Logic

1. Propositions

2. Predicates

3. Quantifiers

Proposition

- Proposition: a statement that is either true or false
  - Must always be one or the other!
  - Example: “The sky is red”
  - Not a proposition: $x + 3 > 4$

proposition

- Consider $P(x) = x < 5$
  - P(x) has no truth values (x is not given a value)
  - P(1) is true
    - The proposition $1<5$ is true
  - P(10) is false
    - The proposition $10<5$ is false
- Thus, P(x) will become a proposition when given a value

Anatomy of a propositional function

$$P(x) = \underbrace{x + 5 > x}_{\text{predicate}}$$

variable

Propositional functions

- Functions with multiple variables:
  - $P(x,y) = x + y == 0$
    - $P(1,2)$ is false, $P(1,-1)$ is true
  - $P(x,y,z) = x + y == z$
    - $P(3,4,5)$ is false, $P(1,2,3)$ is true
  - $P(x_1,x_2,x_3 \ldots x_n) = \ldots$

Quantifier

Quantification

- Create a proposition from a propositional function
- Expresses the extent to which a predicate is true over a range of elements
- In English we use the terms *all*, *some*, *many*, *none*, and *few*.

Quantifiers

- A quantifier is “an operator that limits the variables of a proposition”

- Two types:
  - Universal
  - Existential

Universal quantifiers 1

- Represented by an upside-down A: $\forall$
  - It means "for all"
  - Let $P(x) = x+1 > x$

- We can state the following:
  - $\forall x\, P(x)$
  - English translation: "for all values of x, P(x) is true"
  - English translation: "for all values of x, x+1>x is true"

Universal quantifiers 2

- But is that always true?
  - $\forall x\, P(x)$
- Let x = the character ‘a’
  - Is ‘a’+1 > ‘a’?
- Let x = the state of Virginia
  - Is Virginia+1 > Virginia?
- You need to specify your universe!
  - What values x can represent
  - Called the “domain” or “universe of discourse” by the textbook

Universal quantifiers 3

- Let the universe be the real numbers.
  - Then, $\forall x\ P(x)$ is true
- Let $P(x) = x/2 < x$
  - Not true for the negative numbers!
  - Thus, $\forall x\ P(x)$ is false
    - When the domain is all the real numbers
- In order to prove that a universal quantification is true, it must be shown for ALL cases
- In order to prove that a universal quantification is false, it must be shown to be false for only ONE case

Universal quantification 4

- Given some propositional function P(x)
- And values in the universe $x_1 .. x_n$
- The universal quantification $\forall x \; P(x)$ implies:

$$P(x_1) \land P(x_2) \land \dots \land P(x_n)$$

Universal quantification 5

- Think of ∀ as a for loop:
- $\forall x\ P(x)$, where $1 \leq x \leq 10$
- … can be translated as …

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for ( x = 1; x <= 10; x++ )
    is P(x) true?
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- If P(x) is true for all parts of the for loop, then $\forall x\ P(x)$
  - Consequently, if P(x) is false for any one value of the for loop, then $\forall x\ P(x)$ is false

Existential quantification 1

- Represented by an bacwards E: $\exists$
  - It means “there exists”
  - Let $P(x) = x+1 > x$

- We can state the following:
  - $\exists x\ P(x)$
  - English translation: “there exists (a value of) x such that P(x) is true”
  - English translation: “for at least one value of x, x+1>x is true”

Existential quantification 2

- Let $P(x) = x+1 < x$
  - There is no numerical value x for which $x+1<x$
  - Thus, $\exists x\, P(x)$ is false

Existential quantification 3

- Let $P(x) = x+1 > x$
  - There is a numerical value for which $x+1>x$
    - In fact, it's true for all of the values of x!
  - Thus, $\exists x\, P(x)$ is true

- In order to show an existential quantification is true, you only have to find ONE value
- In order to show an existential quantification is false, you have to show it's false for ALL values

Existential quantification 4

- Given some propositional function $P(x)$
- And values in the universe $x_1 .. x_n$
- The existential quantification $\exists x \ P(x)$ implies:

$$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

Truth values of Quantifiers

Table 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x.

A note on quantifiers

- Recall that P(x) is a propositional function
  - Let P(x) be "x == 0"
- Recall that a proposition is a statement that is either true or false
  - P(x) is not a proposition
- There are two ways to make a propositional function into a proposition:
  - Supply it with a value
    - For example, P(5) is false, P(0) is true
  - Provide a quantifiaction
    - For example, $\forall x\ P(x)$ is false and $\exists x\ P(x)$ is true
      - Let the universe of discourse be the real numbers

Binding variables

- Let $P(x,y)$ be $x > y$
- Consider: $\forall x\ P(x,y)$
  - This is not a proposition!
  - What is y?
    - If it's 5, then $\forall x\ P(x,y)$ is false
    - If it's x-1, then $\forall x\ P(x,y)$ is true
- Note that y is not "bound" by a quantifier

Binding variables 2

- $(\exists x\, P(x)) \lor Q(x)$
  - The x in Q(x) is not bound; thus not a proposition
- $(\exists x\, P(x)) \lor (\forall x\, Q(x))$
  - Both x values are bound; thus it is a proposition
- $(\exists x\, P(x) \land Q(x)) \lor (\forall y\, R(y))$
  - All variables are bound; thus it is a proposition
- $(\exists x\, P(x) \land Q(y)) \lor (\forall y\, R(y))$
  - The y in Q(y) is not bound; this not a proposition

Negating quantifications

- Consider the statement:
  - All students in this class have red hair
- What is required to show the statement is false?
  - There exists a student in this class that does NOT have red hair
- To negate a universal quantification:
  - You negate the propositional function
  - AND you change to an existential quantification
  - $\neg \forall x\, P(x) = \exists x\, \neg P(x)$

Negating quantifications 2

- Consider the statement:
  - There is a student in this class with red hair
- What is required to show the statement is false?
  - All students in this class do not have red hair
- Thus, to negate an existential quantification:
  - To negate the propositional function
  - AND you change to a universal quantification
  - $\neg \exists x\, P(x) = \forall x\, \neg P(x)$

Negating Quantified Expressions

"Every student in your class has taken a course in calculus."

$\neg \forall x P(x) \quad \equiv \quad \exists x \neg P(x)$

"It is not the case that every student in your class has taken a course in calculus."

Which is the same as

"There is a student in your class who has not taken a course in calculus."

Translating from English into Logical Expressions

Example

Express the statement using predicates and quantifiers:

"Every student in class has studied calculus."

Solution

"Every student in class has studied calculus."

First, rewrite the statement:

"For every student in this class, that student has studied calculus."

Solution

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"For every student in this class, that student has studied calculus."

Next, introduce a variable x:

"For every student x in this class, x has studied calculus."

Solution

"Every student in class has studied calculus."

We introduce $C(x)$, which is the statement "x has studied calculus."

Consequently, if the domain for x consists of the students in the class, we can translate our statement as $\forall x C(x)$.

Solution

"Every student in class has studied calculus."

What if we change the domain to consist of all people?

"Every student in class has studied calculus."

What if we change the domain to consist of all people?

"For every person *x*, if person *x* is a student in this class then *x* has studied calculus."

"Every student in class has studied calculus."

What if we change the domain to consist of all people?

"For every person *x*, if person *x* is a student in this class then *x* has studied calculus."

If $S(x)$ represents the statement that person x is in this class: $\forall x(S(x) \rightarrow C(x))$

[Note: not $\forall x(S(x) \wedge C(x))$ because this statement says that all people are students in this class and have studied calculus!]

Here's the table for logical implication:

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Some students in this class have studied calculus. Domain - People in the world

Ans - $\exists x\ [S(x) \land C(x)]$

Note our statement can not be expressed as $\exists x\ [S(x) \to C(x)]$. This statement is true even if S(x) is false. That means if you randomly pick any person in the world, he has studied calculus.

Translating from English 3

- Consider:
  - "Some students have visited Mexico"
  - "Every student in this class has visited Canada or Mexico"
- Let:
  - $S(x)$ be "x is a student in this class"
  - $M(x)$ be "x has visited Mexico"
  - $C(x)$ be "x has visited Canada"

Translating from English 4

- Consider: “Some students have visited Mexico”
  - Rephrasing: “There exists a student who has visited Mexico”
- $\exists x\, M(x)$
  - True if the universe of discourse is all students
- What about if the universe of discourse is all people?
  - $\exists x\, (S(x) \rightarrow M(x))$
    - This is wrong! Why?
  - $\exists x\, (S(x) \land M(x))$

Translating from English 5

- Consider: "Every student in this class has visited Canada or Mexico"
- $\forall x\ (M(x) \vee C(x))$
  - When the universe of discourse is all students
- $\forall x\ (S(x) \rightarrow (M(x) \vee C(x))$
  - When the universe of discourse is all people
- Why isn't $\forall x\ (S(x) \wedge (M(x) \vee C(x)))$ correct?

Translating from English 6

- Note that it would be easier to define V(x, y) as “x has visited y”
  - $\forall x\ (S(x) \land V(x,\text{Mexico}))$
  - $\forall x\ (S(x) \rightarrow (V(x,\text{Mexico}) \lor V(x,\text{Canada}))$

Translating from English 7

- Translate the statements:
  - “All hummingbirds are richly colored”
  - “No large birds live on honey”
  - “Birds that do not live on honey are dull in color”
  - “Hummingbirds are small”
- Assign our propositional functions
  - Let $P(x)$ be “x is a hummingbird”
  - Let $Q(x)$ be “x is large”
  - Let $R(x)$ be “x lives on honey”
  - Let $S(x)$ be “x is richly colored”
- Let our universe of discourse be all birds

Translating from English 8

- Our propositional functions
  - Let P(x) be "x is a hummingbird"
  - Let Q(x) be "x is large"
  - Let R(x) be "x lives on honey"
  - Let S(x) be "x is richly colored"
- Translate the statements:
  - "All hummingbirds are richly colored"
    - $\forall x\ (P(x) \rightarrow S(x))$
  - "No large birds live on honey"
    - ∀x(Q(x)->~R(x))
    - Alternatively: $\forall x\ (\neg Q(x) \lor \neg R(x))$
  - "Birds that do not live on honey are dull in color"
    - $\forall x\ (\neg R(x) \rightarrow \neg S(x))$
  - "Hummingbirds are small"
    - $\forall x\ (P(x) \rightarrow \neg Q(x))$

Multiple quantifiers

- You can have multiple quantifiers on a statement

- $\forall x \exists y\ P(x, y)$
  - "For all x, there exists a y such that P(x,y)"
  - Example: $\forall x \exists y\ (x+y == 0)$

- $\exists x \forall y\ P(x,y)$
  - There exists an x such that for all y P(x,y) is true"
  - Example: $\exists x \forall y\ (x*y == 0)$

Order of quantifiers

- $\exists x \forall y$ and $\forall x \exists y$ are not equivalent!

- $\exists x \forall y \, P(x,y)$
  - $P(x,y) = (x+y == 0)$ is false

- $\forall x \exists y \, P(x,y)$
  - $P(x,y) = (x+y == 0)$ is true

Negating multiple quantifiers

- Recall negation rules for single quantifiers:
  - $\neg\forall x\, P(x) = \exists x\, \neg P(x)$
  - $\neg\exists x\, P(x) = \forall x\, \neg P(x)$
  - Essentially, you change the quantifier(s), and negate what it's quantifying

- Examples:
  - $\neg(\forall x \exists y\, P(x,y))$

    $= \exists x\, \neg\exists y\, P(x,y)$

    $= \exists x \forall y\, \neg P(x,y)$
  - $\neg(\forall x \exists y \forall z\, P(x,y,z))$

    $= \exists x \neg\exists y \forall z\, P(x,y,z)$

    $= \exists x \forall y \neg\forall z\, P(x,y,z)$

    $= \exists x \forall y \exists z\, \neg P(x,y,z)$

Negating multiple quantifiers 2

- Consider $\neg(\forall x \exists y\, P(x,y)) = \exists x \forall y\, \neg P(x,y)$
  - The left side is saying "for all x, there exists a y such that P is true"
  - To disprove it (negate it), you need to show that "there exists an x such that for all y, P is false"

- Consider $\neg(\exists x \forall y\, P(x,y)) = \forall x \exists y\, \neg P(x,y)$
  - The left side is saying "there exists an x such that for all y, P is true"
  - To disprove it (negate it), you need to show that "for all x, there exists a y such that P is false"

Translating between English and quantifiers

- The product of two negative integers is positive
  - $\forall x \forall y\, ((x<0) \land (y<0) \rightarrow (xy > 0))$
  - Why conditional instead of and?
- The average of two positive integers is positive
  - $\forall x \forall y\, ((x>0) \land (y>0) \rightarrow ((x+y)/2 > 0))$
- The difference of two negative integers is not necessarily negative
  - $\exists x \exists y\, ((x<0) \land (y<0) \land (x-y \geq 0))$
  - Why and instead of conditional?
- The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers
  - $\forall x \forall y\, (|x+y| \leq |x| + |y|)$

Negation examples

- Rewrite these statements so that the negations only appear within the predicates

a) $\neg \exists y \exists x\, P(x,y)$

- $\forall y \neg \exists x\, P(x,y)$
- $\forall y \forall x\, \neg P(x,y)$

b) $\neg \forall x \exists y\, P(x,y)$

- $\exists x \neg \exists y\, P(x,y)$
- $\exists x \forall y\, \neg P(x,y)$

c) $\neg \exists y\, (Q(y) \land \forall x\, \neg R(x,y))$

- $\forall y\, \neg(Q(y) \land \forall x\, \neg R(x,y))$
- $\forall y\, (\neg Q(y) \lor \neg(\forall x\, \neg R(x,y)))$
- $\forall y\, (\neg Q(y) \lor \exists x\, R(x,y))$

Negation examples

- Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\forall x \exists y \forall z\, T(x,y,z)$

- $\neg(\forall x \exists y \forall z\, T(x,y,z))$
- $\neg \forall x \exists y \forall z\, T(x,y,z)$
- $\exists x \neg \exists y \forall z\, T(x,y,z)$
- $\exists x \forall y \neg \forall z\, T(x,y,z)$
- $\exists x \forall y \exists z\, \neg T(x,y,z)$

b) $\forall x \exists y\, P(x,y) \lor \forall x \exists y\, Q(x,y)$

- $\neg(\forall x \exists y\, P(x,y) \lor \forall x \exists y\, Q(x,y))$
- $\neg \forall x \exists y\, P(x,y) \land \neg \forall x \exists y\, Q(x,y)$
- $\exists x \neg \exists y\, P(x,y) \land \exists x \neg \exists y\, Q(x,y)$
- $\exists x \forall y\, \neg P(x,y) \land \exists x \forall y\, \neg Q(x,y)$

Rules of inference for the universal quantifier

- Assume that we know that $\forall x\ P(x)$ is true
  - Then we can conclude that $P(c)$ is true
    - Here c stands for some specific constant
  - This is called “universal instantiation”

- Assume that we know that $P(c)$ is true for any value of c
  - Then we can conclude that $\forall x\ P(x)$ is true
  - This is called “universal generalization”

Rules of inference for the existential quantifier

- Assume that we know that $\exists x\, P(x)$ is true
  - Then we can conclude that $P(c)$ is true for some value of c
  - This is called "existential instantiation"

- Assume that we know that $P(c)$ is true for some value of c
  - Then we can conclude that $\exists x\, P(x)$ is true
  - This is called "existential generalization"

Here's the table for logical implication:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

double implication

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Example. Show that $(P \to Q) \vee (Q \to P)$ is a tautology.

I construct the truth table for $(P \to Q) \vee (Q \to P)$ and show that the formula is always true.

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \vee (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

The last column contains only T's. Therefore, the formula is a tautology.

Example. Construct a truth table for $(P \to Q) \wedge (Q \to R)$.

P	Q	R	$P \to Q$	$Q \to R$	$(P \to Q) \wedge (Q \to R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Determine the truth value of the statement

(10 > 42) -> Aparna likes chocolate cupcakes

The statement (10 > 42) is false.

We can't tell whether the statement "Aparna likes chocolate cupcakes" is true or false.

But it doesn't matter.

If the "if" part of an "if-then" statement is false, then the "if-then" statement is true.

So the given statement must be true.

Example. Show that $P \to Q$ and $\neg P \vee Q$ are logically equivalent.

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the columns for $P \to Q$ and $\neg P \vee Q$ are identical, the two statements are logically equivalent.

Statement	If p, then q.
Converse	If q, then p.
Inverse	If not p, then not q.
Contrapositive	If not q, then not p.

Statement	If two angles are congruent, then they have the same measure.
Converse	If two angles have the same measure, then they are congruent.
Inverse	If two angles are not congruent, then they do not have the same measure.
Contrapositive	If two angles do not have the same measure, then they are not congruent.

since the hypothesis and conclusion are equivalent, all four statements are true. But this will not always be the case!

Statement	If p , then q .
Converse	If q , then p .
Inverse	If not p , then not q .
Contrapositive	If not q , then not p .

Statement	If a quadrilateral is a rectangle, then it has two pairs of parallel sides.
Converse	If a quadrilateral has two pairs of parallel sides, then it is a rectangle. **(FALSE!)**
Inverse	If a quadrilateral is not a rectangle, then it does not have two pairs of parallel sides. **(FALSE!)**
Contrapositive	If a quadrilateral does not have two pairs of parallel sides, then it is not a rectangle.