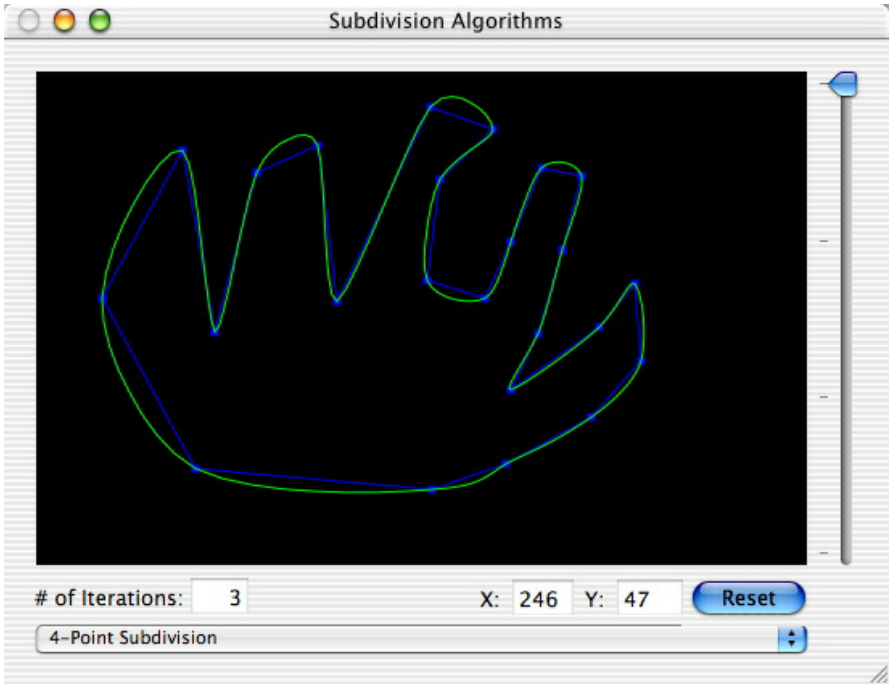


Figure 2. A triangle with several interations of four-point subdivison

Which one is smoother?

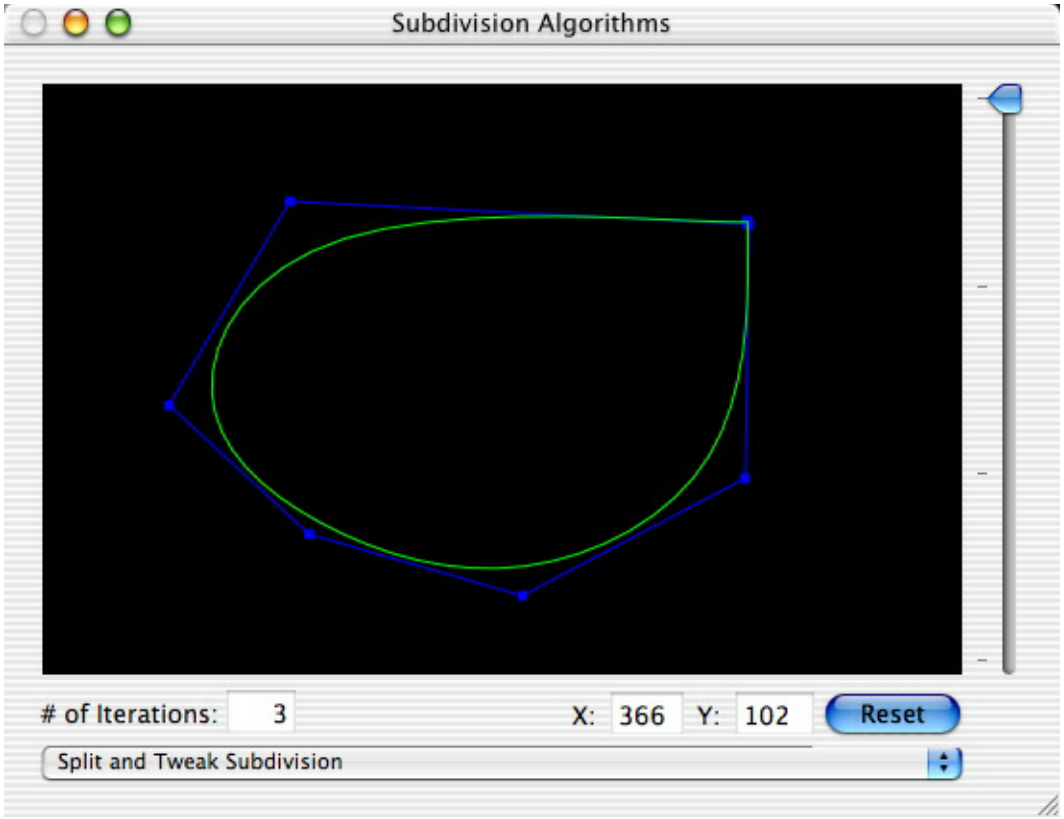
- By Stevie Strickland





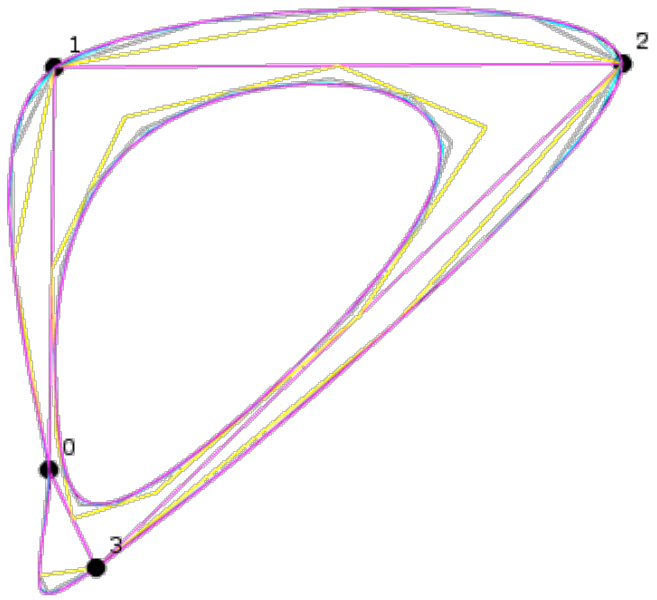
Making a sharp corner in a B-spline curve

- By Stevie Strickland
- HOW?



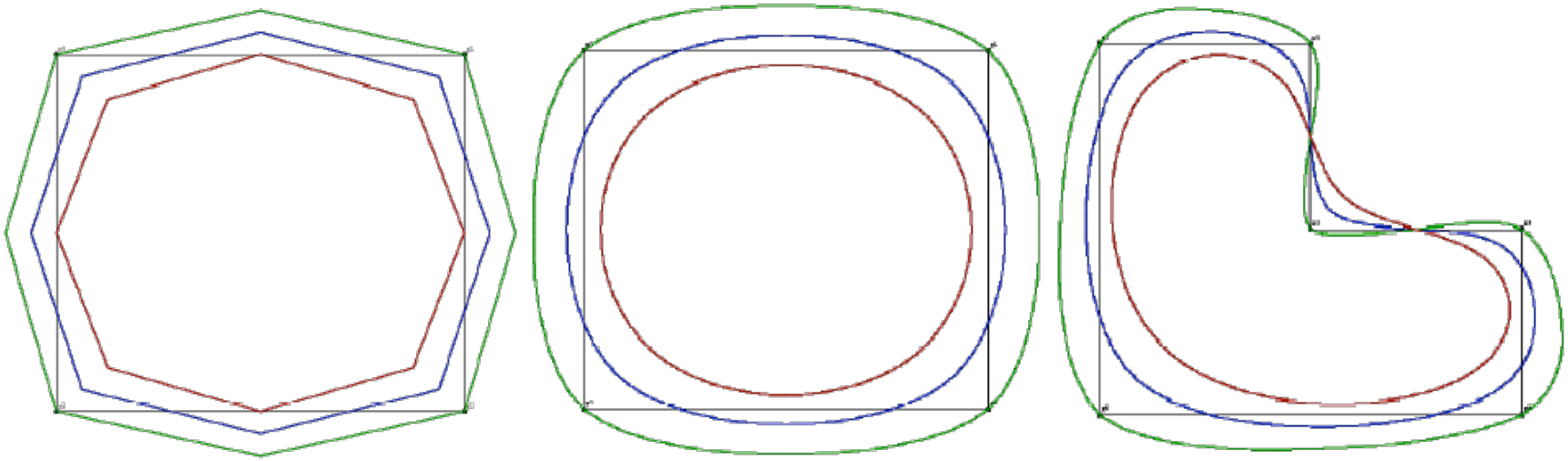
A compromise

- B-spline subdivision rounds the corners and thus shrinks a convex shape
- 4-points subdivision bulges the edges out and hence grows a convex shape
- Both produce curves that are relatively far from the initial control polygon
- We may want a compromise
  - Closer to the polygon
  - Bulge less
  - Cut less



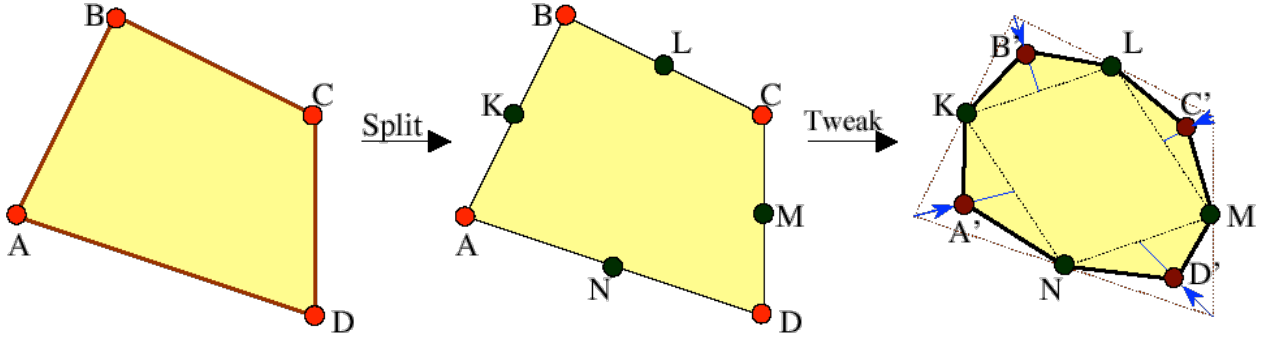
Jarek's compromise: J_4

- Split each edge as before (in both schemes)
- Tweak old vertices by half of the B-spline displacement and the new vertices by half of the 4-point displacement

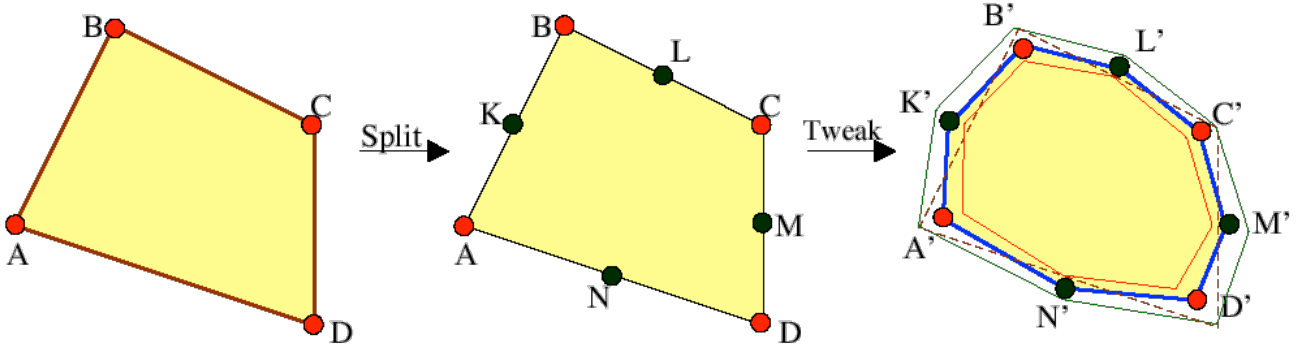


3 subdivision schemes

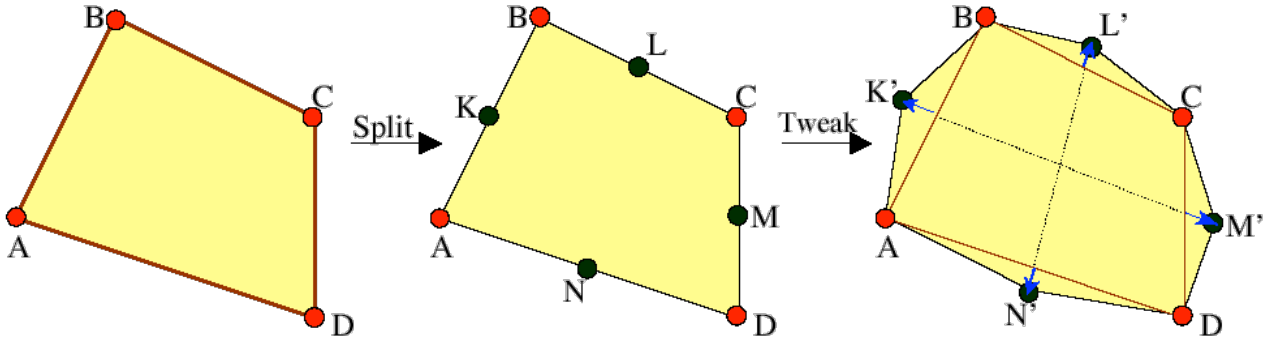
Bezier



J_4



4-points

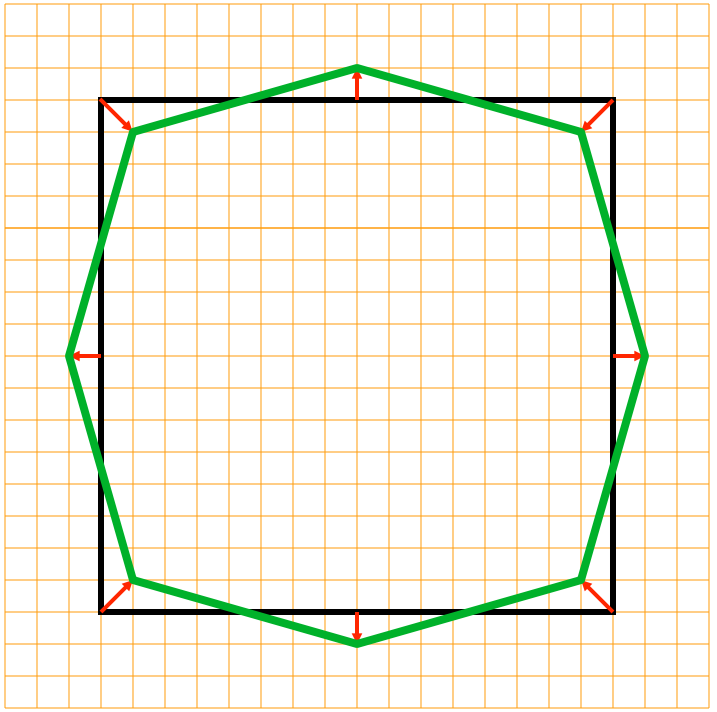


Implementing J_4

1 - For each vertex $\mathbf{v}_i$, compute the average $\mathbf{a}_i$ of its neighbors and store a displacement vector $\underline{\mathbf{D}}_i=(\mathbf{a}_i-\mathbf{v}_i)/8$.

2 - For each edge $(\mathbf{v}_i,\mathbf{v}_{i+1})$, insert a new vertex $\mathbf{m}_i=(\mathbf{v}_i+\mathbf{v}_{i+1})/2$, and displace it by $-(\underline{\mathbf{D}}_i+\underline{\mathbf{D}}_{i+1})/2$.

3 - Displace each old vertex $\mathbf{v}_i$ by $\underline{\mathbf{D}}_i$.

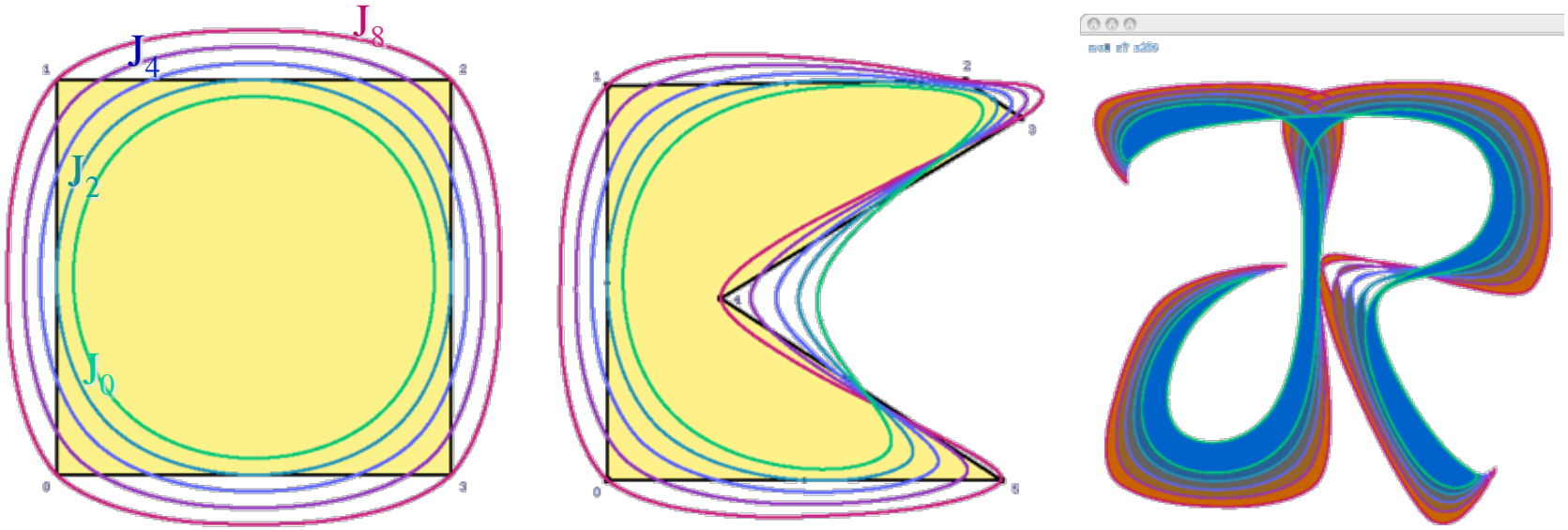


Three subdivision schemes

- B-spline (uniform cubic B-spline), most popular:
  - Introduces new vertices at edge mid-points (split)
  - Tucks in the old vertices by 1/2 the displacement towards the average of their new neighbors (tweak)
  - Very smooth result: piecewise cubic parametric formulation
  - Does not interpolate the original vertices
  - Shrinks convex regions
- 4-points:
  - Leaves the old vertices unchanged
  - Bulges the edges by displacing their mid-point away from the average of the neighbors (split & tweak)
  - Interpolates the original vertices
  - Grows convex regions
- J_4
  - Combines half of the tweaks of the other two
  - Final curve is closer to the original polygon

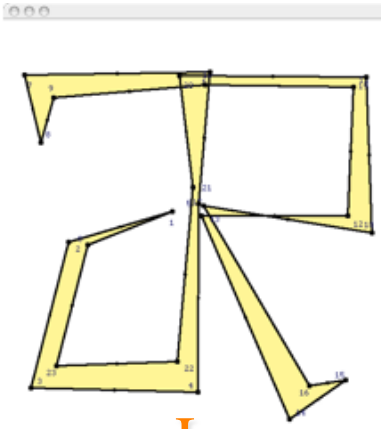
A unifying theory: **Js refinements**

- $J_s = (r , t , u(s/8))$
  - J_0 = cubic B-spline
  - J_2 = osculates edges (smoother than quadratic B-spline)
  - J_4 = in general preserves area well
  - J_5 = reduces discrepancy with L (good for LoD)
  - J_8 = 4-point (interpolating)



Comparing J_s schemes

$J_m = J_2, J_0, J_{-2}, J_{-2}, J_{-2} \ldots$



L

J_0

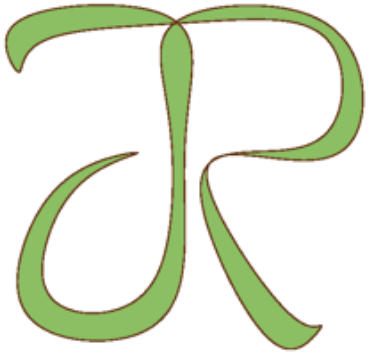
J_2

J_4

J_5

J_6

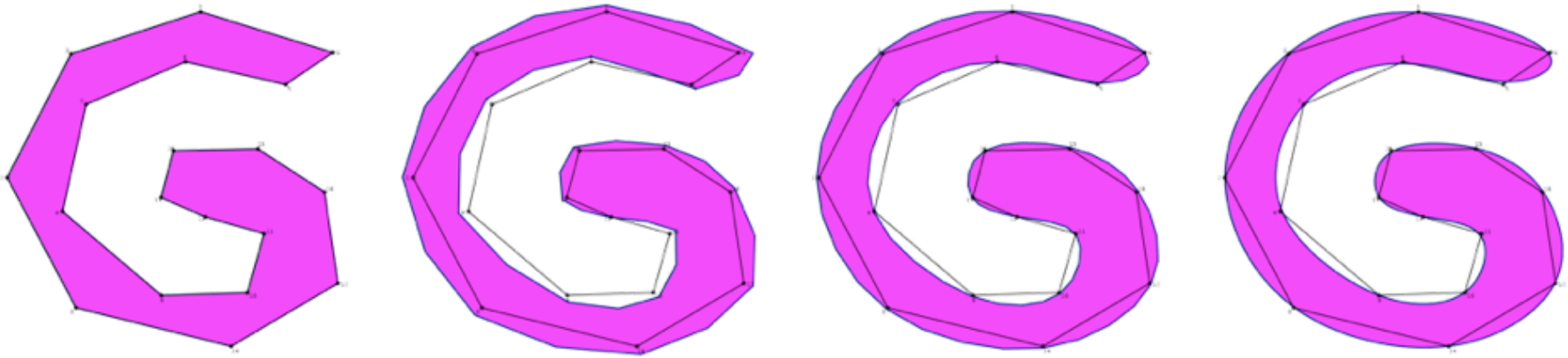
J_8



J_m

Retrofitting

- A new control polygon may be computed so that the subdivision curve passes through, or close to, the original control vertices.



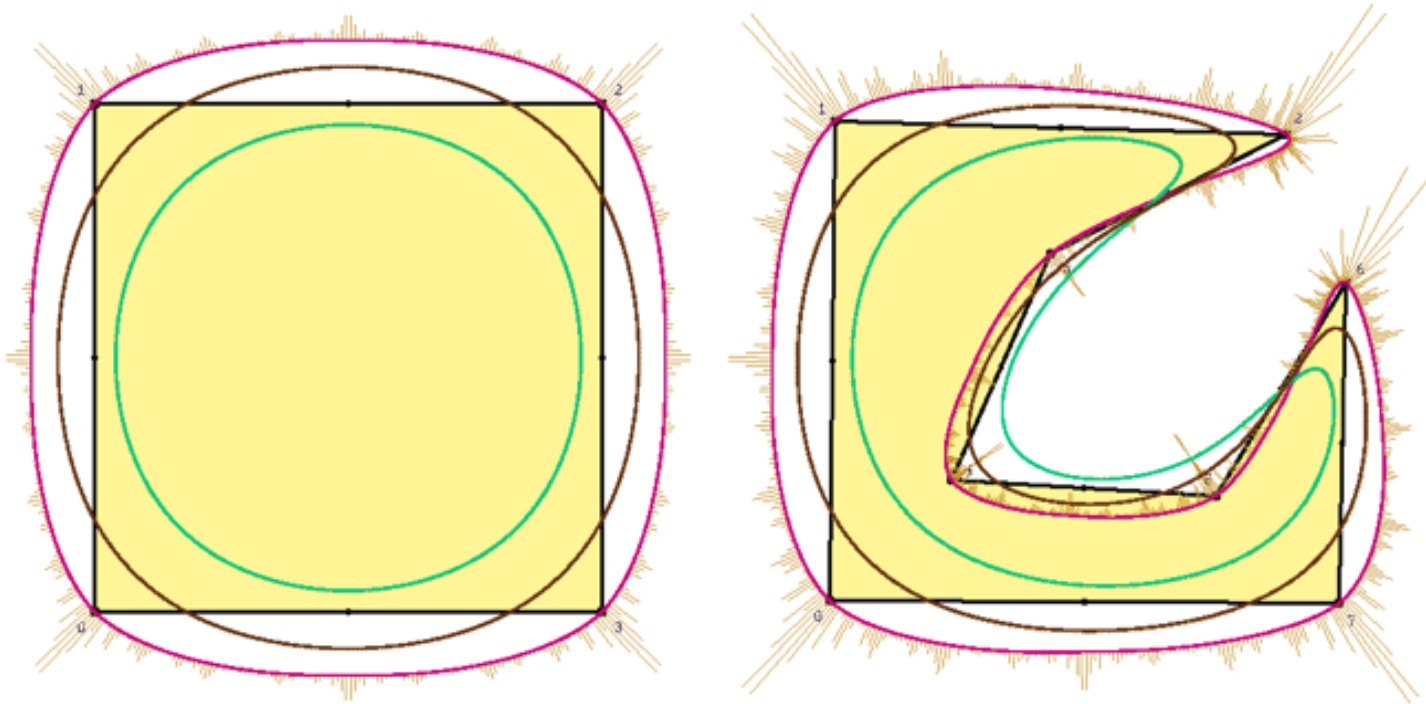
Continuity measures for paths and curves

- Parametric (path)
  - C^0: Continuous trajectory at finite speed (∞ acceleration?)
  - C^1: Velocity is a continuous function of parameter t
  - C^2: Acceleration is a continuous function of parameter t
  - C^3: Jerk is a continuous function of parameter t

- Geometric (curve)
  - G^0: Continuous curve, no gap
  - G^1: Tangent is a continuous function of arclength
  - G^2: Curvature is a continuous function of arclength

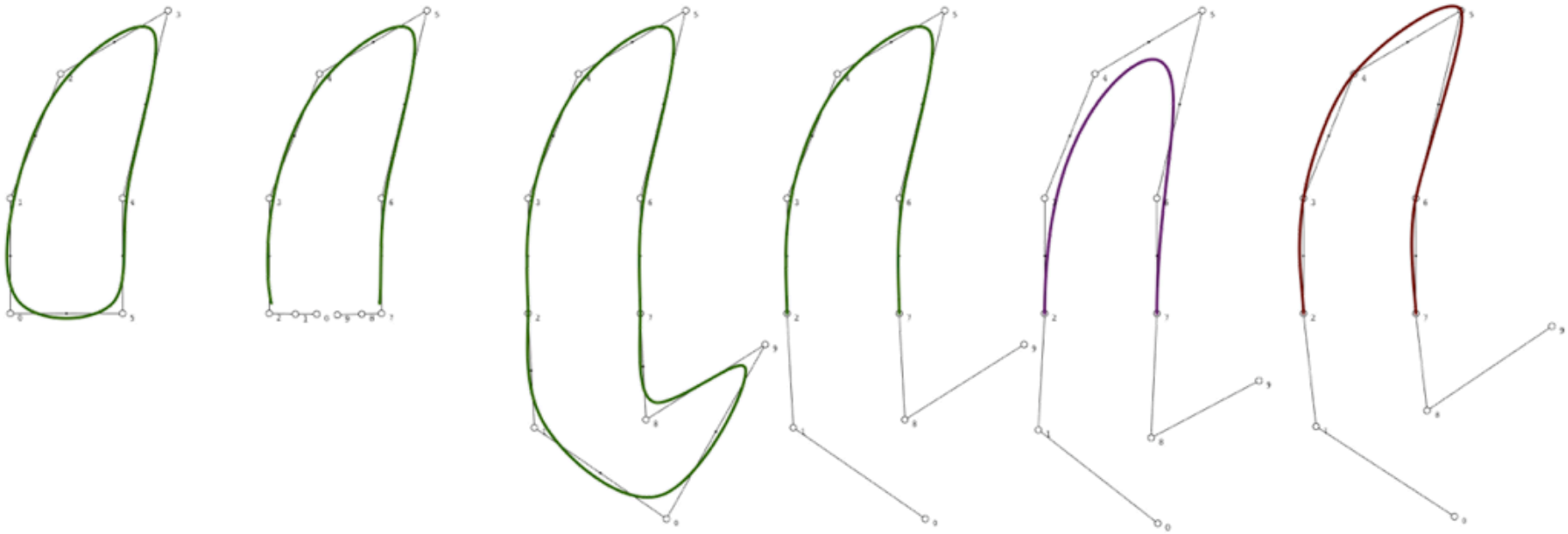
Smoothness

- Visualize (lack of smoothness) using jerk
  - J_0: (no visible jerk) second degree (curvature) continuous, I.e., C^2
  - J_8: (total jerk) first degree (tangent) continuous, I.e. C^1
  - J_m (brown) is C^3



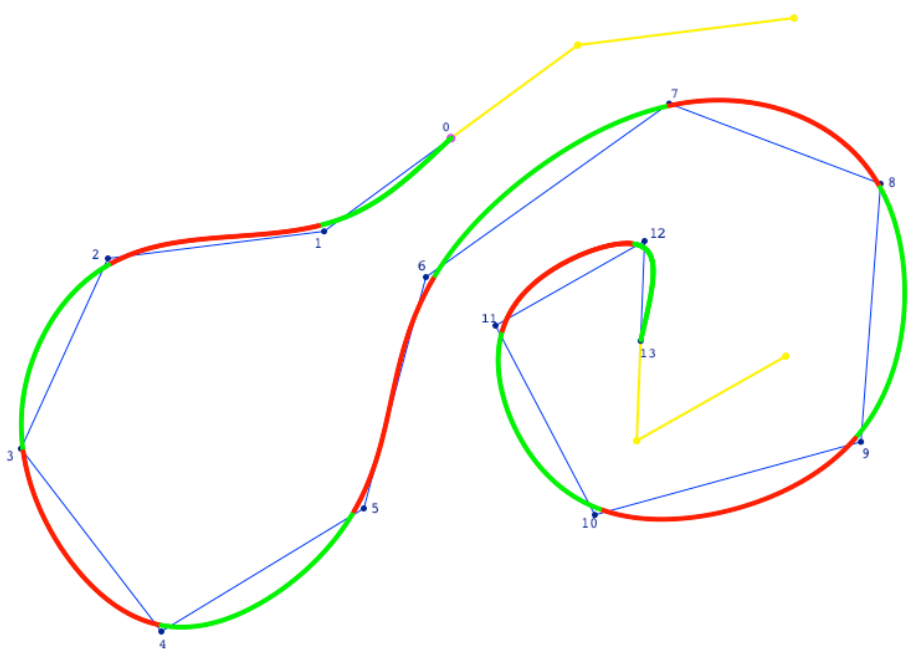
Open curves

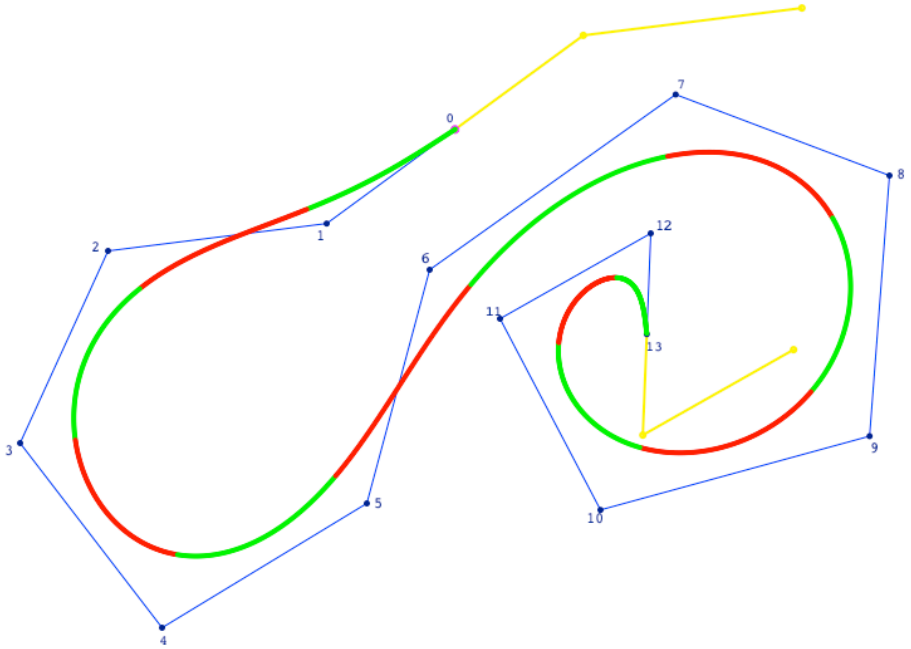
- Temporarily add (guesses of) 2 extra points on each end
- Subdivide as before
- Render curve, except 5 spans
- Remove the extra points



Spans

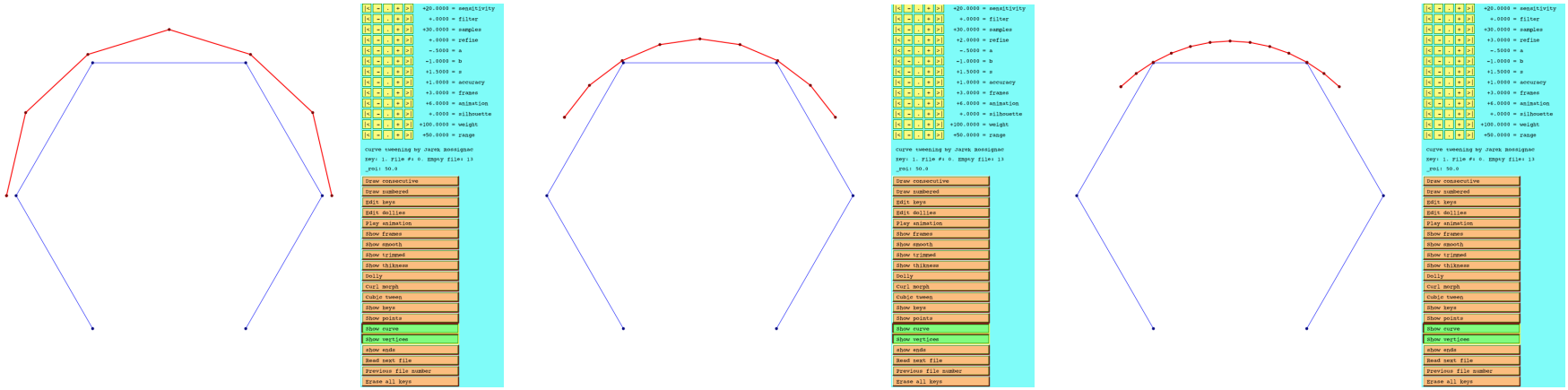
- Subdivision bends each original edge into a span





Per span subdivision

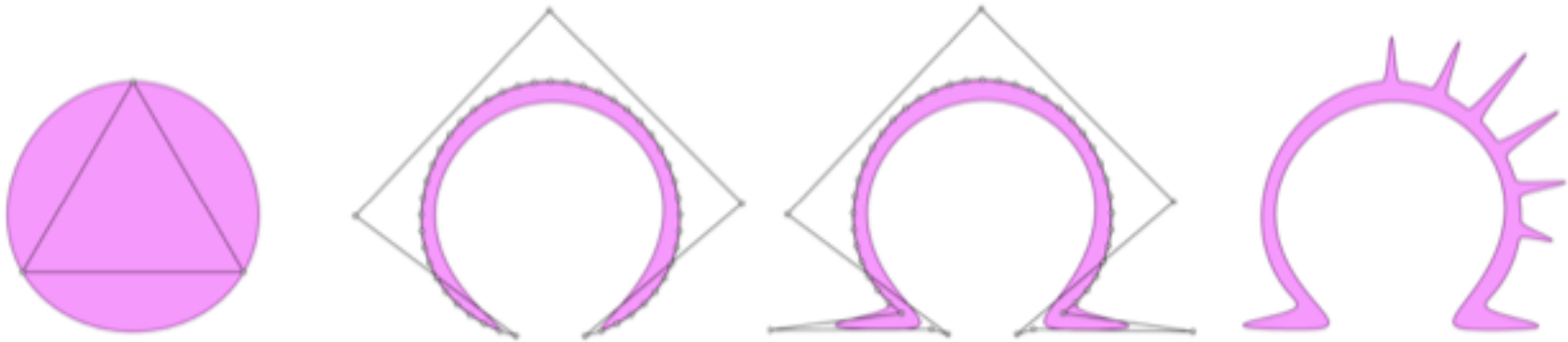
- You do not need to subdivide the entire curve at the same time
- You can subdivide one span at a time (need 6 control vertices)



- Or use a sliding window (ringing), which requires storing a ring of only 4 vertices per level of subdivision (see paper)

Multiresolution design

- At each level of subdivision, let the user specify adjustment vectors for some of the vertices
- Retain these vectors and apply them during rendering
- This lets the user control large and small features with few strokes

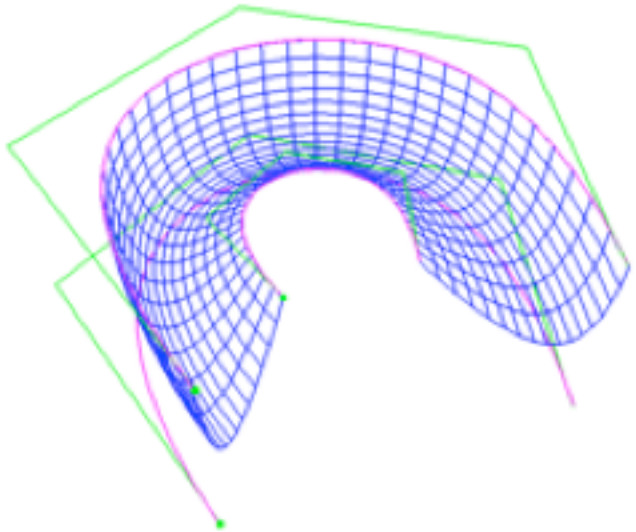


What about curves in 3D?

- The smoothing and subdivision procedures work for any dimension
  - 1D for speed control in P1a
  - 2D for curves, paths, and trajectories of cars
  - 3D for curves, paths, and trajectories of airplanes
  - 2D for designing and smoothing trajectories on surfaces (skier)
  - In the space of screws for smoothing key-frames motions…

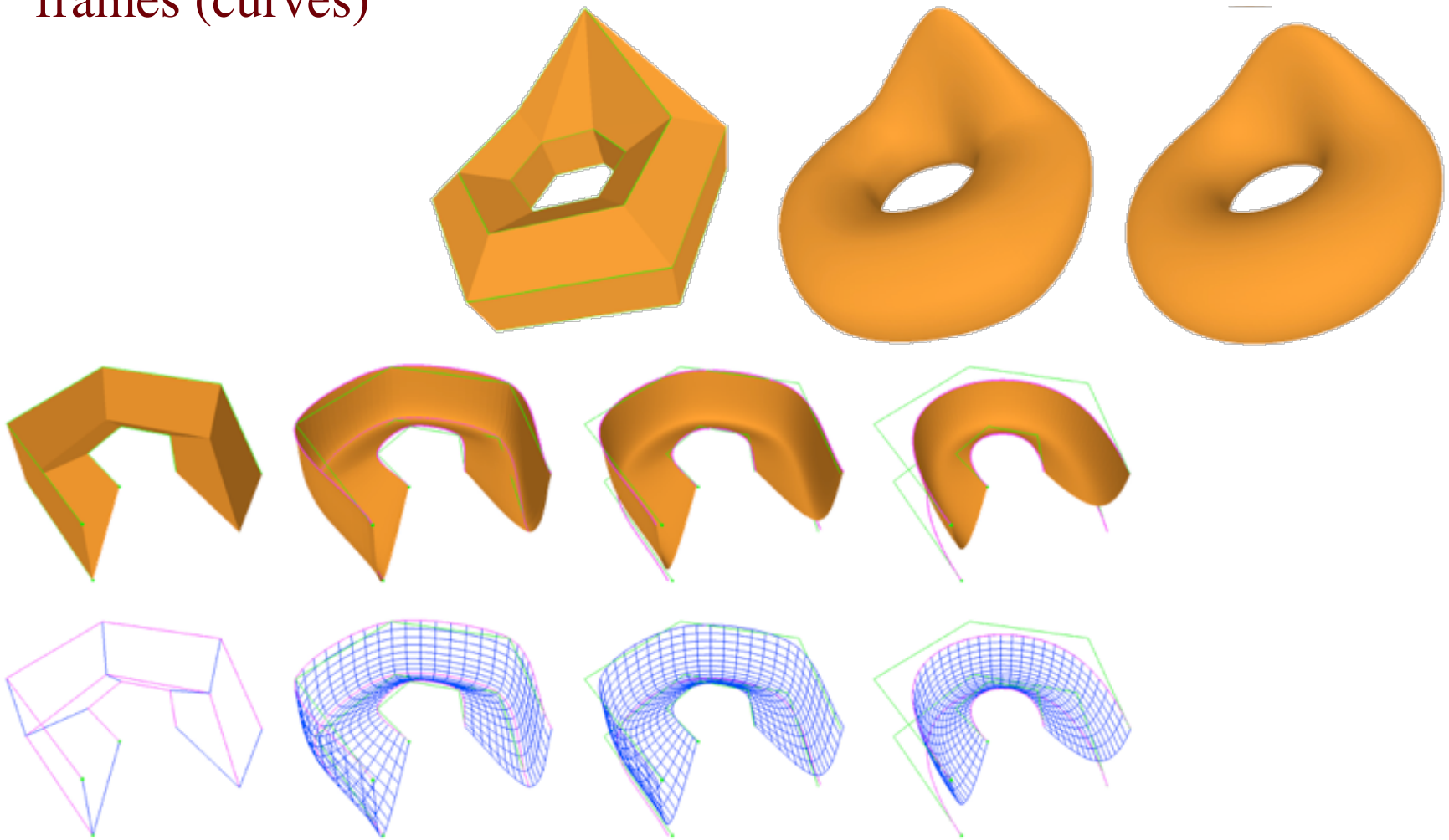
What about animation?

- At each frame, draw subdivision curve
- Its control vertices move with time
- Their trajectories are each defined as a subdivision curve



What about surfaces?

- Same as curve animation, but draw quads between consecutive frames (curves)



Adaptive resolution

- The level of subdivision of a curve or surface should be adjusted to the resolution and the size of its projection on the screen
- When the size changes, as we switch resolution, a popping may be perceived.
- Jarek subdivision exhibits less popping that four-point and cubic B-spline
- Geomorphs are also used to smoothen the popping over several frames

Examples of quiz/exam questions

- Define/illustrate curve, path, motion, trajectory
- Explain interpolating and approximating
- Constructions of velocity, normal, acceleration, radius of curvature, jerk
- Implementation of r, k, d, t, u, u(x) steps
- Write polyloop smoothing using them
- Why tuck-tuck is better then tuck
- Cubic fit formulation for tuck-tuck
- Difference between smoothing and subdivision
- Subdivision formula for quadratic B-spline
- Subdivision formula for cubic B-spline
- Spans. local control, open curves
- Bezier control polygons from polyloop
- Cubic Bezier evaluation and subdivision
- Convex hull of cubic bezier curve
- 4-point subdivision as cubic fit
- Draw 4-points, jarek, and cubic B-spline for a simple polyloop by hand
- General formulation for jarek subdivision J_s
- Properties of J_0, J_3, J_4, J_5, J_8