Column space and Row space

Definition
Let A be an $m \times n$ matrix.
1. The subspace of R^n spanned by the row vectors of A is called the ***row space*** of A.
2. The subspace of R^m spanned by the column vectors of A is called the ***column space*** of A.

OR

ColumnSpace: Let K be a ***field of scalars***. Let A be an $m \times n$ matrix, with column vectors $v_1, v_2, \ldots, v_n$. A linear combination of these vectors is any vector of the form

$$c_1v_1 + c_2v_2 + \ldots\ldots\ldots\ldots + c_nv_n,$$

where $c_1, c_2, \ldots, c_n$ are scalars. The set of all possible linear combinations of $v_1, v_2, \ldots, v_n$ is called the ***column space of A***. That is, the ***column space of A*** is the span of the vectors $v_1, v_2, \ldots, v_n$.

RowSpace: Let K be a ***field of scalars***. Let A be an $m \times n$ matrix, with row vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$. A linear combination of these vectors is any vector of the form

$$c_1r_1 + c_2r_2 + \ldots\ldots\ldots\ldots + c_mr_m,$$

where $c_1, c_2, \ldots, c_m$ are scalars. The set of all possible linear combinations of $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ is called the **row space** of A. That is, the ***row space of A*** is the span of the vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$.

Note: dimension(*rowspace(A)*) = Rank(*A*) = dimension(*columnspace(A)*)

Application
One of the most common application of Null space is in Rocket Thrusters. Here, the column space is the set of directions that we can achieve by the thrusters. If they're all perfectly functional then we can move in any direction. In this case our column space is the entire range. The null space are the set of thruster instructions that completely waste fuel. They're the set of instructions where the thrusters will thrust, but the direction will not be changed at all. Hence, by determining the Null spaces waste of fuel can be avoided. Basically, by finding the Null spaces one can avoid the parameters which show zero correlation with the operation and can achieve optimum results and reduce the wastage of resources.

Significance
Suppose that A is an m×n matrix over a field F. Then its rows are n-component vectors and these are m in number. Similarly there are n columns which are each m-component vectors.

The row-space of A is the subspace of F^n, generated by the row vectors, and its elements are linear combinations of the row vectors. Clearly this space has dimension which is </= both m and n and is known as the row rank of the given matrix.
Similarly the column-space of the matrix is the subspace of F^m generated by the column vectors of the matrix. This space although in general different from the row space, but has the same dimension as that of the row space, because every linear relation among the columns also forces such relations among the rows and vice versa.

* Row and Column Space

1) Find a basis for Row Space of $A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$

We must ~~Red~~ first Reduce A;

$R_4 \rightarrow R_4 - R_1$
$R_5 \rightarrow R_5 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & -4 & -1 & 0 \\ 0 & -1 & -2 & -2 & -3 \end{bmatrix}$$

$R_5 \rightarrow R_5 + R_2$
$R_4 \rightarrow R_4 + R_1$

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$$

$R_4 \rightarrow R_4 - R_3$
$R_5 \rightarrow R_5 + R_3$

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $w_1 = [1, 1, 4, 1, 2]$, $w_2 = [0, 1, 2, 1, 1]$, $w_3 = [0, 0, 0, 1, 2]$ form a basis for the Row space of A.

2) Find basis for the column space of A

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ -2 & 1 & 7 & 2 \\ -1 & -4 & -1 & 3 \\ 3 & 2 & -7 & -1 \end{bmatrix}$$

Reduce the Matrix to Row echelon form

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 + R_1$
$R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & 5 & 10 \\ 0 & -2 & -2 & 7 \\ 0 & -4 & -4 & -13 \end{bmatrix}$$

$R_4 \to R_4 + R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 5 & 5 & 10 \\ 0 & -2 & -2 & 7 \\ 0 & 1 & 1 & -3 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & -2 & -2 & 7 \\ 0 & 5 & 5 & 10 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$
$R_4 \to R_4 - 5R_2$

$$\cancel{\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & -2 & -2 & 7 \\ 0 & 5 & 5 & 10 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 25 \end{bmatrix}$$

$R_4 \to R_4 - 25R_3$ $\begin{bmatrix} 1 & 2 & -16 & 4 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

It is clear that 1st, 2nd and 4th columns are the pivot columns. Thus, 1st, 2nd and 4th columns of the original Matrix A form basis for the column space i.e.

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \\ -1 \end{bmatrix} \right\}$$

3) Find the basis for the column space of A,

$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 3 & 1 & 5 \end{bmatrix}$

Obtaining the Reduced Row echelon form of A,

$R_3 \to R_3 - R_1$
$R_4 \to R_4 - 2R_1$
$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \end{bmatrix}$

$R_4 \to R_4 - 3R_2$
$R_3 \to R_3 - R_2$
$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Since, column 1 and column 2 form the pivot.
Therefore, column space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

HW

4) Find the basis for Row space of A,

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 3 & 1 & 5 \end{bmatrix}$$

Soln: $w_1 = [1 \ 0 \ -1 \ 1]$ and $w_2 = [0 \ 1 \ 1 \ 1]$