Computational Companion to “Flexible 3×3 Nets of Equimodular Elliptic Type” — PROOF OF THE EXISTENCE CRITERION

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Tested on: Mathematica 14.0

Section 1

Results of Section 1 is used in Lemma 1

In[◦]:=
```
(*define*)
σ = (α + β + γ + δ) / 2;
a = Sin[α] / Sin[σ - α];
b = Sin[β] / Sin[σ - β];
c = Sin[γ] / Sin[σ - γ];
d = Sin[δ] / Sin[σ - δ];
M = a b c d;

(*check equalities*)
FullSimplify[1 - a b - Sin[σ] Sin[σ - α - β] / (Sin[σ - α] Sin[σ - β]) == 0]
FullSimplify[1 - b c - Sin[σ] Sin[σ - γ - β] / (Sin[σ - γ] Sin[σ - β]) == 0]
FullSimplify[1 - b d - Sin[σ] Sin[σ - δ - β] / (Sin[σ - δ] Sin[σ - β]) == 0]

FullSimplify[c d - 1 - Sin[σ] Sin[σ - α - β] / (Sin[σ - γ] Sin[σ - δ]) == 0]
FullSimplify[a d - 1 - Sin[σ] Sin[σ - γ - β] / (Sin[σ - α] Sin[σ - δ]) == 0]
FullSimplify[a c - 1 - Sin[σ] Sin[σ - δ - β] / (Sin[σ - α] Sin[σ - γ]) == 0]

FullSimplify[1 - M - Sin[σ] Sin[σ - α - β] Sin[σ - γ - β]
    Sin[σ - δ - β] / (Sin[σ - α] Sin[σ - β] Sin[σ - γ] Sin[σ - δ]) == 0]
```

Out[◦]=
```
True
```

Out[◦]=
```
True
```

Out[ ]=

```
True
```

Out[ ]=

```
True
```

Out[ ]=

```
True
```

Out[ ]=

```
True
```

Out[ ]=

```
True
```

Section 2

Results of Section 2 is used in Proof of Proposition 2

In[∘]:=
```
(*define*)
σ = (α + β + γ + δ) / 2;

(*check equalities*)
(*Cos[α]*)
FullSimplify[
 (ε (Sin[σ - β] Sin[σ] + Sin[σ - α] Sin[σ - α - β] - Sin[σ - γ] Sin[σ - γ - β] -
       Sin[σ - δ] Sin[σ - δ - β])) / (2 Sin[γ] Sin[δ]) == ε Cos[α]]

(*Cos[β]*)
FullSimplify[
 (ε (Sin[σ - α] Sin[σ] + Sin[σ - β] Sin[σ - α - β] + Sin[σ - δ] Sin[σ - γ - β] +
       Sin[σ - γ] Sin[σ - δ - β])) / (2 Sin[γ] Sin[δ]) == ε Cos[β]]

(*Cos[γ]*)
FullSimplify[
 (ε (Sin[σ - β] Sin[σ] - Sin[σ - α] Sin[σ - α - β] + Sin[σ - γ] Sin[σ - γ - β] -
       Sin[σ - δ] Sin[σ - δ - β])) / (2 Sin[α] Sin[δ]) == ε Cos[γ]]

(*Cos[δ]*)
FullSimplify[
 (ε (Sin[σ - β] Sin[σ] - Sin[σ - α] Sin[σ - α - β] - Sin[σ - γ] Sin[σ - γ - β] +
       Sin[σ - δ] Sin[σ - δ - β])) / (2 Sin[α] Sin[γ]) == ε Cos[δ]]

(*Cos[σ]*)
FullSimplify[(Sin[σ - β] Sin[σ] ^2 - Sin[σ - α] Sin[σ - α - β] Sin[σ] -
     Sin[σ - γ] Sin[σ - γ - β] Sin[σ] - Sin[σ - δ] Sin[σ - δ - β] Sin[σ] -
     2 Sin[σ - α] Sin[σ - γ] Sin[σ - δ]) / (2 Sin[α] Sin[γ] Sin[δ]) == Cos[σ]]
```

Out[∘]=
```
True
```

Out[∘]=
```
True
```

Out[∘]=
```
True
```

Out[∘]=
```
True
```

Out[∘]=
```
True
```

Section 3

Results of Section 3 is used in Proof of Lemma 7

In[ ]:=
```
(*define*)
σ = (α + β + γ + δ) / 2;
thetaIndex[k_] := Mod[k - 1, 4] + 1; (*cyclic index:1..4*)
A[i_, j_] /; 1 ≤ i ≤ 4 && 1 ≤ j ≤ 4 :=
  4 Cos[θ[thetaIndex[i]] / 2 + (Pi / 4) i j (j - 1) + (Pi / 2) j] ^2 *
   Cos[θ[thetaIndex[i - 1]] / 2 + (Pi / 4) (i - 1) j (j - 1) + (Pi / 2) j] ^2;
(*Table[A[i,j],{i,1,4},{j,1,4}];*)
ξ[i_?IntegerQ] /; 1 ≤ i ≤ 4 :=
  If[OddQ[i], θ[thetaIndex[i]], θ[thetaIndex[i - 1]]];
η[i_?IntegerQ] /; 1 ≤ i ≤ 4 := If[OddQ[i], θ[thetaIndex[i - 1]], θ[thetaIndex[i]]];
(*Table[{i,ξ[i],η[i]},{i,1,4}];*)

(*check equalities*)
lhs[i_Integer] := ε (A[i, 1] Sin[σ - β] Sin[σ] + A[i, 2] Sin[σ - γ] Sin[σ - γ - β] +
      A[i, 3] Sin[σ - α] Sin[σ - α - β] +
      A[i, 4] Sin[σ - δ] Sin[σ - δ - β]) / (2 Sin[α] Sin[γ]);
rhs[i_Integer] :=
  ε (Cos[β] - Cos[γ] (Cos[α] Cos[δ] + Cos[ξ[i]] Sin[α] Sin[δ]) - Cos[η[i]]
      Sin[γ] (Cos[α] Sin[δ] - Cos[ξ[i]] Sin[α] Cos[δ])) / (Sin[α] Sin[γ]);
expr[i_Integer] := FullSimplify[Equal[lhs[i], rhs[i]]];
(*---Evaluate for i=1..4---*)
Table[{i, expr[i]}, {i, 1, 4}]
```

Out[ ]=
```
{{1, True}, {2, True}, {3, True}, {4, True}}
```

Section 4

Result of Section 4 is used in Proof of Lemma 9 in Appendix D

In[ ]:=
```
(*define*)
σ = (α2 + β2 + γ2 + δ2) / 2;
(*D_{α2 θ1}*)
Dα2θ1 =
  Sqrt[Sin[θ1]^2 Sin[α2]^2 + (Cos[α2] Sin[δ2] - Cos[θ1] Sin[α2] Cos[δ2])^2];
(*R_{α2 γ2 θ1} =
 (cos β2-cos γ2 (cos α2 cos δ2+cos θ1 sin α2 sin δ2))/(D_{α2 θ1} sin γ2)*)
Rα2γ2θ1 = (Cos[β2] - Cos[γ2] (Cos[α2] Cos[δ2] + Cos[θ1] Sin[α2] Sin[δ2])) /
   (Dα2θ1 Sin[γ2]);

(*check equality*)
FullSimplify[Rα2γ2θ1^2 - 1 == (4 Sin[α2]^2 Sin[δ2]^2) / (Dα2θ1^2 Sin[γ2]^2) *
   (Sin[θ1 / 2]^2 - (Sin[σ - α2 - β2] Sin[σ - δ2 - β2]) / (Sin[α2] Sin[δ2])) *
   (Sin[θ1 / 2]^2 - (Sin[σ - α2] Sin[σ - δ2]) / (Sin[α2] Sin[δ2]))]
```

Out[ ]=
```
True
```