Computational Companion to “Quasi-Symmetric Nets: A Constructive Approach to the Equimodular Elliptic Type of Kokotsakis Polyhedra” — Example 1 Helper

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Tested on: Mathematica 14.0

In[5327]:=

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(*===========================================================================
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 =====*)
(*Quit*)
(*All angle sets in degrees*)
alpha1 = 105;
beta1 = 15;
gamma1 = 120;
delta1 = 90;

sigma1 = (alpha1 + beta1 + gamma1 + delta1) / 2;

anglesDeg = {
   {alpha1, beta1, gamma1, delta1}, (*Vertex 1*)
   {delta1, gamma1, beta1, alpha1}, (*Vertex 2*)
   {180 - delta1, 180 - gamma1, 180 - beta1, 180 - alpha1}, (*Vertex 3*)
   {alpha1, beta1, gamma1, delta1} (*Vertex 4*)};

(*-----------------------------------------*)
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(*Function to compute sigma from 4 angles*)
computeSigma[{α_, β_, γ_, δ_}] := (α + β + γ + δ) / 2;

(*------------------------------------------*)
(*Function to compute a,b,c,d from angles*)
computeABCD[{α_, β_, γ_, δ_}] :=
  Module[{alpha = α Degree, beta = β Degree, gamma = γ Degree,
    delta = δ Degree, sigma}, sigma = computeSigma[{α, β, γ, δ} Degree];
   {Sin[alpha] / Sin[sigma - alpha], Sin[beta] / Sin[sigma - beta],
    Sin[gamma] / Sin[sigma - gamma], Sin[delta] / Sin[sigma - delta]}];

(*------------------------------------------*)
(*Compute all σ values*)
sigmas = computeSigma /@ anglesDeg;

(*Compute all {a,b,c,d} values*)
results = computeABCD /@ anglesDeg;

(*Optional:extract individual values*)
{a1, b1, c1, d1} = results⟦1⟧;
{a2, b2, c2, d2} = results⟦2⟧;
{a3, b3, c3, d3} = results⟦3⟧;
{a4, b4, c4, d4} = results⟦4⟧;
{σ1, σ2, σ3, σ4} = sigmas;


(*========================================================================
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(*===============================
   CONDITION (N.0)===============================*)
(*========================================================================
 =====*)
uniqueCombos = {{1, 1, 1, 1}, {1, 1, 1, -1}, {1, 1, -1, -1}, {1, 1, -1, 1},
   {1, -1, 1, 1}, {1, -1, -1, 1}, {1, -1, 1, -1}, {1, -1, -1, -1}};

checkConditionN0Degrees[{α_, β_, γ_, δ_}] :=
  Module[{angles = {α, β, γ, δ}, results},
   results = Mod[uniqueCombos.angles, 360] // Chop;
   ! MemberQ[results, 0]];

conditionsN0 = checkConditionN0Degrees /@ anglesDeg;
allVerticesPass = And @@ conditionsN0;

Column[{TextCell[Style["================= CONDITION (N.0) =================",
    Darker[Green], Bold, 16], "Text"],
  If[allVerticesPass,
   Style["✔ All vertices satisfy (N.0).", Darker[Green], Bold],
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   Style["✘ Some vertices fail (N.0).", Red, Bold]]}]


(*============================================================================
 =====*)
(*================================
   CONDITION (N.3)================================*)
(*============================================================================
 =====*)
Ms = FullSimplify[Times @@@ results];
allEqualQ = Simplify[Equal @@ Ms];

Column[{TextCell[Style["================= CONDITION (N.3) =================",
    Blue, Bold, 16], "Text"], If[allEqualQ,
   Row[{Style["✓ M1 = M2 = M3 = M4 = ", Bold], Ms⟦1⟧}],
   Style["✘ M_i are not all equal.", Red, Bold]]}]


(*============================================================================
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(*===========================CONDITION (N.4)===========================*)
(*============================================================================
 =====*)
aList = results⟦All, 1⟧;
cList = results⟦All, 3⟧;
dList = results⟦All, 4⟧;

rList = FullSimplify /@ (aList * dList); {r1, r2, r3, r4} = rList;
sList = FullSimplify /@ (cList * dList); {s1, s2, s3, s4} = sList;

Column[{TextCell[Style["================= CONDITION (N.4) =================",
    Darker[Blue], Bold, 16], "Text"],
  If[r1 == r2 && r3 == r4 && s1 == s4 && s2 == s3, Column[
    {Row[{Style["✓ r1 = r2 = ", Bold], r1, Style["; ✓ r3 = r4 = ", Bold], r3}],
     Row[{Style["✓ s1 = s4 = ", Bold], s1, Style["; ✓ s2 = s3 = ", Bold],
        s2}]}], Style["✘ Condition (N.4) fails.", Red, Bold]]
 }]


(*============================================================================
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(*==================================
  CONDITION (N.5)===============================*)
(*============================================================================
 =====*)
fList = FullSimplify /@ (aList * cList); {f1, f2, f3, f4} = fList;
M1 = Ms⟦1⟧;
```

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m[M1_] := Piecewise[{{1 - M1, M1 < 1}, {(M1 - 1) / M1, M1 > 1}}] // N;
K = EllipticK[m[M1]] // N;
Kp = EllipticK[1 - m[M1]] // N;

computeTi[i_] :=
  Module[{sigma = sigmas[[i]], r = rList[[i]], s = sList[[i]], f = fList[[i]], base},
   base = I * Im[InverseJacobiDN[Piecewise[
         {{Sqrt[f], M1 < 1}, {1 / Sqrt[f], M1 > 1}}], m[M1]]] / Kp;
   Which[sigma < 180, Which[r > 1 && s > 1, base, (r < 1 && s > 1) || (r > 1 && s < 1),
     1 + base, r < 1 && s < 1, 2 + base], sigma > 180, Which[r > 1 && s > 1,
     2 + base, (r < 1 && s > 1) || (r > 1 && s < 1), 3 + base, r < 1 && s < 1, base]]];

tList = computeTi /@ Range[4];
RoundWithTolerance[x_, tol_ : 10^-6] := Module[{nearest}, nearest = Round[x];
   If[Abs[x - nearest] ≤ tol, nearest, x]];

checkValidCombination[M1_, ε_ : 10^-6] :=
  Module[{i, combo, dotProd, rePart, imPart, n2, expr, foundQ = False},
   Print[Style["⚠ Approximate validation using ε-tolerance. For rigorous
       proof, see the referenced paper.", Darker@Orange, Italic]];
   Do[combo = uniqueCombos[[i]];
    dotProd = tList.combo;
    rePart = Abs[Re[dotProd]];
    imPart = Abs[Im[dotProd]];
    If[M1 < 1,
     If[Mod[RoundWithTolerance[rePart], 4] < ε,
      If[Mod[RoundWithTolerance[imPart], 2] < ε, expr =
        tList[[1]] + combo[[2]] × tList[[2]] + combo[[3]] × tList[[3]] + combo[[4]] × tList[[4]];
       Print[Style["✅ Valid Combination Found (M < 1):",
          Darker[Green], Bold], "\n", Style["e1 = ", Bold], combo[[2]],
        Style[", e2 = ", Bold], combo[[3]], Style[", e3 = ", Bold], combo[[4]],
        "\n", Style["t1 = ", Bold], Re[tList[[1]]], "K + ", Im[tList[[1]]], "iK'",
        "\n", Style["t2 = ", Bold], Re[tList[[2]]], "K + ", Im[tList[[2]]], "iK'",
        "\n", Style["t3 = ", Bold], Re[tList[[3]]], "K + ", Im[tList[[3]]], "iK'",
        "\n", Style["t4 = ", Bold], Re[tList[[4]]], "K + ", Im[tList[[4]]],
        "iK'", "\n", Style["t1 + e1*t2 + e2*t3 + e3*t4 = ", Bold],
        Re[expr], "K + ", Im[expr], "iK'"];
       foundQ = True;
       Break[]]]];
    If[M1 > 1,
     If[Mod[RoundWithTolerance[imPart], 2] < ε,
      n2 = Quotient[RoundWithTolerance[imPart], 2];
      If[Mod[RoundWithTolerance[Abs[rePart - 2 n2]], 4] < ε, expr =
        tList[[1]] + combo[[2]] × tList[[2]] + combo[[3]] × tList[[3]] + combo[[4]] × tList[[4]];
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    Print[Style["✅ Valid Combination Found (M > 1):",
       Darker[Green], Bold], "\n", Style["e1 = ", Bold], combo[[2]],
      Style[", e2 = ", Bold], combo[[3]], Style[", e3 = ", Bold], combo[[4]],
      "\n", Style["t1 = ", Bold], Re[tList[[1]]], "K + ", Im[tList[[1]]], "iK'",
      "\n", Style["t2 = ", Bold], Re[tList[[2]]], "K + ", Im[tList[[2]]], "iK'",
      "\n", Style["t3 = ", Bold], Re[tList[[3]]], "K + ", Im[tList[[3]]], "iK'",
      "\n", Style["t4 = ", Bold], Re[tList[[4]]], "K + ", Im[tList[[4]]],
      "iK'", "\n", Style["t1 + e1*t2 + e2*t3 + e3*t4 = ", Bold],
      Re[expr], "K + ", Im[expr], "iK'"];
    foundQ = True;
    Break[]]]], {i, Length[uniqueCombos]}];
  If[! foundQ, Print[Style["❌ No valid combination found.", Red, Bold], "\n",
    Style["t1 = ", Bold], Re[tList[[1]]], "K + ", Im[tList[[1]]], "iK'", "\n",
    Style["t2 = ", Bold], Re[tList[[2]]], "K + ", Im[tList[[2]]], "iK'", "\n",
    Style["t3 = ", Bold], Re[tList[[3]]], "K + ", Im[tList[[3]]], "iK'", "\n",
    Style["t4 = ", Bold], Re[tList[[4]]], "K + ", Im[tList[[4]]], "iK'", "\n"]];];


Column[{TextCell[Style["================= CONDITION (N.5) =================",
     Purple, Bold, 16], "Text"]}]
res = checkValidCombination[M1];


(*============================================================================
 =====*)
(*================================
   OTHER PARAMETER=================================*)
(*============================================================================
 =====*)
Column[
 {TextCell[Style["================= OTHER PARAMETERS =================",
     Darker[Purple], Bold, 16], "Text"],
  Row[{Style["u = ", Bold], 1 - M1}],
  Row[{Style["σ1 = ", Bold], σ1 Degree, Style[", σ2 = ", Bold], σ2 Degree,
     Style[", σ3 = ", Bold], σ3 Degree, Style[", σ4 = ", Bold], σ4 Degree}],
  Row[{Style["f1 = ", Bold], f1, Style[", f2 = ", Bold],
     f2, Style[", f3 = ", Bold], f3, Style[", f4 = ", Bold], f4}],
  Row[{Style["z1 = ", Bold], FullSimplify[1 / (f1 - 1)], Style[", z2 = ", Bold],
     FullSimplify[1 / (f2 - 1)], Style[", z3 = ", Bold], FullSimplify[1 / (f3 - 1)],
     Style[", z4 = ", Bold], FullSimplify[1 / (f4 - 1)]}],
  Row[{Style["x1 = ", Bold], FullSimplify[1 / (r1 - 1)], Style[", x2 = ", Bold],
     FullSimplify[1 / (r2 - 1)], Style[", x3 = ", Bold], FullSimplify[1 / (r3 - 1)],
     Style[", x4 = ", Bold], FullSimplify[1 / (r4 - 1)]}],
  Row[{Style["y1 = ", Bold], FullSimplify[1 / (s1 - 1)], Style[", y2 = ", Bold],
     FullSimplify[1 / (s2 - 1)], Style[", y3 = ", Bold], FullSimplify[1 / (s3 - 1)],
     Style[", y4 = ", Bold], FullSimplify[1 / (s4 - 1)]}],
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  Row[{Style["p1 = ", Bold], Simplify[Sqrt[r1 - 1]],
    Style[", p2 = ", Bold], Simplify[Sqrt[r2 - 1]], Style[", p3 = ", Bold],
    Simplify[Sqrt[r3 - 1]], Style[", p4 = ", Bold], Simplify[Sqrt[r4 - 1]]}],
  Row[{Style["q1 = ", Bold], Simplify[Sqrt[s1 - 1]],
    Style[", q2 = ", Bold], Simplify[Sqrt[s2 - 1]], Style[", q3 = ", Bold],
    Simplify[Sqrt[s3 - 1]], Style[", q4 = ", Bold], Simplify[Sqrt[s4 - 1]]}],
  Row[{Style["p1·q1 = ", Bold], Simplify[Sqrt[r1 - 1] Sqrt[s1 - 1]],
    Style[", p2·q2 = ", Bold], Simplify[Sqrt[r2 - 1] Sqrt[s2 - 1]],
    Style[", p3·q3 = ", Bold], Simplify[Sqrt[r3 - 1] Sqrt[s3 - 1]],
    Style[", p4·q4 = ", Bold], Simplify[Sqrt[r4 - 1] Sqrt[s4 - 1]]}],
  Row[{Style["\!\(\*OverscriptBox[\(α1\), \(_\)]\) = ", Bold],
    σ1 - anglesDeg⟦1, 1⟧, "°", Style[
     ", \!\(\*OverscriptBox[\(β1\), \(_\)]\) = ", Bold], σ1 - anglesDeg⟦1, 2⟧,
    "°", Style[", \!\(\*OverscriptBox[\(γ1\), \(_\)]\) = ", Bold],
    σ1 - anglesDeg⟦1, 3⟧, "°",
    Style[", \!\(\*OverscriptBox[\(δ1\), \(_\)]\) = ", Bold],
    σ1 - anglesDeg⟦1, 4⟧, "°"}]
 }]


(*=============================================================================
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(*=============================
  BRICARD's EQUATIONS=============================*)
(*=============================================================================
 =====*)

(*Bricard's equation as defined in terms of angles and variables*)
BricardsEquation[{α_, β_, γ_, δ_}, σ_, x_, y_] := Module[
   {c22, c20, c02, c11, c00},
   c22 = Sin[σ - δ] Sin[σ - δ - β];
   c20 = Sin[σ - α] Sin[σ - α - β];
   c02 = Sin[σ - γ] Sin[σ - γ - β];
   c11 = -Sin[α] Sin[γ];
   c00 = Sin[σ] Sin[σ - β];
   c22 x^2 y^2 + c20 x^2 + c02 y^2 + 2 c11 x y + c00
  ];


(*Step 1: Bricard's system of equations*)
Column[{(*Header*)TextCell[
   Style["================= Bricard's System of Equations =================",
    Red, Bold, 16], "Text"], (*Explanatory note*)
  Row[
   {TextCell[Style["We introduce new notation for the cotangents of half of
       the dihedral angles. Denote w₁:= ", GrayLevel[0.3], 13],
     "Text"], TraditionalForm[cot[Subscript[θ, 1] / 2]], TextCell[
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     Style[", t:= ", GrayLevel[0.3], 13], "Text"],
    TraditionalForm[cot[Subscript[θ, 2] / 2]],
    TextCell[Style[", w₂:= ", GrayLevel[0.3], 13], "Text"],
    TraditionalForm[cot[Subscript[θ, 3] / 2]],
    TextCell[Style[", and z:= ", GrayLevel[0.3], 13], "Text"],
    TraditionalForm[cot[Subscript[θ, 4] / 2]]
   }], Spacer[12],
  (*Traditional form results*)Row[{"P₁(w₁, z) = ",
    TraditionalForm[Collect[FullSimplify[BricardsEquation[anglesDeg⟦1⟧ Degree,
        sigmas⟦1⟧ Degree, w₁, z]], w₁]], " = 0"}], Spacer[6],
  Row[{"P₂(w₁, t) = ",
    TraditionalForm[Collect[FullSimplify[BricardsEquation[anglesDeg⟦2⟧ Degree,
        sigmas⟦2⟧ Degree, w₁, t]], w₁]], " = 0"}], Spacer[6],
  Row[{"P₃(w₂, t) = ",
    TraditionalForm[Collect[FullSimplify[BricardsEquation[anglesDeg⟦3⟧ Degree,
        sigmas⟦3⟧ Degree, w₂, t]], w₂]], " = 0"}], Spacer[6],
  Row[{"P₄(w₂, z) = ", TraditionalForm[Collect[FullSimplify[BricardsEquation[
        anglesDeg⟦4⟧ Degree, sigmas⟦4⟧ Degree, w₂, z]], w₂]], " = 0"}]
 }]

(*discriminant*)
Disc[t_] := (Sin[sigma1 Degree] Sin[(sigma1 - alpha1 - beta1) Degree] +
     t^2 Sin[(sigma1 - alpha1) Degree] Sin[(sigma1 - beta1) Degree])
   (Sin[(sigma1 - gamma1) Degree] Sin[(sigma1 - delta1) Degree] + t^2 Sin[
       (sigma1 - beta1 - gamma1) Degree] Sin[(sigma1 - beta1 - delta1) Degree]);

(*==================================
 FLEXION 1==================================*)
e0 = 1;
W2[t_] := t;
Z[t_] := (-t Sin[beta1 Degree] + e0 Sqrt[Disc[t]]) /
   (Sin[(sigma1 - delta1) Degree] Sin[(sigma1 - alpha1 - beta1) Degree] +
     t^2 Sin[(sigma1 - alpha1) Degree] Sin[(sigma1 - beta1 - delta1) Degree]);
U[t_] := (t Sin[beta1 Degree] + e0 Sqrt[Disc[t]]) /
   (Sin[(sigma1 - delta1) Degree] Sin[(sigma1 - alpha1 - beta1) Degree] +
     t^2 Sin[(sigma1 - alpha1) Degree] Sin[(sigma1 - beta1 - delta1) Degree]);
W1[t_] := e0 Sqrt[Disc[t]] /
    (Sin[(sigma1 - gamma1) Degree] Sin[(sigma1 - delta1) Degree] +
      t^2 Sin[(sigma1 - beta1 - gamma1) Degree]
       Sin[(sigma1 - beta1 - delta1) Degree]);

(*==================================
 FLEXION 2==================================*)
e00 = -1;
W22[t_] := t;
Z2[t_] := (-t Sin[beta1 Degree] + e00 Sqrt[Disc[t]]) /
   (Sin[(sigma1 - delta1) Degree] Sin[(sigma1 - alpha1 - beta1) Degree] +
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      t^2 Sin[(sigma1 - alpha1) Degree] Sin[(sigma1 - beta1 - delta1) Degree]);
U2[t_] := ( t Sin[beta1 Degree] + e00 Sqrt[Disc[t]]) /
    (Sin[(sigma1 - delta1) Degree] Sin[(sigma1 - alpha1 - beta1) Degree] +
      t^2 Sin[(sigma1 - alpha1) Degree] Sin[(sigma1 - beta1 - delta1) Degree]);
W12[t_] := e00 Sqrt[Disc[t]] /
     (Sin[(sigma1 - gamma1) Degree] Sin[(sigma1 - delta1) Degree] +
       t^2 Sin[(sigma1 - beta1 - gamma1) Degree]
        Sin[(sigma1 - beta1 - delta1) Degree]);


(*Step 2: Checking that formula from Theorem 1 simplifies to (E.1)*)
Column[
 {(*Header*)TextCell[Style["================= FLEXIONS =================",
    Darker[Red], Bold, 16], "Text"], (*Explanatory note*)
  TextCell[Style["Solutions to Bricard's equations under a free parameter
      t ∈ ℂ using Theorem 1:", GrayLevel[0.3], 13], "Text"], Spacer[12],
  (*Heading for results*)TextCell[Style["Solution 1:", Bold, 14], "Text"],
  Spacer[6], (*Traditional form results*)
  Row[{"w1(t) = ", TraditionalForm[FullSimplify[Z[t]]]}], Spacer[6],
  Row[{"t(t) = ", TraditionalForm[FullSimplify[W2[t]]]}], Spacer[6],
  Row[{"w2(t) = ", TraditionalForm[FullSimplify[U[t]]]}], Spacer[6],
  Row[{"z(t) = ", TraditionalForm[FullSimplify[W1[t]]]}], Spacer[12],
  (*Heading for results*)TextCell[Style["Solution 2:", Bold, 14], "Text"],
  Spacer[6], (*Traditional form results*)
  Row[{"w1(t) = ", TraditionalForm[FullSimplify[Z2[t]]]}], Spacer[6],
  Row[{"t(t) = ", TraditionalForm[FullSimplify[W22[t]]]}], Spacer[6],
  Row[{"w2(t) = ", TraditionalForm[FullSimplify[U2[t]]]}], Spacer[6],
  Row[{"z(t) = ", TraditionalForm[FullSimplify[W12[t]]]}]
 }]


(*Step 3: Checking that (E.1) solves P2(w1, t) =
 0 and P3(w2, t) = 0 even when +- signs do NOT agree*)
Column[
 {(*Header*)TextCell[Style["================= Equations: P2(w1, t) = 0
      and P3(w2, t) = 0 =================",
    Orange, Bold, 16], "Text"], (*Explanatory note*)TextCell[
   Style["Let w1s1 and w1s2 be formulas for w1(t) from solutions
      1 and 2, respectively. Similarly, let w2s1 and w2s2 be
      the formulas for w2(t). We show that all four pairs -
      (w1s1, w2s1), (w1s1, w2s2), (w1s2, w2s1), and (w1s2, w2s2)
      - solve equations P2(w1, t) = 0 and P3(w2, t) = 0.",
    GrayLevel[0.3], 13], "Text"], Spacer[12],
  (*Heading for results*)TextCell[Style["Pair 1:", Bold, 14], "Text"],
  Spacer[6], (*Traditional form results*)
  Row[{"P2(w1s1, t) = ", TraditionalForm[Collect[FullSimplify[
       BricardsEquation[anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z[t], t]], w1]],
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      "  ", Style["✓", Darker[Green], Bold]}], Spacer[6],
    Row[{"P_3(w_{2s1}, t) = ", TraditionalForm[Collect[FullSimplify[
          BricardsEquation[anglesDeg⟦3⟧ Degree, sigmas⟦3⟧ Degree, U[t], t]], w_2]],
      "  ", Style["✓", Darker[Green], Bold]}], Spacer[12],

    TextCell[Style["Pair 2:", Bold, 14], "Text"],
    Spacer[6], (*Traditional form results*)
    Row[{"P_2(w_{1s1}, t) = ", TraditionalForm[Collect[FullSimplify[
          BricardsEquation[anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z[t], t]], w_1]],
      "  ", Style["✓", Darker[Green], Bold]}], Spacer[6],
    Row[{"P_3(w_{2s2}, t) = ", TraditionalForm[Collect[FullSimplify[
          BricardsEquation[anglesDeg⟦3⟧ Degree, sigmas⟦3⟧ Degree, U2[t], t]], w_2]],
      "  ", Style["✓", Darker[Green], Bold]}], Spacer[12],

    TextCell[Style["Pair 3:", Bold, 14], "Text"],
    Spacer[6], (*Traditional form results*)
    Row[{"P_2(w_{1s2}, t) = ", TraditionalForm[Collect[FullSimplify[BricardsEquation[
           anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z2[t], t]], w_1]],
      "  ", Style["✓", Darker[Green], Bold]}], Spacer[6],
    Row[{"P_3(w_{2s1}, t) = ", TraditionalForm[Collect[FullSimplify[
          BricardsEquation[anglesDeg⟦3⟧ Degree, sigmas⟦3⟧ Degree, U[t], t]], w_2]],
      "  ", Style["✓", Darker[Green], Bold]}], Spacer[12],

    TextCell[Style["Pair 4:", Bold, 14], "Text"],
    Spacer[6], (*Traditional form results*)
    Row[{"P_2(w_{1s2}, t) = ", TraditionalForm[Collect[FullSimplify[BricardsEquation[
           anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z2[t], t]], w_1]],
      "  ", Style["✓", Darker[Green], Bold]}], Spacer[6],
    Row[{"P_3(w_{2s2}, t) = ", TraditionalForm[Collect[FullSimplify[
          BricardsEquation[anglesDeg⟦3⟧ Degree, sigmas⟦3⟧ Degree, U2[t], t]], w_2]],
      "  ", Style["✓", Darker[Green], Bold]}]
  }]

(*Step 4: Checking that (E.1) does NOT satisfy w_1(t)/w_2(t) =
  c and w_1(t)w_2(t) =
   c when c is less or greater -1 even when +- signs do NOT agree*)
tMin = 0;
tMax = 5;

expressions = {Z[t] * U[t], Z[t] / U[t], Z[t] * U2[t], Z[t] / U2[t],
    Z2[t] * U[t], Z2[t] / U[t], Z2[t] * U2[t], Z2[t] / U2[t]};
labels = {"w_{1s1} * w_{2s1}", "w_{1s1} / w_{2s1}", "w_{1s1} * w_{2s2}", "w_{1s1} / w_{2s2}",
    "w_{1s2} * w_{2s1}", "w_{1s2} / w_{2s1}", "w_{1s2} * w_{2s2}", "w_{1s2} / w_{2s2}"};
Column[
 {TextCell[Style["============ Plots of w_1(t)w_2(t) and w_1(t)/w_2(t) For All
      Pairs ============", Darker[Orange], Bold, 16], "Text"],
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  (*Explanatory text*)
  TextCell[Style["Checking that all four pairs - (w1s1, w2s1), (w1s1, w2s2),
      (w1s2, w2s1), and (w1s2, w2s2) - does NOT satisfy w1(t)/w2(t) =
      c and w1(t)w2(t) = c when c ≠ -1.", GrayLevel[0.3]], "Text"],
  Spacer[12],

  (*Plots panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions⟦i⟧, {t, tMin, tMax},
       PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → All,
       AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
 }]


(*Step 5: Solving P1(w1, z) = 0 and P4(w2, z) = 0 for w1 and w2*)
(*Define the first Bricard equation P1(Z,z)=0*)
P1 = BricardsEquation[anglesDeg⟦1⟧ Degree, sigmas⟦1⟧ Degree, Z, z];
(*Solve for Z1,Z2*)
solutionZExpressions = (Z /. FullSimplify[Solve[P1 == 0, Z]]);
Zz1[z_] := solutionZExpressions⟦1⟧;
Zz2[z_] := solutionZExpressions⟦2⟧;

(*Define the fourth Bricard equation P4(U,z)=0*)
P4 = BricardsEquation[anglesDeg⟦4⟧ Degree, sigmas⟦4⟧ Degree, U, z];
(*Solve for U1,U2*)
solutionUExpressions = (U /. FullSimplify[Solve[P4 == 0, U]]);
Uu1[z_] := solutionUExpressions⟦1⟧;
Uu2[z_] := solutionUExpressions⟦2⟧;

Column[
 {(*Header*)TextCell[Style["================= Equations: P1(w1, z) = 0
      and P4(w2, z) = 0 =================",
    Darker[Cyan], Bold, 16], "Text"], (*Explanatory note*)
  TextCell[Style["We solve P1(w1, z) = 0 and P4(w2, z) = 0 for w1 and w2.",
    GrayLevel[0.3], 13], "Text"], Spacer[12],
  (*Heading for results*)Row[{"w1,1(z) = ", TraditionalForm[Zz1[z]]}],
  Row[{"w1,2(z) = ", TraditionalForm[Zz2[z]]}],
  Spacer[6], Row[{"w2,1(z) = ", TraditionalForm[Uu1[z]]}],
  Row[{"w2,2(z) = ", TraditionalForm[Uu2[z]]}]
 }]

(*Step 6: Checking that pairs (w1,1(z), w2,1(z)), (w1,1(z), w2,2(z)),
(w1,2(z), w2,1(z)), and (w1,2(z), w2,2(z)) does NOT satisfy w1(z)/w2(z) =
 c and w1(z)w2(z) = c when c is less or greater 1*)
zMin = Sqrt[(3 - Sqrt[3]) / (3 + Sqrt[3])];
zMax = 1;
```

```
expressions = {Zz1[z] * Uu1[z], Zz1[z] / Uu1[z], Zz1[z] * Uu2[z], Zz1[z] / Uu2[z],
   Zz2[z] * Uu1[z], Zz2[z] / Uu1[z], Zz2[z] * Uu2[z], Zz2[z] / Uu2[z]};
labels = {"w₁₁ * w₂₁", "w₁₁ / w₂₁", "w₁₁ * w₂₂",
   "w₁₁ / w₂₂", "w₁₂ * w₂₁", "w₁₂ / w₂₁", "w₁₂ * w₂₂", "w₁₂ / w₂₂"};
Column[
 {TextCell[Style["============ Plots of w₁(z)w₂(z) and w₁(z)/w₂(z) For All
      Pairs ============", Magenta, Bold, 16], "Text"],

  (*Explanatory text*)
  TextCell[Style["Checking that all four pairs - (w₁₁(z),
      w₂₁(z)), (w₁₁(z), w₂₂(z)), (w₁₂(z), w₂₁(z)), and
      (w₁₂(z), w₂₂(z)) - does NOT satisfy w₁(z)/w₂(z) = c and
      w₁(z)w₂(z) = c when c ≠ +1.", GrayLevel[0.3]], "Text"],
  Spacer[12],

  (*Plots panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions⟦i⟧, {z, zMin, zMax},
       PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → All,
       AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
 }]

(*Compute and print all P_i for flexion 1*)
TextCell[
 Style["================ FLEXIBILITY (Double Checking) ================",
  Darker[Magenta], Bold, 16], "Text"]
funcs = {{w₁, z}, {w₁, t}, {w₂, t}, {w₂, z}};
TextCell[Style["Solution 1:", Bold, 14], "Text"]
Do[angles = anglesDeg⟦i⟧ Degree;
  sigma = sigmas⟦i⟧ Degree;
  {α, β, γ, δ} = angles;
  poly = Which[i == 1, BricardsEquation[{α, β, γ, δ}, sigma, Z[t], W1[t]],
    i == 2, BricardsEquation[{α, β, γ, δ}, sigma, Z[t], W2[t]],
    i == 3, BricardsEquation[{α, β, γ, δ}, sigma, U[t], W2[t]],
    i == 4, BricardsEquation[{α, β, γ, δ}, sigma, U[t], W1[t]]];
  Print[Row[{Subscript["P", i], "(", funcs⟦i, 1⟧,
     ", ", funcs⟦i, 2⟧, ") = ", FullSimplify[poly]}]], {i, 1, 4}];

(*Compute and print all P_i for flexion 2*)
TextCell[Style["Solution 2:", Bold, 14], "Text"]
funcs = {{w₁, z}, {w₁, t}, {w₂, t}, {w₂, z}};
Do[angles = anglesDeg⟦i⟧ Degree;
  sigma = sigmas⟦i⟧ Degree;
  {α, β, γ, δ} = angles;
  poly = Which[i == 1, BricardsEquation[{α, β, γ, δ}, sigma, Z2[t], W12[t]],
    i == 2, BricardsEquation[{α, β, γ, δ}, sigma, Z2[t], W22[t]],
    i == 3, BricardsEquation[{α, β, γ, δ}, sigma, U2[t], W22[t]],
```

```
    i == 4, BricardsEquation[{α, β, γ, δ}, sigma, U2[t], W12[t]]];
  Print[Row[{Subscript["P", i], "(", funcs[[i, 1]],
     ", ", funcs[[i, 2]], ") = ", FullSimplify[poly]}]], {i, 1, 4}];


(*==============================================================================
 =====*)
(*==================NOT LINEAR COMPOUND==================*)
(*==============================================================================
 =====*)
(*Define domain limits for t*)
tMin = 2;
tMax = 5;
(*{TraditionalForm[cot[Subscript[θ,1]/2]],
  TraditionalForm[cot[Subscript[θ,2]/2]],
  TraditionalForm[cot[Subscript[θ,3]/2]],
  TraditionalForm[cot[Subscript[θ,4]/2]]};*)

(*==================================
 FLEXION 1==================================*)
(*List of expressions& labels*)
expressions =
  {Z[t] * U[t], Z[t] / U[t], W1[t] * W2[t], W1[t] / W2[t], Z[t] * W2[t], Z[t] / W2[t],
   U[t] * W1[t], U[t] / W1[t], Z[t] * W1[t], Z[t] / W1[t], W2[t] * U[t], W2[t] / U[t]};
labels = {"w₁ * w₂", "w₁ / w₂", "z * t", "z / t", "w₁ * t", "w₁ / t",
   "w₂ * z", "w₂ / z", "w₁ * z", "w₁ / z", "w₂ * t", "t / w₂"};

Column[{TextCell[Style["============= NOT LINEAR COMPOUND =============",
    Darker[Brown], Bold, 16], "Text"],
  (*Explanatory text*)
  TextCell[Style["Above we consider the first pair of equations (P₂(w₁, t) =
      0 and P₃(w₂, t) = 0). Solving them as quadratic equations in w₁
      and w₂, respectively we parametrize the solutions by the first
      three expressions in Solutions 1 and 2 in a neighborhood of any
      point (w₁, t, w₂) such that 2 + 6t² + 3t⁴ ≠ 0. Here, we choose any
      continuous branch of the square root in this neighborhood, and the
      signs in ± need Not agree (this means we consider all 4 pairs we
      describe above). We conclude that one component of the solution
      set of the first pair of Bricard's equations satisfies w₁/w₂=-1,
      and NO other component satisfies w₁/w₂=const NOR w₁w₂=const.

Analogously, one component of the solution set of the other pair of equations
      (P₁(w₁, z) = 0 and P₄(w₂, z) = 0) satisfies w₁/w₂=+1,, and NO
      other component satisfies w₁/w₂=const NOR w₁w₂=const. As a
      result, NO component of the solution set of all four equations
      satisfies w₁/w₂=const NOR w₁w₂=const. So, our example does not
      belong to the linear compound class, even after switching the
```

```
boundary strips.", GrayLevel[0.3]], "Text"],
  Spacer[6],
  TextCell[
   Style["Below, we also present the plots of functions w₁w₂, w₁/w₂, zt,
      z/t, w₁t, w₁/t, w₂z, w₂/z, w₁z, w₁/z,
      w₂t, and t/w₂.", GrayLevel[0.3]], "Text"],
  Spacer[6],
  TextCell[Style["Solution 1:", Bold, 14], "Text"], Spacer[12],
  Panel[GraphicsGrid[Partition[Table[Plot[expressions⟦i⟧, {t, tMin, tMax},
        PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → All,
        AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
 }]


(*=================================
 FLEXION 2=================================*)
(*List of expressions& labels*)
expressions = {Z2[t] * U2[t], Z2[t] / U2[t], W12[t] * W22[t], W12[t] / W22[t],
   Z2[t] * W22[t], Z2[t] / W22[t], U2[t] * W12[t], U2[t] / W12[t],
   Z2[t] * W12[t], Z2[t] / W12[t], W22[t] * U2[t], W22[t] / U2[t]};
labels = {"w₁ * w₂", "w₁ / w₂", "z * t", "z / t", "w₁ * t", "w₁ / t",
   "w₂ * z", "w₂ / z", "w₁ * z", "w₁ / z", "w₂ * t", "t / w₂"};

Column[{ TextCell[Style["Solution 2:", Bold, 14], "Text"], Spacer[12],
  (*Plots panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions⟦i⟧, {t, tMin, tMax},
        PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → All,
        AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
 }]


(*===========================================================================
 =====*)
(*=================================
 NOT TRIVIAL=================================*)
(*===========================================================================
 =====*)
(*Define domain limits for t*)
tMin = 0;
tMax = 5;

(*=================================
 FLEXION 1=================================*)
(*List of expressions to plot*)
expressions = {Z[t], W2[t], U[t], W1[t]};
```

```
labels = {"w₁", "t", "w₂", "z"};

Column[{TextCell[
   Style["================= NOT TRIVIAL (FLEXION 1) =================",
    Darker[Pink], Bold, 16], "Text"],
  (*Explanatory text*)
  TextCell[Style["This configuration does not belong to the trivial class –
      even after switching the boundary strips – since none of the
      functions w₁, t, w₂, or z is constant.", GrayLevel[0.3]], "Text"],
  Spacer[12],
  (*Plots in a light panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions⟦i⟧, {t, tMin, tMax},
       PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → All,
       AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]

 }]

(*==================================
 FLEXION 2==================================*)
(*List of expressions to plot*)
expressions = {Z2[t], W22[t], U2[t], W12[t]};
labels = {"w₁", "t", "w₂", "z"};

Column[{TextCell[
   Style["================= NOT TRIVIAL (FLEXION 2) =================",
    Darker[Green], Bold, 16], "Text"],

  (*Explanatory text*)
  TextCell[Style["This configuration does not belong to the trivial class –
      even after switching the boundary strips – since none of the
      functions w₁, t, w₂, or z is constant.", GrayLevel[0.3]], "Text"],
  Spacer[12],

  (*Plots in a light panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions⟦i⟧, {t, tMin, tMax},
       PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → All,
       AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]

 }]


(*============================================================================
 =====*)
(*==========================
 SWITCHING BOUNDARY STRIPS===========================*)
```

```
(*==============================================================================
 =====*)
SwitchingRightBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices:{row,column}={1,2},{1,3},{4,2},{4,3}*)
  modified⟦1, 2⟧ = 180 - anglesDeg⟦1, 2⟧; (*β1*)
  modified⟦1, 3⟧ = 180 - anglesDeg⟦1, 3⟧; (*γ1*)
  modified⟦4, 2⟧ = 180 - anglesDeg⟦4, 2⟧; (*β4*)
  modified⟦4, 3⟧ = 180 - anglesDeg⟦4, 3⟧; (*γ4*)
  modified]

SwitchingLeftBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices:{row,column}={2,2},{2,3},{3,2},{3,3}*)
  modified⟦2, 2⟧ = 180 - anglesDeg⟦2, 2⟧; (*β2*)
  modified⟦2, 3⟧ = 180 - anglesDeg⟦2, 3⟧; (*γ2*)
  modified⟦3, 2⟧ = 180 - anglesDeg⟦3, 2⟧; (*β3*)
  modified⟦3, 3⟧ = 180 - anglesDeg⟦3, 3⟧; (*γ3*)
  modified]

SwitchingLowerBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices:{row,column}={1,1},{1,2},{2,1},{2,2}*)
  modified⟦1, 1⟧ = 180 - anglesDeg⟦1, 1⟧; (*α1*)
  modified⟦1, 2⟧ = 180 - anglesDeg⟦1, 2⟧; (*β1*)
  modified⟦2, 1⟧ = 180 - anglesDeg⟦2, 1⟧; (*α2*)
  modified⟦2, 2⟧ = 180 - anglesDeg⟦2, 2⟧; (*β2*)
  modified]

SwitchingUpperBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices:{row,column}={3,1},{3,2},{4,1},{4,2}*)
  modified⟦3, 1⟧ = 180 - anglesDeg⟦3, 1⟧; (*α3*)
  modified⟦3, 2⟧ = 180 - anglesDeg⟦3, 2⟧; (*β3*)
  modified⟦4, 1⟧ = 180 - anglesDeg⟦4, 1⟧; (*α4*)
  modified⟦4, 2⟧ = 180 - anglesDeg⟦4, 2⟧; (*β4*)
  modified]


(*==============================================================================
 =====*)
(*==================================NOT CONIC & NOT CHIMERA & NOT LINEAR
     CONJUGATE & NOT ISOGONAL==================================*)
(*==============================================================================
 =====*)
Column[{TextCell[
    Style["================= NOT CONIC & NOT CHIMERA & NOT LINEAR CONJUGATE
       & NOT ISOGONAL=================", Blue, Bold, 16], "Text"],
  TextCell[Style[
    "Condition (N.0) is satisfied for all i=1,...,4 ⇒ NOT equimodular-conic,
      NOT chimera, NOT isogonal and NOT linear conjugate.
```

```
      Applying any boundary-strip switch still preserves
      (N.0), so no conic, no chimera, no isogonal and no
      linear conjugate form emerges.", GrayLevel[0.3]], "Text"]
 }]

(*Now the exact same Module for checking all switch combinations…*)
Module[{angles = anglesDeg, switchers, combinations, results},
 (*Define switch functions*)
 switchers = <|"Right" → SwitchingRightBoundaryStrip,
   "Left" → SwitchingLeftBoundaryStrip, "Lower" → SwitchingLowerBoundaryStrip,
   "Upper" → SwitchingUpperBoundaryStrip|>;
 (*Generate all combinations of switches (from size 1 to 4)*)
 combinations = Subsets[Keys[switchers], {1, Length[switchers]}];
 (*Evaluate condition after each combination of switches*)results = Table[
   Module[{switched = angles, name, passQ}, name = StringRiffle[combo, " + "];
    Do[switched = switchers[sw][switched], {sw, combo}];
    passQ = And @@ (checkConditionN0Degrees /@ switched);
    (*Print the result after switching*)
    (*Print["\nSwitch combination: ",name];
    Print["Switched anglesDeg:"];
    Print[MatrixForm[switched]];*)
    {name, passQ}], {combo, combinations}];
 (*Display results*)
 Column[Prepend[Table[Module[{comboName = res⟦1⟧, passQ = res⟦2⟧},
     Row[{Style[comboName <> ": ", Bold],
       If[passQ,
        Style["Condition (N.0) is still satisfied.", Darker[Green]],
        Style["Condition (N.0) fails.", Red, Bold]
       ]
      }
     ]], {res, results}], TextCell[
    Style["CONDITION (N.0) AFTER SWITCHING BOUNDARY STRIPS", 14], "Text"]]]]


(*============================================================================
 =====*)
(*================================
 NOT ORTHODIAGONAL================================*)
(*============================================================================
 =====*)

(*Column[
  {TextCell[Style["================= NOT ORTHODIAGONAL =================",
     Darker[Blue],Bold,16],"Text"],
   TextCell[Style[
     "cos(αi)·cos(γi) ≠ cos(βi)·cos(δi) for each i = 1 ⇒ NOT orthodiagonal.
       Switching boundary strips does not
```

```
      correct this.",  GrayLevel[0.3]],"Text"]
  }]

Module[{angles=anglesDeg,switchers,combinations,results},
  (*Define switch functions*)switchers=<|"Right"→SwitchingRightBoundaryStrip,
    "Left"→SwitchingLeftBoundaryStrip,"Lower"→SwitchingLowerBoundaryStrip,
    "Upper"→SwitchingUpperBoundaryStrip|>;
  (*Helper function:compute and print difference only*)
  formatOrthodiagonalCheck[quad_List]:=
   Module[{vals},vals=Table[Module[{a,b,c,d,lhs,rhs,diff},{a,b,c,d}=quad[[i]];
       lhs=FullSimplify[Cos[a Degree] Cos[c Degree]];
       rhs=FullSimplify[Cos[b Degree] Cos[d Degree]];
       diff=Chop[lhs-rhs];
       Style[Row[{"cos(α"<>ToString[i]<>")·cos(γ"<>ToString[i]<>") - ",
          "cos(β"<>ToString[i]<>")·cos(δ"<>ToString[i]<>") = ",
          NumberForm[diff,{5,3}]}],If[diff==0,Red,Black]]],{i,Length[quad]}];
    Column[vals]];
  (*Orthodiagonal check for anglesDeg before any switching*)
  Print[TextCell[Style["\nInitial anglesDeg (no switches):",Bold]]];
  Print[MatrixForm[angles]];
  Print[TextCell[Style[
     "Orthodiagonal check: cos(αi)·cos(γi) - cos(βi)·cos(δi) for i = 1..4",
     Italic]]];
  Print[formatOrthodiagonalCheck[angles]];
  (*Generate all combinations of switches (from size 1 to 4)*)
  combinations=Subsets[Keys[switchers],{1,Length[switchers]}];
  (*Evaluate condition after each combination of switches*)results=
   Table[Module[{switched=angles,name,passQ},name=StringRiffle[combo," + "];
     Do[switched=switchers[sw][switched],{sw,combo}];
     passQ=And@@(checkConditionN0Degrees/@switched);
     Print[Style["\nSwitch combination: ", Bold],name];
     Print[Style["Switched anglesDeg:", Italic]];
     Print[MatrixForm[switched]];
     Print[
      TextCell[Style["Orthodiagonal check: cos(αi)·cos(γi) - cos(βi)·cos(δi)
          for i = 1..4",Italic]]];
     Print[formatOrthodiagonalCheck[switched]];
     {name,passQ}],{combo,combinations}];]*)


Column[
 {TextCell[Style["================= ORTHOGONALITY CHECK =================",
    Darker[Blue], Bold, 16], "Text"],
  TextCell[Style["cos(αi)·cos(γi) ≠ cos(βi)·cos(δi) for at least
      one i = 1,..., 4 ⇒ NOT orthodiagonal. Switching boundary
      strips does not correct this.", GrayLevel[0.3]], "Text"]}]
```

```
(*Helper
 function:Returns True if at least one cosine product difference is non-
   zero.Returns False if all differences are zero.*)
isNotOrthodiagonal[quad_List] :=
  Or @@ Table[Module[{a, b, c, d, diff}, {a, b, c, d} = quad[[i]];
      diff = Chop[Cos[a Degree] Cos[c Degree] - Cos[b Degree] Cos[d Degree]];
      diff ≠ 0], {i, Length[quad]}];

(*First,check the initial,unswitched angles*)
Print[TextCell[Style["\nInitial anglesDeg (no switches):", Bold]]];
If[isNotOrthodiagonal[anglesDeg], Print[Style[
     " -> Condition met: At least one difference is non-zero.", Darker@Green]],
  Print[Style[" -> Condition NOT met: All differences are zero.", Red]]];

(*Now,use your desired module to check all switch combinations*)
Module[{angles = anglesDeg, switchers, combinations, results},
 (*Define switch functions*)
 switchers = <|"Right" → SwitchingRightBoundaryStrip,
   "Left" → SwitchingLeftBoundaryStrip, "Lower" → SwitchingLowerBoundaryStrip,
   "Upper" → SwitchingUpperBoundaryStrip|>;
 (*Generate all combinations of switches (from size 1 to 4)*)
 combinations = Subsets[Keys[switchers], {1, Length[switchers]}];
 (*Evaluate condition after each combination
  of switches and store in'results'*)results = Table[
   Module[{switched = angles, name, passQ}, name = StringRiffle[combo, " + "];
    Do[switched = switchers[sw][switched], {sw, combo}];
    (* ***THIS IS THE KEY CHANGE****)(*Set passQ using our
     new helper function*)passQ = isNotOrthodiagonal[switched];
    {name, passQ}], {combo, combinations}];
 (*Display results in the specified column format*)
 Column[Prepend[Table[Module[{comboName = res[[1]], passQ = res[[2]]},
     Row[{Style[comboName <> ": ", Bold], If[passQ,
        Style["Condition met (at least one difference is non-zero).", Darker[
          Green]], Style["Condition NOT met (all differences are zero).",
         Red, Bold]]}]], {res, results}], TextCell[
    Style["\nNON-ORTHOGONALITY CHECK AFTER SWITCHING BOUNDARY STRIPS", 14],
    "Text"]]]]



(*=============================================================================
 =====*)
(*============================
   NOT CONJUGATE-MODULAR=============================*)
(*=============================================================================
 =====*)
(*Column[{TextCell[
```

```
      Style["================= NOT CONJUGATE-MODULAR =================",
       Purple,Bold,16],"Text"],
     TextCell[Style["M1 = M2 = M3 = M4 = M
         and M ≠ 2 ⇒ NOT conjugate-modular. Boundary-strip
         switches preserve this.",GrayLevel[0.3]],"Text"]
    }]
   Ms=FullSimplify[Times@@@results];
allEqualQ=Simplify[Equal@@Ms];

Module[{angles=anglesDeg,switchers,combinations,results,
   computeConjugateModularInfo},(*Define switch functions*)
  switchers=<|"Right"→SwitchingRightBoundaryStrip,
    "Left"→SwitchingLeftBoundaryStrip,"Lower"→SwitchingLowerBoundaryStrip,
    "Upper"→SwitchingUpperBoundaryStrip|>;
  (*Computes Mᵢ and pᵢ and prints them,
  with classification*)computeConjugateModularInfo[quad_List]:=
   Module[{abcdList,Ms,summary},abcdList=computeABCD/@quad;
    Ms=FullSimplify[Times@@@abcdList];
    summary=If[Simplify[Equal@@Ms]&&Ms[[1]]=!=2,
      Style["M1 = M2 = M3 = M4 = M and M ≠ 2",Bold],
      Style["M1 = M2 = M3 = M4 = M and M = 2",Red,Bold]];
    Column[{Style["Mᵢ values:",Bold],Row[{"M1 = ",Ms[[1]],
        ", M2 = ",Ms[[2]],", M3 = ",Ms[[3]],", M4 = ",Ms[[4]]}],summary}]];
  (*Original anglesDeg check*)
  Print[
   TextCell[Style["\nInitial configuration (no switches applied):",Bold]]];
  Print[MatrixForm[angles]];
  Print[computeConjugateModularInfo[angles]];
  (*Generate all switch combinations (from size 1 to 4)*)
  combinations=Subsets[Keys[switchers],{1,Length[switchers]}];
  (*Evaluate each switched configuration*)results=
   Table[Module[{switched=angles,name,passQ},name=StringRiffle[combo," + "];
     Do[switched=switchers[sw][switched],{sw,combo}];
     passQ=And@@(checkConditionN0Degrees/@switched);
     Print[Style["\nSwitch combination: ", Bold],name];
     Print[Style["Switched anglesDeg:", Italic]];
     Print[MatrixForm[switched]];
     Print[computeConjugateModularInfo[switched]];
     {name,passQ}],{combo,combinations}];]*)


Column[{TextCell[
    Style["================= CONJUGATE-MODULAR CHECK =================",
     Purple, Bold, 16], "Text"],
   TextCell[Style["M1 = M2 = M3 = M4 = M and M ≠ 2 ⇒ NOT conjugate-modular.
       Boundary-strip switches preserve this.", GrayLevel[0.3]], "Text"]}]
```

```
(*Helper Function:Returns True if all M_i values are equal
   AND their common value is not 2. Returns False otherwise.*)
isNotConjugateModular[quad_List] :=
  Module[{abcdList, Ms}, abcdList = computeABCD /@ quad;
   Ms = FullSimplify[Times @@@ abcdList];
   (*The condition is met if they are all equal AND the value isn't 2*)
   Simplify[Equal @@ Ms] && (Ms[[1]] ≠ 2)];

(*First,check the initial,unswitched angles*)
Print[TextCell[Style["\nInitial anglesDeg (no switches):", Bold]]];
If[isNotConjugateModular[anglesDeg], Print[
   Style[" -> Condition met: All Mᵢ are equal and M ≠ 2.", Darker@Green]],
  Print[Style[" -> Condition NOT met.", Red]]];

(*Now,use the clean module to check all switch combinations*)
Module[{angles = anglesDeg, switchers, combinations, results},
 (*Define switch functions*)
 switchers = <|"Right" → SwitchingRightBoundaryStrip,
   "Left" → SwitchingLeftBoundaryStrip, "Lower" → SwitchingLowerBoundaryStrip,
   "Upper" → SwitchingUpperBoundaryStrip|>;
 (*Generate all combinations of switches (from size 1 to 4)*)
 combinations = Subsets[Keys[switchers], {1, Length[switchers]}];
 (*Evaluate condition after each combination
  of switches and store the result*)results = Table[
   Module[{switched = angles, name, passQ}, name = StringRiffle[combo, " + "];
    Do[switched = switchers[sw][switched], {sw, combo}];
    (*Set passQ using our new helper function for this check*)
    passQ = isNotConjugateModular[switched];
    {name, passQ}], {combo, combinations}];
 (*Display results in the specified column format*)
 Column[Prepend[Table[Module[{comboName = res[[1]], passQ = res[[2]]},
     Row[{Style[comboName <> ": ", Bold], If[passQ,
        Style["Condition met (All Mᵢ are equal and M ≠ 2).", Darker[Green]],
        Style["Condition NOT met.", Red, Bold]]}]], {res, results}],
   TextCell[Style["\nCONJUGATE-MODULAR CHECK AFTER SWITCHING BOUNDARY STRIPS",
     14], "Text"]]]]
```

Out[5346]=

================= **CONDITION (N.0)** =================

✔ **All vertices satisfy (N.0).**

Out[5349]=

================= **CONDITION (N.3)** =================

✔ **M1 = M2 = M3 = M4 =** $-1+\sqrt{3}$

Out[5355]=

================== CONDITION (N.4) ==================

✔ **r1** = **r2** = $\frac{2}{\sqrt{3}}$; ✔ **r3** = **r4** = $\frac{2}{\sqrt{3}}$

✔ **s1** = **s4** = $3-\sqrt{3}$; ✔ **s2** = **s3** = $\frac{1}{\sqrt{3}}$

Out[5365]=

================== CONDITION (N.5) ==================

⚠ *Approximate validation using*
ε-tolerance. For rigorous proof, see the referenced paper.

✅ **Valid Combination Found (M < 1):**

e1 = −1, **e2** = −1, **e3** = 1

t1 = 0.K + 0.446521iK'

t2 = 1.K + 0.446521iK'

t3 = 3.K + 0.446521iK'

t4 = 0.K + 0.446521iK'

t1 + e1*t2 + e2*t3 + e3*t4 = −4.K + 0.iK'

Out[5367]=

================== OTHER PARAMETERS ==================

u = $2-\sqrt{3}$

σ1 = 165 °, **σ2** = 165 °, **σ3** = 195 °, **σ4** = 165 °

f1 = $\frac{1}{2}\left(1+\sqrt{3}\right)$, **f2** = $4-2\sqrt{3}$, **f3** = $4-2\sqrt{3}$, **f4** = $\frac{1}{2}\left(1+\sqrt{3}\right)$

z1 = $1+\sqrt{3}$, **z2** = $-1-\frac{2}{\sqrt{3}}$, **z3** = $-1-\frac{2}{\sqrt{3}}$, **z4** = $1+\sqrt{3}$

x1 = $3+2\sqrt{3}$, **x2** = $3+2\sqrt{3}$, **x3** = $3+2\sqrt{3}$, **x4** = $3+2\sqrt{3}$

y1 = $2+\sqrt{3}$, **y2** = $\frac{1}{-1+\frac{1}{\sqrt{3}}}$, **y3** = $\frac{1}{-1+\frac{1}{\sqrt{3}}}$, **y4** = $2+\sqrt{3}$

p1 = $\sqrt{-1+\frac{2}{\sqrt{3}}}$, **p2** = $\sqrt{-1+\frac{2}{\sqrt{3}}}$, **p3** = $\sqrt{-1+\frac{2}{\sqrt{3}}}$, **p4** = $\sqrt{-1+\frac{2}{\sqrt{3}}}$

q1 = $\sqrt{2-\sqrt{3}}$, **q2** = $i\sqrt{1-\frac{1}{\sqrt{3}}}$, **q3** = $i\sqrt{1-\frac{1}{\sqrt{3}}}$, **q4** = $\sqrt{2-\sqrt{3}}$

p1·q1 = $\sqrt{-4+\frac{7}{\sqrt{3}}}$, **p2·q2** = $i\sqrt{-\frac{5}{3}+\sqrt{3}}$, **p3·q3** = $i\sqrt{-\frac{5}{3}+\sqrt{3}}$, **p4·q4** = $\sqrt{-4+\frac{7}{\sqrt{3}}}$

$\overline{\alpha 1}$ = 60°, $\overline{\beta 1}$ = 150°, $\overline{\gamma 1}$ = 45°, $\overline{\delta 1}$ = 75°

Out[5369]=

================= Bricard's System of Equations =================

We introduce new notation for the cotangents of half of the dihedral angles. Denote $w_1 := \cot\left(\frac{\theta_1}{2}\right)$, $t := \cot\left(\frac{\theta_2}{2}\right)$, $w_2 := \cot\left(\frac{\theta_3}{2}\right)$, and $z := \cot\left(\frac{\theta_4}{2}\right)$

$$P_1(w_1, z) = \frac{w_1^2\left(\left(3+\sqrt{3}\right)z^2+2\sqrt{3}\right)}{4\sqrt{2}} - \frac{\left(3+\sqrt{3}\right)w_1 z}{2\sqrt{2}} + \frac{2z^2+\sqrt{3}-1}{4\sqrt{2}} = 0$$

$$P_2(w_1, t) = \frac{1}{4}\left(-3t^2-\sqrt{3}-1\right)w_1^2 + \frac{1}{4}\left(t^2+\sqrt{3}-1\right) - \frac{\left(\sqrt{3}-1\right)t\,w_1}{\sqrt{2}} = 0$$

$$P_3(w_2, t) = \frac{1}{4}\left(3t^2+\sqrt{3}+1\right)w_2^2 + \frac{1}{4}\left(-t^2-\sqrt{3}+1\right) - \frac{\left(\sqrt{3}-1\right)t\,w_2}{\sqrt{2}} = 0$$

$$P_4(w_2, z) = \frac{w_2^2\left(\left(3+\sqrt{3}\right)z^2+2\sqrt{3}\right)}{4\sqrt{2}} - \frac{\left(3+\sqrt{3}\right)w_2 z}{2\sqrt{2}} + \frac{2z^2+\sqrt{3}-1}{4\sqrt{2}} = 0$$

Out[5381]=

================= FLEXIONS =================

Solutions to Bricard's equations under a free parameter $t \in \mathbb{C}$ using Theorem 1:

Solution 1:

$$w_1(t) = \frac{\sqrt{3t^4+6t^2+2}-\sqrt{6}\,t+\sqrt{2}\,t}{3t^2+\sqrt{3}+1}$$

$$t(t) = t$$

$$w_2(t) = \frac{\sqrt{3t^4+6t^2+2}+\sqrt{2}\left(\sqrt{3}-1\right)t}{3t^2+\sqrt{3}+1}$$

$$z(t) = \frac{\sqrt{3t^4+6t^2+2}}{\sqrt{3}\,t^2+\sqrt{3}+1}$$

Solution 2:

$$w_1(t) = -\frac{\sqrt{3t^4+6t^2+2}+\sqrt{2}\left(\sqrt{3}-1\right)t}{3t^2+\sqrt{3}+1}$$

$$t(t) = t$$

$$w_2(t) = -\frac{\sqrt{3t^4+6t^2+2}-\sqrt{6}\,t+\sqrt{2}\,t}{3t^2+\sqrt{3}+1}$$

$$z(t) = -\frac{\sqrt{3t^4+6t^2+2}}{\sqrt{3}\,t^2+\sqrt{3}+1}$$

Out[5382]=

================= Equations: $P_2(w_1, t) = 0$ and $P_3(w_2, t) = 0$ =================

Let w_{1s1} and w_{1s2} be formulas for $w_1(t)$ from solutions 1 and 2, respectively. Similarly, let w_{2s1} and w_{2s2} be the formulas for $w_2(t)$. We show that all four pairs – (w_{1s1}, w_{2s1}), (w_{1s1}, w_{2s2}), (w_{1s2}, w_{2s1}), and (w_{1s2}, w_{2s2}) – solve equations $P_2(w_1, t) = 0$ and $P_3(w_2, t) = 0$.

Pair 1:

$P_2(w_{1s1}, t) = 0$ ✓

$P_3(w_{2s1}, t) = 0$ ✓

Pair 2:

$P_2(w_{1s1}, t) = 0$ ✓

$P_3(w_{2s2}, t) = 0$ ✓

Pair 3:

$P_2(w_{1s2}, t) = 0$ ✓

$P_3(w_{2s1}, t) = 0$ ✓

Pair 4:

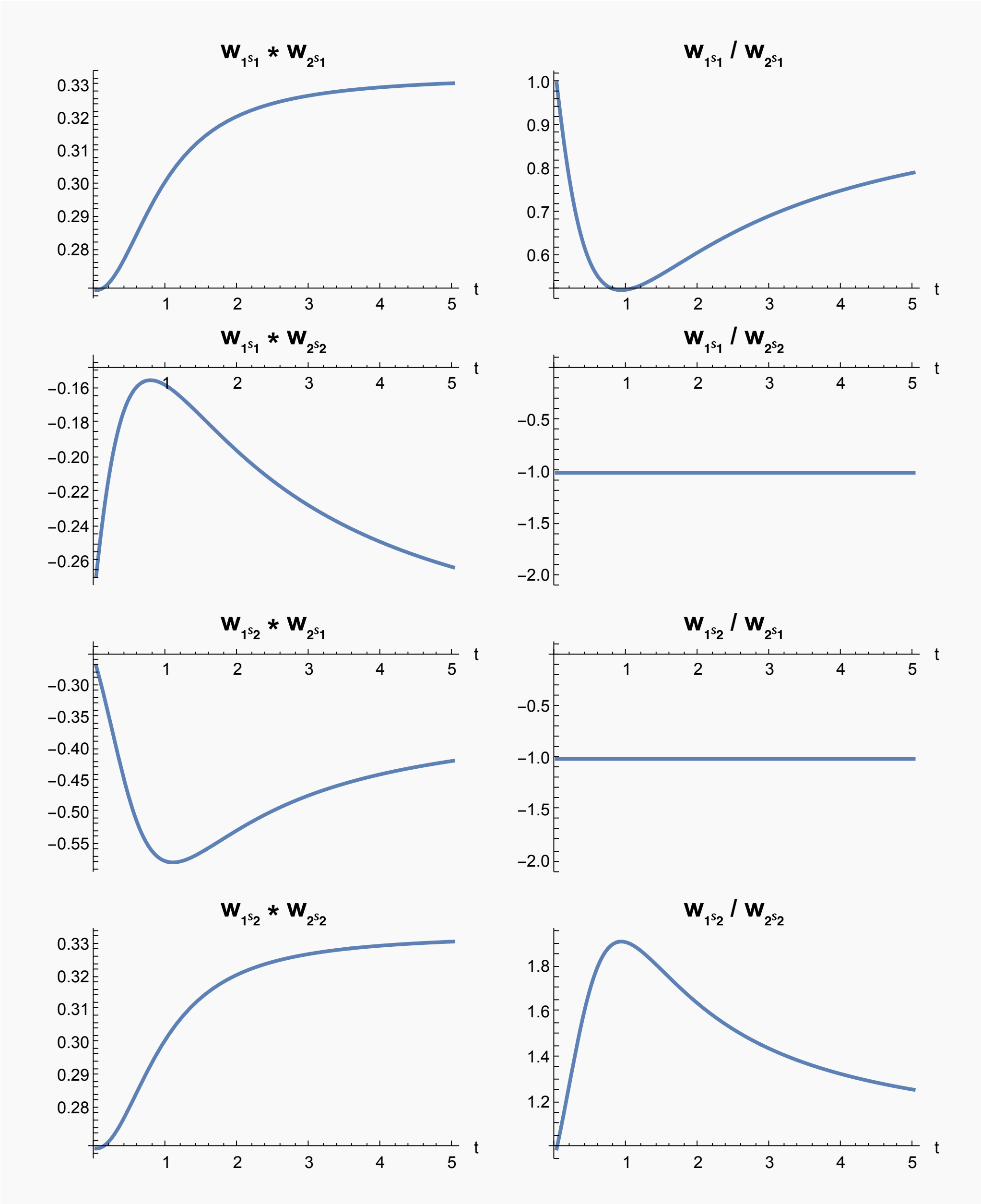
$P_2(w_{1s2}, t) = 0$ ✓

$P_3(w_{2s2}, t) = 0$ ✓

Out[5387]=

============= Plots of $w_1(t)w_2(t)$ and $w_1(t)/w_2(t)$ For All Pairs =============

Checking that all four pairs – (w_{1s1}, w_{2s1}), (w_{1s1}, w_{2s2}), (w_{1s2}, w_{2s1}), and (w_{1s2}, w_{2s2}) – does NOT satisfy $w_1(t)/w_2(t) = c$ and $w_1(t)w_2(t) = c$ when $c \neq -1$.



Out[5396]=

================= Equations: $P_1(w_1, z) = 0$ and $P_4(w_2, z) = 0$ =================

We solve $P_1(w_1, z) = 0$ and $P_4(w_2, z) = 0$ for w_1 and w_2.

$$w_{1\,1}(z) = \frac{\left(3+\sqrt{3}\right) z - \sqrt{2}\ \sqrt{-\left(\left(3+\sqrt{3}\right) z^4\right) + 6 z^2 + \sqrt{3} - 3}}{\left(3+\sqrt{3}\right) z^2 + 2\ \sqrt{3}}$$

$$w_{1\,2}(z) = \frac{\sqrt{2}\ \sqrt{-\left(\left(3+\sqrt{3}\right) z^4\right) + 6 z^2 + \sqrt{3} - 3} + \left(3+\sqrt{3}\right) z}{\left(3+\sqrt{3}\right) z^2 + 2\ \sqrt{3}}$$

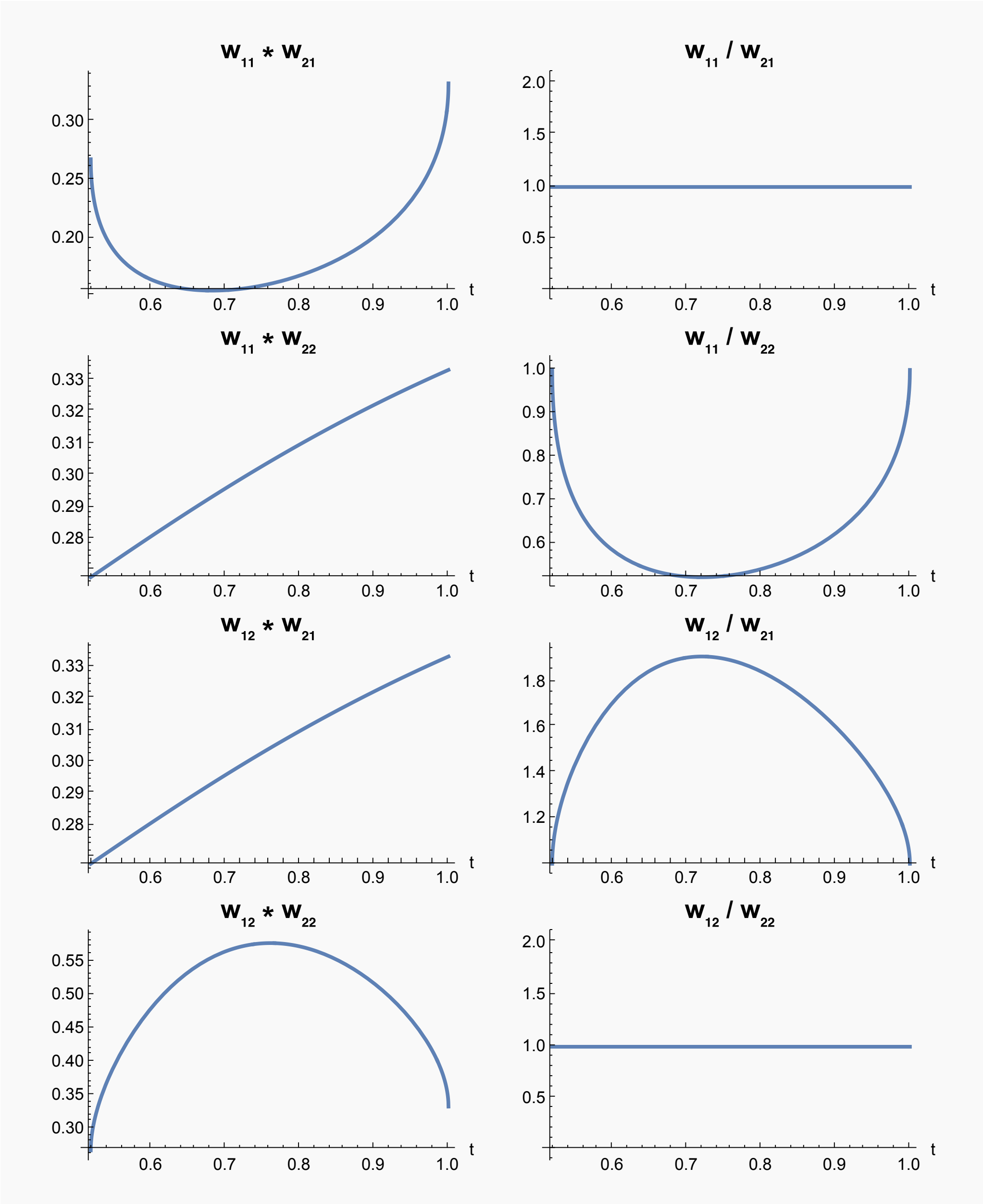
$$w_{2\,1}(z) = \frac{\left(3+\sqrt{3}\right) z - \sqrt{2}\ \sqrt{-\left(\left(3+\sqrt{3}\right) z^4\right) + 6 z^2 + \sqrt{3} - 3}}{\left(3+\sqrt{3}\right) z^2 + 2\ \sqrt{3}}$$

$$w_{2\,2}(z) = \frac{\sqrt{2}\ \sqrt{-\left(\left(3+\sqrt{3}\right) z^4\right) + 6 z^2 + \sqrt{3} - 3} + \left(3+\sqrt{3}\right) z}{\left(3+\sqrt{3}\right) z^2 + 2\ \sqrt{3}}$$

Out[5401]=

============= Plots of $w_1(z)w_2(z)$ and $w_1(z)/w_2(z)$ For All Pairs =============

Checking that all four pairs – $(w_{11}(z), w_{21}(z))$, $(w_{11}(z), w_{22}(z))$, $(w_{12}(z), w_{21}(z))$, and $(w_{12}(z), w_{22}(z))$ – does NOT satisfy $w_1(z)/w_2(z) = c$ and $w_1(z)w_2(z) = c$ when $c \neq +1$.



Out[5402]=

================= FLEXIBILITY (Double Checking) =================

Out[5404]=

Solution 1:

$P_1(w_1, z) = 0$

$P_2(w_1, t) = 0$

$P_3(w_2, t) = 0$

$P_4(w_2, z) = 0$

Out[5406]=

Solution 2:

$P_1(w_1, z) = 0$

$P_2(w_1, t) = 0$

$P_3(w_2, t) = 0$

$P_4(w_2, z) = 0$

Out[5413]=

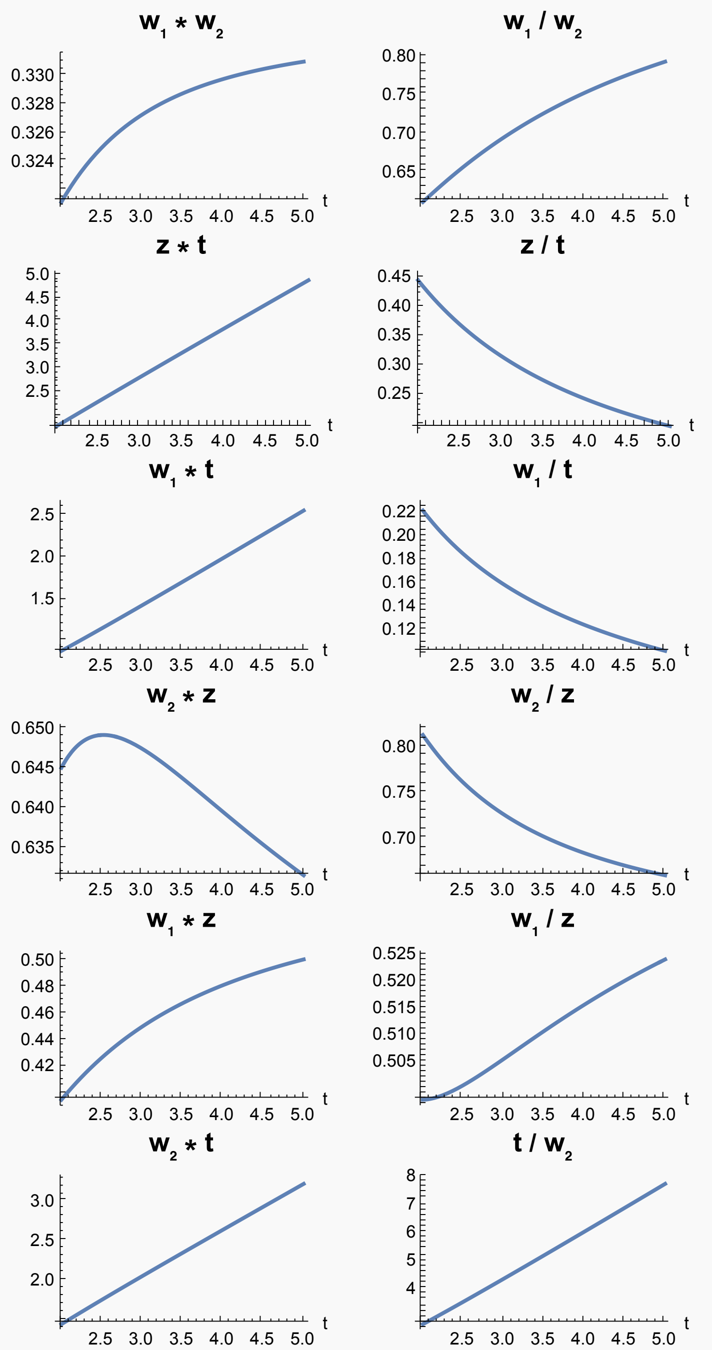
============= **NOT LINEAR COMPOUND** =============

Above we consider the first pair of equations ($P_2(w_1, t) = 0$ and $P_3(w_2, t) = 0$). Solving them as quadratic equations in w_1 and w_2, respectively we parametrize the solutions by the first three expressions in Solutions 1 and 2 in a neighborhood of any point (w_1, t, w_2) such that $2 + 6t^2 + 3t^4 \neq 0$. Here, we choose any continuous branch of the square root in this neighborhood, and the signs in $\pm$ need Not agree (this means we consider all 4 pairs we describe above). We conclude that one component of the solution set of the first pair of Bricard's equations satisfies $w_1/w_2=-1$, and NO other component satisfies $w_1/w_2=$const NOR $w_1 w_2=$const.

Analogously, one component of the solution set of the other pair of equations ($P_1(w_1, z) = 0$ and $P_4(w_2, z) = 0$) satisfies $w_1/w_2=+1$,, and NO other component satisfies $w_1/w_2=$const NOR $w_1 w_2=$const. As a result, NO component of the solution set of all four equations satisfies $w_1/w_2=$const NOR $w_1 w_2=$const. So, our example does not belong to the linear compound class, even after switching the boundary strips.

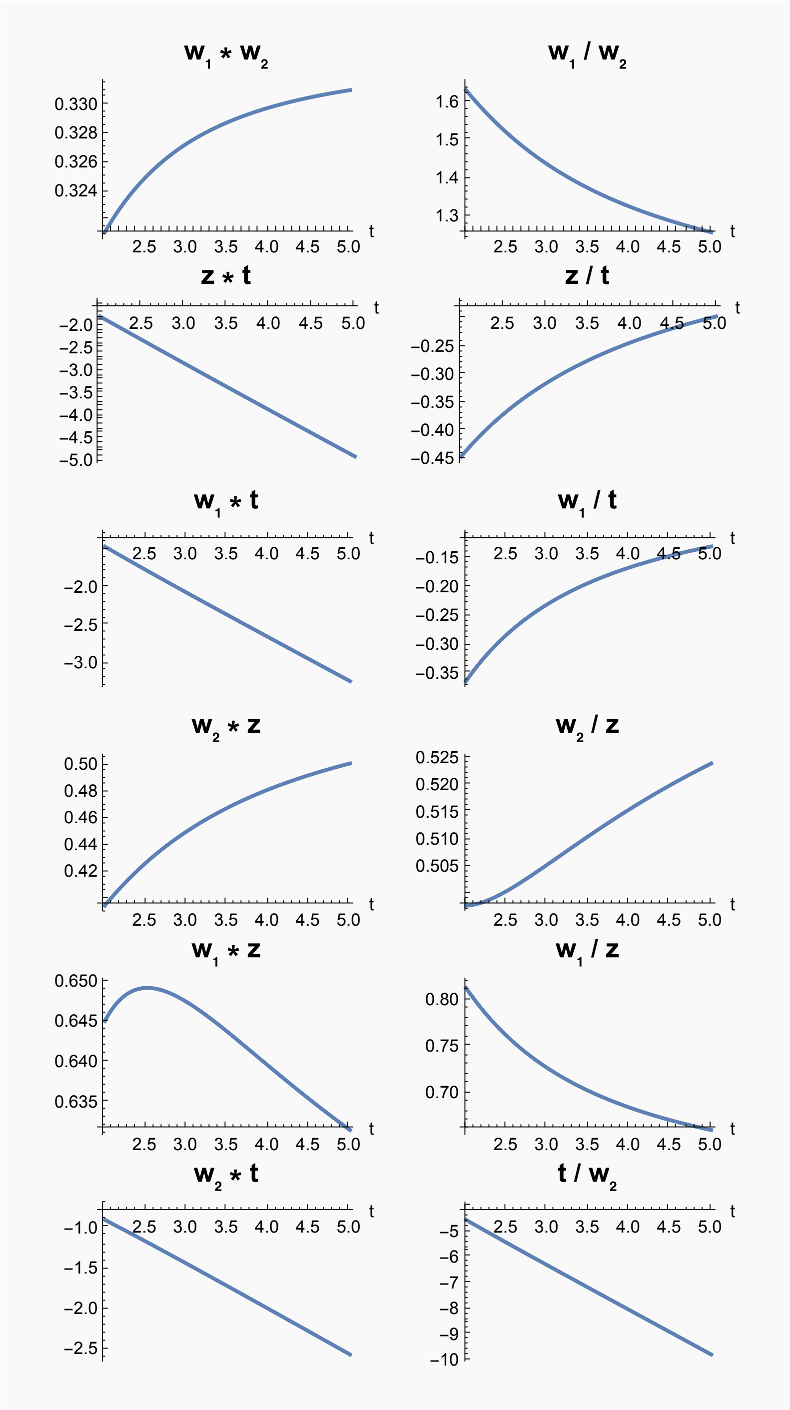
Below, we also present the plots of functions $w_1 w_2$, w_1/w_2, zt, z/t, $w_1 t$, w_1/t, $w_2 z$, w_2/z, $w_1 z$, w_1/z, $w_2 t$, and t/w_2.

Solution 1:


$w_1 * w_2$
w_1 / w_2
$z * t$
z / t
$w_1 * t$
w_1 / t
$w_2 * z$
w_2 / z
$w_1 * z$
w_1 / z
$w_2 * t$
t / w_2
t

Out[5416]=

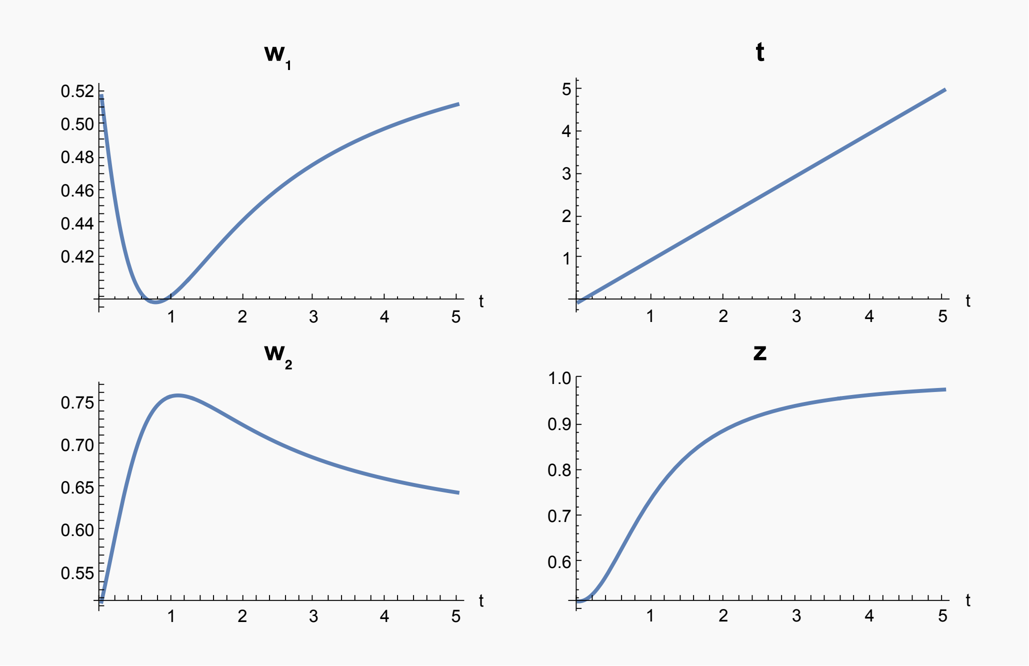
Solution 2:



Out[5421]=

================= **NOT TRIVIAL (FLEXION 1)** =================

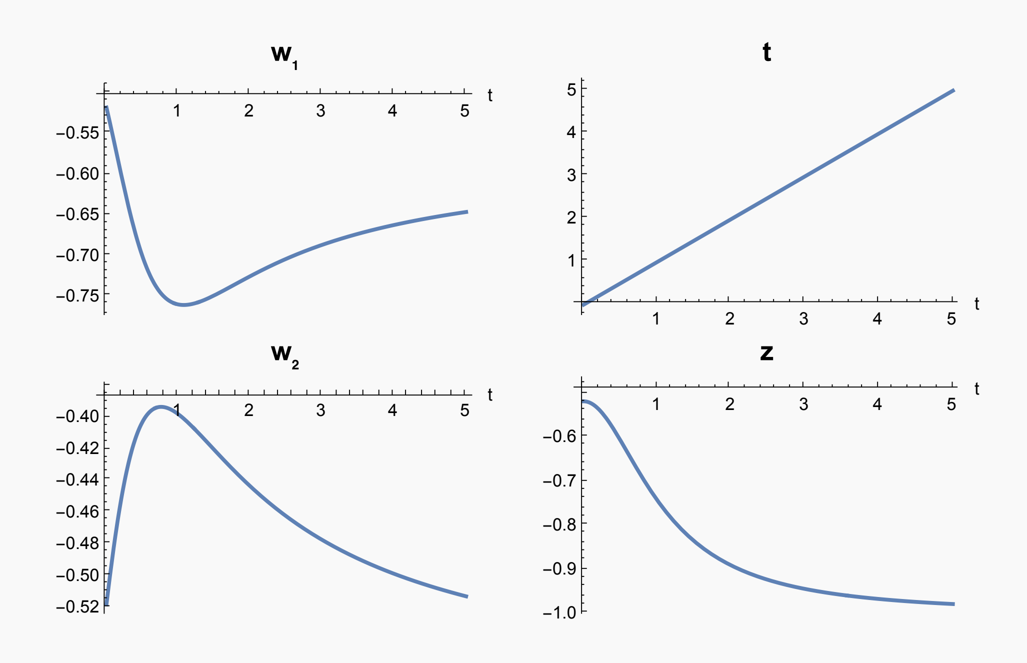
This configuration does not belong to the trivial class – even after switching the boundary strips – since none of the functions w_1, t, w_2, or z is constant.



Out[5424]=

================= NOT TRIVIAL (FLEXION 2) =================

This configuration does not belong to the trivial class – even after switching the boundary strips – since none of the functions w_1, t, w_2, or z is constant.



Out[5429]=

================= NOT CONIC & NOT CHIMERA & NOT LINEAR CONJUGATE & NOT ISOGONAL=================

Condition (N.0) is satisfied for all i=1,...,4 ⇒ NOT equimodular-conic, NOT chimera, NOT isogonal and NOT linear conjugate. Applying any boundary-strip switch still preserves (N.0), so no conic, no chimera, no isogonal and no linear conjugate form emerges.

Out[5430]=

CONDITION (N.0) AFTER SWITCHING BOUNDARY STRIPS
Right: Condition (N.0) is still satisfied.
Left: Condition (N.0) is still satisfied.
Lower: Condition (N.0) is still satisfied.
Upper: Condition (N.0) is still satisfied.
Right + Left: Condition (N.0) is still satisfied.
Right + Lower: Condition (N.0) is still satisfied.
Right + Upper: Condition (N.0) is still satisfied.
Left + Lower: Condition (N.0) is still satisfied.
Left + Upper: Condition (N.0) is still satisfied.
Lower + Upper: Condition (N.0) is still satisfied.
Right + Left + Lower: Condition (N.0) is still satisfied.
Right + Left + Upper: Condition (N.0) is still satisfied.
Right + Lower + Upper: Condition (N.0) is still satisfied.
Left + Lower + Upper: Condition (N.0) is still satisfied.
Right + Left + Lower + Upper: Condition (N.0) is still satisfied.

Out[5431]=

================== ORTHOGONALITY CHECK ==================

$\cos(\alpha_i)\cdot\cos(\gamma_i) \neq \cos(\beta_i)\cdot\cos(\delta_i)$ for
at least one i = 1,..., 4 ⇒ NOT orthodiagonal.
Switching boundary strips does not correct this.

Initial anglesDeg (no switches):

-> Condition met: At least one difference is non-zero.

Out[5435]=

NON-ORTHOGONALITY CHECK AFTER SWITCHING BOUNDARY STRIPS
Right: Condition met (at least one difference is non-zero).
Left: Condition met (at least one difference is non-zero).
Lower: Condition met (at least one difference is non-zero).
Upper: Condition met (at least one difference is non-zero).
Right + Left: Condition met (at least one difference is non-zero).
Right + Lower: Condition met (at least one difference is non-zero).
Right + Upper: Condition met (at least one difference is non-zero).
Left + Lower: Condition met (at least one difference is non-zero).
Left + Upper: Condition met (at least one difference is non-zero).
Lower + Upper: Condition met (at least one difference is non-zero).
Right + Left + Lower: Condition met (at least one difference is non-zero).
Right + Left + Upper: Condition met (at least one difference is non-zero).
Right + Lower + Upper: Condition met (at least one difference is non-zero).
Left + Lower + Upper: Condition met (at least one difference is non-zero).
Right + Left + Lower + Upper:
Condition met (at least one difference is non-zero).

Out[5436]=

================== CONJUGATE-MODULAR CHECK ==================

M1 = M2 = M3 = M4 = M and M ≠ 2 ⇒ NOT
conjugate-modular. Boundary-strip switches preserve this.

Initial anglesDeg (no switches):

-> Condition met: All M_i are equal and $M \neq 2$.

Out[5440]=

CONJUGATE-MODULAR CHECK AFTER SWITCHING BOUNDARY STRIPS

Right: Condition met (All M_i are equal and $M \neq 2$).
Left: Condition met (All M_i are equal and $M \neq 2$).
Lower: Condition met (All M_i are equal and $M \neq 2$).
Upper: Condition met (All M_i are equal and $M \neq 2$).
Right + Left: Condition met (All M_i are equal and $M \neq 2$).
Right + Lower: Condition met (All M_i are equal and $M \neq 2$).
Right + Upper: Condition met (All M_i are equal and $M \neq 2$).
Left + Lower: Condition met (All M_i are equal and $M \neq 2$).
Left + Upper: Condition met (All M_i are equal and $M \neq 2$).
Lower + Upper: Condition met (All M_i are equal and $M \neq 2$).
Right + Left + Lower: Condition met (All M_i are equal and $M \neq 2$).
Right + Left + Upper: Condition met (All M_i are equal and $M \neq 2$).
Right + Lower + Upper: Condition met (All M_i are equal and $M \neq 2$).
Left + Lower + Upper: Condition met (All M_i are equal and $M \neq 2$).
Right + Left + Lower + Upper: Condition met (All M_i are equal and $M \neq 2$).