Rank Nullity Theorem

Definition

The null space of a real $m \times n$ matrix A to be the set of all real solutions to the associated homogeneous linear system $Ax = 0$. Thus,

$$nullspace(A) = \{x \in Rn : Ax = 0\}.$$

The dimension of $nullspace(A)$ is referred to as the nullity of A and is denoted $nullity(A)$. In order to find nullity(A), we need to determine a basis for $nullspace(A)$. Recall that if $rank(A) = r$, then any row-echelon form of A contains r leading ones, which correspond to the bound variables in the linear system. Thus, there are $n-r$ columns without leading ones, which correspond to free variables in the solution of the system $Ax = 0$. Hence, there are $n - r$ free variables in the solution of the system $Ax = 0$. Therefore $nullity(A) = n - r$.

Theorem:

For any $m \times n$ matrix A,

$$rank(A) + nullity(A) = n \text{.................}[1]$$

Proof:

If $rank(A) = n$, then by the Invertible Matrix Theorem, the only solution to $Ax = 0$ is the trivial solution $x = 0$. Hence, in this case, $nullspace(A) = \{0\}$, so $nullity(A) = 0$ and Equation 1 holds.

Now suppose $rank(A) = r < n$. In this case $n - r > 0$ free variables in the solution to $Ax = 0$.

Let $t_1, t_2, \ldots, t_{n-r}$ denote these free variables (chosen as those variables not attached to a leading one in any row-echelon form of A), and let $x_1, x_2, \ldots, x_{n-r}$ denote the solutions obtained by sequentially setting each free variable to 1 and the remaining free variables to zero.

Note that $\{x_1, x_2, \ldots, x_{n-r}\}$ is linearly independent. Moreover, every solution to $Ax = 0$ is a linear combination of $x_1, x_2, \ldots, x_{n-r}$: $x = t_1x_1 + t_2x_2 + \cdots + t_{n-r}x_{n-r}$, which shows that $\{x_1, x_2, \ldots, x_{n-r}\}$ spans $nullspace(A)$. Thus, $\{x_1, x_2, \ldots, x_{n-r}\}$ is a basis for $nullspace(A)$, and $nullity(A) = n - r$.

Applications

1. It is easy to highlight the need for linear algebra for physicists - Quantum Mechanics is entirely based on it. Also important for time domain (state space) control theory and stresses in materials using tensors.
2. In circuit theory, matrices are used to solve for current or voltage. In electromagnetic field theory which is a fundamental course for communication engineering, conception of divergence, curl are important. For other fields of engineering, computer memory extensively uses the conception of partition of matrices. If the matrices size gets larger than the space of computer memory it divides the matrices into submatrices and does calculation.
3. Matrices can be cleverly used in cryptography. Exchanging secret information using matrix is very robust and easy in one sense. How about MATLAB? This software is widely used in engineering fields and MATLAB's default data type is matrix. And, of course, Linear Algebra is the underlying theory for all of linear differential equations. In electrical engineering filed, vector spaces and matrix algebra come up often.
4. Least square estimation has a nice subspace interpretation. Many linear algebra texts show this. This kind of estimation is used a lot in digital filter design, tracking (Kalman filters), control systems, etc.

Significance

The rank-nullity theorem is useful in calculating either one by calculating the other instead, which is useful as it is often much easier to find the rank than the nullity (or vice versa).

* Rank Nullity Theorem

1) If A is a 5×6 Matrix with Rank 2, what is the dimension of the Nullspace of A?

Since,

Nullity is the difference between the number of columns of A and the rank of A,

$\therefore$ Nullity $= 6 - 2$
$= 4$

$\therefore$ Nullspace A is a 4-dimensional subspace of $\mathbb{R}^6$

2) Verify Rank Nullity Theorem for $A = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$

The Rank of Matrix is 1, since the basis

$\left\{ \begin{pmatrix} 3 \\ -6 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$ can be reduced to $\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$.

Obtaining the Reduced Row echelon form of A,

$\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$ $R_2 \to \frac{-R_2}{2}$

$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$

Then $AX = 0$,

$$\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 = -\frac{x_2}{3}$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{x_2}{3} \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

$$\therefore \text{Nullspace}(A) = \text{span} \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

dimension of Nullspace A = Nullity A
= 1

Hence, the rank and Nullity are both 1 and sum to 2, i.e., the number of column of A.

Hence, we have verified the theorem.

3) Find the Nullity of $A = \begin{bmatrix} -1 & 0 & -1 & 2 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix}$

Converting Matrix A to Row echelon form.

$R_2 \rightarrow R_2 + 2R_1$ $\begin{bmatrix} -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 4 \\ 1 & 0 & 1 & -1 \end{bmatrix}$

$R_3 \to R_3 + R_1$ $\begin{bmatrix} -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_2 \to R_2 - 4R_3$ $\begin{bmatrix} -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_2 \leftrightarrow R_3,\ R_1 \to -R_1$ $\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ The Rank is 2

By Rank - Nullity Theorem,

Rank(A) + Nullity(A) = No. of columns A

2 + Nullity(A) = 4

$\therefore$ Nullity(A) = 2

HW 4) Find Nullity for $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

Soln: Rank (A) = 2

Nullity(A) = 1