Mathematical Induction

Domino Effect

Have you heard of the "Domino Effect"?

- Step 1. The **first** domino falls
- Step 2. When **any** domino falls, the **next** domino falls

So ... **all dominos will fall!**

That is how Mathematical Induction works.

In the world of numbers we say:

- Step 1. Show it is true for first case, usually **n=1**
- Step 2. Show that if **n=k** is true then **n=k+1** is also true

Example: is 3^n-1 a multiple of 2?

1. Show it is true for **n=1**

$$3^1-1 = 3-1 = 2$$

Yes 2 is a multiple of 2. That was easy.

3^1-1 is true

2. Assume it is true for **n=k**

3^k-1 is true

(Hang on! How do we know that? We don't!
It is an **assumption** ... that we treat
as a fact for the rest of this example)

Now, prove that $\mathbf{3^{k+1}-1}$ is a multiple of 2

$$3^{k+1}-1$$

$$3\times3^k-1$$

$$\underbrace{2\times3^k}_{\text{Multiple of 2}\ \checkmark} + \underbrace{3^k-1}_{\mathbf{n=k}\text{ case}\ \checkmark}$$

Prove that $1 + 3 + 5 + ... + (2n-1) = n^2$

1. Show it is true for **n=1**

$$1 = 1^2 \text{ is True}$$

2. Assume it is true for **n=k**

$$1 + 3 + 5 + ... + (2k-1) = k^2 \text{ is True}$$

(An assumption!)

3. Now, prove it is true for "k+1"

$$1 + 3 + 5 + ... + (2k-1) + (2(k+1)-1) = (k+1)^2 \quad \textbf{?}$$

We know that $\mathbf{1 + 3 + 5 + \ldots + (2k-1) = k^2}$ (the assumption above), so we can do a replacement for all but the last term:

$$\mathbf{k^2} + (2(k+1)-1) = (k+1)^2$$

Now expand all terms:

$$k^2 + 2k + 2 - 1 = k^2 + 2k+1$$

And simplify:

$$k^2 + 2k + 1 = k^2 + 2k + 1$$

$1 + 3 + 5 + \ldots + (2(k+1)-1) = (k+1)^2$ is True

Example: Triangular Numbers

Triangular numbers are numbers that can make a triangular dot pattern.

Prove that the **n-th** triangular number is:

$$T_n = n(n+1)/2$$

1. Show it is true for **n=1**

$$T_1 = 1 \times (1+1) / 2 = 1 \text{ is True}$$

2. Assume it is true for **n=k**

$$T_k = k(k+1)/2 \text{ is True (An assumption!)}$$

Now, prove it is true for "k+1"

$$T_{k+1} = (k+1)(k+2)/2 \quad ?$$

We know that $T_k = k(k+1)/2$ (the assumption above)

T_{k+1} has an extra row of $(k + 1)$ dots

So, $T_{k+1} = T_k + (k + 1)$

$$(k+1)(k+2)/2 = k(k+1) / 2 + (k+1)$$

Multiply all terms by 2:

$$(k + 1)(k + 2) = k(k + 1) + 2(k + 1)$$

$$(k + 1)(k + 2) = (k + 2)(k + 1)$$

They are the same! So it is **true**.

So:

$$T_{k+1} = (k+1)(k+2)/2 \quad \text{is True}$$

Strong Induction

- Sometimes in an induction proof it is hard to show that S_k implies S_{k+1} .

- It may be easier to show some “lower” S_m (with $m<k$) implies S_{k+1} .

- For such situations there is a slight variant of induction called strong induction.

- Strong induction works just like regular induction, except that in Step (2) instead of assuming Sk is true and showing this forces S_{k+1} to be true, we assume that all the statements $S_1, S_2, \cdots, S_k$ are true and show this forces S_{k+1} to be true.

- The idea is that if the first k dominoes falling always forces the $(k+1)$ th domino to fall, then all the dominoes must fall.

Strong Induction - Example 1

Suppose g_n is a recursively defined sequence given by $g_1 = 1$, $g_2 = 2$, $g_3 = 6$, and $g_n = (n^3 - 3n^2 + 2n)g_{n-3}$ for all $n \geq 4$.

Theorem. For all $n \in \mathbb{N}$, $g_n = n!$.

Proof. We verify that $g_1 = 1 = 1!$, so the claim is true when $n = 1$. **Also, $g_2 = 2 = 2!$ and $g_3 = 6 = 3!$, so the claim is true when $n = 2$ and when $n = 3$.**
Now let $k \geq 3$, and assume now that $g_i = i!$ for all integers i such that $1 \leq i \leq k$. Now since $1 \leq k - 2 \leq k$,

$$\begin{aligned} g_{k+1} &= [(k+1)^3 - 3(k+1)^2 + 2(k+1)]g_{k-2} \\ &= [(k+1)^3 - 3(k+1)^2 + 2(k+1)](k-2)! \\ &= [k^3 - k](k-2)! \\ &= [(k+1)k(k-1)](k-2)! \\ &= (k+1)!. \end{aligned}$$

Thus $g_n = n!$ for all integers $n \in \mathbb{N}$.

Proposition If $n \in \mathbb{N}$, then $12|(n^4 - n^2)$.

Proof. We will prove this with strong induction.

1. First note that the statement is true for the first six positive integers:

   For $n = 1$, 12 divides $1^4 - 1^2 = 0$.

   For $n = 2$, 12 divides $2^4 - 2^2 = 12$.

   For $n = 3$, 12 divides $3^4 - 3^2 = 72$.

   For $n = 4$, 12 divides $4^4 - 4^2 = 240$.

   For $n = 5$, 12 divides $5^4 - 5^2 = 600$.

   For $n = 6$, 12 divides $6^4 - 6^2 = 1260$.

2. For $k \geq 6$, assume $12|(n^4 - n^2)$ for $1 \leq m \leq k$ (i.e., $S_1, S_2, \cdots, S_k$ are true). We must show S_{k+1} is true, that is, $12|(k+1)^4 - (k+1)^2$. Now, S_{k-5} being true means $12|(k-5)^4 - (k-5)^2$. To simplify, put $k - 5 = l$ so $12|(l^4 - l^2)$, meaning $l^4 - l^2 = 12a$ for $a \in \mathbb{Z}$, and $k + 1 = l + 6$.Then:

$(k+1)^4 - (k+1)^2 = (l+6)^4 - (l+6)^2$

$= l^4 + 24l^3 + 216l^2 + 864l + 1296 - (l^2 + 12l + 36)$

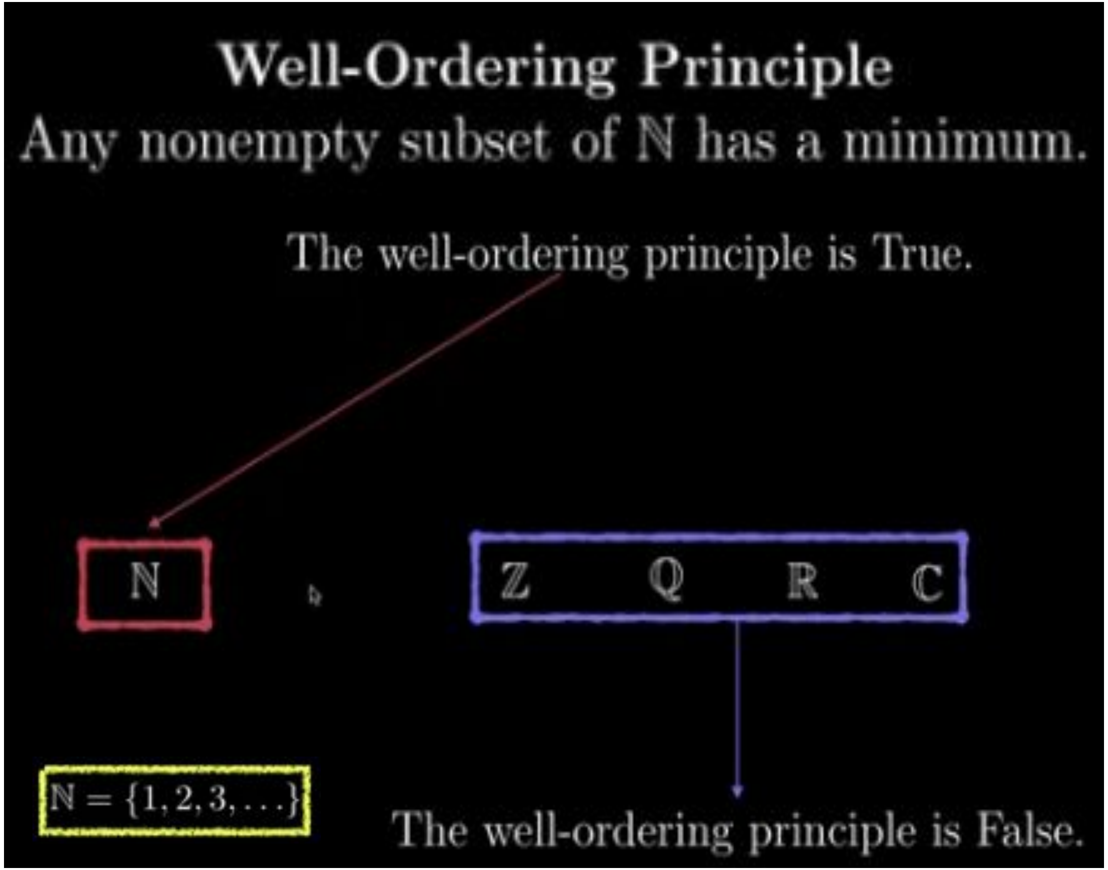
$= (l^4 - l^2) + 24l^3 + 216l^2 + 852l + 1260$

$12a + 24l^3 + 216l^2 + 852l + 1260$

$= 12(a + 2l^3 + 18l^2 + 71l + 105.$

Because $a + 2l^3 + 18l^2 + 71l + 105 \in \mathbb{Z}$, we get $12|(k+1)^4 - (k+1)^2$.

Well Ordering Principle


Well-Ordering Principle
Any nonempty subset of $\mathbb{N}$ has a minimum.
The well-ordering principle is True.
$\mathbb{N}$
$\mathbb{Z}$ $\mathbb{Q}$ $\mathbb{R}$ $\mathbb{C}$
$\mathbb{N} = \{1, 2, 3, \ldots\}$
The well-ordering principle is False.

Example 1

If $A = \{2, 5, 7, 10, 22\}$ then it is clear that

1. $A \subseteq \mathbb{N}$.
2. $A \neq \emptyset$.

The well-ordering principle says that A has a minimum.

In this case, $m = 2$ is the minimum of A.

We can have nonempty subsets of the integers $\mathbb{Z}$

that do NOT have a minimum.

For instance,

$$A = \{-n \mid n \in \mathbb{N}\} = \{-1, -2, -3, \ldots\}$$

$$B = \left\{-n^2 \mid n \in \mathbb{N}\right\} = \{-1, -4, -9, \ldots\}$$

$$C = \{-2n \mid n \in \mathbb{N}\} = \{-2, -4, -6, \ldots\}$$

A similar argument can be made for $\mathbb{Q}$ and $\mathbb{R}$.

Example 2

If $A = \{n \in \mathbb{N} \,|\, n^2 + 3\,n - 200 > 0\}$, then

$\boxed{\begin{array}{l} 1.\ A \subseteq \mathbb{N}. \\ 2.\ A \neq \emptyset. \end{array}}$ $\longrightarrow$ This is True.

By the well-ordering principle, A has a minimum.

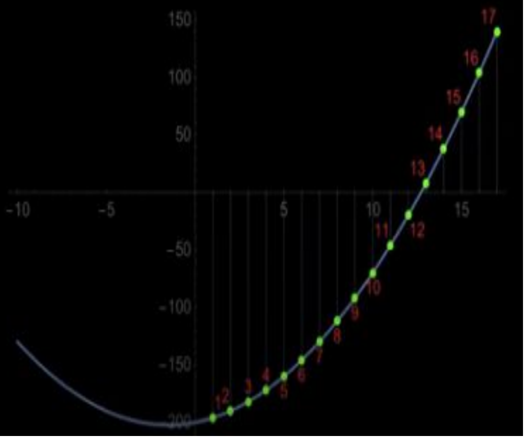
That is, there is $m \in A = \{n \in \mathbb{N} \,|\, n^2 + 3\,n - 200 > 0\}$

such that $m \leq n$ for all n that satisfy the inequality
$n^2 + 3\,n - 200 > 0$

$$m = \min(A).$$

The well-ordering principle does not tell me what "m" is exactly
it just tells me that "m" exists.

We can check that
$A = \{n \in \mathbb{N} \,|\, n^2 + 3n - 200 > 0\} = \{13, 14, 15, \ldots\}$.



$$A = \{n \in \mathbb{N} \mid n \cdot \sin(2\,n) > 8\} = \{10, 13, 16, 17, 20, 23, 26, 29, \ldots\}$$

In this case the minimum $m = 10$.