Gram-Schmidt Process

Definition

We define the projection operator by

$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u},$$

where **<u,v>** denotes the inner product of the vectors **u** and **v**. This operator projects the vector **v** orthogonally onto the line spanned by vector **u**. If **u = 0**, we define **$\mathbf{proj_0(v):=0}$**, i.e., the projection map proj_0 is the zero map, sending every vector to the zero vector.

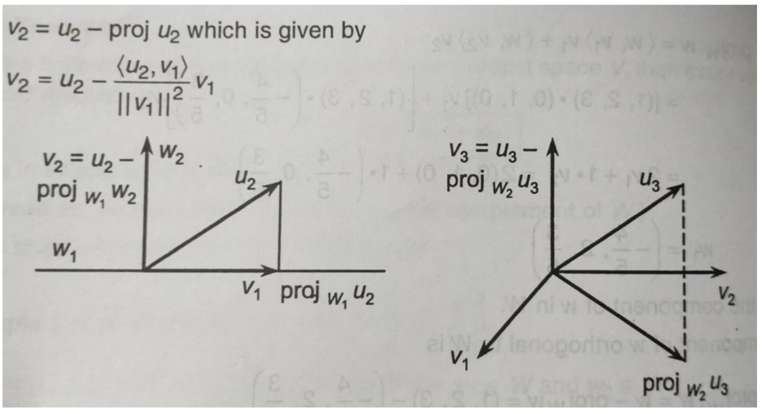
Procedure

Gram-Schmidt process gives us a method of finding orthonormal vectors.

Let for convenience V be a non-zero inner product space in R^3 and let $\boldsymbol{u} = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ be any base for V. The following steps lead to an orthogonal basis $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ for V.

Step 1: Let $v_1 = u_1$.

Step 2: As shown in the figure, we obtain a vector v_2, orthogonal to v_1.



Step 3: To find v_3 that is orthogonal to both v_1 and v_2, we find u_3 orthogonal to space w_2 spanned by v_1 and v_2 which is given by

$$v_3 = v_3 - \text{proj } u_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_2, v_2 \rangle}{\|v_2\|^2} v_2$$

Thus, we get the orthogonal set $\{v_1, v_2, v_3\}$.

History

Jörgen Pederson Gram was an insurance expert in Denmark. He got his master's degree in mathematics with specialisation in the then new developing subject modern algebra. He then joined the Hafnia Life Insurance Company where he developed mathematical Foundations of accident insurance. During this period he got his Ph.D. on "On Series Development Utilising the Least Square Methods". On his thesis Gram-Schmidt process is based. In 1910 he became Director of the Danish (Denmark) Insurance Board. Gram eventually developed interest in number theory and won a gold medal from "The Royal Danish Society of Sciences And Letters". He had keen interest

the interplay between applied mathematics and pure mathematics, which led to four treatises Danish forest management. Unfortunately he died in an accident while going to attend a me the Royal Danish Society.

Erhardt Schmidt was a reputed German mathematician. He got his Ph.D. degree from Göttingen University, Germany in 1905. He was a student of another giant German mathematician David Hilbert. He then went to teach at Berlin University in 1917 and stayed there for the rest of his life. He made important contributions to various mathematical fields. Schmidt first described "Gram-Schmidt process in a paper on integral equations published in 1907. He is credited to gal have developed the general concept "Hilbert Space" which is fundamental in the study of infinite dimensional vector spaces.

Applications

- **QR Decomposition**

  In linear algebra, a QR decomposition, also known as a QR factorization or QU factorization is a decomposition of a matrix A into a product $A = QR$ of an orthogonal matrix Q and an upper triangular matrix R. QR decomposition is often used to solve the linear least squares problem and is the basis for a particular eigenvalue algorithm, the QR algorithm.

- **Functional Analysis**

  Functional analysis is a branch of mathematical analysis, the core of which is formed by the study of vector spaces endowed with some kind of limit-related structure (e.g. inner product, norm, topology, etc.) and the linear functions defined on these spaces and respecting these structures in a suitable sense. The historical roots of functional analysis lie in the study of spaces of functions and the formulation of properties of transformations of functions such as the Fourier transform as transformations defining continuous, unitary etc. operators between function spaces.

- **Linear least squares Problem**

  Mathematically, linear least squares is the problem of approximately solving an overdetermined system of linear equations A x = b, where b is not an element of the column space of the matrix A.

Significance

If the entire vector space we wanted a basis for all of R^n or a basis for, say, the xy-plane in R^3, then we could certainly do this. The problem arises when we're looking for an orthonormal basis for subspaces that are more complicated.

Gram-Schmidt is designed to turn a basis into an orthonormal basis *without altering the subspace that it spans*.

Gram-Schmidt is also important in that it preserves the orientation of given basis (roughly speaking, the order in which the basis elements are introduced).

* Gram Schmidt Process

1) Construct an orthonormal basis R^3 by applying Gram-Schmidt Orthogonilisation to $S = \{(3,1),(4,2)\}$.

Let $u_1 = (3,1)$, $u_2 = (4,2)$

Step 1 : $v_1 = u_1 = (3,1)$

Step 2 : $v_2 = u_2 - \text{proj}\ u_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$

Now,

$\langle u_2, v_1\rangle = (4,2).(3,1) = 14$

$\|v_1\|^2 = 9+1 = 10$

$\therefore v_2 = (4,2) - \frac{14}{10}(3,1) = \left(-\frac{1}{5}, \frac{3}{5}\right)$

Hence, $v_1 = (3,1)$, $v_2 = \left(-\frac{1}{5}, \frac{3}{5}\right)$ form orthogonal basis for R^2.

Norms of these vectors are,

$\|v_1\| = \sqrt{9+1} = \sqrt{10}$

$\|v_2\| = \sqrt{\frac{1}{25} + \frac{9}{25}} = \sqrt{\frac{2}{5}}$

Hence, the orthogonal basis are

$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$.

$q_2 = \frac{v_2}{\|v_2\|} = \left(\frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$

2) Find an orthonormal basis for the subspaces of R^3 by applying Gram-Schmidt process where $S = \{(1,2,0), (0,3,1)\}$

Let $u_1 = (1,2,0)$, $u_2 = (0,3,1)$

$v_1 = u_1 = (1,2,0)$

$$v_2 = u_2 - \text{proj } u_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$

Now, $\langle u_2, v_1 \rangle = (0,3,1) \cdot (1,2,0)$
$= 6$

$\|v_1\|^2 = 1+4+0 = 5$

$$\therefore v_2 = (0,3,1) - \frac{6}{5}(1,2,0) = \left(-\frac{6}{5}, \frac{3}{5}, 1\right)$$

Norms of vectors

$\|v_1\| = \sqrt{5}$, $\|v_2\| = \sqrt{\frac{36}{25} + \frac{9}{25} + 1} = \sqrt{\frac{14}{5}}$

Hence, the orthonormal basis are

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \left(\frac{-6}{\sqrt{70}}, \frac{3}{\sqrt{70}}, \frac{5}{\sqrt{70}}\right)$$

3) Let the vector space P_2 have the inner product defined by

$$\langle p, q \rangle = \int_{-1}^{1} p(x) \cdot q(x)\, dx$$

Use Gram-Schmidt process to transform the standard basis $S = \{1, x, x^2\}$ into an orthonormal basis.

We have $u_1 = 1$, $u_2 = x$, $u_3 = x^2$

$\therefore v_1 = u_1 = 1$

$$v_2 = u_2 - \text{proj}\, u_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} \cdot v_1$$

Now, $\langle u_2, v_1 \rangle = \int_{-1}^{1} x \cdot 1\, dx = 0$

$$\|v_1\|^2 = \int_{-1}^{1} [v_1(x)]^2 dx = \int_{-1}^{1} 1^2 dx = 2$$

$$\therefore v_2 = x - \frac{0 \times 1}{2} = x$$

$$v_3 = u_3 - \text{proj}\, u_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} \cdot v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} \cdot v_2$$

Now,

$$\langle u_3, v_1 \rangle = \int_{-1}^{1} x^2 \cdot 1\, dx = \frac{2}{3}$$

$$\therefore \langle u_3, v_2 \rangle = \int_{-1}^{1} x^2 \cdot x\, dx = 0$$

$$\|v_2\|^2 = \int_{-1}^{1} [v_2(x)]^2 dx = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

$\therefore V_3 = x^2 - \frac{2}{3\times 2} \cdot 1 - 0 \quad = x^2 - \frac{1}{3}$

Hence, $v_1 = 1$, $v_2 = x$, $v_3 = x^2 - \frac{1}{3}$

$$\|v_3\|^2 = \int_{-1}^{1} [v_3(x)]^2 \, dx = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{2x^2}{3} + \frac{1}{9}\right) dx$$

$$= \frac{8}{45}$$

Hence, the Orthonormal basis is.

$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}$, $\quad q_2 = \frac{v_2}{\|v_2\|} = \frac{x}{\sqrt{2/3}} = \sqrt{\frac{3}{2}}\, x$

$q_3 = \frac{v_3}{\|v_3\|} = \frac{1}{2}\sqrt{\frac{5}{2}}\,(3x^2 - 1)$

HW 4) Construct an orthonormal basis of R^3 using Gram-Schmidt process to $S = \{(3,0,4), (-1,0,7), (2,9,11)\}$

Soln: $\left[\left(\frac{3}{5}, 0, \frac{4}{5}\right), \left(\frac{-4}{5}, 0, \frac{3}{5}\right), (0,1,0)\right]$