Tarea 1 173605

sábado, 6 de septiembre de 2025 02:58 p. m.

1. Verificar (mediante definición) si la caminata aleatoria simple es un proceso estacionario o no.

Caminata Aleatoria $\{S_t\}$ $\forall\ t = 1, 2, \ldots$

se obtiene sumando v.a.s iid $X_i \sim N(0, \sigma^2)$

$S_0 = 0$ $\quad S_t = X_1 + X_2 + \ldots + X_t \quad S_t = \sum_{i=1}^{t} X_i$

$t = 1, 2 \ldots$ Donde X_t es un ruido iid

1) $\mu_x(t)$ independiente de t

$E[S_t] = E[\sum_{i=1}^{t} X_i] = \sum_{i=1}^{t} E[X_i] = \sum_{i=1}^{t} 0 = 0$

Cumple que $\mu_x(t) = 0$ es independiente de t

2) $\gamma_x(h)$ independiente de t $\forall\ h, t \in \mathbb{Z}$

$\gamma_x(h) = Cov(S_t, S_{t+h})$

$= Cov(\sum_{i=1}^{t} X_i, \sum_{i=1}^{t} X_i + \sum_{i=t+1}^{t+h} X_i)$

$= Cov(\sum_{i=1}^{t} X_i, \sum_{i=1}^{t} X_i) + Cov(\sum_{i=1}^{t} X_i, \sum_{i=t+1}^{t+h} X_i)$

$= Cov(S_t, S_t) + Cov(S_t, S_{h-t})$

$= VAR[S_t]$

$VAR[S_t] = VAR[\sum_{i=1}^{t} X_i] = \sum_{i=1}^{t} VAR[X_i] = t\sigma^2$

El proceso no es estacionario. $\gamma_x(h)$ depende de t.

2. Probar que la función de autocovarianza de un proceso $MA(1)$ es igual a cero para retrasos $|k| \geq 2$.

Demostración Directa.

$MA(1)$ $\quad \tilde{X}_t = (1 - \theta B) Z_t,$ tomamos $\tilde{X}_t = X_t - \mu$, $\mu = 0$ para no

para no arrastrar el término

$X_t = Z_t + \theta Z_t$

Por definición $\quad \gamma_x(h) = \gamma_x(t+h, t) = Cov(X_{t+h}, X_t)$

$\gamma_x(h) = Cov(Z_t - \theta Z_{t-1}, Z_{t+h} - \theta Z_{t-1+h})$

$= Cov(Z_t, Z_{t+h} - \theta Z_{t-1+h}) - \theta\, Cov(Z_{t-1}, Z_{t+h} - \theta Z_{t-1+h})$

$= Cov(Z_t, Z_{t+h}) - \theta\, Cov(Z_t, Z_{t-1+h}) - \theta\,[Cov(Z_{t-1}, Z_{t+h}) - \theta\, Cov(Z_{t-1}, Z_{t-1+h})]$

dado que Z_t es ruido blanco e independiente

$Cov(Z_h, Z_g)$, $h \neq g$ es 0

$Cov(Z_h, Z_g)$, $h = g$ es $VAR[Z_t] = \sigma_z^2$

$\Rightarrow$ si $|h| > 1$

Si se evalua en cada termino de la expresion de $\gamma_x(h)$

- $Cov(Z_t, Z_{t+h})$ $\quad t \neq t+h$
- $-\theta\, Cov(Z_t, Z_{t-1+h})$ $\quad t \neq t-1+h$
- $-\theta\, Cov(Z_{t-1}, Z_{t+h})$ $\quad t-1 \neq t+h$
- $\theta^2\, Cov(Z_{t-1}, Z_{t-1+h})$ $\quad t-1 \neq t-1+h$

$\Rightarrow$ si $|h|>1$ $\gamma_x(h) = 0+0+0+0=0$ $\forall h$, $\forall$ $|h|>1 \Rightarrow \forall k$ $|k|\geq 2$ se cumple $\gamma_x(k)=0$

3. Sea $\{\tilde{X}_t\}$ un proceso $MA(2)$, pruebe que la media y la varianza de dicho proceso están dadas por las expresiones
    - $E(\tilde{X}_t) = 0.$
    - $Var(\tilde{X}_t) = (1+\theta_1^2+\theta_2^2)\sigma_Z^2$

    En donde $\{Z_t\}$ representa un proceso de ruido blanco con media cero y varianza constante σ_Z^2, además θ_1 y θ_2 son los parámetros del proceso de media móvil de orden 2.

Modelo: $\tilde{X}_t = (1+\theta_1 B + \theta_2 B^2) = (1-\theta_1 B - \theta_2 B^2) Z_t$

$X_t = Z_t - \theta_1 Z_{t-1} - \theta_2 Z_{t-2}$

Considerando $\tilde{X}_t = X_t - \mu$; $\mu = 0$

$Z_t \sim N(0, \sigma_Z^2)$

- $E[X_t] = E[Z_t] - \theta_1 E[Z_{t-1}] - \theta_2 E[Z_{t-2}]$
  $= 0 - \theta_1[0] - \theta_2[0] = 0$
  $\therefore E[X_t] = 0$ si $\mu = 0$

- $V_{AR}[X_t] = V_{AR}[Z_t - \theta_1 Z_{t-1} - \theta_2 Z_{t-2}]$
  $= V_{AR}[Z_t] + \theta_1^2 V_{AR}[Z_{t-1}] + \theta_2^2 V_{AR}[Z_{t-2}]$
  $= \sigma_Z^2 + \theta_1^2\sigma_Z^2 + \theta_2^2\sigma_Z^2$
  $= (1+\theta_1^2+\theta_2^2)\sigma_Z^2$

4. Sea $\{\tilde{X}_t\}$ un proceso $MA(2)$, calcule la función de autocovarianza y de autocorrelación de dicho proceso para todos sus retrasos k.

$\gamma_x(h) = Cov(X_t, X_{t+h})$

Caso 1: $h=0$

$\gamma_x(0) = Cov(X_t, X_t) = V_{AR}[X_t] = (1+\theta_1^2+\theta_2^2)\sigma_Z^2$

del resultado del ejercicio anterior.

Caso 2: $|h|=1$

$\gamma_x(1) = Cov(X_t, X_{t+1}) = Cov(Z_t - \theta_1 Z_{t-1} - \theta_2 Z_{t-2},\ Z_{t+1} - \theta_1 Z_t - \theta_2 Z_{t-1})$

$= Cov(Z_t, Z_{t+1} - \theta_1 Z_t - \theta_2 Z_{t-1}) - \theta_1 Cov(Z_{t-1}, Z_{t+1} - \theta_1 Z_t - \theta_2 Z_{t-1}) - \theta_2(Z_{t-2}, Z_{t+1} - \theta_1 Z_t - \theta_2 Z_{t-1})$

Considerando que solo hay un Z_{t-2} todas sus $cov = 0 \Rightarrow$ eliminamos el último término

$= Cov(Z_t, Z_{t+1}) - \theta_1 Cov(Z_t, Z_t) - \theta_2 Cov(Z_t, Z_{t-1}) - \theta_1[Cov(Z_{t-1}, Z_{t+1}) - \theta_1 Cov(Z_{t-1}, Z_t) - \theta_2(Z_{t-1}, Z_{t-1})]$

$= \theta_1\sigma_Z^2 - \theta_1[\theta_2\sigma_Z^2] = \theta_1\sigma_Z^2(1-\theta_2)$

Caso 3 $|h|=2$

$\gamma_x(2) = Cov(Z_t - \theta_1 Z_{t-1} - \theta_2 Z_{t-2},\ Z_{t+2} - \theta_1 Z_{t+1} - \theta_2 Z_t)$

$= Cov(Z_t, Z_{t+2} - \theta_1 Z_{t+1} - \theta_2 Z_t) - 0 - 0$

No hay mas terminos Z_{t-1} ni Z_{t-2}

$= -\theta_2 Cov(Z_t, Z_t) = -\theta_2\sigma_Z^2$

Funcion de autocovarianza:

$\Big\{ (1+\theta_1^2+\theta_2^2)\sigma_Z^2 \qquad h=0$

Funcion de autocovarianza:

$$\gamma_X(h) = \begin{cases} (1+\theta_1^2+\theta_2^2)\,\sigma_Z^2 & h = 0 \\ (1-\theta_2)\,\theta_1\sigma_Z^2 & |h| = 1 \\ \theta_2\,\sigma_Z^2 & |h| = 2 \\ 0 & |h| > 2 \end{cases}$$

Ahora para la función de autocorrelacion

$$\rho_X(h) = \gamma_X(h) / \gamma_X(0)$$

$$\rho_X(0) = \gamma_X(0) / \gamma_X(0) = 1$$

$$\rho_X(1) = \gamma_X(1) / \gamma_X(0) = [(1-\theta_2)\,\theta_1\sigma_Z^2] / [(1+\theta_1^2+\theta_2^2)\,\sigma_Z^2]$$

$$= [(1-\theta_2)\,\theta_1] / (1+\theta_1^2+\theta_2^2)$$

$$\rho_X(2) = \gamma_X(2)/\gamma_X(0) = (-\theta_2\sigma_Z^2) / [(1+\theta_1^2+\theta_2^2)\,\sigma_Z^2]$$

$$= -\theta_2 / (1+\theta_1^2+\theta_2^2)$$

Función de autocorrelación

$$\rho_X(h) \begin{cases} 1 & h = 0 \\ \dfrac{(1-\theta_2)\,\theta_1}{(1+\theta_1^2+\theta_2^2)} & |h| = 1 \\ \dfrac{-\theta_2}{(1+\theta_1^2+\theta_2^2)} & |h| = 2 \\ 0 & |h| > 2 \end{cases}$$

5. Considere un proceso $MA(1)$, a partir de la función de autocorrelación, demuestre que

$$-0.5 \leq \rho_1 \leq 0.5$$

$\tilde{X}_t = Z_t - \theta Z_{t-1}$ Modelo.

consideramos $\tilde{X}_t = X_t - \mu$ $\quad \mu = 0$ $\quad Z_t \sim N(0, \sigma_Z^2)$

Sabemos que $\rho_X(h) = \dfrac{\gamma_X(h)}{\gamma_X(0)} \Rightarrow \rho_1(h) = \dfrac{\gamma_X(1)}{\gamma_X(0)}$

$$\gamma_X(0) = Cov(X_t, X_t) = VAR[X_t] = VAR[Z_t - \theta_1 Z_{t-1}]$$

$$= VAR[Z_t] + \theta_1^2\, VAR[Z_{t-1}]$$

$$= \sigma_Z^2 + \theta_1^2\,\sigma_Z^2$$

$$= \sigma_Z^2(1+\theta_1^2)$$

$$\gamma_X(1) = Cov(X_t, X_{t+1}) = Cov(Z_t - \theta_1 Z_{t-1}, Z_{t+1} - \theta_1 Z_t)$$

$$= Cov(Z_t, Z_{t+1} - \theta Z_t) - \theta_1 Cov(Z_{t-1}, Z_{t+1} - \theta_1 Z_t)$$

$$= -\theta_1\, Cov(Z_t, Z_t)$$

$$= -\theta_1\sigma_Z^2$$

$$\Rightarrow \rho_X(1) = (-\theta_1\sigma_Z^2) / (\sigma_Z^2(1+\theta_1^2))$$

$$= -\theta_1 / (1 + \theta_1^2)$$

para encontrar mínimos / máximos se deriva la función e igualamos a 0

$$f'(\theta) = \frac{u'v - vu'}{v^2} = \frac{1(1+\theta^2) - (2\theta)(\theta)}{(1+\theta^2)} = \frac{1 + \theta^2 - 2\theta^2}{1 + 2\theta + \theta^2} = \frac{1 - \theta^2}{1 + 2\theta + \theta^2}$$

$$1 - \theta^2 = 0 \qquad\qquad 1 - 2\theta - \theta^2 = 0$$
$$1 = \theta^2 \qquad\qquad (1 + \theta)^2 = 0$$
$$\sqrt{1} = \theta \qquad\qquad -1 = \theta$$
$$1 = \theta$$

Evaluar

$$\frac{-(-1)}{1 + (-1)^2} = \frac{1}{2} = 0.5 \qquad\qquad \frac{-(1)}{1 + (1)^2} = \frac{-1}{1+1} = \frac{-1}{2} = -0.5$$

al evaluar en min / max de la función se observa

$$-0.5 \leq \rho_1 \leq 0.5$$

6. Considere un proceso $MA(2)$, a partir de la función de autocorrelación, demuestre que $\rho_1^2 \leq 0.5$ y $|\rho_2| \leq 0.5$.

$MA(2)$ $\qquad \tilde{X}_t = Z_t - \theta_1 Z_{t-1} - \theta_2 Z_{t-2}$

se considera $\tilde{X}_t = X_t - \mu$ y $\mu = 0$

Del ejercicio 4:

Función de autocorrelación $MA(2)$

$$\rho_x(h) \begin{cases} 1 & h = 0 \\ \dfrac{(1-\theta_2)\theta_1}{(1 + \theta_1^2 + \theta_2^2)} & |h| = 1 \\ \dfrac{-\theta_2}{(1 + \theta_1^2 + \theta_2^2)} & |h| = 2 \\ 0 & |h| > 2 \end{cases}$$

Parte 1 $\quad \rho_1^2 \leq 0.5 \qquad \rho_1 = \dfrac{\theta_1(1 - \theta_2)}{(1 + \theta_1^2 + \theta_2^2)}$

$$\rho_1 (1 + \theta_2^2 + \theta_1^2) - \theta_1(1 - \theta_2) = 0$$
$$\rho_1 + \rho_1\theta_1^2 + \rho_1\theta_2^2 - \theta_1 + \theta_1\theta_2 = 0$$

Como ordenar el polinomio

$\theta_2: \quad \rho_1\theta_2^2 + \theta_1\theta_2 + [\rho_1 + \rho_1\theta_1^2 - \theta_1] \qquad\qquad \theta_1: \quad \rho_1\theta_1^2 - (1-\theta_2)\theta_1 + [\rho_1 + \rho_1\theta_2^2]$

$a = \rho_1 \qquad\qquad a = \rho_1$

$b = \theta \qquad\qquad b = -(1 - \theta_2)$

$\theta_2: \ \ell_1\theta_2^2 + \theta_1\theta_2 + [\ell_1 + \ell_1\theta_1^2 - \theta_1]$

$a = \ell_1$

$b = \theta_1$

$c = [\ell_1 + \ell_1\theta_1^2 - \theta_1]$

$b^2 \pm 4ac$

$\theta_1^2 \pm 4\ell_1[\ell_1 + \ell_1\theta_1^2 - \theta_1] \geq 0$

$4\ell_1[\ell_1 + \ell_1\theta_1^2 - \theta_1] \geq -\theta_1^2$

$\theta_1: \ \ell_1\theta_1^2 - (1-\theta_2)\theta_1 + [\ell_1 + \ell_1\theta_2^2]$

$a = \ell_1$

$b = -(1-\theta_2)$

$c = [\ell_1 + \ell_1\theta_2^2]$

$b^2 \pm 4ac$

$\Delta = (1-\theta_2)^2 - 4\ell_1^2[1-\theta_2^2] \geq 0$

$$\frac{(1-\theta_2)^2}{4(1-\theta_2^2)} \geq \ell_1^2$$

Queda más facil el polinomio de θ_2

A esa expresion $\leq \ell_1^2$ se le sacan min/max

$$h(\theta_2) = \frac{(1-\theta_2)^2}{4(1+\theta_2^2)} \Rightarrow h'(\theta_2) = \frac{u'v - uv'}{v^2}$$

$$h'(\theta) = \frac{-2(1-\theta)4(1+\theta^2) - (1-\theta)^2 8\theta}{[4(1-\theta^2)]^2}$$

$$= \frac{-8(1-\theta)[(1-\theta^2) + \theta(1-\theta)]}{16(1+\theta^2)^2}$$

al igualar a 0 las raices:

$x_1 = -1$

al sustituir

$$\frac{(1--1)^2}{4(1+(-1)^2)} = \frac{2^2}{4(2)} = \frac{4}{8} = \frac{1}{2} = 0.5$$

en su maximo la funcion $= 0.5$

$\Rightarrow$

$0.5 \geq \ell_1^2$

Parte 2 $|\ell_2| \leq 0.5$

$$\ell_2 = \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}$$

$\ell_2(1+\theta_1^2+\theta_2^2) + \theta_2 = 0$

$\ell_2 + \ell_2\theta_1^2 + \ell_2\theta_2^2 + \theta_2 = 0$

$\underbrace{\ell_2}_{a}\theta_2^2 + \underbrace{\theta_2}_{b} + \underbrace{[\ell_2 + \ell_2\theta_1^2]}_{c}$

$$\theta_2 = \frac{b \pm \sqrt{b^2 \pm 4ac}}{2a}$$

$$\theta_2 = \frac{1 \pm \sqrt{1^2 \pm 4\ell_2(\ell_2+\ell_2\theta_1^2)}}{2\ell_2}$$

Si la solución es real la raíz es mayor a 0

$1^2 \pm 4\ell_2(\ell_2 + \ell_2\theta_1^2) \geq 0$

tomamos el caso de $+4$

$1^2 + 4\ell_2^2 + 4\ell_2^2\theta_1^2 \geq 0$

$1 + 4\ell_2^2(1+\theta_1^2) \geq 0$

$$\frac{1}{4(1+\theta_1^2)} \leq -\ell_2^2$$

$$\sqrt{\frac{1}{4}} \leq \sqrt{\ell_2^2}$$

$$\frac{1}{2} \leq |\ell_2|$$

7. Pruebe que el proceso $MA(q)$ es estacionario y calcule su función de autocovarianza y autocorrelación.

$MA(q)$ $\quad \tilde{X}_t = (1 - \theta_1 B - \theta_2 B^2 - \dots - \theta_q B^q) Z_t$

$\tilde{X}_t = Z_t - \theta_1 Z_{t-1} - \dots - \theta_q Z_{t-q}$

tomando $\tilde{X}_t = X_t - \mu$ y $\mu = 0$

todos los procesos $MA(q)$ son estacionarios

- $\mu_x(t)$ debe ser independiente de t

$E[X_t] = E[Z_t - \theta_1 Z_{t-1} - \dots - \theta_q Z_{t-q}]$

$= E[Z_t] - \theta E[Z_{t-1}] - \dots - \theta_q E[Z_{t-q}]$

$= 0 - 0 - \dots - 0 = 0$

$\mu_x(t)$ es independiente de t

- $\gamma_x(h)$ debe ser independiente de t $\forall\, h, t$

$V_{AR}[X_t] = V_{AR}[Z_t - \theta_1 Z_{t-1} - \dots - \theta_q Z_{t-q}]$

$= V_{AR}[Z_t] + \theta_1^2 V_{AR}[Z_{t-1}] + \dots + \theta_q^2 V_{AR}[Z_{t-q}]$

$= \sigma_Z^2 (1 + \theta_1^2 + \dots + \theta_q^2)$

entonces en este caso de $Cov(Z_t, Z_t)$ es independiente de t

$= \sigma_Z^2 (1 + \sum_{i=1}^{q} \theta_i^2)$

$\gamma_x(h) = Cov(Z_t - \theta_1 Z_{t-1} - \dots - \theta_q Z_{t-q},\ Z_{t+h} - \theta_1 Z_{t-1+h} - \dots - \theta_q Z_{t-q+h})$

$= Cov(Z_t,\ Z_{t+h} - \theta_1 Z_{t+h-1} - \dots - \theta_q Z_{t-q+h}) - \theta Cov(Z_{t-1}, Z_{t+h} \dots)$

$- \dots - \theta_q Cov(Z_{t-q}, Z_{t+h} - \theta_1 Z_{t+h-1} - \dots$

de cada termino anterior solo sobrevive $\neq 0$

$= \theta_k Cov(Z_t, Z_{t+h-k}) + \theta_1 \theta_{k-1} Cov(Z_{t-1}, Z_{t+h-k-1}) + \dots - \theta_{q-k} Cov(Z_{t-q}, Z_{t-h-k})$

$= \sigma_Z^2 [-\theta_k + \theta_1\theta_{k-1} + \dots + \theta_{q+1}\theta_{k-(q-1)} + \theta_q \theta_{q-k}]$

$h = K$

8. Considere los siguientes procesos, para cada uno de ellos
   a) Verifique si el proceso es estacionario e invertible.
   b) Calcule las cinco primeras autocorrelaciones (aparte de la autocorrelación cero) y grafíquelas.

i. $X_t - 0.9X_{t-1} = c + Z_t; c = 4$

a) Estacionario : $|\phi| < 1$ AR(1) es estacionario $\mu = \frac{4}{1-0.9} = 40$

Invertible : Es un AR(1) siempre es invertible

$X_t(1-0.9B) = 4 + Z_t$

$\phi = 0.9 \quad |0.9| < 1$

b) Para AR(1) $\rho_x(h) = \phi^{|h|}$

$\rho_0 = 1$

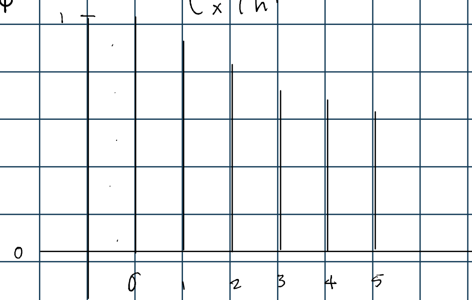
$\rho_1 = 0.9$

$\rho_2 = 0.81$

$\rho_3 = 0.729$

$\rho_4 = 0.6561$

$\rho_5 = 0.59049$



ii. $X_t + 0.1X_{t-1} = c + Z_t; c = 0$ AR(1)

a) Estacionario $|\phi| < 1$ AR(1)

Invertible todos los AR(q) son estacionarios

b) $\rho_x(h)$ para AR(1) son $\rho_x(h) = \phi^{|h|}$

$\rho_0 = 1$

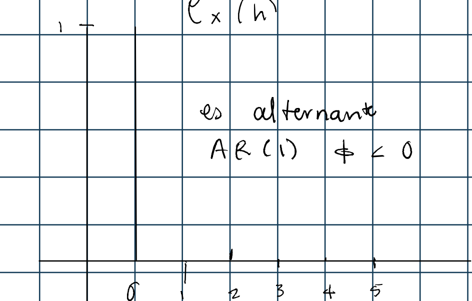
$\rho_1 = -0.1$

$\rho_2 = 0.01$

$\rho_3 = -0.001$

$\rho_4 = 0.0001$

$\rho_5 = -0.00001$



es alternante

AR(1) $\phi < 0$

iii. $X_t - 0.9X_{t-1} + 0.1X_{t-2} = c + Z_t\,; c = 10$ $\mu = \frac{10}{1-0.9+0.1} = \frac{10}{.2} = \frac{100}{2} = 50$

AR(2)

a) Estacionariedad sus parametros cumplen con la región admisible.

Invertibilidad todo proceso AR es invertible.

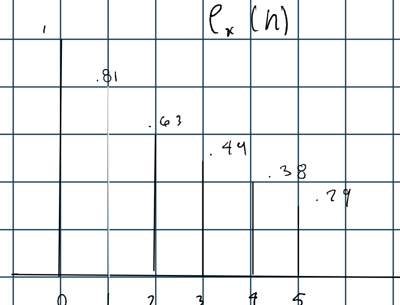
AR(2) tiene region admisible

$\phi_1 = 0.9 \quad \phi_2 = -0.1$

$|\phi_1| < 1 \quad |0.9| < 1$

$\phi_1 + \phi_2 < 1 \quad 0.9 - 0.1 = 0.8 < 1$

$\phi_2 - \phi_1 < 1 \quad -0.1 - 0.9 = -1 < 1$



b) AR(2) $\rho_x(h)$

$\rho_0 = 1$

$\rho_1 = \frac{\phi_1}{1-\phi_2} = \frac{0.9}{1+0.1} = \frac{0.9}{1.1} = 0.\overline{81}$

$\rho_2 = \frac{\phi_1^2}{1-\phi_2} + \phi_2 = \frac{0.9^2}{1.1} - 0.1 = 0.\overline{63}$

Recordar

$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \dots + \phi_q \rho_{k-q}$

$\rho_3 = \phi_1 \rho_2 + \phi_2 \rho_1 = 0.9\left(\frac{0.9^2}{1.1} - 0.1\right) - 0.1\left(\frac{0.9}{1.1}\right) = 0.4\overline{90}$

$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \dots + \phi_q \rho_{k-q}$

$\rho_3 = \phi_1 \rho_2 + \phi_2 \rho_1 = 0.9\left(\frac{0.9^2}{1.1} - 0.1\right) - 0.1\left(\frac{0.9}{1.1}\right) = 0.490$

$\rho_4 = .38$

$\rho_5 = .29$

iv. $X_t + 0.1X_{t-1} - 0.9X_{t-2} = c + Z_t; c = 3$

$\mu = \frac{3}{1 + 0.9 - .1} = \frac{3}{.2} = \frac{30}{2} = 15$

AR(2)

a) Estacionariedad: No es estacionario no cumple con las condiciones

Invertibilidad todo proceso AR es invertible

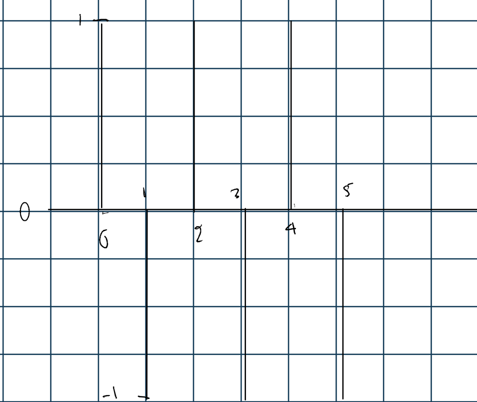
$\phi_1 = -0.1 \quad \phi_2 = 0.9$

$|\phi_1| < 1 \quad |-0.1| < 1$

$\phi_1 + \phi_2 < 1 \quad -0.1 + 0.9 = 0.8 < 1$

$\phi_2 - \phi_1 < 1 \quad 0.9 - -0.1 = 1 \not< 1$

$\therefore$ No estacionario



b) $\rho_x(h)$

$\rho_0 = 1$

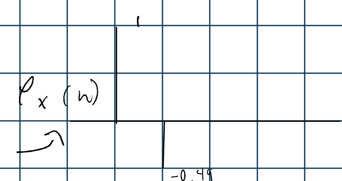
$\rho_1 = \frac{\phi_1}{(1-\phi_2)} = \frac{-0.1}{(1-0.9)} = -1$

$\rho_2 = \frac{\phi_1^2}{(1-\phi_2)} + \phi_2 = \frac{0.01}{0.1} + 0.9 = 1$

$\rho_3 = \phi_1 \rho_2 + \phi_2 \rho_1 = -0.1(1) + 0.9(-1) = -1$

$\rho_4 = 1$

$\rho_5 = -1$



v. $\tilde{X}_t = Z_t + 0.8Z_{t-1}$

MA(1)

a) Estacionariedad todo proceso MA es estacionario

Invertibilidad $|\theta| < 1 \quad 0.8 < 1$ es invertible

b) ρ_x para MA(1) solo hay 2 dif de 0

$\rho_0 = 1$

$\rho_1 = \frac{-\theta}{1+\theta^2} = \frac{-0.8}{1+0.8^2} = 0.49$

$\forall |h| > 1 \quad \rho_x(h) = 0$

vi. $\tilde{X}_t = Z_t + 0.7Z_{t-1} - 0.2Z_{t-2}$

MA(2)

a) Como todo MA es estacionario

Cumple con las condiciones: es invertible

$|\theta_1| < 1 \qquad |-0.7| < 1$

$\theta_1 + \theta_2 < 1 \qquad -0.7 + 0.2 = -0.5 < 1$

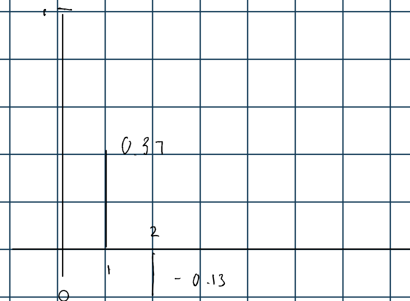
$\theta_2 - \theta_1 < 1 \qquad 0.2 - -0.7 = 0.9 < 1$

b) $\rho_x(h)$ solo $\rho_{0,1,2} \neq 0$

$\rho_0 = 1$

$\rho_1 = \dfrac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2} = .37$

$\rho_2 = \dfrac{-\theta_2}{1+\theta_1^2+\theta_2^2} = -.13$



vii. $\tilde{X}_t = Z_t - 0.7Z_{t-1} + 0.2Z_{t-2}$

MA(2)

a) Como todo MA es estacionario

Como cumple las condiciones es estacionario

$|\theta_1| < 1 \qquad |0.7| < 1$

$\theta_1 + \theta_2 < 1 \qquad 0.7 + (-.2) = 0.5$

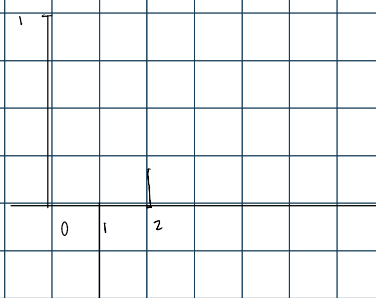
$\theta_2 - \theta_1 < 1 \qquad -0.2 - 0.7 < 1 \qquad -0.9 < 1$

b) $\rho_x(h)$

$\rho_x(0) = 1$

$\rho_x(1) = \dfrac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2} = -0.549$

$\rho_x(2) = \dfrac{-\theta_2}{1+\theta_1^2+\theta_2^2} = 0.13$



viii. $\tilde{X}_t = Z_t + 0.2Z_{t-1} - 0.2Z_{t-2}$

MA(2)

a) Como todo MA es estacionario

Como cumple las condiciones es invertible

$|\theta_1| < 1$

$\theta_1 + \theta_2 < 1 \qquad -0.2 + 0.2 = 0 < 1$

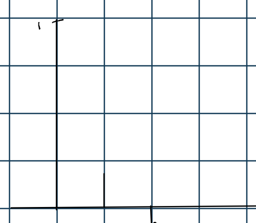
$\theta_2 - \theta_1 < 1 \qquad 0.2 + 0.2 = 0.4 < 1$

b) $\rho_x(h)$

$\rho_x(0) = 1$

$\rho_x(1) = \dfrac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2} = 0.15$

$\rho_x(2) = \dfrac{-\theta_2}{1+\theta_1^2+\theta_2^2} = -0.14$



ix. $\tilde{X}_t = Z_t - 0.1Z_{t-1} + 0.67Z_{t-2} - 0.13Z_{t-3}$

MA(3)

a) Como todo MA es estacionario

los modulos cumplen entonces es invertible

polinomio

$1 - 0.1x + 0.67x^2 - 0.13x^3 = 0$

$x_1 = 5.28 \qquad |z| > 1$

$x_2, x_3 = 0.6 \ldots \qquad \sqrt{a^2 + b^2} = 1.26 > 1$

$1 + 0.1x + 0.67x^2 - 0.13x^3 = 0$

$x_1 = 5.28 \qquad |z| > 1$

$x_2, x_3 = .06 \pm 1.26i \qquad \sqrt{a^2 + b^2} = 1.26 > 1$

Del resumen $\rho_x(h) \;\forall\; MA(q) = \dfrac{-\theta_k + \theta_1\theta_{k-1} + \dots + \theta_{q-k}\theta_q}{(1 + \theta_1^2 + \theta_2^2 + \dots + \theta_q^2)}$

Denominador $= 1 + 0.1^2 + 0.67^2 + 0.13^2 = 1.4758$

$\rho_0 = 1$

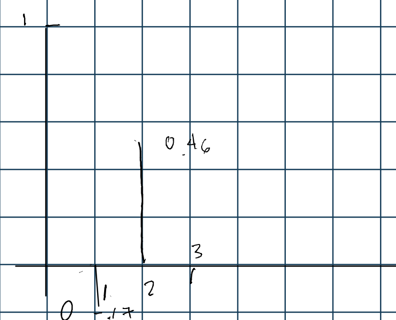
$\rho_1 = (\theta_1^k + \theta_1^1 \theta_2^{k-1} + \theta_2^2 \theta_3^{k-2}) / D = -0.1722$

$\rho_2 = (\theta_2 + \theta_1\theta_3) / D = 0.4627$

$\rho_3 = \theta_3 / D = -0.081$

$\rho_4 = 0$

$\rho_5 = 0$



X. $X_t = Z_t + 0.8X_{t-1} - 0.25X_{t-2} + 0.9X_{t-3}$

AR(3)

a) Como todo AR es invertible

No cumplen los modulos, el proceso no es estacionario

polinomio $-Z_t = -X_t + 0.8Z_{t-1} - 0.25X_{t-2} + 0.9X_{t-3}$

$Z_t = X_t - 0.8X_{t-1} + 0.25X_{t-2} - 0.9X_{t-3}$

$0 = 1 - 0.8x + 0.25x^2 - 0.9x^3$

$x_1 = 0.83 \qquad |z| < 1 \quad \Rightarrow$ no cumple

$x_2, x_3 = -0.27 \pm 1.12i$

El proceso no es estacionario $\Rightarrow$ No aplica Yule-Walker

Se debe hacer una transformación del proceso.

9. Los procesos i-iv del ejercicio 8 podrían escribirse como $X_t = \psi(B)Z_t$. Para cada uno de estos procesos, encuentre las ponderaciones $\psi_1, \dots, \psi_5$ y diga si la suma $\sum_{i=1}^{\infty} \psi_i$ es convergente.

$X_t = \psi(B)Z_t = \sum_{i=0}^{\infty} \psi Z_{t-i}$

Formula de ψ_is

i) $X_t - 0.9X_{t-1} = 4 + Z_t$

$(1 - 0.9B)X_t = 4 + Z_t$

$X_t = 0.9X_{t-1} + 4 + Z_t$

$X_t = 0.9[0.9X_{t-2} + Z_{t-1} + 4] + Z_t + 4$

$X_t = 0.9^2 X_{t-2} + 0.9Z_{t-1} + 0.9(4) + Z_t + 4$

$X_t = 0.9^2[0.9X_{t-3} + Z_{t-2} + 4] + 0.9Z_{t-1} + 0.9(4) + Z_t + 4$

$X_t = 0.9^3 X_{t-3} + 0.9Z_{t-2} + 4(0.9)^2 + 0.9Z_{t-1} + 0.9(4) + Z_t + 4$

$X_t = 0.9^k[X_{t-k}] + \sum_{i=0}^{k-1} (0.9^i Z_{t-i} + 0.9^i(4))$

Si $k \to \infty$

$X_t = 1 + 0.9 + 0.9^2 + \dots$

es una serie geometrica $\quad X_t = \sum_{i=0}^{\infty} \psi_i Z_{t-1} = \sum_{i=0}^{\infty} (0.9)^i (Z_{t-i} + \mu)$

$X_t = 1 + 0.9 + 0.9^2 + \ldots$

es una serie geometrica $\quad X_t = \sum_{i=0}^{\infty} \psi_i Z_{t-1} = \sum_{i=0}^{\infty} (0.9)^i (Z_{t-1} + \mu)$

$\sum \psi_i = \sum_{i=0}^{\infty} (0.9)^i = \frac{0.9}{1-0.9} = 9$

$\psi_1 = 1 \quad \psi_2 = 0.9 \quad \psi_3 = 0.9^2 \quad \psi_4 = 0.9^3 \quad \psi_5 = 0.9^4$

ii) $X_t + 0.1 X_{t-1} = Z_t$

$X_t = -0.1 X_{t-1} + Z_t$

$X_t = -0.1[-0.1 X_{t-2} + Z_{t-1}] + Z_t$

$X_t = 0.1^2 X_{t-2} - 0.1 Z_{t-1} + Z_t$

$X_t = 0.1^2[-0.1 X_{t-3} + Z_{t-2}] - 0.1 Z_{t-1} + Z_t$

$X_t = -0.1^3 X_{t-3} + 0.1^2 Z_{t-2} - 0.1 Z_{t-1}$

$X_t = -0.1^3[-0.1 X_{t-4} + Z_t] + 0.1^2 Z_{t-2} - 0.1 Z_{t-1} + Z_t$

$X_t = 0.1^4 X_{t-4} - 0.1^3 Z_t + 0.1^2 Z_{t-2} - 0.1 Z_{t-1} + Z_t$

$X_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i} \Rightarrow X_t = \sum_{i=0}^{\infty} \phi^i Z_{t-i}$

$\psi_0 = 1 \quad \psi_1 = -0.1 \quad \psi_2 = -0.1^2 \quad \psi_3 = -0.1^3 \quad \psi_4 = -0.1^4 \quad \psi_5 = -0.1^5$

$\sum \psi_i = \sum_{i=0}^{\infty} \phi^i = \sum_{i=0}^{\infty} -0.1^i = \frac{-0.1}{1-(-.1)} = 0.\overline{09}$

iii) $X_t - 0.9 X_{t-1} + 0.1 X_{t-2} = Z_t + 10$

$X_t = 0.9 X_{t-1} - 0.1 X_{t-2} + Z_t + 10$

$X_t = 0.9(0.9 X_{t-2} - 0.1 X_{t-3} + Z_{t-1} + 10] - 0.1[0.9 X_{t-3} - 0.1 X_{t-4} + Z_{t-2} + 10] + Z_t + 10$

$X_t = 0.9^2 X_{t-2} - 0.09 X_{t-3} + 0.9 Z_{t-1} + 9 - 0.09 X_{t-3} + 0.01 X_{t-4} - 0.1 Z_{t-2} + 1 + Z_t + 10$

$X_t = 0.9^2 X_{t-2} - 0.18 X_{t-3} + 0.01 X_{t-4} - 0.1 Z_{t-2} - 1 + 0.9 Z_{t-1} + 9 + Z_t + 10$

$X_t = \sum_{i=0}^{\infty} \psi_i (Z_t - \mu)$

$\psi_0 = 1 \qquad = 1$

$\psi_1 = \phi_1 \qquad = 0.9$

$\psi_2 = \phi_1 \psi_1 + \phi_2 \psi_0 \qquad = 0.9^2 + (-.1)(1) = .71$

$\psi_3 = \phi_1 \psi_2 + \phi_2 \psi_1 \qquad = 0.9(.71) - (0.1)(0.9) = .549$

$\psi_4 = \phi_1 \psi_3 + \phi_2 \psi_2 \qquad = 0.9(.549) - (0.1)(.71) = 0.4231$

$\psi_5 = \phi_1 \psi_4 + \phi_2 \psi_3 \qquad = 0.9(.4231) - (0.1)(.549) = 0.32589$

Como el proceso se demostro estacionario

esta representación MA(∞) es valida

y la media finita;

de esta manera la propiedad

$E[X_t] = \sum_{i=0}^{\infty} \psi_i \mu = \mu \sum_{i=0}^{\infty} \psi_i$

demuestra que la suma converge

de esta ... propiedad

$$E[X_t] = \sum_{i=0}^{\infty} \psi_i \mu = \mu \sum_{i=0}^{\infty} \psi_i$$

demuestra que la suma converge

(v) No es estacionario!! la suma no converge.

$$X_t + 0.1 X_{t-1} - 0.9 X_{t-2} = Z_{t-3}$$

$$X_t = -0.1 X_{t-1} + 0.9 X_{t-2} + Z_{t-3}$$

con lo establecido anteriormente

$$X_t = \sum_{i=0}^{\infty} \psi_i (Z_t + \mu)$$ y sabemos

$$\phi_1 = 0.1 \quad \phi_2 = -0.9$$

$\psi_0 = 1 \qquad = 1$

$\psi_1 = \phi_1 \qquad = 0.1$

$\psi_2 = \phi_1 \psi_1 + \phi_2 \psi_0 = 0.1^2 + (-0.9)(1) = -0.89$

$\psi_3 = \phi_1 \psi_2 + \phi_2 \psi_1 = 0.1(-.89) + (-0.9)(0.1) = -.179$

$\psi_4 = \phi_1 \psi_3 + \phi_2 \psi_2 = 0.1(-.179) + (-0.9)(-.89) = .7831$

$\psi_5 = \phi_1 \psi_4 + \phi_2 \psi_3 = 0.1(.7831) + (-0.9)(-.179) = .23941$

10. Obtener los parámetros de un proceso $AR(3)$ cuyas primeras autocorrelaciones son $\rho_1 = 0.2, \rho_2 = 0.5, \rho_3 = 0.7$. Verificar si el proceso es estacionario.

Se utilizan las ecuaciones de Yule-Walker

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \dots + \phi_p \rho_{k-p}$$

$$\left|\begin{array}{ccc|c} 1 & \rho_1 & \rho_2 & \rho_1 \\ \rho_1 & 1 & \rho_1 & \rho_2 \\ \rho_2 & \rho_1 & 1 & \rho_3 \end{array}\right|$$

resuelto con calculadora

$\phi_1 = -\frac{18}{71} = -0.2535$

$\phi_2 = \frac{57}{142} = 0.4014$

$\phi_3 = \frac{53}{71} = 0.7465$

Es un proceso AR(3)

$$Z_t = X_t (1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)$$

$$1 + 0.2535 X - 0.4014 X^2 - 0.7465 X^3 = 0$$

es el polinomio que corresponde al proceso

$X_1 \quad 1.036 \qquad |z| > 1$

$X_2, X_3 \quad -0.7871 \pm 0.8201i \qquad |z| = \sqrt{a^2 + b^2} > 1$

El proceso es estacionario