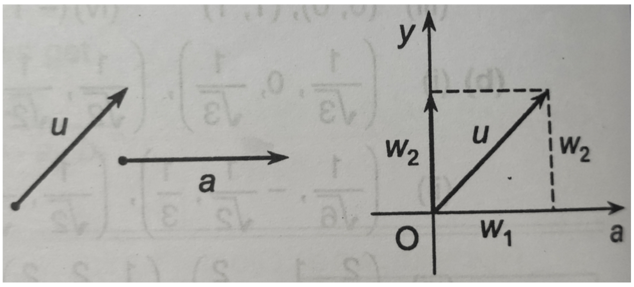
Orthogonal Projections

Definition

Sometimes we need to 'decompose' a vectors ***u*** into sum of two vectors, one in the direction of a given non-zero vector ***a*** and the other in the direction perpendicular to the vector ***a.***

Suppose we have ***u*** and ***a*** as shown in the figure. We now find a from the initial point of ***u.*** Now, drop a perpendicular from the tip of ***u*** to ***a***. Now, find one component $\boldsymbol{w_1}$ from the initial point of ***u*** to the foot of the perpendicular. We also find another vector component $\boldsymbol{w_2}$ from the foot of the perpendicular to the tip of the vector ***u***.



Thus, $\boldsymbol{w_1} + \boldsymbol{w_2} = \boldsymbol{u}$.

The vector component $\boldsymbol{w_1}$ in the direction of ***a*** is called **the orthogonal projection of *u* on** or the **vector component of *u* along *a***. It is denoted by $\boldsymbol{proj_a\, u}$. The vector component $\boldsymbol{w_2}$ in the direction perpendicular to ***a*** is called the vector component of ***u*** **orthogonal** to ***a***. Since $\boldsymbol{w_2} = \boldsymbol{u} - \boldsymbol{w_1}$, it can be denoted by $\boldsymbol{u - proj_a u}$. Vector component of u along a i.e. the projection of ***u*** on ***a*** is given by

$$\text{proj}_a\, u = \frac{u \cdot a}{||a||^2}\, a$$

and the vector component of ***u*** orthogonal to ***a*** is given by

$$u - \text{proj}_a\, u = u - \frac{u \cdot a}{||a||^2}\, a$$

Application

Projections (orthogonal) play a major role in algorithms for certain linear algebra problems:

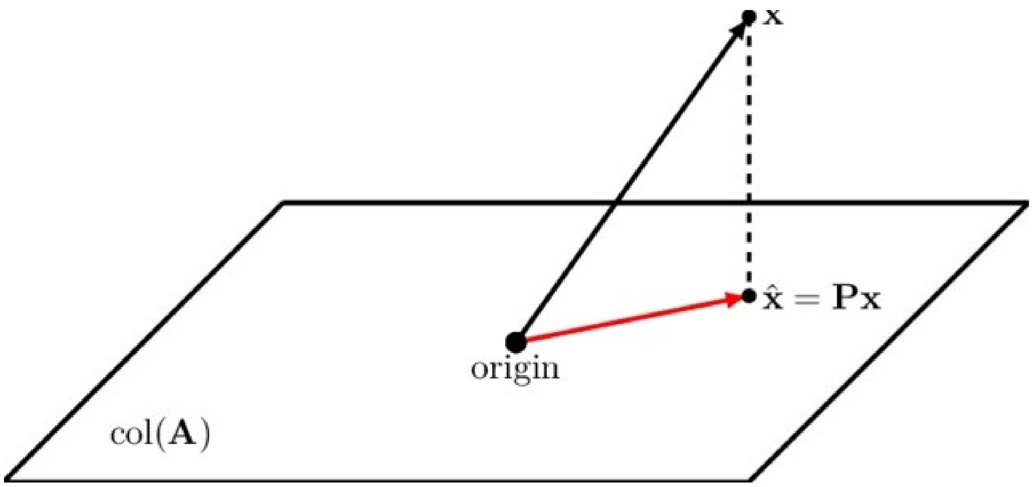
•QR decomposition (Householder transformation and Gram–Schmidt decomposition);
•Singular value decomposition
•Reduction to Hessenberg form (the first step in many eigenvalue algorithms)
•Linear regression
•Projective elements of matrix algebras are used in the construction of certain K-groups in Operator K-theory
•It is also used in the separating axis theorem to detect whether two convex shapes intersect.

Significance

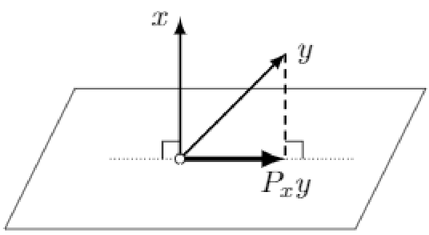
Orthogonal projection methods adopt a different approach to reduce data dimensionality compared to transformation methods or variable selection methods.

These methods have the advantage of better defining the dimensions of the subspace that describe the maximum of variations related or unrelated to the entity of interest, **y**, in the multivariate space. Therefore, to perform a better extraction of the maximum of information most related to **y**, orthogonal projection methods have the advantage of making the regression model independent of the influence of the variations in the data not related to **y**. Hence, orthogonal projection methods show an improvement in data and model interpretability. It also show an improvement of the model predictive performance outside the calibration range and in the presence of different influence factors that may affect the spectral signal.

Geometric Representation



Geometric illustration of the orthogonal projection operator P. A vector x ∈ R m is restricted to the column space of A, where $\boldsymbol{P_x} \in$ col(A).



Similarly, the orthogonal projection of $\boldsymbol{y}$ on the plane $\boldsymbol{R^2}$ is shown above, denoted by $\boldsymbol{P_x y}$.

* Orthogonal Projections.

1) Find the projection of $u=(1,-2,3)$ along $v=(1,2,1)$ in R^3.

We have the dot product i.e.

$$u \cdot v = 1\times 1 + (-2)(2) + 3\times 1.$$
$$= 0$$

$\therefore$ u and v are perpendicular to each other.
Hence, projection of u along v is zero.

2) Find the projection of $u=(3,1,3)$ along and perpendicular to $v=(4,-2,2)$

We have $u \cdot v = 3\times 4 + 1(-2) + 3\times 2$
$$= 16$$

$$\therefore \ \|v\|^2 = \left(\sqrt{16+4+4}\right)^2$$
$$= 24$$

$$\therefore \ \text{proj}_v u = \frac{u\cdot v}{\|v\|^2} v = \frac{16}{24}(4,-2,2) = \left(\frac{8}{3}, \frac{-4}{3}, \frac{4}{3}\right).$$

$$u - \text{proj}_v u = (3,1,3) - \left(\frac{8}{3}, \frac{-4}{3}, \frac{4}{3}\right)$$

$$= \left(\frac{1}{3}, \frac{7}{3}, \frac{5}{3}\right)$$

3) Find $\|proj_v u\|$ where $u = (3,0,4)$ and $v = (2,3,3)$.

We have $u.v = 3\times2 + 0\times3 + 4\times3$

$= 6 + 0 + 12$

$= 18$

$\|v\|^2 = 4 + 9 + 9 = 22$

$\therefore proj_v u. = \frac{u.v}{\|v\|^2} v$

$= \frac{\cancel{18}9}{\cancel{22}11}(2,3,3) = \left(\frac{18}{11}, \frac{27}{11}, \frac{27}{11}\right)$

$\therefore \|proj_v u\| = \sqrt{\frac{18^2}{11^2} + \frac{27^2}{11^2} + \frac{27^2}{11^2}} = 9\sqrt{\frac{2}{11}}.$

4) Compute the orthogonal projection of $x = \begin{pmatrix}-6\\4\end{pmatrix}$ onto the line L spanned by $u = \begin{pmatrix}3\\2\end{pmatrix}$ and find the distance from x to L.

We have $proj_L x = \frac{x.L}{\|L\|^2} L = \frac{(-6\times3 + 4\times2)}{9+4}\begin{pmatrix}3\\2\end{pmatrix} = \frac{-10}{13}\begin{pmatrix}3\\2\end{pmatrix}$

$\therefore$ Orthogonal component of x i.e. $x - proj_L x = \frac{1}{13}\begin{pmatrix}-48\\72\end{pmatrix}$

$\therefore$ The distance from x to L,

$\|x - proj_L x\| = \frac{1}{13}\sqrt{48^2 + 72^2} = \underline{6.65}$

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