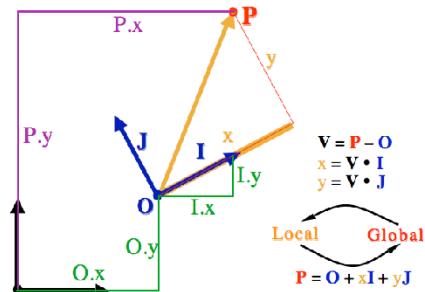


2D geometry



- Vectors, points, dot-product
- Coordinates, transformations
- Lines, edges, intersections
- Triangles
- Circles

Updated Sept 2010



Motivation

- The algorithms of Computer Graphics, Video Games, and Digital Animations are about modeling and processing geometric descriptions of shapes, animations, and light paths.

Vectors (Linear Algebra)

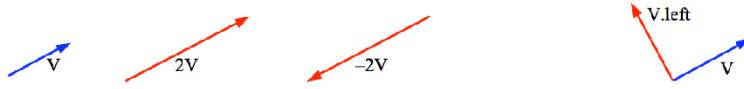
- A vector is defined by a **direction** and a...?
magnitude (also called norm or length)
- A vector may be used to represent what?
displacement, force...
- What is a **unit** vector?
a vector with **magnitude 1** (measured in chosen unit)
- What does a **unit** vector represent?
a **direction** (tangent, outward normal)
- What is sV , where s is a scalar?
a vector with direction of V , but norm **scaled** by s
- What is $U+V$?
the **sum** of the displacements of U and of V

Unary vector operators

- Defines a direction (and displacement magnitude)
- Used to represent a basis vector, tangent, normal, force...

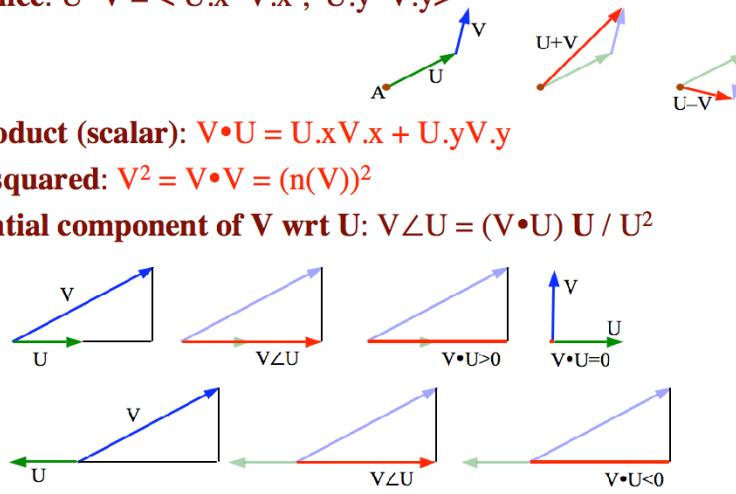


- **Coordinates:** $\mathbf{V} = \langle V.x, V.y \rangle$
- **Opposite:** $-\mathbf{V} = \langle -V.x, -V.y \rangle$
- **Norm** (or length, or magnitude): $n(\mathbf{V}) = \sqrt{(V.x^2 + V.y^2)}$
- **Null vector:** $\mathbf{o} = \langle 0, 0 \rangle, n(\mathbf{o})=0$
- **Scaling:** $s\mathbf{V} = \langle sV.x, sV.y \rangle, \mathbf{V}/s = \langle V.x/s, V.y/s \rangle$
- **Direction** (unit vector): $\mathbf{U(V)} = \mathbf{V}/n(\mathbf{V}), \text{assume } n(\mathbf{V}) \neq 0$
- **Rotated 90 degrees:** $R(\mathbf{V}) = \langle -V.y, V.x \rangle$



Binary vector operators

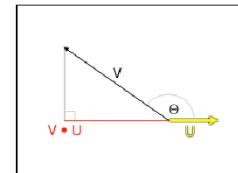
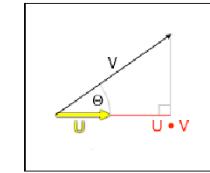
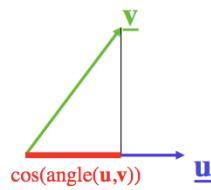
- **Sum:** $U+V = \langle U.x+V.x, U.y+V.y \rangle$,
- **Difference:** $U-V = \langle U.x-V.x, U.y-V.y \rangle$
- **Dot product (scalar):** $V \cdot U = U.x V.x + U.y V.y$
- **Norm squared:** $V^2 = V \cdot V = (n(V))^2$
- **Tangential component of V wrt U :** $V \angle U = (V \cdot U) U / U^2$



Dot Product: “Your best friend”

$$\mathbf{U} \cdot \mathbf{V} = \|\mathbf{U}\| \cdot \|\mathbf{V}\| \cdot \cos(\text{angle}(\mathbf{U}, \mathbf{V}))$$

- $\mathbf{U} \cdot \mathbf{V}$ is a scalar.
- if \mathbf{U} and \mathbf{V} are orthogonal \Rightarrow then $\mathbf{U} \cdot \mathbf{V} == 0$
- $\mathbf{U} \cdot \mathbf{V} == 0 \Rightarrow \mathbf{U} == 0$ or $\mathbf{V} == 0$ or (\mathbf{U} and \mathbf{V} are orthogonal)
- $\mathbf{U} \cdot \mathbf{V}$ is positive if the angle between \mathbf{U} and \mathbf{V} is less than 90°
- $\mathbf{U} \cdot \mathbf{V} = \mathbf{V} \cdot \mathbf{U}$, because: $\cos(a) = \cos(-a)$.
- $\|\mathbf{u}\| = \|\mathbf{v}\| = 1 \Rightarrow \mathbf{u} \cdot \mathbf{v} = \cos(\text{angle}(\mathbf{u}, \mathbf{v}))$ # unit vectors
- $\mathbf{V} \cdot \mathbf{u} = \text{signed length of projection of } \mathbf{V} \text{ onto the direction (unit vector)}$



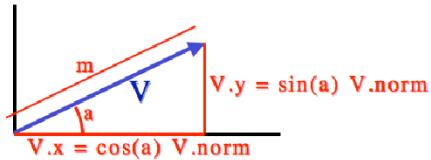
Dot product quiz

- What does the dot product $V \cdot U$ measure when U is unit?
The projected displacement of V onto U
- What is $V \cdot U$ equal to when U and V are unit?
 $\cos(\text{angle}(U,V))$
- What is $V \cdot U$ equal to for general U and V ?
 $\cos(\text{angle}(U,V)) n(V) n(U)$
- When is $V \cdot U = 0$?
 $n(U)=0$ OR $n(V)=0$ OR U and V are orthogonal
- When is $V \cdot U > 0$?
the angle between them is less than 90°
- How to compute $V \cdot U$?
 $U.xV.x + U.yV.y$
- What is V^2 ?
 $V^2 = V \cdot V = \text{sq}(n(V))$

Angles between vectors

- **Polar coordinates of a vector:** ($m=n(V)$, $a=\text{atan}2(V.y, V.x)$)

$$a \in [-\pi, \pi]$$



Assume $V \neq 0, U \neq 0$

- **Angle between two vectors:** $\cos(a) = V \cdot U / (n(V) * n(U))$

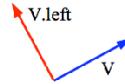
Use difference between polar coordinates to sort vectors by angle

- V and U are **orthogonal** (i.e. perpendicular) when $V \cdot U = 0$

$$V.x U.x + V.y U.y = 0$$

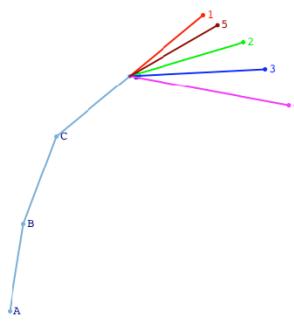
- V and U are **parallel** when $V \cdot R(U) = 0$

$$V.x U.y = V.y U.x$$



Application: Motion prediction

- Based on last 4 positions, how to predict the next one?



$a=2.6665165, b=-6.0261025, c=4.359586$

Change of orthonormal basis — important!

- A **basis** is two non-parallel vectors $\{I, J\}$
- A basis is **orthonormal** is $I^2=1$ and $J=I \cdot J$
- What is the vector with coordinates $\langle x, y \rangle$ in basis $\{I, J\}$?
 $xI+yJ$
- What is the vector $\langle x, y \rangle$ if we do not specify a basis?
 $xX+yY$, X is the horizontal, Y vertical unit vector
- What are the coordinates $\langle x, y \rangle$ of V in orthonormal basis (I, J) ?
 $x=V \cdot I$, $y=V \cdot J$

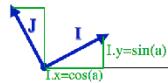
Rotating a vector

- What is the **rotation** (I, J) of basis (X, Y) by angle a ?

$$(I, J) = (\langle \cos(a), \sin(a) \rangle, \langle -\sin(a), \cos(a) \rangle)$$

- How to rotate vector $\langle x, y \rangle$ by angle a ?

compute (I, J) as above, then compute $xI + yJ$



- What are the **coordinates** of V rotated by angle a ?

$$V.\text{rotate}(a) =$$

$$= V.x \langle \cos(a), \sin(a) \rangle + V.y \langle -\sin(a), \cos(a) \rangle$$

$$= \langle \cos(a) V.x - \sin(a) V.y, \sin(a) V.x + \cos(a) V.y \rangle$$

- What is the **matrix form** of this **rotation**?

$$\begin{pmatrix} \cos(a) V.x - \sin(a) V.y \\ \sin(a) V.x + \cos(a) V.y \end{pmatrix} = \begin{pmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{pmatrix} \begin{pmatrix} V.x \\ V.y \end{pmatrix}$$

Vector coordinates quiz

- When are two vectors **orthogonal** (to each other)
When the angle between their directions is $\pm 90^\circ$
- What is an **orthonormal basis**?
two orthogonal unit vectors (\mathbf{I}, \mathbf{J})
- What is the vector with **coordinates** $\langle V.x, V.y \rangle$ in (\mathbf{I}, \mathbf{J})?
 $V.x \mathbf{I} + V.y \mathbf{J}$
- What are the coordinates of vector **combination** $U+sV$?
 $\langle U.x+sV.x, U.y+sV.y \rangle$
- What is the **norm** of V ?
 $n(V) = V.\text{norm} = \sqrt{V.x^2+V.y^2}$ (always ≥ 0)
- What are the coordinates of V **rotated** by 90°
 $R(V)=\langle -V.y, V.x \rangle$, *verify that $V \bullet R(V) = 0$*

Radial coordinates and conversions

- What are the **radial** coordinates {r,a} of V?
 $\{ V.\text{norm}, \text{atan2}(V.y, V.x) \}$
- What are the **Cartesian** coordinates of {r,a} ?
 $< r \cos(a), r \sin(a) >$

Reflection: used in collision and ray tracing

Consider a **line L** with **tangent** direction T

- What is the **normal** N to L?

$$N=R(T)$$

- What is the **normal component** of V?

$$(V \cdot N)N \quad (\text{it is a vector})$$

- What is the **tangent component** of V?

$$(V \cdot T)T$$

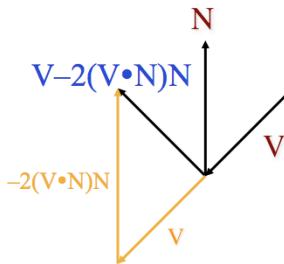
- What is the **reflection** of V on L?

$$(V \cdot T)T - (V \cdot N)N \quad (\text{reverse the normal component})$$

- What is the **reflection** of V on L (simpler form not using T)?

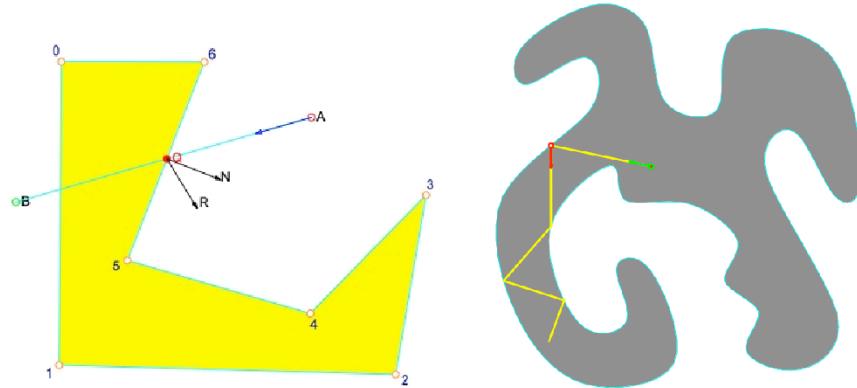
$$V - 2(V \cdot N)N \quad (\text{cancel, then subtract normal component})$$

This one works in 3D too (where T is not defined)



Application of reflection: Photon tracing

- Trace the path of a photon as it bounces off mirror surfaces (or mirror edges for a planar version)



Cross-product: “Your other best friend”

- The **cross-product** $U \times V$ of two vectors is a vector that is orthogonal to both, U and V and has for magnitude the product of their lengths and of the sine of their angle
$$\|U \times V\| = \|U\| \|V\| \sin(\text{angle}(U, V))$$
- Hence, the cross product of two vectors in the plane of the screen is a vector orthogonal to the screen.

OPERATOR OVERLOADING FOR 2D CONSTRUCTIONS

- When dealing with 2D constructions, we define $U \times V$ as a scalar:

$$U \times V = \|U\| \|V\| \sin(\text{angle}(U, V))$$

The 2D cross product is the z-component of the 3D cross-product.

Verify that in 2D: $U \times V = U \cdot R(V)$

Change of arbitrary basis

- What is the vector with coordinates $\langle x, y \rangle$ in basis $\{I, J\}$?

$$xI + yJ$$

- What are the coordinates $\langle x, y \rangle$ of V in basis (I, J) ?

Solve $V = xI + yJ$,

a system of two linear equations with variables x and y

(two vectors are equal if their x and y coordinates are)

The solution (using Cramers rule):

$$x = V \times J / I \times J \quad \text{and} \quad y = V \times I / J \times I$$

Proof

$$V = xI + yJ \rightarrow V \times J = xI \times J + yJ \times J \rightarrow V \times J = xI \times J \rightarrow V \times J / I \times J = x$$

Points (Affine Algebra)

- Define a location
- Coordinates $P = (P.x, P.y)$
- Given origin O : P is defined by vector $OP = P - G = \langle P.x, P.y \rangle$
- Subtraction: $PQ = Q - P = \langle Q.x - P.x, Q.y - P.y \rangle$
- Translation (add vector): $Q = P + V = (P.x + V.x, P.y + V.y)$

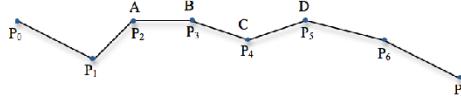


Incorrect but convenient notation:

- Average: $(P+Q)/2 = ((P.x+Q.x)/2, (P.y+Q.y)/2)$
correct form: $P + PQ/2$
- Weighted average: $\sum w_i P_i$, with $\sum w_i = 1$
correct form: $O + \sum w_j OP_j$

Practice with points

- What does a **point** represent?
a location
- What is $P+V$?
 P translated by displacement V
- What is the **displacement** from P to Q ?
 $PQ = Q - P$ (vector)
- What is the midpoint between P and Q ?
 $P + 0.5PQ$ (also written $P+PQ/2$ or wrongly $(P+Q)/2$)
- What is the center of mass G of triangle area (A,B,C)?
 $G=(A+B+C)/3$, properly written $G=A+(AB+AC)/3$



vector \neq point

	Vectors U,V,W	Points P,Q
Meaning	displacement	location
Translation	forbidden	$P+V$
Addition	$U+V$	forbidden
Subtraction	$W=U-V$	$V (= PQ) = Q-P$
Dot product	$s=U \bullet V$	forbidden
Cross product	$W=U \times V$	forbidden

Orientatin and point-triangle inclusion

- When is the sequence A,B,C a **left turn**?

$$cw(A,B,C) = AB \times BC > 0$$

(also = $R(AB) \cdot BC > 0$ and also $AB \times AC > 0 \dots$)

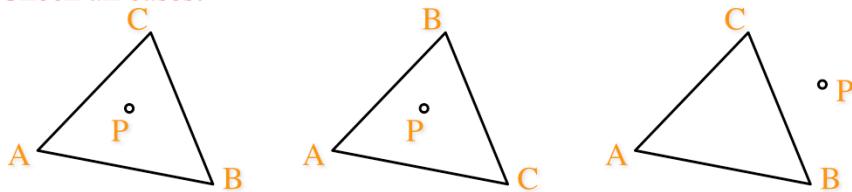
- When is triangle(A,B,C) **cw** (clockwise)?

$$cw(A,B,C)$$

- When is point **P** in triangle(A,B,C)?

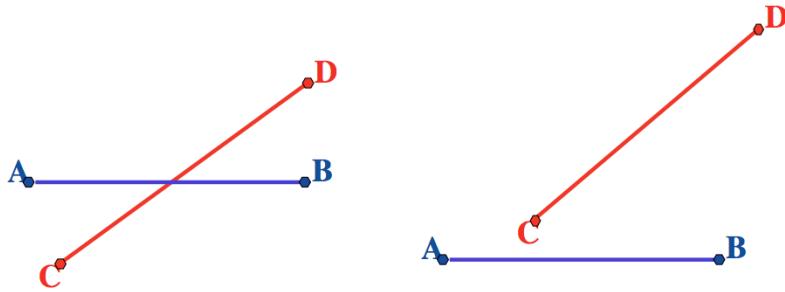
$$cw(A,B,P) == cw(B,C,P) \&& cw(A,B,P) == cw(C,A,P)$$

Check all cases:



Edge intersection test

- Linear parametric expression of the **point P(s) on edge(A,B)?**
 $P(s) = A + sAB$ (also written $(1-s)A + sB$) for s in $[0,1]$
my Processing implementation is called **L(A,s,B)**
- When do edge(A,B) and edge(C,D) intersect?
 $cw(A,B,C) \neq cw(A,B,D) \text{ } \&\& (cw(C,D,A) \neq cw(C,D,B))$
(special cases of collinear triplets require additional tests)

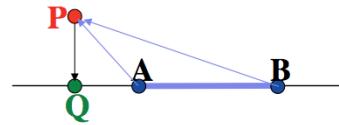


Normal projection on edge

- When does the projection Q of point P onto Line(A,B) fall between A and B
i.e.: when does P project onto edge(A,B)?

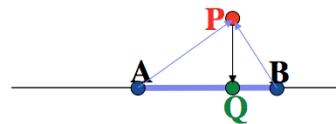
when:

$$0 \leq AP \cdot AB \leq AB \cdot AB$$

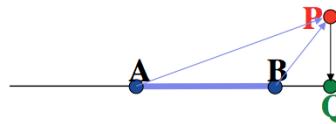


or equivalently, when:

$$0 \leq AP \cdot AB \quad \& \quad 0 \leq BP \cdot BA$$



explain why:



PinE(point,edge): Point-in-edge test

When is point P in edge(a,b)?

when $\|ab \times ap\| < \epsilon \|ab\| \text{ && } ab \cdot ap > 0 \text{ && } ba \cdot bp > 0$

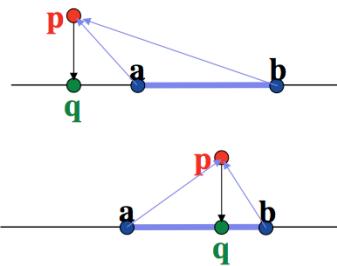
PROOF:

q = projection of p onto the line (a,b)

The distance $\|pq\|$ from p to q is

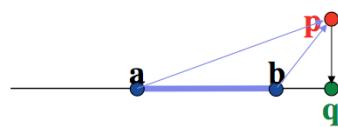
$$\|ab \times ap\| / \|ab\|$$

It needs to be less than a threshold ϵ



We also want the projection q to be
inside edge(a,b), hence:

$$ab \cdot ap > 0 \text{ && } ba \cdot bp > 0$$



Parallel lines

- When are line(P,T) and line(Q,U) **parallel**

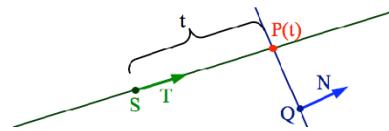
$$T \times U == 0$$

or equivalently when

$$T \bullet R(U) == 0$$

Ray/line intersection

- What is the expression of **point on ray**(S,T)?
 $P(t) = S + tT$, ray starts at S and has tangent T
- What is the constraint for **point P** to be **on line**(Q,N)?
 $QP \cdot N = 0$, normal component of vector QP is zero
- What is the intersection X of ray(S,T) with line(Q,N)?
 $X = P(t) = S + tT$, with t defined as the solution of
 $QP(t) \cdot N = 0$
- How to compute parameter t for the intersection above?
 $(P(t) - Q) \cdot N = 0$
 $(S + tT - Q) \cdot N = 0$
 $(QS + tT) \cdot N = 0$
 $QS \cdot N + tT \cdot N = 0$, distributing \cdot over $+$
 $t = - (QS \cdot N) / (T \cdot N)$



Lines intersections

Two useful representations of a line:

- **Parametric** form, LineParametric(S, T): $P(t) = S + tT$
- **Implicit** form, LineImplicit(Q, N): $QP \cdot N = 0$
 $\text{LineParametric}(S, T) = \text{LineImplicit}(S, R(T))$

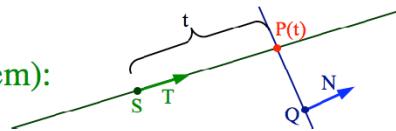
Intersection: $\text{LineParametric}(S, T) \cap \text{LineImplicit}(Q, N)$

- Substitute $P(t) = S + tT$ for P into $QP \cdot N = 0$
- Solve for the parameter value: $t = (SQ \cdot N) / (T \cdot N)$
- Substitute back: $P(t) = S + (SQ \cdot N) / (T \cdot N) T$

Other approaches (solve linear system):

$$S + tT = S' + uT \quad \text{or} \quad QP \cdot N = 0$$

$$Q'P \cdot N' = 0$$



Half-space

- Linear **half-space** $H(S,N) = \{P : SP \cdot N < 0\}$
set of points P such that they are “behind” S with respect to N
N is the **outward normal** to the half-space
 $H(S,N)$ does not contain line $\{P : SP \cdot N == 0\}$ (is **topologically open**)



- $L = \text{line}(S,T)$ (through S with tangent T)
- $L.\text{right} = H(S,R(T))$
 - $N=R(T)$ is the **outward normal to the half-space**
 - $L.\text{right}$ is shown on the left in a Processing canvas (Y goes down)
 - $L.\text{right}$ does not contain L (**topologically open**)

Transformations

- **Translation** of $P=(x,y)$ by vector V : $T_V(P) = P+V$
- **Rotation** : $R_a(P) = (x \cos(a) - y \sin(a) , x \sin(a) + y \cos(a))$
by angle a around the origin
- **Composition**: $T_V(R_a(P))$, rotates by a , then translates by V
- **Translations commute**: $T_U(T_V(P))=T_V(T_U(P))=T_{U+V}(P)$
- **2D rotations commute**: $R_b(R_a(P))=R_a(R_b(P))=R_{a+b}(P)$
- **Rotations/translations do not commute**: $T_V(R_a(P)) \neq R_a(T_V(P))$
- **Canonical representation of compositions of transformations**:
Want to represent ... $T_W(R_c(T_U(R_b(P))) \dots$ as $T_V(R_a(P))$
How to compute V and a ?
How to apply it to points and vectors?
Answer: represent a composed transformation by a **coordinate system**

Coordinate system (“frame”)

- **Coordinate system [I,J,O]**

O is the origin of the coordinate system (a translation vector)

{I,J} is an **ortho-normal basis**: I.norm=1, J=R(I)

{I,J} captures the rotation part of the transformation

- Given **local coordinates** (x,y) of P in [I,J,O]

$P = O + xI + yJ$, “start at O, move by x along I, move by y along J”

- Given P, O, I, J, compute (x,y)

$$x = OP \cdot I, y = OP \cdot J$$

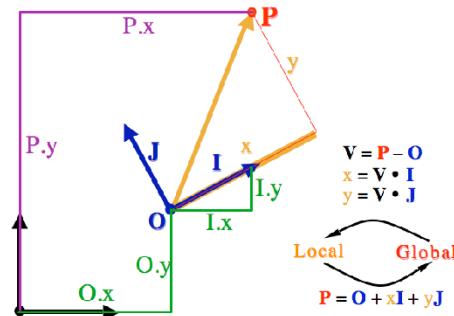
proof: $OP \cdot I = xI + yJ \cdot I = xI \cdot I$

- For a **vector V, no translation**

Local coordinates <x,y>

$$\text{Vector } V = xI + yJ$$

Inverse: $x = V \cdot I, y = V \cdot J$



Rotation around center C

- What is the **result P'** of rotatin a point P by angle a around C?

Rotate vector CP and add it to C

$$P' = C + CP.\text{rotate}(a)$$

$$\text{Hence: } P' = C + CP.x \langle \cos a, \sin a \rangle + CP.y \langle \sin a, \cos a \rangle$$

This can be executed in Processing (and OpenGL) as 3 transforms:

- Translate by CO (now C is at the origin and P is at O+CP)
- Rotate by angle a (rotates CP around origin: O+CP.rotate(a))
- Translate by OC (to put things back: O+CP.rotate(a)+OC)

A different (faster?) implementation

$(\cos(a) P.x - \sin(a) P.y, \sin(a) P.x + \cos(a) P.y)$

may also be implemented as:

$$P.x -= \tan(a/2) P.y$$

$$P.y += \sin(a) P.x$$

$$P.x -= \tan(a/2) P.y$$

- *Which one is it faster to compute (this or the matrix form)?*

For animation, or to trace a circle:

- *pre-compute $\tan(a/2)$ and $\sin(a)$*
- *at each frame,*
 - update $P.x$ and $P.y$*
 - add displacement OC if desired before rendering*

Practice with Transforms

- What is the **translation of point P by displacement V?**

$$P+V$$

- What is the **translation of vector U by displacement V?**

$$U \quad (\text{vectors do not change by translation})$$

- What is the rotation (around origin) of point P by angle a?

same as $O + \text{rotation of } OP$

$$(\cos(a) P.x - \sin(a) P.y, \sin(a) P.x + \cos(a) P.y)$$

- What is the matrix form of this rotation?

$$\begin{pmatrix} \cos(a) P.x - \sin(a) P.y \\ \sin(a) P.x + \cos(a) P.y \end{pmatrix} = \begin{pmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{pmatrix} \begin{pmatrix} P.x \\ P.y \end{pmatrix}$$

Change of frame

Let (x_1, y_1) be the coordinates of P in $[I_1 J_1 O_1]$

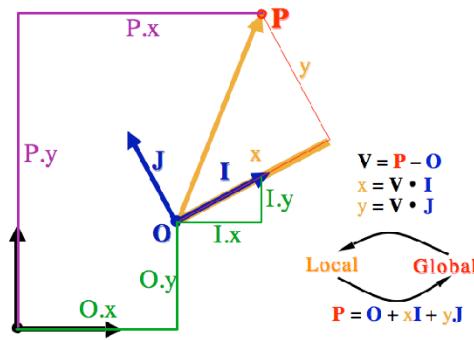
What are the coordinates (x_2, y_2) of P in $[I_2 J_2 O_2]$?

$$P = O_1 + x_1 I_1 + y_1 J_1 \quad (\text{convert local to global})$$

$$x_2 = O_2 P \cdot I_2 \quad (\text{convert global to local})$$

$$y_2 = O_2 P \cdot J_2$$

Applications:



What is in a rigid body transform matrix?

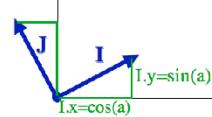
▪ Why do we use homogeneous transforms?

To be able to represent the cumulative effect of rotations, translations, (and scalings) into a single matrix form

$$\begin{pmatrix} P.x \\ P.y \\ 1 \end{pmatrix} = \begin{pmatrix} I.x & J.x & O.x \\ I.y & J.y & O.y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

What do the **columns of M** represent?

- A canonical transformation $T_O(R_a(P))$
 $O = \langle O.x, O.y \rangle$ is the translation vector
- $[I, J]$ is the local basis (image of the global basis)
- (I, J) is a 2×2 rotation matrix: $I.x = J.y = \cos(a)$, $I.y = -J.x = \sin(a)$
- $a = \text{atan2}(I.y, I.x)$ is the rotation angle, with $a \in [-\pi, \pi]$



Homogeneous matrices

- Represent a 2D coordinate system by a 3×3 homogeneous matrix
- Transform points and vectors through matrix-vector multiplication
 - For point P with local coordinate (x,y) use $\langle x,y,1 \rangle$
 - For vector V with local coordinate $\langle x,y \rangle$, use $\langle x,y,0 \rangle$
- Computing the global coordinates of P from local (x,y) ones

$$\langle P.x, P.y, 1 \rangle = [I.h \ J.h \ O.h](x, y, 1) = xI.h + yJ.h + O.h.$$

$$\begin{pmatrix} P.x \\ P.y \\ 1 \end{pmatrix} = \begin{pmatrix} I.x & J.x & O.x \\ I.y & J.y & O.y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- Vectors are not affected by origin (no translation)

Inverting a homogeneous matrix

- The inverse of R_a is R_{-a}
- The inverse of a rotation matrix is its **transpose**
 $I.x = J.y = \cos(a)$ remain unchanged since $\cos(-a) = \cos(a)$
 $I.y = -J.x = \sin(a)$ change sign (swap places) since $\sin(-a) = -\sin(a)$

$$\begin{pmatrix} P.x \\ P.y \\ 1 \end{pmatrix} = \begin{pmatrix} I.x & J.x & O.x \\ I.y & J.y & O.y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- The inverse of T_v is T_{-v}
- The inverse of $T_v(R_a(P))$ is $R_{-a}(T_{-v}(P))$
It may also be computed directly as $x=OP \cdot I$, $y=OP \cdot J$

Examples of questions for quiz...

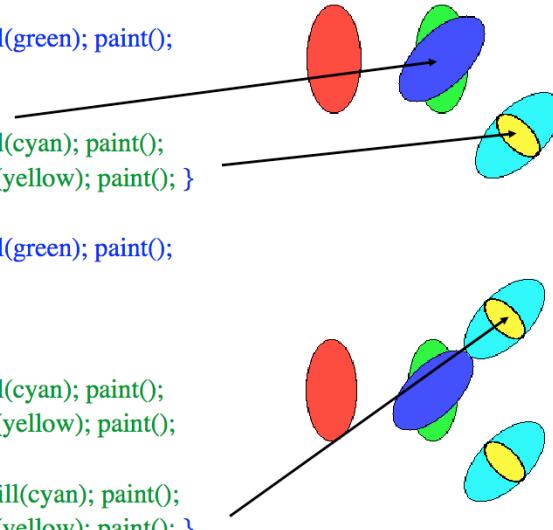
- What is the dot product $\langle 1,2 \rangle \bullet \langle 3,4 \rangle$?
- What is $R(\langle 1,2 \rangle)$?
- What is V^2 , when $V=\langle 3,4 \rangle$?
- What is the rotation by -30° of point P around point C?
- Let (x_1, y_1) be the coordinates of point P in $[I_1, J_1, O_1]$. How would you compute its coordinates (x_2, y_2) in $[I_2, J_2, O_2]$? (Do not use matrices, but combinations of points and vectors.)
- Point P will travel at constant velocity V. When will it hit the line passing through Q and tangent to T?

Transformations in graphics libraries

- `translate(V.x,V.y);` # implements $T_V(P)$
- `rotate(a);` # implements $R_a(P)$
- `translate(V.x,V.y); rotate(a);` # implements $T_V(R_a(P))$
Notice left-to-right order. Think of moving global CS.
- `Scale(u,v);` # implements $(uP.x,vP.y)$

Push/pop operators

- {
 fill(red); paint();
 translate(100,0); fill(green); paint();
 rotate(PI/4);
 fill(blue); paint();
 translate(100,0); fill(cyan); paint();
 scale(1.0,0.25); fill(yellow); paint(); }
- {
 fill(red); paint();
 translate(100,0); fill(green); paint();
 rotate(PI/4);
 fill(blue); paint();
 pushMatrix();
 translate(100,0); fill(cyan); paint();
 scale(1.0,0.25); fill(yellow); paint();
 popMatrix();
 translate(0, -100); fill(cyan); paint();
 scale(1.0,0.25); fill(yellow); paint(); }



Circles and disks

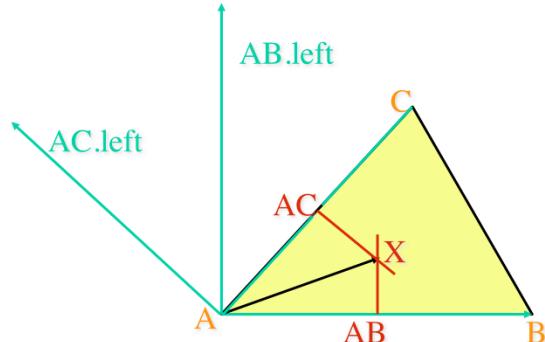
- How to identify all points P on circle(C,r) of center C and radius r?
 $\{ P : PC^2 = r^2 \}$
- How to identify all points P in disk(C,r)?
 $\{ P : PC^2 \leq r^2 \}$
- When do disk(C₁,r₁) and disk(C₂,r₂) interfere?
 $C_1C_2^2 < (r_1 + r_2)^2$

Circles and intersections

- **Circle** of center C and radius r, $\text{Circle}(C,r)$: $\{P: CP^2=r^2\}$
where $CP^2 = CP \bullet CP$
- **Disk** of center C and radius r, $\text{Disk}(C,r)$: $\{P: CP^2 \leq r^2\}$
- $\text{Disk}(C_1,r_1)$ and $\text{Disk}(C_2,r_2)$ **interfere** when $(C_1 \bullet C_2)^2 < (r_1 + r_2)^2$
- The intersection of $\text{LineParametric}(S,T)$ with $\text{Circle}(C,r)$:
Replace P in $CP^2 = r^2$ by $S+tT$
 $CP = P-C = P-S-tT = SP-tT$
 $(SP-tT) \bullet (SP-tT) = r^2$
 $(SP \bullet SP) - 2(SP \bullet T)t + (T \bullet T)t^2 = r^2$
 $t^2 - 2(SP \bullet T)t + (SP^2 - r^2) = 0$
Solve for t: real roots, t_1 and t_2 , assume $t_1 < t_2$
Points $S+tT$ when $t \in]t_1, t_2[$ are in $\text{Disk}(C,r)$

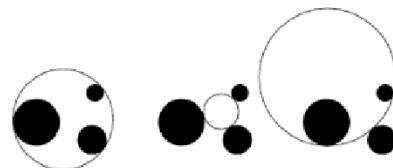
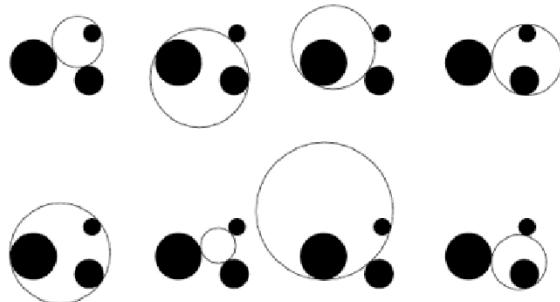
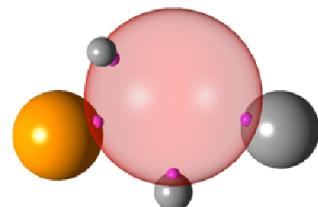
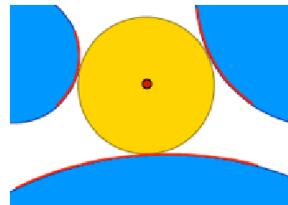
Circumcenter

```
pt centerCC (pt A, pt B, pt C) { // circumcenter to triangle (A,B,C)
    vec AB = A.vecTo(B);
    float ab2 = dot(AB,AB);
    vec AC = A.vecTo(C); AC.left();
    float ac2 = dot(AC,AC);
    float d = 2*dot(AB,AC);
    AB.left();
    AB.back(); AB.mul(ac2);
    AC.mul(ab2);
    AB.add(AC);
    AB.div(d);
    pt X = A.makeCopy();
    X.addVec(AB);
    return(X);
}
```

$$\left\{ \begin{array}{l} 2AB \cdot AX = AB \cdot AB \\ 2AC \cdot AX = AC \cdot AC \end{array} \right.$$


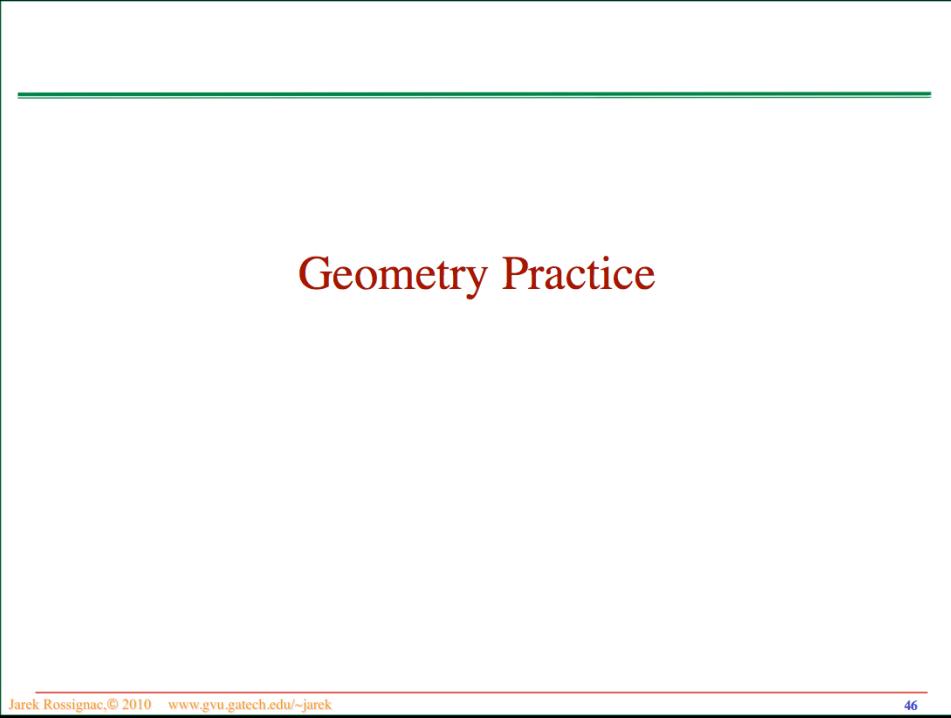
Circles & spheres tangent to others

- Compute circle tangent to 3 given ones
- In 3D, compute sphere tangent to 4 given ones.



Examples of questions for quiz...

- What is the implicit equation of circle with center C and radius r ?
- What is the parametric equation of circle (C,r) ?
- How to test whether a point is in circle (C,r) ?
- How to test whether an edge intersects a circle?
- How to compute the intersection between an edge and a circle?
- How to test whether two circles intersect?
- How to compute the intersection of two circles
- Assume that disk(C_1, r_1) starts at $t=0$ and travels with constant velocity V . When will it collide with a static disk(C_2, r_2)?
- Assume that a disk(C_1, r_1) arriving with velocity V has just collided with disk(C_2, r_2). Compute its new velocity V' .



Geometry Practice

1) Point on line

When is a point P on the line passing through point Q and having unit normal vector N?

$QP \bullet N = 0$, the vector from Q to a point on the line is orthogonal to N

2) Linear motion of point

- Point P starts at S and moves with constant velocity V. Where is it after t time units?

$P(t)=S+tV$, the displacement is time*velocity

3) Collision

- When will $P(t)$ of question 2 collide with the line of question 1 line through Q with unit normal vector N

$QP(t)=P(t)-Q=S+tV-Q=(S-Q)+tV=QS+tV$
 $(QS+tV) \cdot N=0$, condition for $P(t)$ to be on the line
Solving for t by distributing \cdot over $+$
 $t=(SQ \cdot N)/(V \cdot N)$, notice that $SQ=-QS$
When $V \cdot N=0$: no collision
 Q may already be on the line

4) Intersection

- Compute the intersection of a line through S with tangent V with line L' through Q with **normal** N .

Compute $t = (SQ \cdot N) / (V \cdot N)$, as in the previous slide and substitute this expression for t in $P = S + tV$, yielding:

$$P = S + ((SQ \cdot N) / (V \cdot N))V$$

If $V \cdot N = 0$: no intersection

5) Medial

- When is $P(s)=S+tV$, with $|V|=1$ at the same distance from S as from the line L' through Q with **normal N**

Since $|V|=1$, $P(t)$ has traveled a distance of t from S.

The distance between $P(t)$ and the line through Q with normal N is $QP(t) \cdot N$
Hence, we have two equations: $P(t)=S+tV$ and $QP(t) \cdot N=t$

Solve for t by substitution: $t = (QS \cdot N) / (1 - V \cdot N)$

If $V \cdot N=0$, use $-N$ instead of N

6) Point/line distance

- What is the distance between point P and the line through Q with normal N

$d = \mathbf{QP} \cdot \mathbf{N}$, as used in the previous question

7) Tangent circle

- Compute the radius r and center G of the circle tangent at S to a line with normal V and tangent to a line going through Q with normal N

From question 5:

$$r = (QS \cdot N) / (1 - V \cdot N)$$

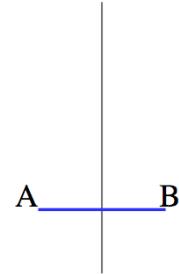
When $V \cdot N = 0$, use $-N$

$$G = S + rV$$

8) Bisector

- What is the bisector of points A and B?

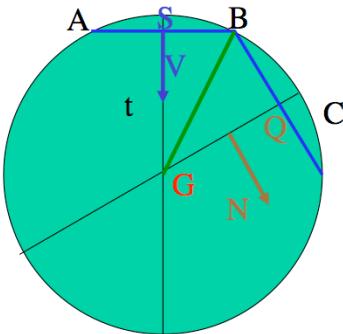
Line through $(A+B)/2$
With normal $N=AB.left.unit$



9) Radius

- Compute the radius and center of the circle passing through the 3 points: A, B, and C

We compute the bisectors of AB and BC
and use the result of question 3
 $S = (A+B)/2$; $V = BA.\text{left.unit}$
 $Q = (B+C)/2$; $N = BC.\text{unit}$
 $t = (SQ \cdot N) / (V \cdot N)$ (*from question 3*)
 $G = S + tV$
 $r = GB.\text{unit}$



10) Distance

- What is the square distance between points P and Q

PQ•PQ

11) Equidistant

- Let $P(t)=S+tV$, with $|V|=1$. When will $P(t)$ be equidistant from points S and Q ?

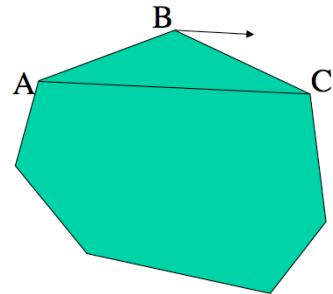
Similarly to question t, we have $P(t)=S+tV$
and want t such that $(QP(t))^2=t^2$

using W^2 is $W \bullet W$
 $(QS+tV) \bullet (QS+tV)=t^2$
Distributing and using $V \bullet V = 1$ permits to eliminate t^2

Solving for t :
 $t = QS^2/(2QS \bullet V)$

12) Tangent

- Estimate the tangent at B to the curve that interpolates the polyloop
... A, B, C...

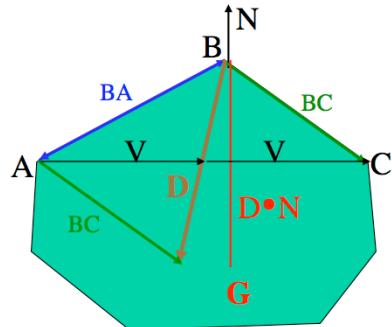


AC.unit

13) Center of curvature

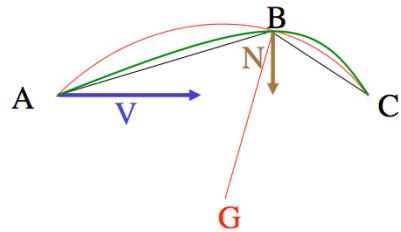
- Estimate the radius r and center G of curvature at point B the **curve approximated** by the polyline containing vertices A, B, C

Velocity: $V = AC/2$
Normal: $N = V.left.unit$
Acceleration: $D = BA+BC$
Normal acceleration: $-D \cdot N$
 $r = -V^2/D \cdot N$
The center of the osculating circle
 $G = B - rN$



Vector formulae for G in 2D and 3D?

- How to compute the center of curvature G in the previous question



$$V := AC/2 ;$$

$$N := BA + ((AB \cdot V) / (V \cdot V)) V ;$$

$$G := B + ((V \cdot V) / (2N \cdot N)) N ;$$

Practice: Circle/line intersection

- When does line(P,T) intersect disk(C,r)?

$$|PC \bullet (T.\text{left})| \leq r$$

- Where does line(S,T) intersect disk(C,r)?

$$CP = P - C = P - S - tT = SP - tT$$

$$(SP - tT) \bullet (SP - tT) = r^2$$

$$(SP \bullet SP) - 2(SP \bullet T)t + (T \bullet T)t^2 = r^2 \quad (\text{distribute } \bullet \text{ over } -)$$
$$t^2 - 2(SP \bullet T)t + (SP^2 - r^2) = 0$$

Solve for t: real roots, t_1 and t_2 , assume $t_1 < t_2$

Points $S + tT$ when $t \in]t_1, t_2[$ are in Disk(C,r)