Set Theory

Finite and infinite Sets

- **Infinite Set:** An infinite set is one in which it is not possible to list and count all the members of the set.
- **Example: 1**
- E = {even numbers greater than 9}
- E = {10, 12, 14, 16...}
- Here n(E) = infinite
- **Example: 2**
- G = { whole numbers greater than 2000 }
- G = { 2001, 2002, 2003, 2004, ...}
- Here n(G) = infinite

- **Classify the following as finite and infinite sets.**

a. A = {x : x ∈ N and x is even}
b. B = {x : x ∈ N and x is composite}
c. C = {x : x ∈ N and $3x - 2 = 0$}
d. D = {x : x ∈ N and $x^2 = 9$ }
e. E = {The set of numbers which are multiple of 3}
f. F = {The set of letters in English alphabets}
g. G = {The set of persons living in a house}
h. H = {x : x ∈ P, P is a number}
i. I = {The set of fractions with numerator 3}

a. Infinite
b. Infinite
c. Finite
d. Finite
e. Infinite
f. Finite
g. Finite
h. Infinite
i. Infinite

De Morgan’s law

For any two finite sets A and B

(i) $(A \cup B)' = A' \cap B'$ (which is a De Morgan's law of union).

(ii) $(A \cap B)' = A' \cup B'$ (which is a De Morgan's law of intersection).

Cartesian Product of 2 non empty set

sets $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$

Cartesian product $A \times B =$

$\{(a_1,b_1), (a_1,b_2), (a_1,b_3), (a_2,b_1), (a_2,b_2),(a_2,b_3), (a_3,b_1), (a_3,b_2), (a_3,b_3)\}$.

If the number of elements of A is h i.e., $n(A) = h$ & that of B is k i.e., $n(B) = k$, then the number of ordered pairs in Cartesian product will be

$n(A \times B) = n(A) \times n(B) = hk$.

Cartesian Product

The Cartesian product of sets refers to the product of two non-empty sets in an ordered way. Or, in other words, the assortment of all ordered pairs attained by the product of two non-empty sets. An ordered pair basically means that two elements are taken from each set.

Let P & Q be two sets such that $n(P) = 4$ and $n(Q) = 2$. If in the Cartesian product we have $(m,1)$, $(n,-1)$, $(x,1)$, $(y, -1)$. Find P and Q, where m, n, x, and y are all distinct.

Answer :

P = set of first elements = $\{m, n, x, y\}$ and Q = set of second elements = $\{1, -1\}$

Subsets

- A set A is a subset of a set B, if all the elements of A are contained in/members of the larger set B.
- set A is a subset of B if and only if every element of A is also an element of B.
- We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.
- The empty set({ } or ϕ) is a subset of every set.

- $A = \{3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$?
- **Answer:** Yes

- $A = \{3, 3, 3, 9\}$, $B = \{5, 9, 1, 3\}$, $A \subseteq B$?
- **Answer:** Yes

- $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $A \subseteq B$?
- **Answer:** No

Number of Subsets

- If, M = {a, b, c}
- Then, the subsets of M are:
- {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}, {}
- Therefore, the number of subsets, S = 8
- And the formula, $S = 2^n$
- Where, S is the number of sets And, n is the number of elements of the set.
- Is the formula used to calculate the number of subsets of a given set.
- So from above, M = {a, b, c}
- $S = 2^n$, $2^3 = 2 \times 2 \times 2 = 8$

Solution

- **Determine whether each of the following statements is true or false.**
  - $x \in \{x\}$ TRUE
    - (Because x is the member of the singleton set { x })
  - $\{x\} \subseteq \{x\}$ TRUE
    - (Because Every set is the subset of itself. Note that every Set has necessarily two subsets Ø and the Set itself, these two subset are known as Improper subsets and any other subset is called Proper Subset)
  - $\{x\} \in \{x\}$ FALSE
    - (Because { x} is not the member of {x}) Similarly other
  - $\{x\} \in \{\{x\}\}$ TRUE
  - $\varnothing \subseteq \{x\}$ TRUE
  - $\varnothing \in \{x\}$ FALSE

Power Sets

- A Power Set is a set of all the subsets of a set.
- The power set of S is denoted by P(S).
- **Notation:**
- The number of members of a set is often written as $|S|$, so we can write:

$$|P(S)| = 2^n$$

Power Sets

- **A= { a, b, }**
- The power set of A is $2^4 = 16$
- P(A)={}, {a}, {b}, {a, b}, {c}, {a, c}, {b, c}, {a, b, c}, {d}, {a, d}, {b, d}, {a, b, d}, {c, d}, {a, c, d}, {b, c, d}, {a, b, c, d}.
- **B={1, 2, 3}**
- The power set of B is $2^3 = 8$
- P(B)={}, {1}, {2}, {1, 2}, {3}, {1, 3}, {2, 3}, {1, 2, 3}

Power Set

- **C={a,1,b,2,c}**
- The power set of B is $2^4 = 16$

- P(C)={}, {a}, {1}, {a, 1}, {b}, {a, b}, {1, b}, {a, 1, b}, {2}, {a, 2}, {1, 2}, {a, 1, 2}, {b, 2}, {a, b, 2}, {1, b, 2}, {a, 1, b, 2}, {c}, {a, c}, {1, c}, {a, 1, c}, {b, c}, {a, b, c}, {1, b, c}, {a, 1, b, c}, {2, c}, {a, 2, c}, {1, 2, c}, {a, 1, 2, c}, {b, 2, c}, {a, b, 2, c}, {1, b, 2, c}, {a, 1, b, 2, c}.

Partition of a set

Partition of a set, say *S*, is a collection of *n* disjoint subsets, say P_1, P_1, ... P_n that satisfies the following three conditions −

- P_i does not contain the empty set.

  $$[\ P_i \neq \{\ \emptyset\ \} \text{ for all } 0 < i \leq n\ ]$$

- The union of the subsets must equal the entire original set.

  $$[\ P_1 \cup P_2 \cup \ldots \cup P_n = S\ ]$$

- The intersection of any two distinct sets is empty.

  $$[\ P_a \cap P_b = \{\ \emptyset\ \}, \text{ for } a \neq b \text{ where } n \geq a, b \geq 0\ ]$$

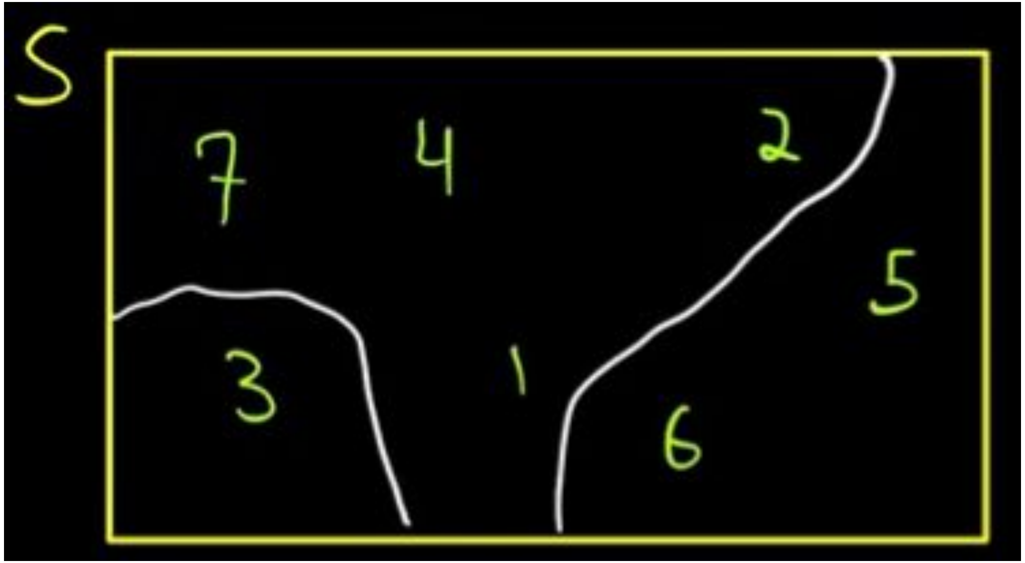
Example

Let S = { a, b, c, d, e, f, g, h }

One probable partitioning is { a }, { b, c, d }, { e, f, g, h }

Another probable partitioning is { a, b }, { c, d }, { e, f, g, h }

Partition of a set

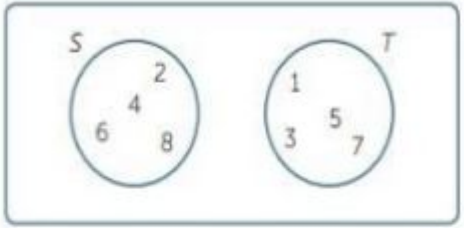


P = { {3} , {1,2,4,7} , {5,6} }

Disjoint Sets

- **Disjoint sets:**
- Two sets are called disjoint if they have no elements in common.

- **For Example:**
- The sets S = {2, 4, 6, 8} and T = {1, 3, 5, 7} are disjoint.



Disjoint Sets

- Another way to define disjoint sets is to say that their intersection is the empty set,
- Two sets A and B are disjoint if $A \cap B = \{ \}$.
- In the example above,
- $S \cap T = \{ \}$ because no number lies in both sets.
- The overlapping region of two circles represents the intersection of the two sets.
- When two sets are disjoint, we can draw the two circles without any overlap.

- **Which of the following sets are disjoint or overlapping:**

a. A = {The set of boys in the school}

   B = {The set of girls in the school}

b. P = {The set of letters in the English alphabets}

   Q = {The set of vowels in the English alphabets}

c. $X = \{x : x \text{ is an odd number}, x < 9\}$

   $Y = \{x : x \text{ is an even number}, x < 10\}$

d. $E = \{9, 99, 999\}$

   $F = \{1, 10, 100\}$

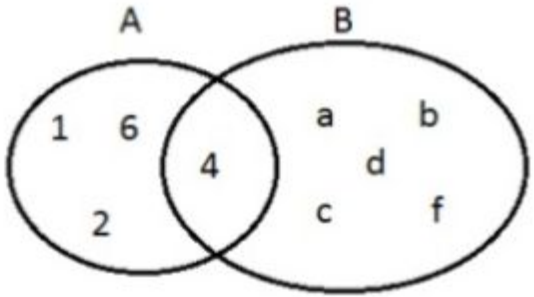
a. Disjoint Sets
b. Overlapping Sets
c. Disjoint Sets
d. Disjoint Sets

Union of Sets

- Combining all the elements of any two sets is called the Union of those sets.
- Union of two sets A and B is obtained by combining all the members of the sets and is represented as $A \cup B$
- **Examples of Union of Sets**
- If $A = \{1, 2, 3, 4, 5\}$ and
- $B = \{2, 4, 6\}$,
- Then the union of these sets is $A \cup B = \{1, 2, 3, 4, 5, 6\}$

Examples of Union of Sets

- A = {1, 2, 4, 6} and B = {4, a, b, c, d, f}
- Then the union of these sets is $A \cup B$ = {1, 2, 4, 6, a, b, c, d, f}



Union of Sets

- **Examples of Union of Sets**
- A = {x / x is a number bigger than 4 and smaller than 8}
- B = {x / x is a positive number smaller than 7}

- A = {5, 6, 7} and B = {1, 2, 3, 4, 5, 6}
- $A \cup B$ = {1, 2, 3, 4, 5, 6, 7}
- Or $A \cup B$ = {x / x is a number bigger than 0 and smaller than 8}

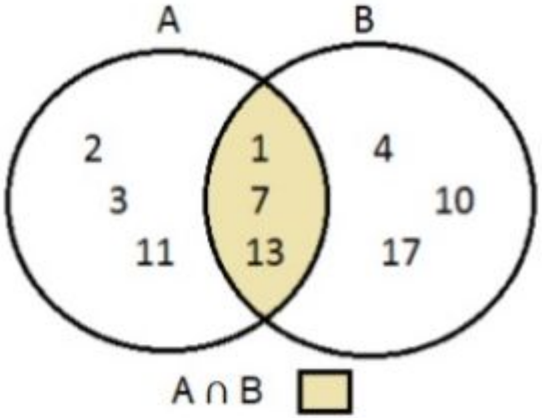
Union of Sets

- **Examples of Union of Sets**
- A = {#, %, $}
- B = { }
- Then, A ∪ B = {#, %, $}

- **Examples of Union of Sets**
- N = {-5, -4, 0, 6, 8}   and O = {-4, 0, 8, 9}
- Then, N ∪ O = {-5, -4, 0, 6, 8, 9}

Intersection of Sets

- Intersection of Sets is defined as the grouping up of the common elements of two or more sets.
- It is denoted by the symbol ∩
- **Example of Intersection of Sets**
- When Set A = {1, 2, 3, 7, 11, 13} and Set B = {1, 4, 7, 10, 13, 17},
- A n B is all the common elements of the set A and B.
- Therefore, A ∩ B = {1, 7, 13}.
- This can be shown by using Venn diagram as:



Difference of Sets

- The difference set of any two sets A and B. is the set of the members of set A which is not the members of set B.
- **Example of Difference of Sets**
- A = {0, 1, 2, 3}
- B = {2, 3}
- The difference set is {0, 1}.
- We can write it as A – B or A \ B. We say: ‘A difference B’.
- B – A or B \ A = { }

- A = {1, 2, 3} and B = {4, 5, 6}.
- Find the difference between the two sets:
- (i) A and B
- (ii) B and A

- The two sets are disjoint as they do not have any elements in common.
- (i) A - B = {1, 2, 3} = A
- (ii) B - A = {4, 5, 6} = B

- **Given three sets P, Q and R such that:**
- P = {x : x is a natural number between 10 and 16},
- Q = {y : y is a even number between 8 and 20} and
- R = {7, 9, 11, 14, 18, 20}
-
- (i) Find the difference of two sets P and Q
- (ii) Find Q - R
- (iii) Find R - P
- (iv) Find Q – P

- According to the given statements:
- $P = \{11, 12, 13, 14, 15\}$
- $Q = \{10, 12, 14, 16, 18\}$
- $R = \{7, 9, 11, 14, 18, 20\}$
- (i) $P - Q$ = {Those elements of set P which are not in set Q}

  $= \{11, 13, 15\}$
- (ii) $Q - R$ = {Those elements of set Q not belonging to set R}

  $= \{10, 12, 16\}$
- (iii) $R - P$ = {Those elements of set R which are not in set P}

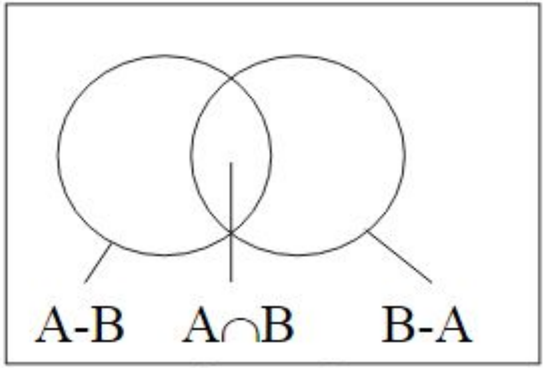
  $= \{7, 9, 18, 20\}$
- (iv) $Q - P$ = {Those elements of set Q not belonging to set P}

  $= \{10, 16, 18\}$

Universal Set

- The universal set is the set of all elements that are considered in a specific theory. We'll note the universal set with U.
- We'll choose as universal set: U = {6,7,8,9,15,16,17,18,20,21}.
- We have to determine the sets:
- M = {x / x are the multiple of 3}
- N = {x / x are the multiple of 5}
- The elements of M and N have to be chosen from the universal set U.
- To determine M, we'll identify the multiples of 3 from U: {6,9,15,18,21}
- M = {6,9,15,18,21}
- To determine N, we'll identify the multiples of 5 from U: {15,20}.
- N = {15,20}

Principle of Inclusion Exclusion



S: universal set

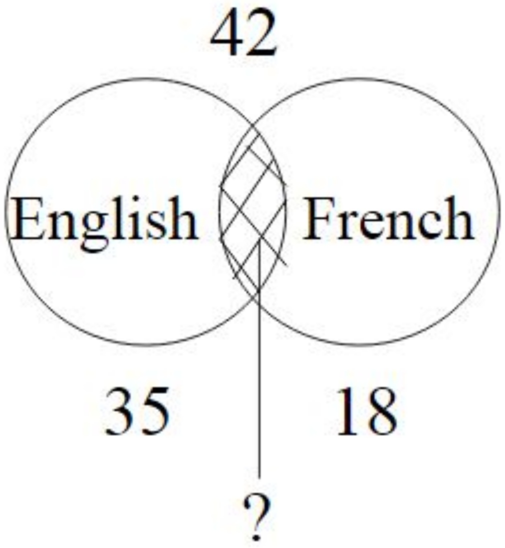
A-B, A∩B, B-A are mutually disjoint sets.

i.e. $x \in A\text{-}B$, then $x \notin B$,
and therefore $x \notin B\text{-}A$,
$x \notin A \cap B$.

$$|\mathbf{A} \cup \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}| - |\mathbf{A} \cap \mathbf{B}|$$
$$|\mathbf{A}\text{-}\mathbf{B}| = |\mathbf{A}| - |\mathbf{A} \cap \mathbf{B}|$$
$$|\mathbf{B}\text{-}\mathbf{A}| = |\mathbf{B}| - |\mathbf{A} \cap \mathbf{B}|$$

$\therefore$ e.g. $|A \cap B|$ can be computed in several ways depends on the information given.

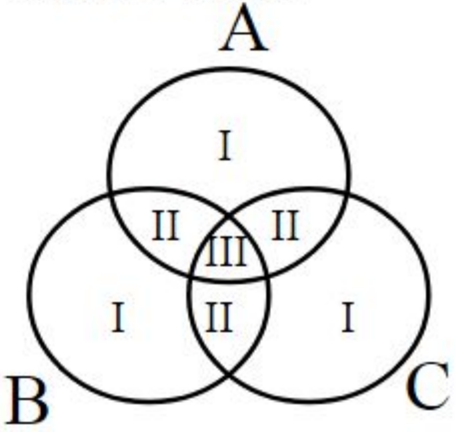
e.g. In a group of 42 tourists, everyone speaks English or French; there are 35 English speakers and 18 French speakers. How many speak both English & French?



$$\underset{42}{|A \cup B|} = \underset{35}{|A|} + \underset{18}{|B|} - |A \cap B|$$

$$\therefore\ |A \cap B| = 11$$

e.g. What if we have 3 sets:



For $|A| + |B| + |C|$:

I: counted 1 time

II: counted 2 times

III: counted 3 times

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$\begin{aligned}
\text{L.H.S.} &= |A \cup (B \cup C)| \\
&= |A| + |B \cup C| - |A \cap (B \cup C)| \\
&= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C| \\
&= \text{R.H.S.}
\end{aligned}$$

eg: a survey of 150 college students reveals that:

83 own <u>**C**</u>ars,

97 own <u>**B**</u>ikes,

28 own <u>**M**</u>otorcycles,

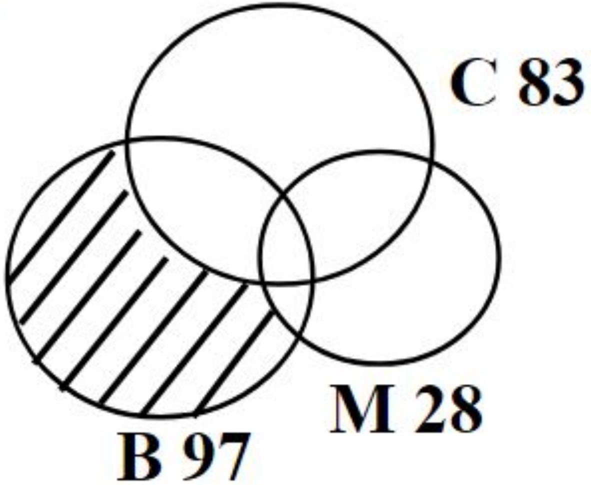
53 own a car and a bike,

14 own a car and a motorcycle,

7 own a bike and a motorcycle,

2 own all three.

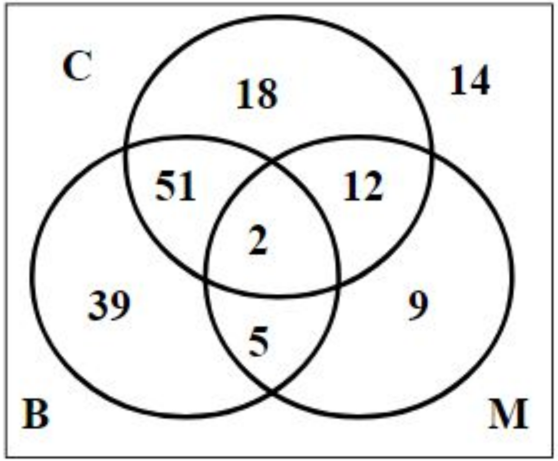
a. How many own a bike and nothing else?



$$
\begin{aligned}
&|\mathbf{B} - (\mathbf{C} \cup \mathbf{M})| \\
&= |B| - |B \cap (C \cup M)| \\
&= |B| - |(B \cap C) \cup (B \cap M)| \\
&= |B| - (|B \cap C| + |B \cap M| - |B \cap C \cap M|) \\
&= 97 - (53 + 7 - 2) \\
&= 39
\end{aligned}
$$

b. How many students do not own any of the three?

150



$$150 - |C \cup B \cup M|$$
$$= 150 - (83+97+28-53-14-7+2)$$
$$= 150 - 136$$
$$= 14$$

Find the no of integers between 1 to 100 which are divisible by 3 and 5

Let’s see how many numbers between 1 to 100 are divisible by 3 and 5. To be divisible by 3 and 5 simultaneously, a number has to be divisible by 15 (3x5). Between 1 and 100, there are 100/15=6.66 and so 6 numbers (you can check that by multiplying 15 by 6 and 7 which gives us 90 and 105 respectively)-15,30,45,60,75 and 90.

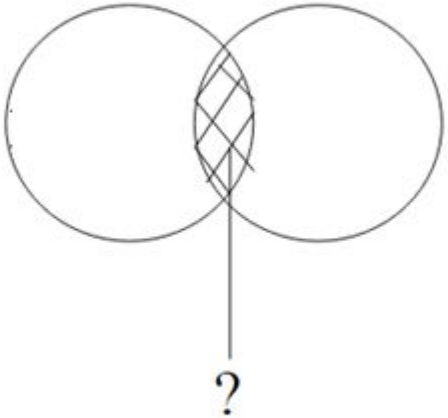
Therefore, there are 94 (100–6) numbers between 1 and 100 that are not divisible by 3 and 5 simultaneously.

Find the no of integers between 1 to 100 which are divisible by 3 or 5

A = divisible by 3, n(A) = 33

B = divisible by 5, n(B) = 20

Divisible by 3 and 5 = 6



$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$\therefore\ |A \cap B| = 6$$

33+20-6=47

Find the no of integers between 1 to 100 which are neither divisible by 3 nor by 5

n(Total) - n(A U B)

100 - 47 = 53

Find the number of positive integers from 1 to 100, which are **neither divisible by 2 nor by 3 nor by 5.**

A : Divisible by 2
B : Divisible by 3
C : Divisible by 5

$A \cap B$: Divisible by 2 and 3 $\Rightarrow$ Divisible by 6
$B \cap C$: Divisible by 3 and 5 $\Rightarrow$ Divisible by 15
$C \cap A$: Divisible by 5 and 2 $\Rightarrow$ Divisible by 10

$A \cap B \cap C$: Divisible by 2, 3 and 5 $\Rightarrow$ Divisible by 30

$n(A \cup B \cup C) = (103) - (32) + (3)$

$= 74$

Final Ans $=$ Total $- \; n(A \cup B \cup C)$

$= 100 - 74$

$= 26$