Logical Inference

“It’s not sunny and it’s colder than yesterday”

“We will go swimming only if it’s sunny.”

“If we don’t go swimming then we will take canoe trip.”

“If we take a canoe trip, then we will be home by sunset.”

Therefore the Conclusion is:
“We will be home by sunset.”

What is a inference rule?

Definition:

$$\frac{\begin{array}{l}P_1\\P_2\end{array}}{Q}$$

is a inference rule if $(P_1 \wedge P_2) \rightarrow Q$ is a tautology.

Example. Show that $P \to Q$ and $\neg P \vee Q$ are logically equivalent.

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the columns for $P \to Q$ and $\neg P \vee Q$ are identical, the two statements are logically equivalent.

Implication rule

The first (and simplest) rule is:

Modus Ponens (MP) Rule:

$$\begin{array}{l} P \\ \underline{P \rightarrow Q} \\ Q \end{array}$$

- This rule is called Modus Ponens (MP). Intuitively, if we have the condition of an implication, then we can obtain its consequence.

Example:

P	means: “there is a storm.”
$P \rightarrow Q$	means: “if there is a storm, then the office is closed.”
Q	means: “the office is closed.”

Exercise: Show $[P \wedge (P \rightarrow Q)] \rightarrow Q \equiv T$.

Another implication rule

- Recall that $P \to Q \equiv \neg P \vee Q \equiv Q \vee \neg P \equiv \neg Q \to \neg P$.
- The MP rule just studied above tells us that:

$$\begin{array}{l} \neg Q \\ \underline{\neg Q \to \neg P} \\ \neg P \end{array}$$

- If we replace the $\neg Q \to \neg P$ in the above with the logically equivalent proposition $P \to Q$, then we get another implication rule:

Modus Tonens (MT) Rule:

$$\begin{array}{l} \neg Q \\ \underline{P \to Q} \\ \neg P \end{array}$$

Exercise: Show $[\neg Q \wedge (P \to Q)] \to \neg P \equiv T$.

Logical equivalence vs. inference

By using inference rules, we can "prove" the conclusion follows from the premises. In inference, we can always replace a logic formula with another one that is logically equivalent, just as we have seen for the implication rule.

Example:

Suppose we have: $P \to (Q \to R)$ and $Q \wedge \neg R$. Use inference to show $\neg P$.

- First, we note $Q \wedge \neg R \equiv \neg(\neg Q \vee R) \equiv \neg(Q \to R)$.
- So we have the following inference:

(1)	$P \to (Q \to R)$	Premise
(2)	$Q \wedge \neg R$	Premise
(3)	$\neg(Q \to R)$	Logically equivalent to (2)
(4)	$\neg P$	Applying the second implication rule (Modus Tonens) to (1) and (3)

Hypothetical Syllogism

if P implies Q and Q implies R, then we can get that P implies R.

Example:

$P \to Q$	means	"if there is a storm, then the office is closed."
$Q \to R$	means	"if the office is closed, then I don't go to work."
$P \to R$	means	"if there is a storm, then I don't go to work."

Conjunction rule

$$\frac{\begin{array}{l} P \\ Q \end{array}}{P \wedge Q}$$

Intuitively, this means when you have P and Q both being true, then $P \wedge Q$ is also true.

Simplification Rule

$$\frac{P \wedge Q}{P}$$

Intuitively, this means when you have $P \wedge Q$ being true, clearly P is also true.

Disjunctive Syllogism (DS)

$$\frac{\begin{array}{l} P \vee Q \\ \neg P \end{array}}{Q}$$

Addition Rule:

$$\frac{P}{P \vee Q}$$

Resolution Rule

$$\frac{\begin{array}{l} P \vee Q \\ \neg P \vee R \end{array}}{Q \vee R}$$

Rule of Inference	Tautology	Name
$\begin{array}{l} p \\ p\to q \\ \hline \therefore q \end{array}$	$(p\wedge(p \to q)) \to q$	Modus Ponens
$\begin{array}{l} \neg q \\ p\to q \\ \hline \therefore \neg p \end{array}$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus Tollens
$\begin{array}{l} p\to q \\ q\to r \\ \hline \therefore p \to r \end{array}$	$((p\to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$\begin{array}{l} \neg p \\ p\vee q \\ \hline \therefore q \end{array}$	$(\neg p \wedge (p \vee q)) \to q$	Disjunctive Syllogism
$\begin{array}{l} p \\ \hline \therefore (p \vee q) \end{array}$	$p\to (p \vee q)$	Addition
$\begin{array}{l} (p\wedge q) \to r \\ \hline \therefore p \to (q \to r) \end{array}$	$((p\wedge q) \to r) \to (p \to (q \to r))$	Exportation
$\begin{array}{l} p\vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$	$((p\vee q) \wedge (\neg p \vee r)) \to q \vee r$	Resolution

Example:

Premises:

a. "It's not sunny and it's colder than yesterday"
b. "We will go swimming only if it's sunny."
c. "If we don't go swimming then we will take canoe trip."
d. "If we take a canoe trip, then we will be home by sunset."

Conclusion: "We will be home by sunset." t.

propositions:

p: It's sunny this afternoon.
q: It's colder than yesterday.
r: We will go swimming.
s: We will take a canoe trip.
t: We will be home by sunset.

Example:

Premises:

a. "It's not sunny and it's colder than yesterday" $\neg p \wedge q$
b. "We will go swimming only if it's sunny." $r \rightarrow p$
c. "If we don't go swimming then we will take canoe trip." $\neg r \rightarrow s$
d. "If we take a canoe trip, then we will be home by sunset." $s \rightarrow t$

Conclusion: "We will be home by sunset." t.

(1)	$\neg p \wedge q$	Premise
(2)	$\neg p$	Simplification rule using (1)
(3)	$r \rightarrow p$	Premise
(4)	$\neg r$	MT using (2) (3)
(5)	$\neg r \rightarrow s$	Premise
(6)	s	MP using (4) (5)
(7)	$s \rightarrow t$	Premise
(8)	t	MP using (6) (7)

This is a valid argument showing that from the premises (a), (b), (c) and (d), we can prove the conclusion t.

Example:

Suppose $P \rightarrow Q$; $\neg P \rightarrow R$; $Q \rightarrow S$. Prove that $\neg R \rightarrow S$.

(1)	$P \rightarrow Q$	Premise
(2)	$\neg P \vee Q$	Logically equivalent to (1)
(3)	$\neg P \rightarrow R$	Premise
(4)	$P \vee R$	Logically equivalent to (3)
(5)	$Q \vee R$	Apply resolution rule to (2)(4)
(6)	$\neg R \rightarrow Q$	Logically equivalent to (5)
(7)	$Q \rightarrow S$	Premise
(8)	$\neg R \rightarrow S$	Apply HS rule to (6)(7)

Example: Suppose:

(1) If it is Saturday today, then we play soccer or basketball.
(2) If the soccer field is occupied, we dont play soccer.
(3) It is Saturday today, and the soccer field is occupied.
Prove: “we play basketball or volleyball”.

First we formalize the problem:
P: It is Saturday today.
Q: We play soccer.
R: We play basketball.
S: The soccer field is occupied.
T: We play volleyball.
Premise: $P \to (Q \vee R),\ S \to \neg Q,\ P,\ S$
Need to prove: $R \vee T$.

(1)	$P \rightarrow (Q \vee R)$	Premise
(2)	P	Premise
(3)	$Q \vee R$	Apply MP rule to (1)(2)
(4)	$S \rightarrow \neg Q$	Premise
(5)	S	Premise
(6)	$\neg Q$	Apply MP rule to (4)(5)
(7)	R	Apply DS rule to (3)(6)
(8)	$R \vee T$	Apply Addition rule to (7)

Example: Suppose:

- all natural numbers are integers;
- there exists a natural number;

Prove that there exists an integer.

We can formalize this problem as follows. (Let the universe of discourse be all real numbers.)

$N(x)$: x is a natural number.
$I(x)$: x is an integer.
Premise: $\forall x\,(N(x) \to I(x)),\ \exists x\, N(x)$
Need to prove: $\exists x\, I(x)$

(1)	$\exists\, x\, N(x)$	Premise
(2)	$N(c)$	Apply existential instantiation rule to (1)
(3)	$\forall\, x\, (N(x) \rightarrow I(x))$	Premise
(4)	$N(c) \rightarrow I(c)$	Apply universal instantiation rule to (3)
(5)	$I(c)$	Apply MP rule to (2)(4)
(6)	$\exists\, x\, I(x)$	Apply existential generalization rule to (5)

Solving a murder case

The following is a murder case solved by Sherlock Holmes, in "A Study in Scarlet" (a detective mystery novel by Sir Arthur Conan Doyle).

Quote from "A Study in Scarlet"

> *"And now we come to the great question as to the reason why. Robbery has not been the object of the murder, for nothing was taken. Was it politics, then, or was it a woman? That is the question which confronted me. I was inclined from the first to the latter supposition. Political assassins are only too glad to do their work and fly. This murder had, on the contrary, been done most deliberately, and the perpetrator has left his tracks all over the room, showing he had been there all the time."*

From these, Sherlock Holmes concluded: "It was a woman".

Solving a murder case

Known premises:

1. If it's a robbery, something would have been taken.
2. Nothing was taken.
3. If it's not a robbery, it must be politics or a woman.
4. It it's politics, the assassin would have left immediately.
5. If assassin left tracks all over the room, he cannot have left immediately.
6. The assassin left tracks all over the room.

Show the conclusion: "It was a woman".