RELATION

Relation has ordered pair of set :-
{1, 3} & {3, 1} are not same

$x R y$ $<$

R = {1, 3} , {2, 4}

I = {1, 2, 3, ... }

→ Power Set : It is the set of all subsets.

S = {0, 1} → n = 2

P(S) = {{0}, {1}, {0, 1}, {ϕ}} $2^n = 4$

S = {ϕ}

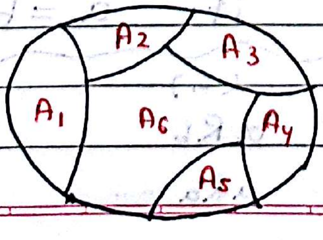
P(S) = {ϕ} → 1

→ Class of a set : Collection of some of the subsets of a set.

Eg : S = {1, 2, 3, 4}

→ Find class of 2 elements : A = {1, 2}, {1, 3}, {1, 4}
{2, 3}, {2, 4}, {3, 4}

→ Partition of a set : It is a collection of non-empty ; mutually disjoint subsets, whose union is A. Patition of set A → $A_1, A_2, A_3, \dots, A_6$



$A_1 \cap A_2 \dots = \{\phi\}$

$A_1 \cup A_2 \dots \cup A_6 = A$

→ Cartesian Product

$A = \{1, 2, 3\}$
$B = \{x, y, z\}$

$$A \times B = \left\{ \begin{array}{lll} (1,x), & (1,y), & (1,z), \\ (2,x), & (2,y), & (2,z), \\ (3,x), & (3,y), & (3,z) \end{array} \right\}$$

[Ordered]

* Properties of a Relation :-

1] Reflexive [$x R x$]

If x is related to some element then that some element has to be x.

Not all optr are reflexive

→ Definition :
A relation R on set A is called reflexive if $a, a \in R$ for every element $a \in A$

$\forall a \in A$
then $(a, a) \in R$

Eg: '=' optr : $a R b$
if $a = b$ then $b = a$ ✓

'<' optr : $a R b$ (<) $(3 < 4)$
but $a R a$ $(3 < 3)$ ✗

2] Irreflexive :
Opposite of Reflexive

3] Symmetric : if $x R y$ then $y R x$

→ Only Logical optrs

Eg : '=' $x R y$ then $y R x$
$(5, 5)$ ✓

'<' $x R y$ then $y R x$ ✗
$(4, 5)$
$4 < 5$ $5 < 4$

4] Assymmetric :
Opp. of symmetric

5] Anti-symmetric : $x R y$ and $y R x$ then $x = y$

Eg : '=' ✓
'<' ✗
'≤' ✓

6] Transitive property : $x R y$ and $y R z$ then $x R z$

Eg : '<' ✓

Q] Is 'Is divides' is a Relation on a set of +ve integers. [$x \in N$]

Sol^n: i] Reflexive xRx

Every number divides itself ✓

ii] Symmetric xRy then yRx

$x \% y$ is NOT $= y \% x$ [$x \in N$]

Asymmetric

iii] Antisymmetric if xRy and yRx then $x = y$

Because when $x = y$ then each number can divide itself.

iv] Transitive if xRy and yRz then xRz

$x = 8,\ y = 4,\ z = 2$

$\frac{8}{4}$ and $\frac{4}{2} \longrightarrow \frac{8}{2}$

2 4 8

Q] $S = \{1, 2, 3, 4\}$; $R2 = \{(1,1), (1,2), (2,1)\}$

$R1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

Sol^n: i] Reflexive : ✗ R_1 [(3,3) is missing] ; ✗ R2

ii] Symmetric : ✗ R1 , ✓ R2

$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$

Reflexive : ✓

symm : ✓

AntiSymm : ✗

$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

Transitive : ✓

$(3-5)\%5$ $(3-6)\%5$

$3-0=3$ $3-1=2$ $(10-2)\%5$

3

$-0 = 2$ -2 $+5$ $=3$

* Equivalence Relation

- Reflexive (aRa)
- Symmetric [aRb and bRa]
- Transitive [aRb and bRc then aRc]

→ Mod operator on set of ~~+ve integers~~ whole numbers

~~Reflexive : $5 \% 5 = 0$ (True) ✓~~

~~Symmetric : $5 \% 2 = 1$ (True) ✓~~

~~$2 \% 5 = 2$ (True)~~

~~Transitive : $5 \% 10$ and $10 \% 15$~~

~~5 and 10 ⟶ $5 \% 15$~~

~~(True) 5~~

Neither [Fails at 0]

→ Congruence mod $m > 1$, set of whole number

⇓

$a \equiv b \ \% \ m$

$(a-b) \% m$ [$m > 1$]

Reflexive : $a - a = 0 \ \% \ m = 0$

Symmetric : ~~$(a-b) \% m$~~ True

Transitive : aRb and bRc then aRc True

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→ R is a relation on set of strings of english letters such that aRb If len(a) = len(b)

Equivalence

* Partial Order of Relation (POR)

- reflexive
- anti-symmetric   aRb and bRa then a=b
- transitive

→ Is divides [/] on set of integers except zero

- reflexive   a/a = 1
- anti-symmetric   a has to be equal to b
  ∴ aRa and bRa then a = b
  true
- transitive : ~~True~~ ~~False~~ a = 20, b = 10, c = 5
  True

→ Let R be ~~the~~ relation on set of people such that xRy if x & y are people & x is older than y. POS ?

Optr : >

POS : NO

→ Consider a Power Set P(S) show that inclusion relation is POS on power set P(S)

- Every set is a subset of itself

$\therefore$ Reflexive

- Antisymm

$a \subseteq b$, $b \subseteq a$ then $a = b$ (subset of)

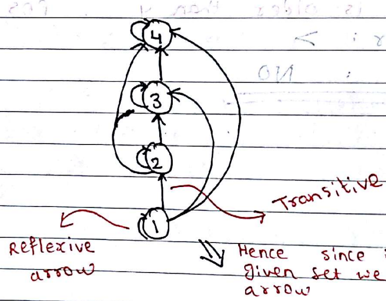
$\therefore$ Antisymm

* Hasse Diagram for Partially ordered set:

$\{1, 2, 3, 4\}$

$\{(a,b) \mid a \le b\}$

$\{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4),$
$(1,1), (2,2), (3,3), (4,4)\}$

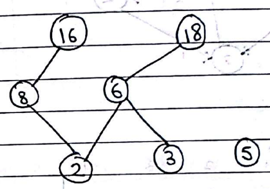


⇒ Hence since in exists in given set we exclude this arrow

→ Draw Hasse Diagram for a PO set x/y on a set $S = \{2, 3, 5, 6, 8, 16, 18\}$

Sol^n: x divides y

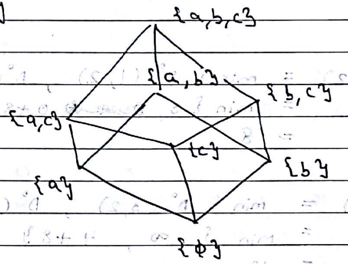
$(2,2)$,
$\{(2,6), (2,8), (2,16), (2,18), (3,3), (3,6), (3,18),$
$(5,5), (6,6), (6,18), (8,8), (8,16), (16,16),$
$(18,18)\}$



→ $\subset$ operator Improper subset

$S = \{a, b, c\}$

Sol^n: $\{(\phi), (a), (b), (c), (a,b), (b,c), (a,c),$
$(a,b,c)\}$



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* Transitive closure - Floyd Warshall algo

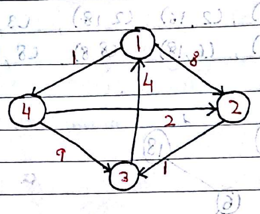
$$D^{k}(i,j) = \min\{D^{k-1}(i,j),\ D^{k-1}(i,k) + D^{k-1}(k,j)\}$$

↑ Matrix

Shortest route between i, j via k

ie $i \longrightarrow j$, $i \to k \to j$

$k = 0$
↳ Direct Path



$D^{4}(1,2) = 3$

$D^{0}(1,2) = 8$

i] $K = 0$

$$D^{(0)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \boxed{0} & 8 & \infty & 1 \\ 2 & \boxed{\infty} & 0 & 1 & \infty \\ 3 & \boxed{4} & \infty & 0 & \infty \\ 4 & \boxed{\infty} & 2 & 9 & 0 \end{array}$$

↑ Adjacency matrix

ii] $k = 1$

$$D^{1}(1,2) = \min\{D^{0}(1,2),\ D^{0}(1,1) + D^{0}(1,2)\}$$
$$= \min\{8,\ 0+8\}$$
$$= 8$$

$$D^{1}(3,2) = \min\{D^{0}(3,2),\ D^{0}(3,1) + D^{0}(1,2)\}$$
$$= \min\{\infty,\ 4+8\}$$
$$= 12$$

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→ Shortcut Method to generate Matrix :

i] Circle K^{th} row

ii] Square (i, K)

iii] add square + circle to update i^{th} row

$$D^{1} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0+0 & 0+8 & 0+\infty & 0+1 \\ 2 & & 0 & & \\ 3 & & & 0 & \\ 4 & & & & 0 \end{array}$$

copy diagonal elements as it is

Add sq + circled elements in row to get the new row.

↳ Then compare previous row + new row and write minimum elements

$$D^{1} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 8 & \infty & 1 \\ 2 & & 0 & & \\ 3 & & & 0 & \\ 4 & & & & 0 \end{array}$$

Now second row :

$$D^{1} = \begin{bmatrix} 0 & 8 & \infty & 1 \\ \infty & \min(\infty,0) & \min(\infty,1) & \infty \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$D^{1} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 8 & \infty & 1 \\ 2 & \infty & 0 & 1 & \infty \\ 3 & & & 0 & \\ 4 & & & & 0 \end{array}$$

Now third row:

$$D' = \begin{bmatrix} 0 & 8 & \infty & 1 \\ \infty & 0 & 1 & \infty \\ 4 & 12 & 0 & 5 \\ & & & 0 \end{bmatrix}$$

Now Fourth row:

$$D' = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & \boxed{8} & \infty & 1 \\ 2 & \infty & \boxed{0} & 1 & \infty \\ 3 & 4 & \boxed{12} & 0 & 5 \\ 4 & \infty & \boxed{2} & 9 & 0 \end{array}$$

iii] K = 2

$$D^2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 8 & \boxed{9} & 1 \\ 2 & \infty & 0 & \boxed{1} & \infty \\ 3 & 4 & 12 & \boxed{0} & 5 \\ 4 & \infty & 2 & \boxed{3} & 0 \end{array}$$

iv] K = 3

$$D^3 = \begin{bmatrix} 0 & 8 & 9 & \boxed{1} \\ 5 & 0 & 1 & \boxed{6} \\ 4 & 12 & 0 & \boxed{5} \\ 7 & 2 & 3 & \boxed{0} \end{bmatrix}$$

v] K = 4

$$D^4 = \begin{bmatrix} 0 & 3 & 4 & 1 \\ 5 & 0 & 1 & 6 \\ 4 & 7 & 0 & 5 \\ 7 & 2 & 3 & 0 \end{bmatrix}$$

Transitive Closure:-

- Path (4,3) → directed path costs 9, but path via 2 costs less (2+1 = 3).
We replace all non-zero and non-infinity values by 1 to get transitive closure.
Because we are just interested in shortest path & not the cost in transitive closure.

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* Equivalence Class :

R is an equivalence relation on set 'A', the set of all elements that are related to 'a' of 'A' is called equivalence class of 'a' & it is denoted by [a]

R — equivalence Relation

1] aRa

2] aRb then bRa

3] aRb and bRc then aRc

Q] Let R be a relation on set of integers such that aRb if $a = b$ or $a = -b$.
Find equivalence class of R for a given a

Sol^n: $[a] = \{-a, a\}$

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Q] What are the equivalence class of 0 & 1 for congruence mod4 $(a \equiv 0) \bmod 4$ on a set of integers

Sol^n: $(a \equiv 0) \bmod 4 \longrightarrow (a-0)\,\%\,4$

$[0] = \{ \ldots, -8, -4, 0, 4, 8, \ldots \}$

$(a \equiv 1) \bmod 4 \longrightarrow (a-1)\,\%\,4$

$[1] = \{ \ldots, -7, -3, 1, 5, 9, \ldots \}$

Q] Let $R = \{a, b\}$ such that $a\,\%\,2 = b\,\%\,2$. $\rightarrow aRb$
Find ~~A/R~~ A/R, equivalence class of ~~...~~ a & b

Sol^n:

$[a], [b] = \{$ Even nos $\}$ OR $\{$ Odd nos $\}$

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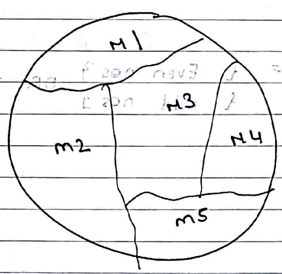
Q] Let A be the set of all students that in a school such that they major in exactly 1 subject.
Let R be a relation in 'a' consisting of (x, y) & majoring in the same subjects.
Check if R is an equivalence relation and find the equivalence class.

Sol^n: Equivalence classes of R is a collection of subset where each subset contains students with a specified major.
Relation R splits all students in a into a collection of disjoint subsets, where each subset contains students with a specified major.



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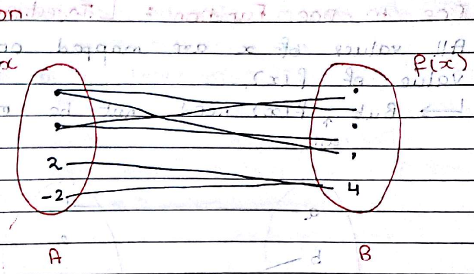
* Function



x	$f(x)$
2	$x^2 = 4$
-2	$= 4$

Q] Determine which of the relations are functions on the domain $= \{1, 2, 3, 4\}$

i] $R = \{(1,1), (2,1), (4,1), (3,3), (3,1)\}$

Sol^n: Not a function [X one $\rightarrow$ many]

ii] $R = \{(1,2), (2,3), (4,2)\}$

Sol^n: Not a function $\rightarrow$ since 3 has not mapping

iii] $R = \{x, y\}$ and $x, y \in R$ & $R \Rightarrow y^2 = x$

Sol^n: Not a function because x points to two values.

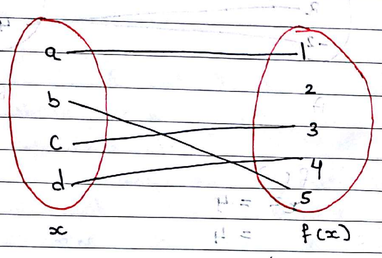
iv] $R = \{x, y\}$ and $x, y \in Q$ & $R \Rightarrow x^2 + y^2 = 1$

Sol^n: Not a function

① One to one Function : [Injection]

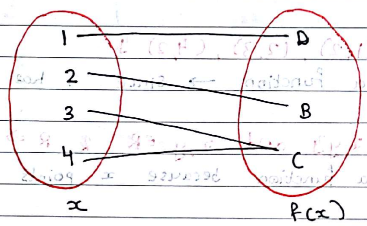
All values of x are mapped only to one value of $f(x)$.

↳ But all $f(x)$ need not be mapped to x



② Surjective or Onto

All values in $f(x)$ has a mapping of one or more in x



→ Function which is both one to one & onto is Bijective

* Composition of a Function :-

Function of a function.

$A \xrightarrow{g} B \xrightarrow{f} C$ ~~fog(x)~~ gof (c)

$A = \{1, 2, 3, 4\}$

$f(x) = x^2$

$g(y) = y^3$

$x = 2$; $f(x) = 4$

gof $= g[f(x)] = g(4) = 64$

Q] Find composition of fog & gof :-

$g = \{a, b, c\}$.

$g(a) = b$, $g(b) = c$, $g(c) = a$

$f(a) = 3$, $f(b) = 2$, $f(c) = 1$

Sol^n:

fog$(a) = f[g(a)] = f(b) = 2$

fog$(b) = f[g(b)] = f(c) = 1$

fog$(c) = f[g(c)] = f(a) = 3$

gof$(a) = g[f(a)] = g(3)$ ✗
gof$(b) = g[f(b)] = g(2)$ ✗
gof$(c) = g[f(c)] = g(1)$ ✗
} undefined

* Pigeon-Hole Principle

There are 5 holes and 6 pigeons and all pigeons have to go in one of the hole.

↳ Then we can say that there is at-least one hole having more than 1 pigeon.

Q] Suppose a bag has an infinite number of R, B, G, Black socks, what is the least number of socks you must remove to ensure you have a pair

Sol^n: 5

↳ In case of 2, 3, 4 socks there may be a pair but we can't say that with absolute certainty.

↳ In case of 5 socks, we can have 2 cases.

↳ 1^{st} 4 are diff. colours, then the 5^{th} one will pair with one of the first 4

↳ 1^{st} 4 itself have a pair.

Hence, min 5 socks can ensure that pair is obtained.

Q] A bag contains 10 R marbles, 10 W, 10 BL marbles. What is min no. of marbles that I should randomly from the bag to ensure that I get atleast four marbles of same colour.

Sol^n: 3 x 3 colours = 9 + 1 colour

= 3 colours 1

3 colour 2 + 1 any one of three

3 colour 3

Q] Box contains 68 R, 8 G, 10 B, 12 Y, 15 W balls. What is Min no. of balls to choose randomly from the box to ensure we get 9 balls of same colour

Sol^n: 39 ⇒ 6 x 5 = 30 + 2x4 = 38 + 1 = 39

(Red over) (Green over) (8 green, yellow, white) (these colours)

Q] There are 100 balls in a bucket numbered 1 - 100, 51 balls are chosen randomly from the bucket. PT that atleast 2 of the balls chosen will have consecutive nos.

Sol^n: 50 balls → all alternative

+

1 balls → In between any of the above consecutive number.

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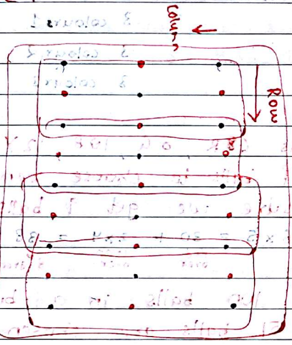
Q] 27 pts are aligned so that each row has 9 pts and each column has 3 pts. Each pt is painted with either red or blue colour. ~~There~~ PT there exist atleast one mono-chromatic rectangle.

Soln:



Two colours and 3 points in a row can be painted in $2^3 = 8$ ways which means out of 9 columns 8 columns have different colour scheme. But the 9th one will be a repeat.

> 6 pts in row

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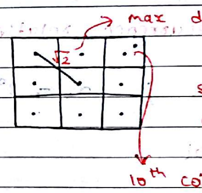
Q] PT 6 int randomly selected from set $S = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. There must be 2 ints whose sum is 15.

Soln: 3, 4, 5, $\underbrace{6, 7, 8}_{=15}$

Q] 3 x 3 cm square. 10 coins placed in sq then atleast 2 coins must have less than sqrt (2cm) apart.

Soln:



so atleast one pair having dist $\sqrt{2}$ or less

Q] At a business meeting no one shakes their own hand and no one shakes another persons hand more than once. PT: there are 2 ppl who have shaken hand for the same no. of times.

Soln: worst case 2 ppl shake hand with each other.

Recurrence Relation

→ For any given sequence if a given term can be expressed using previous elements/terms then it is a recurrence relation.

Solution of the relation is the series.

Eg: Fibonacci series.

$a_n = a_{n-1} + a_{n-2}$ ... Recurrence Relation

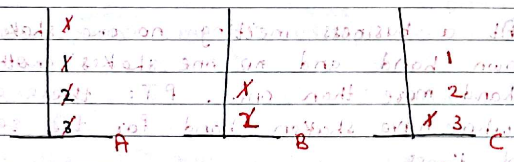
↳ with initial condn $a_0 = 0, a_1 = 1$

Q What will be recurrence relation for: $3^0, 3^1, 3^2, 3^3, \ldots$

Soln: $a_n = 3 \cdot a_{n-1}$; $a_0 = 3^0$, $a_1 = 3^1$

Q Tower of Hanoi

Soln:



① (n−1) discs transfered from A → B in a_{n-1} moves

② 1 move req to transfer the largest disc from A → C

③ Now (n−1) discs are on B and largest on C
(n−1) discs req. a_{n-1} moves to be transfered from B → C

Total = $a_{n-1} + 1 + a_{n-1}$

$a_n = 2a_{n-1} + 1$

Q Find RR for 3, 6, 12, 24, 48

Soln: $a_n = a_{n-1} * 2$; $a_0 = 3$; $n \geq 1$

Q 0, 2, 6, 12, 20, 30, 42

Soln: $a_n = n * 2 + a_{n-1}$; $a_0 = 0$, $n \geq 1$

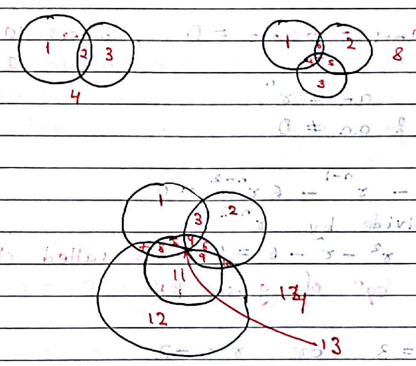
Q Consider intersecting ovals in the plane:

Condn 1: Every oval cuts every other oval in two places

Condn 2: No 2 pts of intersection coincide.

Formulate a RR for the number of regions created.

Soln:



- Let a_r be no. of regions into which plane is divided when we have r ovals.
- so a_{r-1} gives no. of regions when into which plane is divided when we have $r-1$ ovals.
- when we draw r^{th} oval, it will cut each previously drawn oval i.e. $r-1$ ovals in 2 places. Therefore $2 * (r-1)$ pts of intersection for r^{th} oval.

Therefore recurrence relation can be formulated as : $a_r = a_{r-1} + 2(r-1)$

$r = 4$: $a_4 = 2 * 3$
$= 8 + 6$
$a_4 = 14$

* Solving Recurrence Relationship :-

Q] $a_n - a_{n-1} - 6a_{n-2} = 0$, where $a_0 = 1$ & $a_1 = 8$

Soln: Let $a_n = r^n$
& $a_n \neq 0$

$r^n - r^{n-1} - 6r^{n-2} = 0$
divide by r^{n-2}
$r^2 - r - 6 = 0$ ← is called characteristic eqn of given RR

$r = 3$ or $r = -2$

Formula

Soln of RR is given by :
$a_n = \alpha \cdot 3^n + \beta(-2)^n$
given : $a_0 = 1$ & $a_1 = 8$

$1 = \alpha + \beta$ & $8 = 3\alpha - 2\beta$

$\therefore$
$3\alpha - 2\beta = 8$
$2\alpha + 2\beta = 2$
$5\alpha = 10$
$\alpha = 2$ & $\beta = -1$

$\therefore a_n = 2 \cdot 3^n - (-2)^n$

Q] $a_n - 4a_{n-1} + 4a_{n-2} = 0$, where $a_0 = 1$, $a_1 = 3$

Soln: Characteristic eqn :
$r^2 - 4r + 4 = 0$

$\therefore a_n = \alpha \cdot (2^n) + \beta n (2^n)$
↑ Bcoz of repeated root

* Roots are different :
$r = 1, 2, 3, 4, \ldots, K$
Then the solution is given by formula:
$a_n = \alpha_1(1^n) + \alpha_2(2^n) + \alpha_3(3^n) + \ldots + \alpha_K(K^n)$

* Roots are same :
$r = \underbrace{3, 3, 3, 3, \ldots, 3}_{K \text{ times}}$; $\underbrace{2, 2, 2, \ldots, 2}_{m \text{ times}}$

$a_n = \alpha_1(3^n) + \alpha_2 n(3^n) + \alpha_3 n^2(3^n) + \ldots + \alpha_K n^K(3^n) + B_1(2^n) + B_2 n(2^n) + \beta_3 n^2(2^n) + \ldots + \beta_m n^m(2^n)$