Overview of Advanced Calculus (Folland)

Brett Saiki

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This is a condensed version of Gerald B. Folland's *Advanced Calculus* containing only definitions, propositions, theorems, etc. For proofs and detailed explanations, refer to the actual text.

1 Introduction / Basic Topology

1.1 Euclidean Spaces and Vectors

Definition. If $\mathbf{x} \in \mathbb{R}^n$, the **norm** of $\mathbf{x}$ is defined to be

$$|\mathbf{x}| = \sqrt{x_1^2 + \ldots + x_n^2} = \sqrt{\mathbf{x} \cdot \mathbf{x}}$$

Definition. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the **distance** between $\mathbf{x}$ and $\mathbf{y}$ is defined to be

$$|\mathbf{x} - \mathbf{y}| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$$

1.1 Proposition (Cauchy's Inequality). For any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

1.2 Proposition (The Triangle Inequality). For any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

1.3 Equation.

$$\max(|x_1|, \ldots, |x_n|) \leq |\mathbf{x}| \leq \sqrt{n} \max(|x_1|, \ldots, |x_n|)$$

1.2 Subsets of Euclidean Space

Definition. A set $S \subset \mathbb{R}^n$ is **bounded** if there exists a ball that contains every point in S. In other words, there exists a contant C such that $|\mathbf{x}| < C$ for every $\mathbf{x} \in S$.

Definition. Let S be a subset of $\mathbb{R}^n$. Then the **complement** of S is the set of all points in $\mathbb{R}^n$ not in S. The complement is denoted $\mathbb{R}^n \setminus S$ or S^c.

Definition. A point $\mathbf{x} \in \mathbb{R}^n$ is called an **interior point** of S if all points sufficiently close to $\mathbf{x}$ are also in $\S$. The set of all interior points of S is called the **interior** of S and is denoted S^{int}.

Definition. A point $\mathbf{x} \in \mathbb{R}^n$ is called an **boundary point** of S if every ball centered at $\mathbf{x}$ contains both points in $\S$ and in $\S^c$. The set of all boundary points is called the **boundary** of S and is denoted ∂S. S is **open** if it contains none of its boundary points. S is **closed** if it contains all of its boundary points.

Definition. The **closure** of S is the union of S and all its boundary points and is given by

$$\overline{S} = S \cup \partial S$$

Defintiion. A **neigborhood** of a point $\mathbf{x} \in \mathbb{R}^n$ is a set of which $\mathbf{x}$ is an interior point.

1.4 Proposition. Suppose $S \subset \mathbb{R}^n$. Then, the following are true:

a. S is open $\Leftrightarrow$ every point of S is an interior point
b. S is closed $\Leftrightarrow$ S^c is open

1.3 Limits and Continuity

1.6 Definition. The **limit** of $f(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$ is L if for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$|f(\mathbf{x}) - L| < \epsilon \text{ whenever } 0 < |\mathbf{x} - \mathbf{a}| < \delta$$

1.7 Definition. Alternatively, the limit of $f(x)$ exists if for every $\epsilon > 0$, there exists a $\delta' > 0$ such that

$$|f(\mathbf{x}) - L| < \epsilon \text{ whenever } 0 < \max(|x_1 - a_1|, \ldots, |x_n - a_n|) < \delta'$$

1.8 Definition. If f is a function on $U \subset \mathbb{R}^n$, then f is **continuous** on U if for every $\mathbf{x} \in U$,

$$|f(\mathbf{x}) - f(\mathbf{a})| < \epsilon \text{ whenever } |\mathbf{x} - \mathbf{a}| < \delta$$

1.9 Theorem. Suppose $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $U \subset \mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}^k$ is continuous on $\mathbf{g}(U) \subset \mathbb{R}^m$. Then, $\mathbf{g} \circ \mathbf{f} : \mathbb{R}^n \to \mathbb{R}^k$ is continuous on U.

1.10 Theorem. Let $f_1(x, y) = x + y$, $f_2(x, y) = xy$, and $g(x) = \frac{1}{x}$. Then, f_1 and f_2 are continuous on $\mathbb{R}^2$ and g is continuous on $\mathbb{R} \setminus \{0\}$.

1.11 Corollary. The function $f_3(x, y) = x - y$ is continuous on $\mathbb{R}^2$, and the funcction $f_4(x, y) = \frac{x}{y}$ is continuous on $\{(x, y) : y \neq 0\}$.

1.12 Corollary. Result of 1.10 and 1.11: the sum, difference, and product of two continuous functions is continuous while the quotient of two continuous functions is continuous where the denominator is zero.

1.13 Theorem. Suppose $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^k$ is continuous and $U \subset \mathbb{R}^k$, and let $S = \{x \in \mathbb{R}^n : \mathbf{f}(x) \in U\}$. Then, S is open if U is open and S is closed if U is closed. In brief, continuous functions map open sets to open sets and closed sets to closed sets.

1.4 Sequences

1.14 Theorem. Suppose $S \subset \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. Then, $\mathbf{x}$ belongs to $\overline{S}$ if and only if there is a sequence in S that converges to $\mathbf{x}$.

1.15 Theorem. Given $S \subset \mathbb{R}^n$, $\mathbf{a} \in S$, and $\mathbf{f} : S \to \mathbb{R}^m$, the following are equivalent:
(a) $\mathbf{f}$ is continuous at $\mathbf{a}$
(b) for any sequence $\{\mathbf{x}_k\}$ in S that converges to $\mathbf{a}$, the sequence $\{\mathbf{f}(\mathbf{x}_k)\}$ converges to $\mathbf{f}(\mathbf{a})$.

Definition. A point $\mathbf{a} \in \mathbb{R}^n$ is an **accumulation point** of $S \subset \mathbb{R}^n$ if every neighborhood of $\mathbf{a}$ contains infinitely many points in S.

1.5 Completeness

The Completeness Axiom. Let S be a nonempty set of real numbers If S has an upper bound, then S has a least upper bound, called the **supremum** of S (sup S) is the smallest number u such that $x \leq u$ for all $x \in S$. If S has a lower bound, the **infimum** of S (inf S) is the largest number u such that $x \geq u$ for all $x \in S$. If S has no upper bound, we shall define sup S to be $+\infty$, and if S has no lower bound, we shall define inf S to be $-\infty$.

1.16 Theorem (The Monotone Sequence Theorem). Every bounded monotone sequence in $\mathbb{R}$ in convergent. More precisely, the limit of an increasing (resp. decreasing) sequence is the supremum (resp. infimum) of its set of values.

1.17 Theorem (The Nested Interval Theorem). Let $I_1 = [a_1, b_1], I_2 = [a_2, b_2], \ldots$ be a sequence of closed, bounded, intervals in $\mathbb{R}$. Suppose that (a) $I_1 \supset I_2 \supset \ldots$ and (b) $b_k - a_k$ tends to 0 as $k \to \infty$. Then, there is exactly one points in every I_k.

1.18 Theorem. Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

1.19 Theorem. Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Definition. A sequence $\{x_k\}$ is **Cauchy** if $x_k - x_j \to 0$ as $k, j \to \infty$. In other words, a sequence is Cauchy if for every $\epsilon > 0$, there exists an integer K such that $|x_k - x_j| < \epsilon$ whenever $k > K$, $j > K$.

1.20 Theorem A sequence is Cauchy if and only if it is convergent.

1.6 Compactness

1.21 Theorem (Bolzano-Weierstrauss Theorem). If S is a subset of $\mathbb{R}^n$, the following are equivalent:
(a) S is compact
(b) Every sequence of points in S has a convergent subsequence that converges to a point in S.

Definition. A set is **compact** if it is closed and bounded.

1.22 Theorem. Suppose $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and $U \subset \mathbb{R}^k$, and let $S = \{x \in \mathbb{R}^n : \mathbf{f}(x) \in U\}$. If S is compact, then $\mathbf{f}(S)$ is also compact. In brief, continuous functions map compact sets to compact sets.

1.23 Corollary (Extreme Value Theorem). Suppose $S \subset \mathbb{R}^n$ is compact and $f : S \to \mathbb{R}$ is continuous. Then f has an absolute minimum and maximum on S. In other words, there exist points $a, b \in S$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in S$.

Definition. A collection $\mathcal{U}$ of subsets of $\mathbb{R}^n$ is called a **covering** of S if S is contained in the union of the sets in $\mathcal{U}$.

1.24 Theorem (Heine-Borel Theorem). If S is a subset of $\mathbb{R}^n$, the following are equivalent:
(a) S is compact
(b) Every open convering of S has a finite subconvering.

Definition. The **distance** between two sets $U, V \subset \mathbb{R}^n$ is defined to be

$$d(U, V) = \inf\{|\mathbf{x} - \mathbf{y}| : \mathbf{x} \in U, \mathbf{y} \in V\}$$

1.7 Connectedness

Definition. A set is **disconnected** if it is the union of two non-empty subsets, neither of which intersects the closure of the other. More formally, a set S is disconnected if there exist two sets U, V such that

$$S = U \cup V \text{ and } \overline{U} \cap \overline{V} = \varnothing$$

Any set is **connected** if it is not disconnected.

1.25 Theorem. The connected subsets of $\mathbb{R}$ are precisely the intervals in $\mathbb{R}$.

1.26 Theorem. Suppose $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and $U \subset \mathbb{R}^k$, and let $S = \{x \in \mathbb{R}^n : \mathbf{f}(x) \in U\}$. If S is connected, then $\mathbf{f}(S)$ is also connected. In brief, continuous functions map connected sets to connected sets.

1.27 Theorem (Intermediate Value Theorem). Suppose $f : S \to \mathbb{R}$ is continuous at every point of S and $V \subset S$ is connected. If $a, b \in V$ and $f(a) < t < f(b)$ or $f(b) < t < f(a)$, then there is a point $c \in V$ such that $f(c) = t$.

1.28 Theorem. If $S \in \mathbb{R}^n$ is arcwise connected, then S is connected.

1.30 Theorem. If $S \in \mathbb{R}^n$ is open and connected, then S is arcwise connected.

1.8 Uniform Continuity

Definition. A function $\mathbf{f} : S \to \mathbb{R}^m$ is **uniformly continuous** on S if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \epsilon \text{ whenever } \mathbf{x}, \mathbf{y} \in S \text{ and } |\mathbf{x} - \mathbf{y}| < \delta$$

1.33 Theorem. Suppose $S \subset \mathbb{R}^n$ and $\mathbf{f} : S \to \mathbb{R}^m$ is continuous at every point of S. If S is compact, then f is uniformly continuous on S.

2 Differential Calculus

2.1 Differentiability of One Variable

2.1, 2.2 Definition. A function $f : S \to \mathbb{R}$ is **differentiable** at a if there is a number m, called the **derivative**, such that

$$f(a + h) = f(a) + mh + E(h) \text{ where } \lim_{h \to 0} \frac{E(h)}{h} = 0$$

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

2.5 Proposition. Suppose f is defined on an open interval I. If f has a local maximum or minimum at the point $a \in I$ and f is differentiable at a, then $f'(a) = 0$.

2.6 Theorem (Rolle's Theorem). Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, there is at least one point $c \in (a, b)$ such that $f'(c) = 0$.

2.7 Theorem (Mean Value Theorem I). Suppose f is continuous on $[a,b]$ and differentiable on (a,b). There is at least one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

2.8 Theorem. Suppose f is differentiable on I. Then, the following are true:
(a) If $|f'(x)| \leq C$ for all $x \in I$, then $|f(b)-f(a)| \leq C|b-a|$ for all $a, b \in I$.
(b) If $f'(x) = 0$ for all $x \in I$, then f is constant on I.
(c) If $f'(x) \geq 0$ (resp. $f'(x) > 0$, $f'(x) \leq 0$, $f'(x) < 0$) for all $x \in I$, then f is increasing (resp. strictly increasing, decreasing, strictly decreasing) on I.

2.9 Theorem (Mean Value Theorem II). Suppose that f and g are continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) \neq 0$ for all $x \in (a,b)$. Then, there exists $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

2.10 Theorem (L'Hôpital's Rule I). Suppose f and g are differentiable functions on (a,b) and $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$. If g' never vanishes on (a,b) and $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$.
,
2.11 Theorem (L'Hôpital's Rule II). Theorem 2.10 (L'Hôpital's Rule I) remains valid if $\lim |f(x)| = \lim |g(x)| = \infty$.

2.12 Corollary. For any $a > 0$, we have

$$\lim_{x\to\infty} \frac{x^a}{e^x} = \lim_{x\to\infty} \frac{\log x}{x^a} = \lim_{x\to 0^+} \frac{\log x}{x^{-a}}$$

Definition. The **partial derivative** of a function $f(x_1, \dots, x_n)$ with respect to x_j is

$$\lim_{h\to 0} \frac{f(x_1, \dots, x_j + h, \dots, x_n) - f(x_1, \dots, x_j, \dots, x_n)}{h}$$

2.2 Differentiability of Several Variables

Partial derivatives are generally notated $\frac{\partial f}{\partial x_j}$, f_j or $\partial_j f$.

2.15 Definition. A function f defined on an open set $S \subset \mathbb{R}^n$ is called **differentiable** at a point $\mathbf{a} \in S$ if there is a vector $\mathbf{c} \in \mathbb{R}^n$ such that

$$\lim_{h\to 0} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{c}\cdot\mathbf{h}}{|\mathbf{h}|} = 0$$

The vector $\mathbf{c}$ is called the **gradient** of f at $\mathbf{a}$ and is generally denoted $\nabla f(\mathbf{a})$.

2.16 Equation. $f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \nabla(\mathbf{a})\cdot\mathbf{h} + E(\mathbf{h})$ where $\frac{E(\mathbf{h})}{|\mathbf{h}|} \to 0$ as $\mathbf{h} \to \mathbf{0}$.

2.17 Theorem. If f is differentiable at $\mathbf{a}$, then the partial derivatives $\partial_j f(\mathbf{a})$ all exist, and they are the components of the vector $\nabla f(\mathbf{a})$.

2.18 Theorem. If f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.

2.19 Theorem. Let f be a function defined on an open set in $\mathbb{R}^n$ that contains the point $\mathbf{a}$. Suppose that the partial deriatives $\partial_j f$ all exist on some neighborhood if $\mathbf{a}$ and that they are continuous at $\mathbf{a}$. Then, f is differentiable at $\mathbf{a}$.

Definition. Given a unit vector $\mathbf{u}$, the **directional derivative** of f at $\mathbf{a}$ in the direction of $\mathbf{u}$ is defined by

$$\partial_{\mathbf{u}} f(\mathbf{a}) = \frac{d}{dt} f(\mathbf{a} + t\mathbf{u})\Big|_{t=0} = \lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{u}) - f(\mathbf{a})}{t}$$

2.23 Theorem. If f is differentiable at $\mathbf{a}$, then the directional derivatives of f at $\mathbf{a}$ all exist, and they are given by

$$\partial_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}$$

2.3 Chain Rule

2.26 Theorem (Chain Rule I). Suppose that $\mathbf{g}(t)$ is differentiable at $t = a$ $f(\mathbf{x})$ is differentiable at $\mathbf{x} = \mathbf{b}$, and $\mathbf{b} = \mathbf{g}(a)$. Then the composite function $\varphi(t) = f(\mathbf{g}(t))$ is differentiable at $t = a$, and its derivative is given by

$$\varphi'(a) = \nabla f(\mathbf{b}) \cdot \mathbf{g}'(a)$$

or in Leibniz notation where $w = f(\mathbf{x})$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}$$

2.29 Theorem (Chain Rule II). Suppose that $g_1, \ldots, g_n$ are functions of $\mathbf{t} = (t_1, \ldots, t_n)$ and f is a function of $\mathbf{x} = (x_1, \ldots, x_n)$. Let $\mathbf{b} = \mathbf{g}(\mathbf{a})$ and $\varphi = f \circ \mathbf{g}$. If $g_1, \ldots, g_n$ are differentiable at $\mathbf{a}$ (resp. of class C^1 near $\mathbf{a}$) and f is differentiable at $\mathbf{b}$ (resp. of class C^1 near $\mathbf{b}$), then φ is differentiable at $\mathbf{a}$ (resp. of class C^1 near $\mathbf{a}$), and its partial derivatives are given by

$$\frac{\partial \varphi}{\partial t_k} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_k}$$

where the derivatives $\partial f/\partial x_1$ are evaluated at $\mathbf{b}$ and the derivatives $\partial\varphi/\partial t_k$ and $\partial x_j/\partial t_k = \partial g_j/\partial t_k$ are evaluated at $\mathbf{a}$.

Definition. A function f on $\mathbb{R}^n$ is called (positively) **homogeneous** of degree a ($a \in \mathbb{R}$) if $f(t\mathbf{x}) = t^a f(\mathbf{x})$ for all $t > 0$ and $\mathbf{x} \neq \mathbf{0}$.

2.36 Theorem (Euler's Theorem). If f is homogeneous of degree a, then at any point $\mathbf{x}$ where f is differentiable we have

$$x_1\partial_1 f(\mathbf{x}) + x_2\partial_2 f(\mathbf{x}) + \ldots + x_n\partial_n f(\mathbf{x}) = af(\mathbf{a})$$

2.37 Theorem. Suppose that F is a differentiable function on some open set $U \subset \mathbb{R}^3$ and that the set

$$S = \big\{(x, y, z) \in U : F(x, y, z) = 0\big\}$$

is a smooth surface. If $\mathbf{a} \in S$ and $\nabla F(\mathbf{a}) \neq \mathbf{0}$, then the vector $\nabla F(\mathbf{a})$ is perpindicular, or normal, to the surface S at $\mathbf{a}$.

2.38 Corollary. Under the conditions of the theorem, the equations of the tangent plane to S at $\mathbf{a}$ is $\nabla F(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0$.

2.4 The Mean Value Theorem

2.39 Theorem (Mean Value Theorem III). Let S be a region in $\mathbb{R}^n$ that contains the points $\mathbf{a}$ and $\mathbf{b}$ as well as the line segment L that joins them. Suppose that f is a differentiable on every point of L except perhaps the endpoints $\mathbf{a}$ and $\mathbf{b}$. Then, there is a point $\mathbf{c}$ on L such that

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})$$

Definition. A set $S \subset \mathbb{R}^n$ is called **convex** if whenever $\mathbf{a}, \mathbf{b} \in S$, the line segment from $\mathbf{a}$ to $\mathbf{b}$ also lies in S.

2.40 Corollary. Suppose that f is differentiable on an open convex set S and $|\nabla f(\mathbf{x})| \leq M$ for every $\mathbf{x} \in S$. Then, $|f(\mathbf{b}) - f(\mathbf{a})| \leq M|\mathbf{b} - \mathbf{a}|$ for all $\mathbf{a}, \mathbf{b} \in S$.

2.41 Corollary. Suppose that f is differentiable on an open, convex set S and $|\nabla f(\mathbf{x})| = 0$ for every $\mathbf{x} \in S$. Then, f is constant on S.

2.42 Theorem. Suppose that f is differentiable on an open, connected set S and $|\nabla f(\mathbf{x})| = 0$ for every $\mathbf{x} \in S$. Then, f is constant on S.

2.5 Functional Relations and Implicit Functions

2.6 Higher-Order Partial Derivatives

Definition. The partial derivatives of the partial derivatives of some function f are referred to as *second order derivatives* and are generally denoted $f_{j,i}$, $\partial_i, \partial_j f$. A function is said to be of **class** C^k on an open set U if all of its partial derivatives $\partial_{i_1}, \partial_{i_2}, \ldots \partial_{i_k} f$ exist and are continuous on U. If the partial derivatives of f of *all* orders exist and are continuous on U, f is said to be of **class** C^∞ on U. It is common to refer to the derivatives $\partial_j^2 f$ and $\partial_i \partial_j f$ $(i \neq j)$ as **pure** and **mixed** second-order partial derivatives of f.

2.45 Theorem. Let f be a function defined in an open set $S \in \mathbb{R}^n$. Suppose $\mathbf{a} \in S$ and $i, j \in \{1, \ldots, \}$. If the derivatives ∂_i, $\partial_j f$, $\partial_i \partial_j f$, and $\partial_j \partial_i f$ exist in S, and if $\partial_i \partial_j f$ and $\partial_j \partial_i f$ are continuous at $\mathbf{a}$, then $\partial_i \partial_j f(\mathbf{a}) = \partial_j \partial_i f(\mathbf{a})$.

2.46 Corollary. If f is of class C^2 on an open set S, then $\partial_i \partial_j f = \partial_j \partial_i f$ on S, for all i and j.

2.47 Theorem. If f is of class C^k on an open set S, then

$$\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f = \partial_{j_1} \partial_{j_2} \cdots \partial_{j_k} f \text{on } S$$

whenever the sequence $\{j_1, \ldots, j_k\}$ is a reordering of the sequence $\{i_1, \ldots, i_k\}$.

2.51 Proposition. Suppose u is a C^2 function of (x, y) in some open set in $\mathbb{R}^2$. If (x, y) is related to (r, θ) by $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Definition. A **multi-index** is an n-tuple of nonnegative integers usually denoted

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

The number $|\alpha| = \alpha_1 + \ldots + \alpha_n$ is called the **order** or **degree** of α.

2.52 Theorem (The Multinomial Theorem). For any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and any positive number k,

$$(x_1 + x_2 + \ldots + x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!}\mathbf{x}^\alpha$$

2.7 Taylor's Theorem

Definition. Let F be a function of class C^k on an interval I containing the point $x = a$. The k-th order **Taylor polynomial** for f based at a is given by

$$P_{a,k}(h) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!} h^j$$

and the difference

$$R_{a,k}(h) = f(a+h) - P_{a,k}(h) = f(a+h) - \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!} h^j$$

is called the k-th order **Taylor remainder**.

2.55 Theorem (Taylor's Theorem with Integral Remainder I). Suppose that f is of class C^{k+1} $(k \geq 0)$ on an interval $I \subset \mathbb{R}$, and $a \in I$. Then, the remainder $R_{a,k}$ is given by

$$R_{a,k}(h) = \frac{h^{k+1}}{k!} \int_0^1 (1-t)^k f^{(k+1)}(a+th)\, dt$$

2.58 Theorem (Taylor's Theorem with Integral Remainder II). Suppose that f is of class C^{k+1} $(k \geq 1)$ on an interval $I \subset \mathbb{R}$, and $a \in I$. Then, the remainder $R_{a,k}$ is given by

$$R_{a,k}(h) = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} [f^{(k)}(a+th) - f^{(k)}(a)]\, dt$$

2.60 Corollary. If f is of class C^k on I, then $R_{a,k}(h)/h^k \to 0$ as $h \to 0$.

2.61 Corollary. If f is a class C^{k+1} on I and $|f^{(k+1)}(x)| \leq M$ for $x \in I$, then

$$|R_{a,k}(h)| \leq \frac{M}{(k+1)!}|h|^{k+1} \quad (a+h \in I)$$

2.62 Lemma. Suppose g is $k+1$ times differentiable on $[a,b]$. If $g(a) = g(b)$ and $g^{(j)}(a) = 0$ for $1 \leq j \leq k$, then there is a point $c \in (a,b)$ such that $g^{(k+1)}(c) = 0$.

2.63 Theorem (Taylor's Theorem with Lagrange's Remainder). Suppose f is $k+1$ times differentiable on an interval $I \in \mathbb{R}$, and $a \in I$. For each $h \in R$ such that $a + h \in I$, there is a point c between 0 and h such that

$$R_{a,k}(h) = f^{(k+1)}(a+c)\frac{h^{k+1}}{(k+1)!}$$

2.65 Proposition. The Taylor polynomials of degree k about $a = 0$ of the functions

$$e^x, \qquad \cos x, \qquad \sin x, \qquad (1-x)^{-1}$$

are, respectively

$$\sum_{j=0}^{k} \frac{x^k}{j!}, \qquad \sum_{j=0}^{k} \frac{(-1)^j x^{2j}}{(2j)!}, \qquad \sum_{j=0}^{k} \frac{(-1)^j x^{2j+1}}{(2j+1)!}, \qquad \sum_{j=0}^{k} x^j$$

2.68 Theorem (Taylor's Theorem in Several Variables). Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is of class C^k on an open convex set S. If $\mathbf{a} \in S$ and $\mathbf{a}+\mathbf{h} \in S$, then

$$f(\mathbf{a}+\mathbf{h}) = \sum_{|\alpha|\le k} \frac{\partial^\alpha f(\mathbf{a})}{\alpha!}\mathbf{h}^\alpha + R_{a,k}(\mathbf{h})$$

where

$$R_{\mathbf{a},k}(h) = k \sum_{|\alpha|\le k} \frac{\mathbf{h}^\alpha}{\alpha!} \int_0^1 (1-t)^{k-1}[\partial^\alpha f(\mathbf{a}+t\mathbf{h}) - \partial^\alpha f(\mathbf{a})]\, dt$$

If f is a of class C^{k+1} on S, we also have

$$R_{\mathbf{a},k}(h) = (k+1) \sum_{|\alpha|\le k+1} \frac{\mathbf{h}^\alpha}{\alpha!} \int_0^1 (1-t)^k \partial^\alpha f(\mathbf{a}+t\mathbf{h})\, dt$$

and for some $c \in (0,1)$

$$R_{\mathbf{a},k}(h) = \sum_{|\alpha|\le k+1} \partial^\alpha f(\mathbf{a}+c\mathbf{h}) \frac{\mathbf{h}^\alpha}{\alpha!}$$

2.75 Corollary. If f is of class C^k on S, then $R_{\mathbf{a},k}(\mathbf{h})/|\mathbf{h}|^k \to 0$ as $\mathbf{h} \to \mathbf{0}$. If f is of class C^{k+1} on S and $|\partial^\alpha f(\mathbf{x})| \le M$ for $x \in S$ and $|\alpha| = k+1$, then

$$|R_{\mathbf{a},k}(\mathbf{h})| \le \frac{M}{(k+1)!}||\mathbf{h}||^{k+1}$$

where $||\mathbf{h}|| = |h_1| + |h_2| + \ldots + |h_n|$.

2.76. Lemma. If $P(\mathbf{h})$ is a polynomial of degree $\le k$ that vanishes to order $>k$ as $\mathbf{h} \to \mathbf{0}$, i.e. $P(\mathbf{h})/|\mathbf{h}|^k \to 0$, then $P \equiv 0$.

2.77 Theorem. Suppose f is of class $C^{(k)}$ near $\mathbf{a}$. If $f(\mathbf{a}+\mathbf{h}) = Q(\mathbf{h}) + E(\mathbf{h})$ where Q is a polynomial of degree $\le k$ as $E(\mathbf{h})/|\mathbf{h}|^k \to 0$ as $\mathbf{h} \to 0$, then Q is the Taylor polynomial $P_{\mathbf{a},k}$.

2.8 Critical Points

Definition. Suppose f is a differentiable function on some open set $S \subset \mathbb{R}^n$. The point $\mathbf{a} \in S$ is called a **critical point** of f if $\nabla f(\mathbf{a}) = \mathbf{0}$.

2.78 Proposition. If f has a local maximum or minimum at $\mathbf{a}$ and f is differentiable at $\mathbf{a}$ then $\nabla f(\mathbf{a}) = \mathbf{0}$.

Definition. Suppose f is a real-valued function of class C^2 on some open set $S \subset \mathbb{R}$ and f has a critical point at $\mathbf{a}$. It is often to examine the **Hessian** of f at $\mathbf{a}$:

$$H = H(\mathbf{a}) = \begin{pmatrix} \partial_1^2 f(\mathbf{a}) & \partial_1\partial_2 f(\mathbf{a}) & \cdots & \partial_1\partial_n f(\mathbf{a}) \\ \partial_2\partial_1 f(\mathbf{a}) & \partial_2^2 f(\mathbf{a}) & \cdots & \partial_2\partial_n f(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_n\partial_1 f(\mathbf{a}) & \partial_n\partial_2 f(\mathbf{a}) & \cdots & \partial_n^2 f(\mathbf{a}) \end{pmatrix}$$

2.80 Equation. We can use the Hessian to express f around $\mathbf{a}$.

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \frac{1}{2}H\mathbf{h}\cdot\mathbf{h} + R_{\mathbf{a},2}(\mathbf{h})$$

2.81 Theorem. Suppose f is of class C^2 at $\mathbf{a}$ and that $\nabla f(\mathbf{a}) = 0$, and let H be the Hessian matrix. For f to have a local minimum at $\mathbf{a}$, it is necessary for the eigenvalues of H all to be nonnegative and sufficient for them all to be strictly positive. For f to have a local maximum at $\mathbf{a}$, it is necessary for the eigenvalues of H all to be nonpositive and sufficient for them all to be strictly negative.

Definition. If there are two eigenvalues of oppositive signs, then f is said to have a **saddle point**. If there is a zero eigenvalue, then H is singular or **degenerate**.

2.82 Theorem Suppose f is of class C^2 on an open set in $\mathbb{R}^2$ containing the point $\mathbf{a}$, and suppose $\nabla f(\mathbf{a}) = \mathbf{0}$. Let $\alpha = \partial_1^2 f(\mathbf{a}), \beta = \partial_1\partial_2 f(\mathbf{a}), \gamma = \partial_2^2 f(\mathbf{a})$. Then,

(a) If $\alpha\gamma - \beta^2 < 0$, f has a saddle point at $\mathbf{a}$.

(b) If $\alpha\gamma - \beta^2 > 0$ and $\alpha < 0$, f has a local minimum at $\mathbf{a}$.

(c) If $\alpha\gamma - \beta^2 > 0$ and $\alpha < 0$, f has a local maximum at $\mathbf{a}$.

(d) If $\alpha\gamma - \beta^2 = 0$, f has a saddle point at $\mathbf{a}$.

2.9 Extreme Value Problems

2.83 Theorem. Let f be a continuous function on an unbounded closed set $S \subset \mathbb{R}^n$.

(a) If $f(\mathbf{x}) \to +\infty$ as $|\mathbf{x}| \to \infty$ $(\mathbf{x} \in S)$, then f has an absolute minimum but no absolute maximum on S.

(b) If $f(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ $(\mathbf{x} \in S)$ and there is a point $\mathbf{x}_0 \in S$ where $f(\mathbf{x}_0) > 0$ (resp. $f(\mathbf{x}_0) < 0$), then f has an absolute minimum (resp. minimum) on S.

Definition. Suppose we wish to maximize or minimize a differentiable functon f on the set

$$S = \{\mathbf{x} : G(\mathbf{x}) = 0\}$$

where G is of class C^1 and $\nabla G(\mathbf{x}) \neq \mathbf{0}$ on S. **Lagrange's method** requires us to solve the equation

$$\nabla f(\mathbf{a}) = \lambda \nabla G(\mathbf{a}) \text{ for some } \lambda \in \mathbb{R}$$

The parameter λ is called the **Lagrange's multiplier**. Similarly, if the constraint were defined with two functions

$$S = \{\mathbf{x} : G_1(\mathbf{x}) = G_2(\mathbf{x}) = 0\}$$

Then, the equation becomes

$$\nabla f(\mathbf{a}) = \lambda \nabla G_1(\mathbf{a}) + \mu \nabla G_2(\mathbf{a})$$

2.10 Vector-Valued Functions and Their Derivatives

Definition. A mapping $\mathbf{f}$ from an open set $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is said to be **differentiable** at $\mathbf{a} \in S$ if there is an $m \times n$ matrix L such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|\mathbf{f}(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - L\mathbf{h}|}{|\mathbf{h}|} = 0$$

The matrix L is unique and is called the (**Fréchet**) **derivative** of $\mathbf{f}$ at $\mathbf{a}$ and is denoted $D\mathbf{f}(\mathbf{a})$.

2.85 Proposition. An $\mathbb{R}^m$-valued function $\mathbf{f}$ is differentiable at $\mathbf{a}$ precisely when each of its components $f_1, \ldots, f_m$ is differentiable at $\mathbf{a}$. In this case, $D\mathbf{f}(\mathbf{a})$ is the matrix whose jth row vector $\nabla f_j(\mathbf{a})$. In other words,

$$D\mathbf{f} = \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f_m/\partial x_1 & \cdots & \partial f_m/\partial x_n \end{pmatrix}$$

2.86 Theorem (Chain Rule III). Suppose $\mathbf{g} : \mathbb{R}^k \to \mathbb{R}^n$ is differentiable at $\mathbf{a} \in \mathbb{R}^k$ and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{g}(\mathbf{a}) \in \mathbb{R}^n$. Then $\mathbf{H} = \mathbf{f} \circ \mathbf{g} : \mathbb{R}^k \to \mathbb{R}^m$ is differentiable at $\mathbf{a}$ and

$$D\mathbf{H}(\mathbf{a}) = D\mathbf{f}(\mathbf{g}(\mathbf{a}))D\mathbf{g}(\mathbf{a})$$

where the expression on the right is the product of the matrices $D\mathbf{f}(\mathbf{g}(\mathbf{a}))$ and $D\mathbf{g}(\mathbf{a})$.

Definition. Suppose $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping. Then, the **norm** is smallest constant C such that $|A\mathbf{x}| \leq C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$. The norm of A is denoted by $||A||$. Thus,

$$|A\mathbf{x}| \leq ||A||\,|\mathbf{x}| \qquad (\mathbf{x} \in \mathbb{R}^n)$$

2.88 Theorem. Suppose $\mathbf{f}$ is a differentiable $\mathbb{R}^m$-valued function on an open convex set $S \in \mathbb{R}^n$, and suppose that $||D\mathbf{f}(\mathbf{x})|| \leq M$ for all $\mathbf{x} \in S$. Then,

$$|\mathbf{f}(\mathbf{b}) - \mathbf{f}(\mathbf{a})| \leq M|\mathbf{b} - \mathbf{a}| \text{ for all } \mathbf{a}, \mathbf{b} \in S$$

3 The Implicit Function Theorem and Its Applications

3.1 The Implicit Function Theorem

3.1 Theorem (Implicit Function Theorem for a Single Equation). Let $F(\mathbf{x}, y)$ be a function of class C^1 on some neighborhood of a point $(\mathbf{a}, b) \in \mathbb{R}^{n+1}$. Suppose that $F(\mathbf{a}, b) = 0$ and $\partial_y F(\mathbf{a}, b) \neq 0$. Then there exist positive numbers r_0, r_1 such that the following conclusions are valid.

(a) For each $\mathbf{x}$ in the ball $|\mathbf{x} - \mathbf{a}| < r_0$ there is a unique $|y - b| < r_1$ and $F(\mathbf{x}, y) = 0$. We denote this y by $f(\mathbf{x})$; in particular, $f(\mathbf{a}) = b$

(b) The function f thus defined for $|\mathbf{x} - \mathbf{a}| < r_0$ is of class C^1, and its partial derivative are given by

$$\partial_j f(\mathbf{a}) = -\frac{\partial_j F(\mathbf{x}, f(\mathbf{x}))}{\partial_y F(\mathbf{x}, f(\mathbf{x}))}$$

3.3 Corollary. Let F be a function of class C^1 on $\mathbb{R}^n$, and let $S = \{\mathbf{x} : F(\mathbf{x}) = 0\}$. For every $\mathbf{a} \in S$ such that $\nabla F\mathbf{a}) \neq \mathbf{0}$ there is a neighborhood N of $\mathbf{a}$ such that $S \cap N$ is the graph of a C^1 function.

3.9 Theorem (The Implicit Function Theorem for a System of Equations). Let $\mathbf{F}(\mathbf{x}, \mathbf{y})$ be an $\mathbb{R}^k$-valued function of class C^1 on some neighborhood of a point $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{n+k}$ and let $B_{ij} = (\partial F_i/\partial y_j)(\mathbf{a}, \mathbf{b})$. Suppose that $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$ and $\det(B) \neq 0$. Then there exist positive numbers r_0, r_1 such that the following conclusions are valid.

(a) For each $\mathbf{x}$ in the ball $|\mathbf{x} - \mathbf{a}| < r_0$ there is a unique $|\mathbf{y} - \mathbf{b}| < r_1$ and $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$. We denote this $\mathbf{y}$ by $f(\mathbf{x})$; in particular, $f(\mathbf{a}) = \mathbf{b}$

(b) The function $\mathbf{f}$ thus defined for $|\mathbf{x} - \mathbf{a}| < r_0$ is of class C^1, and its partial derivatives $\partial_{x_j}\mathbf{f}$ can be computed by differentiating the equations $\mathbf{F}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0}$ with respect to x_j and solving the resulting linear system of equations for $\partial_{x_j} f_1, \ldots, \partial_{x_j} f_k$.

3.A Theorem (Implicit Function Theorem for a Linear System). Let $L : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and assume that L_x is invertible. Then for every $\mathbf{k} \in \mathbb{R}^m$, there exists a unique $\mathbf{h} \in \mathbb{R}^n$ such that $L(\mathbf{h}, \mathbf{k}) = 0$, and h can be written as $\mathbf{h} = -(L_x)^{-1} L_y \mathbf{k}$.

3.B Theorem (Implicit Function Theorem). Let f be a C^1 mapping on an open set $S \in \mathbb{R}^{n+m} \to \mathbb{R}^n$, such that $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$ for some point $(\mathbf{a}, \mathbf{b}) \in S$. Let $L = \mathbf{f}'(\mathbf{a}, \mathbf{b})$ and assume L_x is invertible. Then there exists open sets $U \in \mathbb{R}^{n+m}$ and $W \in \mathbb{R}^m$ with $(\mathbf{a}, \mathbf{b}) \in U$ and $\mathbf{b} \in W$ such that to every $\mathbf{y} \in W$, there exists a unique $\mathbf{x}$ such that

$$(\mathbf{x}, \mathbf{y}) \in U \quad \text{and} \quad \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$$

If $\mathbf{x} = \mathbf{g}(\mathbf{y})$, then $\mathbf{g} \in C^1(W)$, $\mathbf{g}(\mathbf{b}) = \mathbf{a}$, $\mathbf{f}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$, and $\mathbf{g}'(\mathbf{b}) = -(L_x)L_y$.

3.2 Curves in the Plane

Definition. There are three ways common ways of representing a smooth curve in $\mathbb{R}^2$

(i) as the graph of the function, $y = f(x)$ or $x = f(y)$, where f is of class C^1;

(ii) as the locus of an equation $F(x, y) = 0$, where F is of class C^1;

(iii) parametrically, as the range of a C^1 function $\mathbf{f} : (a, b) \in \mathbb{R}^2$.

3.11 Theorem. The representations (ii) and (iii) are locally equivalent to (i) if (a) and (b) are satisfied, respectively.

(a) Let F be a real-valued function of class C^1 on an open set in $\mathbb{R}^2$, and let $S = \{(x, y) : F(x, y) = 0\}$. If $\mathbf{a} \in S$ and $\nabla F(\mathbf{a}) \neq \mathbf{0}$, there is neighborhood N of $\mathbf{a}$ in $\mathbb{R}^2$ such that $S \cap N$ is the graph of a C^1 function f (either $y = f(x)$ or $x = f(y)$).

(b) Let $\mathbf{f} : (a, b) \to \mathbb{R}^2$ be a function of class C^1. If w$\mathbf{f}'(t_0) \neq \mathbf{0}$, there is an open interval I containing t_0 such that the set $\{\mathbf{f}(t) : t \in I\}$ is the graph of a C^1 function f (either $y = f(x)$ or $x = f(y)$).

Definition. More formally, a set $S \in \mathbb{R}^2$ is a **smooth curve** if (a) S is connected, and (b) every $\mathbf{a} \in S$ satisfies Theorem 3.11.

3.3 Surfaces and Curves in Space

Definition. There are three ways common ways of representing a smooth surface in $\mathbb{R}^3$.

(i) as the graph of the function, $z = f(x, y)$, $y = f(x, z)$ or $x = f(y, z)$, where f is of class C^1;

(ii) as the locus of an equation $F(x, y, z) = 0$, where F is of class C^1;

(iii) parametrically, as the range of a C^1 function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$.

3.15 Theorem. The representations (ii) and (iii) are locally equivalent to (i) if (a) and (b) are satisfied, respectively.

(a) Let F be a real-valued function of class C^1s on an open set in $\mathbb{R}^3$, and let $S = \{(x, y, z) : F(x, y, z) = 0\}$. If $\mathbf{a} \in S$ and $\nabla F(\mathbf{a}) \neq \mathbf{0}$, there is neighborhood N of $\mathbf{a}$ in $\mathbb{R}^3$ such that $S \cap N$ is the graph of a C^1 function f (either $z = f(x, y)$, $y = f(x, z)$, or $x = f(y, z)$).

(b) Let $\mathbf{f}$ be a C^1 mapping from an open set in $\mathbb{R}^2$ into $\mathbb{R}^3$. If $[\partial_u \mathbf{f} \times \partial_v \mathbf{f}](u_0, v_0) \neq \mathbf{0}$, there is a neighborhood N of (u_0, v_0) in $\mathbb{R}^2$ such that the set $\{\mathbf{f}(u, v) : (u, v) \in N\}$ is the graph of a C^1 function.

Definition. A set $S \in \mathbb{R}^3$ is a **smooth surface** if (a) S is connected, and (b) every $(u, v) \in S$ satisfies Theorem 3.15.

Definition. There are three ways common ways of representing a smooth curve in $\mathbb{R}^3$

(i) as the graph of the function, $y = f(x)$ or $z = g(x)$ (or similar expressions with coordinates permuted) where f and g are C^1 functions;

(ii) as the locus of an equation $F(x, y, z) = G(x, y, z) = 0$, where F and G are C^1 functions;

(iii) parametrically, as the range of a C^1 function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$.

Representation (ii) is locally equivalent to (i) when $\nabla F(\mathbf{x})$ and $\nabla G(\mathbf{x})$ are linearly independent at every $\mathbf{x}$ where $F(\mathbf{x}) = G(\mathbf{x}) = 0$, and (iii) is locally equivalent to (i) when $\mathbf{f}'(t) \neq \mathbf{0}$.

3.4 Transformations and Coordinate Systems

3.18 Theorem (The Inverse Mapping Theorem). Let U and V be open sets in $\mathbb{R}^n$, $\mathbf{a} \in U$, and $\mathbf{b} = \mathbf{f}(\mathbf{a})$. Suppose that $\mathbf{f} : U \to V$ is a mapping of class C^1 and the Fréchet's derivative $D\mathbf{f}(\mathbf{a})$ is invertible. Then, there exist neighborhoods $M \subset U$ and $N \subset V$ of $\mathbf{a}$ and $\mathbf{b}$, respectively, so that $\mathbf{f}$ is a one-to-one map from M onto N, and the inverse map $\mathbf{f}^{-1}$ from N to M is also of class C^{-1}. Moreover, if $\mathbf{y} = \mathbf{f}(\mathbf{x}) \in N, D(\mathbf{f}^{-1})(\mathbf{y}) = [D\mathbf{f}(\mathbf{x})]^{-1}$.

4 Integral Calculus

4.1 Integration on a Line

Definition. A **partition** P of the interval $[a, b]$ is a subdivision of $[a, b]$ into non-overlapping subintervals specified by $x_1, \ldots, x_{j-1}$ with $x_0 = a$ and $x_j = b$. We denote a partition with $P = \{x_0, x_1, \ldots, x_J\}$. We say that P' is a **refinement** of P if P' is obtained from P by adding more subdivision points, that is $P \subset P'$.

Definition. Given a partition P and a bounded real-valued function on $[a, b]$, the minimum and maximum of f on each subinterval is given by

$$m_j = \inf\{f(x) : x_{j-1} \le x \le x_j\} \qquad M_j = \sup\{f(x) : x_{j-1} \le x \le x_j\}$$

and the **lower Riemann sum** $s_P f$ and the **upper Riemann sum** $S_P f$ corresponding to the partition P are defined by

$$s_P f = \sum_1^J m_j(x_j - x_{j-1}) \qquad S_P f = \sum_1^J M_j(x_j - x_{j-1})$$

4.3 Lemma. If P' is a refinement of P, then $s_{P'} f \ge s_P f$ and $S_{P'} f \le S_P f$.

4.4 Lemma. If P and Q are partitions of $[a, b]$, then $s_P f \le S_Q f$.

Definition. The **lower integrals** and **upper integrals** of f on $[a, b]$ by

$$\underline{I}_a^b(f) = \sup s_P f \qquad \overline{I}_a^b(f) = \inf S_P f$$

If the upper and lower integrals coincide, f is called **Riemann integrable** on $[a, b]$ and the common value of the upper and lower integrals is the **Riemann integral** $\int_a^b f(x)\, dx$.

4.5 Lemma. If f is a bounded function on $[a, b]$, the following conditions are equivalent:

(a) f is integrable on $[a,b]$

(b) For every $\epsilon > 0$ there is a partition P of $[a,b]$ such that $S_P f - s_P f < \epsilon$.

4.6 Theorem.

(a) Suppose $a < b < c$. If f is integrable on $[a,b]$ and on $[b,c]$, then f is integrable on $[a,c]$ and

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

(b) If f and g are integrable on $[a,b]$, then so is $f+g$ and

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Equation. For any $a, b \in \mathbb{R}$

$$\int_a^b f(x)\,dx = -\int_a^b f(x)\,dx$$

4.9 Theorem. Suppose f is integrable on $[a,b]$.

(a) If $c \in \mathbb{R}$, then cf is integrable on $[a,b]$, and $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

(b) If $[c,d] \subset [a,b]$, then f is integrable $[c,d]$.

(c) If g is integrable on $[a,b]$ and $f(x) \leq g(x)$ for $x \in [a,b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

(d) $|f|$ is integrable on $[a,b]$, and $|\int_a^b f(x)\,dx| \leq \int_a^b |f(x)|\,dx$.

4.10 Theorem. If f is bounded and monotone on $[a,b]$, then f is integrable on $[a,b]$.

4.11 Theorem. If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

4.12 Theorem. If f is bounded on $[a,b]$ and continuous at all except finitely many points in $[a,b]$, then f is integrable on $[a,b]$.

Definition. A set $Z \subset \mathbb{R}$ is said to have **zero content** if for any $\epsilon > 0$ there is finite collection of intervals $I_1, \ldots I_L$ such that (i) $Z \subset \bigcup_1^L I_l$, and (ii) the sum of the lengths of the I_l's is less than ϵ.

4.13 Theorem. If f is bounded on $[a,b]$ and the set of points in $[a,b]$ at which f is discontinuous has zero content, then f is integrable on $[a,b]$.

4.14 Proposition. Suppose f and g are integrable on $[a,b]$ and $f(x) = g(x)$ for all except finitely many points $x \in [a,b]$. Then $\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$.

4.15 Theorem (The Fundamental Theorem of Calculus).

(a) Let f be an integrable function on $[a,b]$. For $x \in [a,b]$, let $F(x) = \int_a^x f(t)\,dt$. Then F is continuous on $[a,b]$; moreover, $F'(x)$ exists and equals $f(x)$ at every x at which f is continuous.

(b) Let F be a continuous function on $[a,b]$ that is differentiable except perhaps at finitely many points in $[a,b]$, and let f be a function on $[a,b]$ that agrees with F' at all points where the latter is defined. If f is integrable on $[a,b]$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.

Definition. If $P = \{x_0, \dots, x_J\}$ is a partition of $[a,b]$ and t_j is any point in the interval $[x_{j-1}, x_j]$, then the quantity

$$\sum_1^J f(t_j)(x_j - x_{j-1})$$

is called a **Riemann sum** for f associated to the partition P.

4.16 Proposition. Suppose f is integrable on $[a,b]$. Given $\epsilon > 0$, there exists $\delta > 0$ such that if $P = \{x_0, \dots, x_J\}$ is any partition of $[a,b]$ satisfying

$$\max_{1 \le j \le J}(x_j - x_{j-1}) < \delta,$$

the sums $s_P f$ and $S_P f$ differ from $\int_a^b f(x)\,dx$ by at most ϵ.

4.2 Integration in Higher Dimensions

Definition. A **partition** P of the rectangle $R = [a,b] \times [c,d]$ is a subdivision of R into non-overlapping subrectangles specified by $x_0, \dots x_J$ and $y_0, \dots, y_K$. Thus, we write $P = \{x_0, \dots, x_J; y_0, \dots, y_K\}$. Each subtrectangle is defined by $R_{jk} = [x_{j-1}, x_j] \times [y_{k-1}, y_k]$ where the are $\Delta A_{jk} = (x_j - x_{j-1})(y_k - y_{k-1})$.

Definition. Given a partition P and a bounded real-valued function on the rectangle R, the minimum and maximum of f on each subrectangle R_{jk} is given by

$$m_{jk} = \inf\{f(x,y) : (x,y) \in R_{jk}\} \qquad M_{jk} = \sup\{f(x,y) : (x,y) \in R_{jk}\}$$

and the **lower Riemann sum** $s_P f$ and the **upper Riemann sum** $S_P f$ corresponding to the partition P are defined by

$$s_P f = \sum_1^J \sum_1^K m_{jk} \Delta A_{jk} \qquad S_P f = \sum_1^J \sum_1^K M_{jk} \Delta A_{jk}$$

Definition. The **lower integrals** and **upper integrals** of f on R by

$$\underline{I}_R(f) = \sup s_P f \qquad \overline{I}_R(f) = \inf S_P f$$

If the upper and lower integrals coincide, f is called **Riemann integrable** on R, and the common value of the upper and lower integrals is the **Riemann integral**, denoted by

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$

The characteristic function of S is the function χ_S defined by

$$\chi_S(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in S \\ 0 & \text{otherwise} \end{cases}$$

Suppose S is a bounded set of $\mathbb{R}^2$ and f is a bounded function on $\mathbb{R}^2$. Let R be the rectangle that contains S. We say f is **integrable** on S if $f\chi_S$ is integrable on R. We define the integral of f over S by

$$\iint_S f\,dA = \iint_R f\chi_S\,dA$$

4.17 Theorem.

(a) If f_1 and f_2 are integrable on the bounded set S and $c_1, c_2 \in \mathbb{R}$, then $c_1 f_1 + c_2 f_2$ is integrable on S, and

$$\iint_S [c_1 f_1 + c_2 f_2]\, dA = c_1 \iint_S f_1\, dA + c_2 \iint_S f_2\, dA$$

(b) Let S_1 and S_2 be bounded sets with no points in common and let f be a bounded function. If f is integrable on S_1 and on S_2, then f is integrable on $S_1 \cup S_2$, and

$$\iint_{S_1 \cup S_2} f\, dA = \iint_{S_1} f\, dA + \iint_{S_2} f\, dA$$

(c) If f and g are integrable on S and $f(\mathbf{x}) \leq g(\mathbf{x})$ for $\mathbf{x} \in S$, then $\iint_S f\, dA \leq \iint_S g\, dA$.

(d) If f is integrable on S, then so is $|f|$, and $|\iint_S f\, dA| \leq \iint_S |f|\, dA$.

Definition. A set $Z \subset \mathbb{R}^2$ is said to have **zero content** if for any $\epsilon > 0$ there is finite collection of rectangles $R_1, \ldots R_M$ such that (i) $Z \subset \bigcup_1^M R_m$, and (ii) the sum of the areas of the R_m's is less than ϵ.

4.18 Theorem. Suppose f is a bounded function on the rectangle R. If the set of points in R at which f is discontinuous has zero content, then f is integrable on R.

4.19 Proposition.

(a) If $Z \subset \mathbb{R}^2$ has zero content and $U \subset Z$, then U has zero content.

(b) If $Z_1, \ldots, Z_k$ have zero content, then so does $\bigcup_1^k Z_j$.

(c) If $\mathbf{f} : (a_0, b_0) \to \mathbb{R}^2$ is of class C^1, then $\mathbf{f}([a, b])$ has zero content whenever $a_0 < a < b < b_0$.

4.20 Lemma. The function χ_S is discontinuous at $\mathbf{x}$ if and only if $\mathbf{x}$ is in the boundary of S.

Definition. A set $S \subset \mathbb{R}^2$ is **(Jordan) measurable** if it is bounded and its boundary has zero content.

4.21 Theorem. Let S be a measureable subset of $\mathbb{R}^2$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded and the set of points in S at which f is discontinuous has zero content. Then f is integrable on S.

4.22 Proposition. Suppose $Z \subset \mathbb{R}^2$ has zero content. If $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded, then f is integrable on Z and $\iint_Z f\, dA = 0$.

4.23 Corollary.

(a) Suppose that f is integrable on the set $S \in \mathbb{R}^2$. If $g(\mathbf{x}) = f(\mathbf{x})$ except for $\mathbf{x}$ in a set of zero content, then g is integrable on S and $\iint_S g\, dA = \iint_S f\, dA$.

(b) Suppose that f is integrable on S and on T and $S \cap T$ has zero content. Then F is integrable on $S \cup T$, and $\iint_{S \cup T} f\, dA = \iint_S f\, dA + \iint_T f\, dA$.

Definition. If S is a Jordan measurable set in the plane, then its **area** is the integral over S of the constant function $f(\mathbf{x}) = 1$

$$\text{area}(S) = \iint_S 1\, dA = \iint \chi_S\, dA$$

4.24 Theorem (The Mean Value Theorem for Integrals). Let S be a compact, connected, measurable subset of $\mathbb{R}^n$, and let f and g be continuous functions on S with $g \geq 0$. Then there is a point $\mathbf{a} \in S$ such that

$$\int \cdots \int_S f(\mathbf{x}) g(\mathbf{x})\, d^n\mathbf{x} = f(\mathbf{a}) \int \cdots \int_S g(\mathbf{x})\, d^n\mathbf{x}$$

4.25 Corollary. Let S be a compact, connected, measurable subset of $\mathbb{R}^n$, and let f be a continuous function on S. Then there is a point $\mathbf{a} \in S$ such that

$$\int \cdots \int_S f(\mathbf{x})\, d^n\mathbf{x} = f(\mathbf{a})|S|$$

where $|S|$ denoted the n-dimensional volume of S.

4.3 Multiple Integrals and Iterated Integrals

4.26 Theorem (Fubini's Theorem). Let $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, and let f be an integrable function on R. Suppose that for each $y \in [c, d]$, the function f_y defined by $f_y = f(x, y)$ is integrable on $[a, b]$, and the function $g(y) = \int_a^b f(x, y)\, dx$ is integrable on $[c, d]$. Then

$$\iint_R f\, dA = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

Likewise, if $f_x = f(x, y)$ is integrable on $[c, d]$ for each $x \in [a, b]$, and $h(x) = \int_c^d f(x, y)\, dy$ is integrable on $[a, b]$, then

$$\iint_R f\, dA = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

Definition. The integrals on the right side of the equations in Theorem 4.26 are called **iterated integrals**. It is customary to omit the brackets and write, for example

$$\int_c^d \int_a^b f(x, y)\, dx\, dy$$

4.4 Changes of Variables for Multiple Integrals

Equation 4.32 (Change of variable in one dimension).

$$\int_a^b f(g(u))g'(u)\, du = \int_{g(a)}^{g(b)} f(x)\, dx$$

Equation 4.34 (Equation 4.32, rewritten).

$$\int_I f(x)\, dx = \int_{g^{-1}(I)} f(g(u))|g'(u)|\, du$$

4.37 Theorem. Let A be an invertible $n \times n$ matrix, and let $\mathbf{G}(\mathbf{u}) = A\mathbf{u}$ be the corresponding linear transformation of $\mathbb{R}^n$. Suppose S is a measurable region in $\mathbb{R}^n$ and f is the integrable function on S. Then $\mathbf{G}^{-1}(S) = \{A^{-1}\mathbf{x} : \mathbf{x} \in S\}$ is measurable and $f \circ \mathbf{G}$ is integrable on $\mathbf{G}^{-1}(S)$, and

$$\int \cdots \int_S f(\mathbf{x})\, d^n\mathbf{x} = |\det A| \int \cdots \int_{\mathbf{G}^{-1}(S)} f(A\mathbf{u})\, d^n\mathbf{u}$$

4.41 Theorem. Given open sets U and V in $\mathbb{R}^n$, let $\mathbf{G} : U \to V$ be a one-to-one transformation of class C^1 whose derivative $D\mathbf{G}(\mathbf{u})$ is invertible for all $\mathbf{u} \in U$. Suppose that $T \subset U$ and $S \subset V$s are measurable sets such that $\mathbf{G}(T) = S$. If f is an integrable function on S, then $f \circ \mathbf{G}$ is integrable on T, and

$$\int \cdots \int_S f(\mathbf{x})\, d^n\mathbf{x} = \int \cdots \int_{\mathbf{G}^{-1}(S)} f(\mathbf{G}(u)|\det D\mathbf{G}(\mathbf{u})|\, d^n\mathbf{u}$$

4.5 Functions defined by Integrals

4.6 Improper Integrals

5 Line and Surface Integrals; Vector Analysis

5.1 Arc Length and Line Integrals

Definition. Let C be a smooth curve in $\mathbb{R}^n$. We consider two points $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$. The distance between them is given by

$$ds = |d\mathbf{x}| = \sqrt{dx_1^2 + \ldots + dx_n^2}$$

Now, let $\mathbf{x} = \mathbf{g}(t), a \leq t \leq b$ be the parameteric form of C. Then the points $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$ are given by $g(t)$ and $g(t+dt)$, so

$$d\mathbf{x} = \mathbf{g}(t+dt) - \mathbf{g}(t) = \mathbf{g}'(t)\ dt = \left(\frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}\right)\ dt$$

and

$$ds = |d\mathbf{x}| = |\mathbf{g}'(t)|\ dt = \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \ldots + \left(\frac{dx_n}{dt}\right)^2}\ dt$$

If we sum up the distances between two close points across all of C, then the total length of the curve, or the **arc length** is given by

$$s = \int_C ds = \int_a^b |\mathbf{g}'(t)|\ dt$$

Definition. We define the **orientation** of C to be the direction of travel along C when we choose some parameterization $\mathbf{x} = \mathbf{g}(t)$.

Definition. A curve in $\mathbb{R}^n$ is defined to be **piecewise smooth** if it can be constructed using finitely many smooth curves joined end-to-end. In other words, there exists some parameterization $\mathbf{x} = \mathbf{g}(t)$ where $\mathbf{g} : [a,b] \to \mathbb{R}^n$ is (i) continuous, and (ii) its derivative exists and is continuous except at finitely many points.

Definition. Let f be a continuous function whose domain includes some piecewise smooth curve C in $\mathbb{R}^n$. If C is parameterized by $\mathbf{x} = \mathbf{g}(t), a \leq t \leq b$, we define the **line integral** of f to be

$$\int_C f\ ds = \int_a^b f(\mathbf{g}(t))|\mathbf{g}'(t)|\ dt$$

Line integrals also extend to vector fields. We define the line integral of a $\mathbb{R}^m$-valued vector field $\mathbf{F}$ to be

$$\int_C \mathbf{F}\ ds = \left(\int_C F_1\ ds, \ldots, \int_C F_n\ ds\right)$$

5.8 Proposition. If $\mathbf{F}$ is a continuous $\mathbb{R}^m$-valued function on $[a,b]$, then

$$\left|\int_a^b \mathbf{F}(t)\ dt\right| \leq \int_a^b |\mathbf{F}(t)|\ dt$$

Definition. If C is a piecewise smooth curve in $\mathbb{R}^n$ and $\mathbf{F}$ is a continuous vector field defined on some neighborhood of C in $\mathbb{R}^n$, the (scalar-valued) **line integral** of $\mathbf{F}$ over C is

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_C (F_1\ dx_1 + \ldots + F_n\ dx_n)$$

If we parameterically descibe C using $\mathbf{x} = \mathbf{g}(t), a \le t \le b$, then

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\ dt$$

We note that $\mathbf{F} \cdot d\mathbf{x}$ is the component of $\mathbf{F}$ tangent to the curve at every point. Thus

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_C F_{\text{tang}}\ ds$$

5.2 Green's Theorem

Definition. A **simple closed curve** in $\mathbb{R}^n$ is a curve whose starting and ending point coincide, but does not intersect itself anywhere else. We can parameterize the curve using $\mathbf{x} = \mathbf{g}(t), a \le t \le b$ such that $\mathbf{g}(a) = \mathbf{g}(b)$.

Definition. We will use the term **regular region** to mean a compact set in $\mathbb{R}^n$ that is the closure of its interior. In other words, a compact set $S \subset \mathbb{R}^n$ is a regular region if every neighborhood of every point on the boundary, contains points in the interior of S.

Definition. We say the a regular region $S \subset \mathbb{R}^2$ has a **piecewise smooth boundary** if the boundary of S is the finite union of disjoing, piecewise smooth simple closed curves. The **positive orientation** on ∂S is the orientation on each of the closed boundary curves such that S is on the left when traveling in the positive direction.

5.12 Theorem (Green's Theorem). Suppose S is a regular region in $\mathbb{R}^2$ with a piecewise smooth boundary. Suppose also that $\mathbf{F}$ is a vector field of class C^1 on $\overline{S}$. Then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right)\ dA$$

5.17 Corollary. Suppose S is a regular region in $\mathbb{R}^n$ with a piecewise smooth boundary, and let $\mathbf{n}(\mathbf{x})$ be the unit outward normal vector to ∂S at $\mathbf{x} \in S$. Suppose also that $\mathbf{F}$ is a vector field of class C^1 on $\overline{S}$. Then

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{n}\ ds = \iint_S \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} \right)\ dA$$

Definition. We define the **normal derivative** of f on ∂S to be $\nabla f \cdot \mathbf{n}$ and denote it $\partial f/\partial n$.

5.3 Surface Area and Surface Integrals

Definition. Let S be a smooth surface in $\mathbb{R}^3$. Assume that S is represented parameterically as the image of a connected open set W in the uv-plane under a one-to-one C^1 map $\mathbf{G} : W \to \mathbb{R}^3$.

$$\mathbf{x} = (x, y, z) = \mathbf{G}(u, v), \qquad (u, v) \in W$$

Consider some point (u,v) and some increments du and dv. Then

$$\mathbf{G}(u + du, v) - \mathbf{G}(u, v) = \frac{\partial \mathbf{G}}{\partial u}\ du \qquad \text{and} \qquad \mathbf{G}(u, v + dv) - \mathbf{G}(u, v) = \frac{\partial \mathbf{G}}{\partial v}\ dv$$

These vectors are tangent to S, so we can compute the area of the parallelogram they span.

$$dA = \left| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right|\ du\ dv$$

If R is a measureable set of W that maps to the surface S, then the surface area of S is given by

$$A = \iint_R \left| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right| \, du \, dv$$

Definition. Let S be a surface in $\mathbb{R}^3$ and $\mathbf{x} = \mathbf{G}(u, v)$ be a parameterization of S where $(u, v) \in W$. Then the **surface integral** of a scalar function f on S is

$$\iint_S f \, dA = \iint_W f(\mathbf{G}(u,v)) \left| \frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right| \, du \, dv$$

Now, suppose $\mathbf{F}$ is a continuous vector field defined on a neighborhood of S. The surface integral of $\mathbf{F}$ over S is defined to be

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = \iint_W \mathbf{F}(\mathbf{G}(u,v)) \cdot \left(\frac{\partial \mathbf{G}}{\partial u} \times \frac{\partial \mathbf{G}}{\partial v} \right) \, du \, dv$$

5.4 Vector Derivatives

Definition. The **gradient** of a C^1 function on $\mathbb{R}^n$ is a vector field defined by

$$\text{grad } f = \nabla f = (\partial_1 f, \ldots, \partial_n f)$$

Definition. If $\mathbf{F}$ is a C^1 vector field on an open subset of $\mathbb{R}^n$, the **divergence** of $\mathbf{F}$ is the function defined by

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \partial_1 F_1 + \ldots + \partial_n F_n$$

Definition. If $\mathbf{F}$ is a C^1 vector field on an open subset of $\mathbb{R}^3$, the **divergence** of $\mathbf{F}$ is the function defined by

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = (\partial_2 F_3 - \partial_3 F_3)\mathbf{i} + (\partial_3 F_1 - \partial_1 F_3)\mathbf{j} + (\partial_1 F_2 - \partial_2 F_1)\mathbf{k}$$

Definition. The **Laplacian** of f, usually denoted $\nabla^2 f$ or Δf, is defined as

$$\nabla^2 f = \Delta f = \text{div}(\text{grad } f) = \partial_1^2 f + \ldots + \partial_n^2 f$$

5.5 The Divergence Theorem

5.34 Theorem (The Divergence Theorem). Suppose R is a regular region in $\mathbb{R}^3$ with a piecewise smooth boundary ∂R, oriented so the positive normal points out of R. Suppose also that $\mathbf{F}$ is a vector field of class C^1 on R. Then

$$\iint_{\partial R} \mathbf{F} \cdot \mathbf{n} \, dA = \iiint_R \text{div } \mathbf{F} \, dV$$

Note that this theorem is also called Gauss's theorem or Ostrogradski's theorem.

5.37 Corollary (Green's Formulas). Suppose R is a regular region in $\mathbb{R}^3$ with a piecewise smooth boundary, and f and g are functions of class C^2 on $\overline{R}$. Then

$$\iint \partial R f \nabla g \cdot \mathbf{n} \, dA = \iiint_R (\nabla f \cdot \nabla g + f \nabla^2 g) \, dV$$

$$\iint \partial R (f \nabla g - g \nabla f) \cdot \mathbf{n} \, dA = \iiint_R (f \nabla^2 g - g \nabla^2 f) \, dV$$

5.6 Applications in Physics

5.7 Stokes's Theorem

5.52 Theorem (Stokes's Theorem). Let S be a surface in R^3 that is bounded by a piecewise smooth curve ∂S. We assume that S is oriented by a choice of normal vector field $\mathbf{n}$. Informally speaking, if we follow ∂S in the positive direction on the positive side of S, then S is on the left. Let $\mathbf{F}$ be a C^1 vector field defined on some neighborhood of S in $\mathbb{R}^3$. Then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dA$$

5.56 Corollary. If S is a closed surface (i.e. a surface with no boundary) in $\mathbb{R}^3$ with unit outward normal $\mathbf{n}$, and $\mathbf{F}$ is a C^1 vector field on S, then

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dA = 0$$

5.8 Integrating Vector Derivatives

5.59 Proposition. Suppose $\mathbf{G}$ is a continuous vector field on an open set R in $\mathbb{R}^n$. The following conditions are equivalent:

(a) If C_1 and C_2 are any two oriented curves in R with the same initial point and the same end point, then $\int_{C_1} \mathbf{G} \cdot d\mathbf{x} = \int_{C_2} \mathbf{G} \cdot d\mathbf{x}$.

(b) If C is any closed curve in R, $\int_C \mathbf{G} \cdot d\mathbf{x} = 0$.

5.60 Proposition. A continuous vector field $\mathbf{G}$ is an open set $R \subset \mathbb{R}^n$ is conservative if and only if $\mathbf{G}$ is the gradient of a C^1 function f on R.

5.62 Theorem. Suppose R is a convex open set in $\mathbb{R}^n$ and $\mathbf{G}$ is a C^1 vector field on R. If $\partial_k G_j - \partial_j G_k = 0$ for all $j \neq k$ (for $n = 3$, this means $\operatorname{curl} \mathbf{G} = \mathbf{0}$), then $\mathbf{G}$ is the gradient of a C^2 function on R.

5.63 Theorem. Suppose R is a convex open set in $\mathbb{R}^3$ and $\mathbf{G}$ is a C^1 vector field on R. If $\mathbf{G}$ satisfies $\operatorname{div} \mathbf{G} = 0$ on R, then $\mathbf{G}$ is the curl of a C^2 vector field on R.

5.64 Theorem. Let R be a bounded convex open set in $\mathbb{R}^3$. For any C^1 function g on $\overline{R}$ and any C^2 vector field $\mathbf{G}$ on $\overline{R}$ such that $\operatorname{div} \mathbf{G} = 0$, there is a C^2 vector field $\mathbf{G}$ on $\overline{R}$ such that $\operatorname{curl} \mathbf{F} = \mathbf{G}$ and $\operatorname{div} \mathbf{F} = g$ on R.

5.65 Proposition. Let R be a bounded convex open set in $\mathbb{R}^3$ with piecewise smooth boundary. Suppose $\mathbf{H}$ is a C^1 vector field on $\overline{R}$ such that $\operatorname{curl} \mathbf{F} = \mathbf{0}$ and $\operatorname{div} \mathbf{F} = 0$ on R and $\mathbf{F} \cdot \mathbf{n} = 0$ on ∂R. Then $\mathbf{H}$ vanishes identically on R.

5.9 Higher Dimensions and Differential Forms

6 Infinite Series

6.1 Definitions and Examples

Definition. An **infinite series** (or series, for short) is an expression of the form

$$\sum_0^\infty a_n = a_0 + a_1 + a_2 + \ldots$$

The **partial sum**, s_k, is the sum of the first k terms in the sequence. If $\{s_k\}$ converges to some limit S, then we say that the series is convergent, and write $\sum_0^\infty a_n = S$. Otherwise, we say that the series is divergent.

6.1 Theorem. Let a_n and b_n be sequences of real numbers.

(a) If the series $\sum_0^\infty a_n$ and $\sum_0^\infty b_n$ are convergent, with sums S and T, then $\sum_0^\infty (a_n + b_n)$ is convergent, with sum $S + T$.

(b) If the series $\sum_0^\infty a_n$ is convergent, with sum S, then for any $c \in \mathbb{R}$, the series $\sum_0^\infty$ is convergent, with sum cS.

(c) If the series $\sum_0^\infty a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. Equivalently, if $a_n \not\to 0$ as $n \to \infty$, then the series $\sum_0^\infty a_n$ is divergent.

Definition. A simple infinite series is the **geometric series** with multiple a and ratio x.

$$a + ax + ax^2 + \ldots = \sum_0^\infty ax^n$$

6.3 Theorem. The geometric series $\sum_0^\infty x^n$ converges if and only if $|x| < 1$, in which case the sum is $(1 - x)^{-1}$.

6.5 Definition. Recall the k-th order Taylor polynomial a C^∞ function. If we let $k \to \infty$, we obtain the **Taylor series** of f (centered at $x = 0$):

$$f(x) = \sum_0^\infty \frac{f^{(n)}(0)}{n!} x^n$$

6.6 Theorem. Let f be a function of class C^∞ on the interva $(-c, c)$ where $0 < c < \infty$.

(a) If there exist constants $a, b > 0$ such that $|f^{(k)}(x) \leq ab^k k!$ for all $|x| < c$ and $k \geq 0$, then (6.5) holds for $|x| < \min(c, b^{-1})$.

(b) If there exist constants $A, B > 0$ such that $|f^{(k)}(x)| \leq AB^k$ for all $|x| < c$ and $k \geq 0$, then (6.5) holds for $|x| < c$.

Definition. Let $\sum_0^\infty a_n$ be some series. Suppose that $\{b_n\}$ is the sequence such that $a_0 = b_0$ and $a_n = b_n - b_{n-1}$ for $n \geq 1$. Then

$$s_k = a_0 + a_1 + \ldots + a_k = b_0 + (b_1 - b_0) + \ldots + (b_k - b_{k-1}) = b_k$$

so the series $\sum_0^\infty a_n$ converges if and only if the sequence $\{b_n\}$ converges, in which case $\sum_0^\infty a_n = \lim b_n$. Such series are called **telescoping series**.

Definition. Given a sequence $\{a_n\}$ of numbers, let $\prod_1^k a_n$ denote the product of the numbers $a_1, \ldots, a_k$. The **infinite product** $\prod_1^\infty a_n$ is said to converge of the number P if the sequence of partial products converges to P:

$$\prod_1^\infty a_n = \lim_{k\to\infty} \prod_1^k a_n = \lim_{k\to\infty} a_1 a_2 \cdots a_k$$

6.2 Series with Nonnegative Terms

6.7 Theorem. Suppose f is a positive, decreasing function on the half-line $[a, \infty)$. Then for any integers j, k with $a \leq j < k$,

$$\sum_{n=j}^{k-1} f(n) \geq \int_j^k f(x)\, dx \geq \sum_{n=j+1}^{k} f(n)$$

6.8 Corollary (Integral Test). Suppose f is a positive, decreasing function on the half-line $[1, \infty)$. Then the series $\sum_1^\infty f(n)$ converges if and only if the improper integral $\int_1^\infty f(x)\,dx$ converges.

6.9 Theorem (p-test). The series $\sum_1^\infty n^{-p}$ converges if $p > 1$ and diverges if $p \geq 1$.

6.11 Theorem (General Comparison Tests). Suppose $0 \leq a_n \leq b_n$ for $n \geq 0$. If $\sum_0^\infty b_n$ converges, then so does $\sum_0^\infty a_n$. If $\sum_0^\infty a_n$ diverges, then so does $\sum_0^\infty b_n$.

6.12 Theorem (Limit Comparison Test). Suppose $\{a_n\}$ and $\{b_n\}$ are sequence of positive numbers and a_n/b_n approaches a positive, finite limit as $n \to \infty$. Then the series $\sum_0^\infty a_n$ and $\sum_0^\infty b_n$ are either both convergent or both divergent.

6.13 Theorem (Ratio Test). Suppose $\{a_n\}$ is a sequence of positive numbers.

(a) If $a_{n+1}/a_n < r$ for all sufficiently large n, where $r < 1$, then the series $\sum_0^\infty a_n$ converges. On the other hand, if $a_{n+1}/a_n \geq 1$ for all sufficiently large n, then the series $\sum_0^\infty a_n$ diverges.

(b) Suppose that $L = \lim_{n\to\infty} a_{n+1}/a_n$ exists. Then the series $\sum_0^\infty a_n$ converges if $L < 1$ and diverges if $L < 1$. No conclusion can be drawn if $L = 1$.

6.14 Theorem (Root Test).

(a) If $a_n^{1/n} < r$ for all sufficiently large n, where $r < 1$, then the series $\sum_0^\infty a_n$ converges. On the other hand, if $a_n^{1/n} \geq 1$ for all sufficiently large n, then the series $\sum_0^\infty a_n$ diverges.

(b) Suppose that $L = \lim_{n\to\infty} a_n^{1/n}$ exists. Then the series $\sum_0^\infty a_n$ converges if $L < 1$ and diverges if $L < 1$. No conclusion can be drawn if $L = 1$.

6.16 Theorem (Raabe's Test). Let $\{a_n\}$ be a sequence of positive numbers. Suppose that

$$\frac{a_{n+1}}{a_n} \to 1 \qquad \text{and} \qquad n\left[1 - \frac{a_{n+1}}{a_n}\right] \to L \text{ as } n \to \infty$$

If $L > 1$, the series $\sum a_n$ converges, and if $L < 1$, the series $\sum a_n$ diverges. If $L = 1$, no conclusion can be drawn.

6.3 Absolute and Conditional Convergence

Definition. A series $\sum a_n$ is called **absolutely convergent** if the series $\sum_0^\infty |a_n|$ conveges.

6.17 Theorem. Every absolutely convergent series is convergent.

6.18 Theorem. Let $a_n^+ = \max(a_n, 0)$ and $a_n^- = \max(-a_n, 0)$ where $\{a_n\}$ is some sequence of real numbers. If $\sum a_n$ is absolutely convergent, the series $\sum a_n^+$ nd $\sum a_n^-$ are both convergent. If $\sum a_n$ is conditionally convergent, the series $\sum a_n^+$ and $\sum a_n^-$ are both divergent.

6.19 Theorem. If $\sum_0^\infty a_n$ is absolutely convergent with sum S, then for every rearrangement of $\sum_0^\infty a_{\sigma(n)}$ is also absolutely concergent with sum S.

6.20 Theorem. Suppose $\sum_0^\infty a_n$ is conditionally convergent. Given any real number S, there is a rearrangement $\sum_0^\infty a_{\sigma(n)}$ that converges to S.

6.4 More Convergence Tests

6.21 Theorem.

(a) If $|a_n| \leq Cn^{-1-\epsilon}$ for some $C, \epsilon >$, then $\sum a_n$ converges absolutely. If $|a_n| \geq C_n^{-1}$ for some $C > 0$, then $\sum a_n$ either converges conditionally or diverges.

(b) (Ratio Test) If $|a_{n+1}/a_n| \to L$ as $n \to \infty$. Then the series $\sum a_n$ converges absolutely if $L < 1$ and diverges if $L < 1$.

(c) (Ratio Test) If $|a_n|^{1/n} \to L$ as $n \to \infty$. Then the series $\sum a_n$ converges absolutely if $L < 1$ and diverges if $L < 1$.

Definition. An **alternating series** is a series whose terms alternate in sign. Such a series is generally denoted $\sum(-1)^n a_n$ or $\sum(-1)^{n-1} a_n$, depending on whether the even or odd terms are positive.

6.22 Theorem (Alternating Series Test). Suppose the sequence $\{a_n\}$ s decreasing and $\lim_{n\to\infty} a_n = 0$. Then the series $\sum_0^\infty (-1)^n a_n$ is convergent. Moreover, if s_k and S denote the kth partial sum and the full sum of this series, we have

$$s_k > S \text{ for even } k, \qquad s_k < S \text{ for odd } k, \qquad |s_k - S| < a_{k+1} \text{ for all } k$$

6.23 Lemma (Summation by Parts). Given two numerical sequences $\{a_n\}$ and $\{b_n\}$, let

$$a'_n = a_n - a_{n-1} \qquad\qquad B_n = b_0 + \ldots + b_n$$

Then

$$\sum_0^k a_n b_n = a_k B_k - \sum_1^k a'_n B_{n-1}$$

6.25 Theorem (Dirichlet's Test). Let a_n and b_n be numerical sequences. Suppose that $\{a_n\}$ is decreasing and tends to 0 as $n \to \infty$, and that the sums $B_n = b_0 + \ldots + b_n$ are bounded in absolute value by a constant C independent o n. Then the series $\sum_0^\infty a_n b_n$ converges.

Corollary (Abel's Test). Suppose $\sum a_n$ is a convergent series and $\{b_n\}$ is a sequence of positive numbers that converges to some number. Then the series $\sum a_n b_n$ converges.

6.5 Double Series; Products of Series

Definition. A **double infinite series** is an expression of the form

$$\sum_{m,n}^\infty a_{mn}$$

Given a one-to-one correspondence $j \to (m, n)$ between the nonnegative integers and a set of ordered pairs of nonnegative integers, we can let $b_j = a_{mn}$ and form an ordinary infinite series $\sum_0^\infty b_j$ to compute the double series. Such a series is called an **ordering** of the double series $\sum_{m,n=0}^\infty a_{mn}$. All orderings of $\sum a_{mn}$ are rearrangements of one another.

Just as before, $\sum a_{mn}$ is called **absolutely convergent** if $\sum |a_{mn}|$ converges. By Theorem 6.19, all orderings of $\sum a_{mn}$ have the same sum.

6.29 Theorem. Suppose that $\sum_0^\infty a_m$ and $\sum_0^\infty b_n$ are both absolutely convergent with sums A and B. Then the double series $\sum_{m,n=0}^\infty a_m b_n$ is absolutely convergent, and its sum is AB.

Definition. Given two convergent series, $\sum_0^\infty a_m$ and $\sum_0^\infty b_n$, we can form the series

$$\sum_{j=0}^{\infty}\left(\sum_{m+n=j} a_n b_m\right) = \sum_{j=0}^{\infty}(a_0 b_j + a_1 b_{j-1} + \ldots + a_{j-1} b_1 + a_j b_0)$$

whose partial sums are the trangular partial sums of the double series $\sum a_m b_n$. We call such a sum the **Cauchy product** of $\sum a_m$ and $\sum b_n$.

7 Functions Defined by Series and Integerals

7.1 Sequences and Series of Functions

Definition. A **sequence** of functions $\{f_k\}_0^\infty$ is a map that assigns to each nonnegative integer$k$ a function f_k. It is implicitly assumed that the functions f_k are defined on some common domain S (usually a subset of $\mathbb{R}$ or $\mathbb{R}^n$) and all take values in the same space.

Definition. A sequence of functions $\{f_k\}$ defined on a set $S \subset \mathbb{R}^n$ is **pointwise** convergent on S if there exists some function f such that

$$f_k \to f(\mathbf{x}) \text{ for every } \mathbf{x} \in S$$

This is denoted $f_k \to f$.

Definition. A sequence of functions $\{f_k\}$ defined on a set $S \subset \mathbb{R}^n$ is **uniformly** convergent on S if for every $\epsilon > 0$ there is an integer K such that

$$|f_k(\mathbf{x}) - f(\mathbf{x})| < \epsilon \text{ whenever } k > K \text{ and } \mathbf{x} \in S$$

Alternatively, for the same K, we can write

$$\sup_{\mathbf{x}\in S} |f_k(\mathbf{x}) - f(\mathbf{x})| < \epsilon \text{ whenever } k > K$$

7.5 Theorem. The sequence $\{f_k\}$ converges to f uniformly on S if and only if there is a sequence $\{C_k\}$ of positive constants such that $|f_k(\mathbf{x}) - f(\mathbf{x})| \leq C_k$ for all $\mathbf{x} \in S$ and $\lim_{k\to\infty} C_k = 0$.

Definition. A sequence $\{f_k\}$ of functions on a set S is **uniformly Cauchy** if for every $\epsilon > 0$ there is an integer K so that

$$|f_j(\mathbf{x}) - f_k(\mathbf{x})| < \epsilon \text{ whenever } j, k > K \text{ and } \mathbf{x} \in S,$$

or alternatively

$$\sup_{\mathbf{x}\in S} |f_j(\mathbf{x}) - f_k(\mathbf{x})| < \epsilon \text{ whenever } j, k < K$$

7.7 Theorem. The sequence $\{f_k\}$ is uniformly Cauchy on S if and only if there is a function f on S such that $f_k \to f$ uniformly on S.

7.8 Theorem. Suppose $f_k \to f$ uniformly on S. If each f_k is continuous on S, then so is f.

Definition. We can also define convergence for series of functions. If a series $\sum f_n(\mathbf{x})$ converges for each $\mathbf{x} \in S$, we say that the series is **pointwise convergent** on S. The resultant sum is a function on S denoted by $\sum f_n$. The series $\sum f_n$ is said to be **uniformly convergent** on S if the sequence of partial sums, $s_k = \sum f_n$ is uniformly convergent on S.

7.9 Theorem (The Weierstrauss M-Test). Let $\{f_n\}_0^\infty$ be a sequence of functions on the set S. Suppose there is a sequence $\{M_n\}_0^\infty$ of positive constants such that (i) $|f_n(x)| \leq M_n$ for all $x \in S$ and all n, and (ii) $\sum_0^\infty M_n < \infty$. Then the series $\sum_0^\infty f_n$ is absolutely and uniformly convergent on S.

7.10 Theorem. Suppose $\{f_n\}$ is a sequence of continuous functions on a set S. If the series $\sum f_n$ converges uniformly on S, its sum is a continuous function on S.

7.2 Integrals and Derivatives of Sequences and Series

7.11 Theorem. Suppose S is a measureable set in $\mathbb{R}^n$ and $\{f_k\}$ is a sequence of integrable functions on S that converges uniformly to an integrable function f on S. Then

$$\int \cdots \int_S f(\mathbf{x})\, d^n\mathbf{x} = \lim_{k\to\infty} \int \cdots \int_S f_k(\mathbf{x})\, d^n\mathbf{x}$$

7.12 Theorem. Let $\{f_k\}$ be a sequence of functions of class C^1 on the interval $[a, b]$. Suppose that $\{f_k\}$ converges pointwise to f and that $\{f_k'\}$ converges uniformly to g on $[a, b]$. Then f is of class C^1 on $[a, b]$, and $g = f'$.

7.13 Theorem. Suppose that $\{f_n\}$ is a sequence of continuous functions on the interval $[a, b]$ and that the series $\sum f_n$ converges pointwise on $[a, b]$.

(a) If $\sum f_n$ converges uniformly on $[a, b]$, then

$$\int_a^b \left[\sum f_n(x)\right]\, dx = \sum \int_a^b f_n(x)\, dx$$

(b) If every f_n is of class C^1 and the series $\sum f_n'$ converges uniformly on $[a, b]$, then the $\sum f_n$ is of class C^1 on $[a, b]$, and

$$\frac{d}{dx}\left[\sum f_n(x)\right] = \sum f_n'(x) \qquad x \in [a, b]$$

7.3 Power Series

Definition. A **power series** is an infinite series of the form

$$\sum_0^\infty a_n(x-b)^n = a_0 + a_1(x-b) + a_2(x-b)^2 + \ldots$$

where x is a real and complex variable.

7.15 Lemma. If the power series $\sum_0^\infty a_n x^n$ converges for $x = x_0$, then it converges absolutely for all x such that $|x| < |x_0|$.

7.16 Theorem. For any power series $\sum_0^\infty a_n x^n$, there is a number $R \in [0, \infty]$, called the **radius of convergence** of the series, such that the series converges absolutely for $|x| < R$ and diverges for $|x| > R$. When $R = 0$, this means that the series converges only for $x = 0$; when $R = \infty$, it means the series converges absolutely for all x.

7.17 Theorem. Let R be the radius of convergence of $\sum_0^\infty a_n x^n$. For any $r < R$, the series $\sum_0^\infty a_n x^n$ converges uniformly on $[-r, r]$. It's sum f is continuous on $(-R, R)$.

7.18 Theorem. Suppose the series $f(x) = \sum_0^\infty a_n x^n$ has radius of convergence $R > 0$.

(a) If $-R < a < b < R$, then $\displaystyle\int_a^b f(x)\,dx = \sum_0^\infty a_n \frac{b^{n+1} - a^{n+1}}{n+1}$.

(b) If F is any antiderivative of f, then $F(x) = F(0) + \displaystyle\sum_0^\infty \frac{a_n}{n+1} x^{n+1}$ for $|x| < R$.

7.19 Theorem. The radius of convergence of any power series $\sum_0^\infty a_n x^n$ is equal to the radius of convergence of the derived series $\sum_0^\infty n a_n x^{n-1}$.

7.20 Theorem. Suppose the radius of convergence of the series $f(x) = \sum a_n x^n$ is $R > 0$. Then the function is of class C^∞ on $(-R, R)$ and its kth derivative may be computed on $(-R, R)$ by differentiating each term k times.

7.21 Corollary. Every power series $\sum a_n x^n$ with a positive radius of convergence is the Taylor series of its sum; that is, if $f(x) = \sum a_n x^n$ for $|x| < R$, then

$$a_n = \frac{f^{(n)}(0)}{n!}$$

7.22 Corollary. If $\sum a_n x^n = \sum b_n x^n$ for $|x| < R$, then $a_n = b_n$ for all n.

7.26 Theorem (Abel's Theorem). If the series $\sum a_n x^n$ converges at $x = R$ (resp. $x = -R$), then it converges uniformly on $[0, R]$ (resp. $[-R, 0]$), and hence defines a continuous function on that interval.

7.28 Corollary. If $\sum a_n$ converges, then $\lim_{x \to 1^-} \sum a_n x^n = \sum a_n$.

7.4 The Complex Exponential and Trig Functions

7.5 Functions Defined by Improper Integrals

Definition. The integral $\int_c^\infty f(x,t)\,dt$ **converges uniformly** for $x \in I \subset \mathbb{R}$, when

$$\sup_{x \in I} \left| \int_d^\infty f(x,t)\,dt \right| \to 0 \quad \text{as} \quad d \to \infty$$

7.38 Theorem. Suppose there is a function $g(t) \geq 0$ on $[c, \infty)$ such that (i) $|f(x,t)| \leq g(t)$ for all $x \in I$ and $t \geq c$, and (ii) $\int_c^\infty g(t)\,dt < \infty$. Then $\int_c^\infty f(x,t)\,dt$ converges absolutely and uniformly for $x \in I$.

7.39 Theorem. Suppose that $f(x,t)$ is a continuous function on the set $S = \{(x,t) \mid x \in I, t \geq c\}$ and the integral $\int_c^\infty f(x,t)\,dt$ is uniformly convergent for $x \in I$. Then

(a) The function $F(x) = \int_c^\infty f(x,t)\,dt$ is continuous on I.

(b) If $[a,b] \subset I$, then $\int_a^b \int_c^\infty f(x,t)\,dt\,dx = \int_c^\infty \int_a^b f(x,t)\,dx\,dt$.

7.40 Theorem. Suppose that $f(x,t)$ and its partial derivative $\partial_x f(x,t)$ are continuous functions on $\{(x,t) \mid x \in I, t \geq c\}$. Suppose also that the integral $\int_c^\infty f(x,t)\ dt$ converges for $x \in I$ and the integral $\sum_c^\infty \partial_x f(x,t)\ dt$ converges uniformly for $x \in I$. Then the former integral is differentiable on I as a function of x and

$$\frac{d}{dx}\int_c^\infty f(x,t)\ dt = \int_c^\infty \frac{\partial f}{\partial x}(x,t)\ dt$$

Theorem (Dirichlet's Test). Let g, g_x, h be continuous on $[a,b) \times S$. If $\lim_{x\to b^-} \sup_{y\in S} |g(x,y)| = 0$, and there exists a constant M such that $\sum_{y\in S} \left|\int_a^x h(u,y)\ du\right| < M$ for $a \leq x \leq b$ and $\int |g_x(x,y)|\ dx$ converges uniformly on S, then $\int_a^b g(x,y)h(x,y)\ dx$ converges uniformly on S.

7.6 The Gamma Function

Definition. The **gamma function** $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\ dt$$

The gamma function is uniquely defined by the following properties:

$$\Gamma(1) = 1 \qquad \Gamma(x+1) = x\Gamma(x) \qquad \Gamma(n+1) = n! \text{ for } n = 1,2,3,\ldots \qquad \log\Gamma \text{ is convex on } (0,\infty)$$

The k-th derivative of the gamma function

$$\Gamma^{(k)}(x) = \int_0^\infty (\log t)^k\, t^{x-1}e^{-t}\ dt$$

Definition. Related to the gamma function is the **beta function**

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\ dt \qquad (x,y>0)$$

7.55 Theorem. For $x, y > 0$,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

7.56 Theorem (The Duplication Formula).

$$\Gamma(2x) = \pi^{-1/2}2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)$$

7.57 Theorem. For $a > 0$,

$$\lim_{x\to\infty}\frac{\Gamma(x+a)}{x^a\Gamma(x)} = 1$$

8 Fourier Series

8.1 Periodic Functions and Fourier Series

Definition. A function f on $\mathbb{R}$ is called **periodic** with period P, or P-periodic, if $f(x+P) = f(x)$ for all x. Thus all of f is completely determined by its values on $[a, a+P]$ for some a.

Definition. A function f defined on an interval $[a, b]$ is **piecewise continuous** if it is continuous at all but finitely many points.

Definition. Given a 2π-periodic, piecewise continuous function f, its **Fourier series** is defined by

$$f(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta} = \frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} \, d\theta$$

are the **Fourier coefficients** of f.

8.2 Convergence of Fourier Series

8.12 Theorem (Bessel's Inequality). If f is 2π-periodic and piecewise continuous, then

$$\sum_{-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 \, d\theta$$

where c_n is the Fourier coefficient of f.

Definition. The partial sum of the Fourier series of f is given by

$$S_N^f = \int_{-\pi}^{\pi} f(\varphi + \theta) D_N(\varphi) \, d\varphi \quad \text{where} \quad D_N(\varphi) = \frac{1}{2\pi} \sum_{-N}^{N} e^{in\varphi}$$

D_N is called the N-th **Dirichlet kernel**.

8.15 Lemma. Let $D_N(\varphi)$ be the N-th Dirichlet kernel. Then

a. $\displaystyle\int_{-\pi}^{0} D_N(\varphi) \, d\varphi = \int_{0}^{\pi} D_N(\varphi) \, d\varphi = \frac{1}{2}$

b. $\displaystyle D_N(\varphi) = \frac{1}{2\pi} \frac{e^{i(N+1)\varphi} - e^{-iN\varphi}}{e^{i\varphi} - 1}$

Definition. We say a periodic function is piecewise smooth if on any bounded interval, it is of class C^1 on all but finitely many points.

8.16 Theorem. Suppose f is 2π-periodic and piecewise smooth. Then the partial sums S_N^f converge pointwise to $\frac{1}{2}[f(\theta-) + f(\theta+)]$. In particular, they converge to $f(\theta)$ at each θ where f is continuous.

Definition. We shall call f standardized if it satisfies $f(\theta) = \frac{1}{2}[f(\theta-) + f(\theta+)]$ for every θ.

8.18 Corollary. If f and g are standardized piecewise-smooth, 2π-periodic functions with the same Fourier coefficients, then $f = g$.

Definition. Abel's summation allows us to find the Fourier series of piecewise continuous functions. Consider the following series for $0 < r < 1$ and its limit as $r \to 1^-$:

$$A_r f(\theta) = \sum_{-\infty}^{\infty} r^{|n|} c_n e^{in\theta}$$

We can re-express the right side with

$$A_r f(\theta) = \int_{-\pi}^{\pi} f(\theta + \varphi) P_r(\varphi)\, d\varphi \quad \text{where} \quad P_r(\varphi) = \sum_{-\infty}^{\infty} r^{|n|} e^{in\theta}$$

where P_r is the **Poisson kernel**. Unlike the Dirchlet kernel, the Poisson kernel uniformly converges on $[-\pi, -\delta]$ and $[\delta, \pi]$ for any $\delta > 0$ as $r \to 1^-$.

8.24 Theorem. Suppose that f is 2π-periodic. If f is piecewise continuous, then

$$\lim_{r \to 1^-} A_r f(\theta) = \frac{1}{2}\left[f(\theta-) + f(\theta+)\right]$$

for every θ. If f is continuous, then $A_r f \to f$ uniformly as $r \to 1^-$.

8.3 Derivatives, Integrals, and Uniform Convergence

8.26 Theorem. Suppose f is 2π-periodic, continuous, and piecewise smooth, and let c_n and c_n' be the Fourier coefficients of f and f'. Then $c_n' = inc_n$. Equivalently, if a_n, b_n and a_n', b_n' are the Fourier coefficients of f and f', then $a_n' = nb_n$ and $b_n' = -na_n$.

8.27 Corollary. Suppose f is 2π-periodic, continuous, and piecewise smooth, and that f' is also piecewise smooth. If

$$\sum_{-\infty}^{\infty} c_n e^{in\theta} = \frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos bn + b_n \sin bn)$$

is the Fourier series of f, then $f'(\theta)$ is the derived series

$$\sum_{-\infty}^{\infty} inc_n e^{in\theta} = \sum_{1}^{\infty} (nb_n \cos bn - na_n \sin bn)$$

at every θ where $f'(\theta)$ exists. At the points where f' jumps, the series converges to $\frac{1}{2}\left[f'(\theta-) + f'(\theta+)\right]$.

8.28 Theorem. Suppose f is a 2π-periodic function and piecewise continuous, with Fourier coefficients c_n or a_n and b_n. Assume that $c_0 = \frac{a_0}{2} = 0$. If F is a continuous, piecewise-smooth function such that $F' = f$ (except where f jumps), then

$$F(\theta) = C_0 + \sum_{n \neq 0} \frac{c_n}{in} e^{in\theta} = C_0 + \sum_{1}^{\infty} \left(\frac{a_n}{n} \sin n\theta - \frac{b_n}{n} \cos n\theta \right)$$

for all θ where C_0 is the mean value of F on $[-\pi, \pi]$.

8.29 Theorem. If f is 2π-periodic, continuous, and piecewise smooth, then the Fourier series of f is absolutely and uniformly convergent.

8.4 Fourier Series on Intervals

Definition. Given a piecewise continuous function f on $[0,\pi]$, we define the even extension of f on $[-\pi,\pi]$ by

$$f_{even} = \begin{cases} f(\theta) & 0 \le \theta \le \pi \\ f(-\theta) & -\pi \le \theta < 0 \end{cases}$$

The Fourier series of f_{even}, also called the **Fourier cosine series** of f, is the series

$$\frac{1}{2}a_0 + \sum_1^\infty a_n \cos n\theta \qquad\qquad a_n = \frac{2}{\pi}\int_0^\pi f(\theta)\cos n\theta \, d\theta$$

Similarly, we can define the odd extension of f on $[-\pi,\pi]$ to be

$$f_{odd} = \begin{cases} f(\theta) & 0 < \theta < \pi \\ -f(-\theta) & -\pi < \theta < 0 \\ 0 & \theta = 0, \pm\pi \end{cases}$$

The Fourier series of f_{odd}, also called the **Fourier sine series** of f, is the series

$$\sum_1^\infty b_n \sin n\theta \qquad\qquad b_n = \frac{2}{\pi}\int_0^\pi f(\theta)\sin n\theta \, d\theta$$

8.30 Theorem. Suppose f is piecewise smooth is on $[0,\pi]$. The Fourier cosine series and the Fourier sine series converge to $\frac{1}{2}[f(\theta-)+f(\theta+)]$ at every $\theta \in (0,\pi)$. The cosine series converges to $f(0+)$ at $\theta = 0$ and to $f(\pi-)$ at $\theta = \pi$; the sine series converges to 0 at both of these points.

We may want to consider periodic function with a period other than 2π. Suppose $f(x)$ is a $2l$-periodic function. Then the Fourier series of f is

$$f(x) = \sum_{-\infty}^\infty c_n e^{in\pi x/l} \qquad\qquad c_n = \frac{1}{2l}\int_{-l}^l f(x)e^{-in\pi x/l}\, dx$$

and the corresponding formula using sines and cosines is

$$f(x) = \frac{1}{2}a_0 + \sum_1^\infty \left[a_n \cos\frac{n\pi x}{l} + b_n \cos\frac{n\pi x}{l}\right]$$

where

$$a_n = \frac{1}{l}\int_{-l}^l f(x)\cos\frac{n\pi x}{l}\, dx \qquad\qquad b_n = \frac{1}{l}\int_{-l}^l f(x)\sin\frac{n\pi x}{l}\, dx$$

Thus for a piecewise smooth function f on $[0,l]$, its Fourier cosine and sine series are

$$f(x) = \frac{1}{2}a_0 + \sum_1^\infty a_n \cos\frac{n\pi x}{l} \qquad\qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\, dx$$

and

$$f(x) = \sum_1^\infty b_n \sin\frac{n\pi x}{l} \qquad\qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\, dx$$

8.5 Applications in Differential Equations

8.6 The Infinite-Dimensional Geometry of Fourier Series

Definition. For complex n-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, we define the **inner product** to be

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{1}^{n} a_j \overline{b_j} \qquad (\mathbf{a}, \mathbf{b} \in \mathbb{C}^n)$$

Now consider piecewise continuous, complex-valued functions f and g on an interval $[a, b]$. We define **inner product** of two functions f and g as

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

We define the **norm** of a function f to be

$$||f|| = \langle f, f \rangle^{1/2} = \left[\int_a^b |f(x)|^2\, dx \right]^{1/2}$$

We call two functions f and g **orthogonal** on $[a, b]$ if $\langle f, g \rangle = 0$. A sequence of functions $\{\varphi_n\}$ is called **orthogonal** if $\langle \varphi_n, \varphi_m \rangle = 0$ for $m \neq n$ and **orthonormal** if, in addition, $||\varphi_n|| = 1$ for all n.

Definition. Since we have norm, we define the **distance** between two functions to be

$$||f - g|| = \left[\int_a^b |f(x) - g(x)|^2\, dx \right]^{1/2}$$

Thus, we have a definition of convergence:

$$f_k \to f \qquad \Longleftrightarrow \qquad ||f_k - f|| \to 0, \text{ i.e., } \int_a^b |f_k(x) - f(x)|^2\, dx \to 0$$

This definition of convergence is called **convergence in norm** or **mean-square convergence**.

8.42 Proposition. If $f_k \to f$ uniformly on $[a, b]$, then $f_k \to f$ in norm in $[a, b]$.

Definition. We define the space of **square-integrable** functions to be

$$L^2(a, b) = \{f \mid f \text{ is Lebesgue measureable on } [a, b] \text{ and } \int_a^b f(x)\, dx < \infty\}$$

8.43 Theorem. Let $e_n(\theta) = e^{in\theta}$.

a. If $f \in L^2(-\pi, \pi)$, the Fourier series

$$\sum_{-\infty}^{\infty} c_n e_n, \qquad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta$$

converges in norm to f, that is,

$$\lim_{N \to \infty} \int_{-\pi}^{\pi} \left| f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right| \, d\theta = 0$$

b. Bessel's inequality is an equality: For any $f \in L^2(-\pi, \pi)$,

$$\sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 \, d\theta$$

This equality is called **Parseval's identity**.

c. If $\{c_n\}_{-\infty}^{\infty}$ is any sequence of complex numbers such that $\sum_{-\infty}^{\infty} |c_n|^2$ converges, then the series $\sum_{-\infty}^{\infty} c_n e_n$ converges in norm to a function in $L^2(-\pi, \pi)$.