Dimension of a Vector Space

Definition
Suppose that **V** is a vector space and $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \ldots, \mathbf{v_t}\}$ is a basis of **V.**

Then the ***dimension*** of **V** is defined by ***dim(V) = t***. If V has no finite bases, we say **V** has infinite dimension.

This is a very simple definition, which belies its power. Grab a basis, any basis, and count up the number of vectors it contains. That is the dimension.

- dim({ 0 }) = 0
- $\dim(R^n) = n$
- $\dim(P_n) = n + 1$
- $\dim(M_{mn}) = mn$

History
The idea of a vector space developed from the notion of ordinary two- and three-dimensional spaces as collections of vectors ***{u, v, w, …}*** with an associated field of real numbers ***{a, b, c, …}***. Vector spaces as abstract algebraic entities were first defined by the Italian mathematician ***Giuseppe Peano in 1888***. Peano called his vector spaces “linear systems” because he correctly saw that one can obtain any vector in the space from a linear combination of finitely many vectors and scalars ***-av + bw + … + cz***. A set of vectors that can generate every vector in the space through such linear combinations is known as a spanning set. The dimension of a vector space is the number of vectors in the smallest spanning set. (For example, the unit vector in the x-direction together with the unit vector in the y-direction suffice to generate any vector in the two-dimensional Euclidean plane when combined with the real numbers.)

Significance
Every vector has a magnitude and a direction. Magnitudes and directions are easy to visualize as actual directions in ordinary space. But vector spaces are not confined to only 3 dimensions. We can generalize them to any dimension. For simplicity let us start from a 3 dimensional case . To describe a 3 d vector we can say "go x units in the +x direction, y units in the +y direction and z units in the+ z direction if you want to reach the destination point." The amazing thing is that we can reach every possible destination by specifying only these three numbers (x, y, z). Now someone might say why stop at 3? Why not use 4 directions? The thing is that 3 is the smallest number of directions that can describe all points in our 3d space. 2 is too few and the 4th one is redundant as we can describe the fourth direction as a combination of the other three directions. Thus we say that we have 3 basis vectors (directions) which span (i.e. can describe every point) a 3 dimensional vector space. (This is no coincidence as the word dimension itself is the total number of directions required or the cardinality of the set of basis vectors) Now we have made a crucial assumption here. The assumption is that the 3 directions that we used are themselves independent of each other. For example N and E are independent directions but N and N-E aren’t. Saying, " go 1.41 units in the NE direction and 1 unit in the N" can be broken down to, " go 2 units towards N and 1 unit towards E. From the above example we may conclude that in order to be mutually independent, the directions should be at right angles to each other. But this simple test cannot work for higher dimensions where the definition of an angle itself needs upgradation.

Applications

Dimensions have many applications as they occur frequently in common circumstances, namely wherever functions with values in some field are involved. They provide a framework to deal with analytical and geometrical problems, or are used in the Fourier transform. This list is not exhaustive: many more applications exist, for example in optimization. The minimax theorem of game theory stating the existence of a unique payoff when all players play optimally can be formulated and proven using vector spaces methods. Representation theory fruitfully transfers the good understanding of linear algebra and vector spaces to other mathematical domains such as group theory.

* Dimension.

1) What is the dimension of the vector space of polynomials in x with real coefficients having degree at most three?

Since, the polynomial is of the form

$$ax^3 + bx^2 + cx + d \quad \text{i.e. } P_3$$

where $a, b, c, d \in \mathbb{R}$,

$\therefore$ The dimension will be 4. i.e. $\dim(V) = 4$. and, has $\{x^3, x^2, x, 1\}$ as a basis.
$[\because \dim(P_n) = n+1]$

2) Find the dimension of $V = \{a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R}\}$

We can easily find the basis for this vector space

i.e. $\{x^2, x, 1\}$ is the basis.

$\therefore$ The dimension of the vector space i.e. $\dim(V) = 3$.

3) Find the dimension of the solution space of the following homogenous system

$$x - y + 2z = 0$$
$$2x + y = 0$$
$$x - 4y + 6z = 0$$

We can write the equations in the form $AX = 0$ i.e.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 1 & -4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By elementary Row Transformation,

$R_2 \leftrightarrow R_3$,

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -4 & 6 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ ~~$R_2 \to R_3$~~ $R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & -1 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $x - y + 2z = 0$; Let $x = t$, $\therefore$ $y = -2t$

$-2x - y = 0$ $\qquad z = -1.5t$

The solution space is $(x, y, z) = (t, -2t, -1.5t)$
$= t(1, -2, -1.5)$

$\therefore$ The vectors $v_1 = (1, -2, -1.5)$ spans the solution space. Thus, dimension is 1.

4) Find the dimension of the plane $x + 2z = 0$ in R^3

Let $z = s$ and $y = t$, $(t, s \in R)$

$\therefore$ The general solution of $x + 2z = 0$ is

$x = -2s$
$y = t$, and
$z = s$

$\therefore (x, y, z) = (-2s, t, s)$
$= t(0, 1, 0) + s(-2, 0, 1)$

Hence, the plane spans the vectors $v_1 = (0, 1, 0)$ and $v_2 = (-2, 0, 1)$.
$\therefore$ The dimension is 2.