Computational Companion to “Quasi-Symmetric Nets: A Constructive Approach to the Equimodular Elliptic Type of Kokotsakis Polyhedra” — Example 4 Helper

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Tested on: Mathematica 14.0

In[233]:=

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(*==========================================================================
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(*Quit*)
(*All angle sets in degrees*)
anglesDeg = {
   {91.32487959870187, 27.53122212644406,
    103.21844931187813, 120}, (*Vertex 1*)
   {115.75047063536742,
    29.335366103921295, 109.7807695499394, 80}, (*Vertex 2*)
   {31.19200181228523,
    113.66642350596918, 89.61760260813426, 85}, (*Vertex 3*)
   {28.19551791700786, 107.67391515450669,
    121.75654544610931, 75} (*Vertex 4*)};


(*------------------------------------------*)
(*Function to compute sigma from 4 angles*)
computeSigma[{α_, β_, γ_, δ_}] := (α + β + γ + δ) / 2;
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(*------------------------------------------*)
(*Function to compute a,b,c,d from angles*)
computeABCD[{α_, β_, γ_, δ_}] :=
  Module[{alpha = α Degree, beta = β Degree, gamma = γ Degree,
    delta = δ Degree, sigma}, sigma = computeSigma[{α, β, γ, δ} Degree];
   {Sin[alpha] / Sin[sigma - alpha], Sin[beta] / Sin[sigma - beta],
    Sin[gamma] / Sin[sigma - gamma], Sin[delta] / Sin[sigma - delta]}];

(*------------------------------------------*)
(*Compute all σ values*)
sigmas = computeSigma /@ anglesDeg;

(*Compute all {a,b,c,d} values*)
results = computeABCD /@ anglesDeg;

(*Optional:extract individual values*)
{a1, b1, c1, d1} = results[[1]];
{a2, b2, c2, d2} = results[[2]];
{a3, b3, c3, d3} = results[[3]];
{a4, b4, c4, d4} = results[[4]];
{σ1, σ2, σ3, σ4} = FullSimplify[sigmas];


(*=========================================================================
 =====*)
(*================================
   CONDITION (N.0)=================================*)
(*=========================================================================
 =====*)
(*uniqueCombos={{1,1,1,1},{1,1,1,-1},{1,1,-1,-1},
  {1,1,-1,1},{1,-1,1,1},{1,-1,-1,1},{1,-1,1,-1},{1,-1,-1,-1}};

checkConditionN0Degrees[{α_,β_,γ_,δ_}]:=Module[
  {angles={α,β,γ,δ},results},results=Mod[uniqueCombos.angles,360]//Chop;
  !MemberQ[results,0]];

conditionsN0=checkConditionN0Degrees/@anglesDeg;
allVerticesPass=And@@conditionsN0;

Column[{TextCell[Style["================= CONDITION (N.0) =================",
    Darker[Green],Bold, 16],"Text"],
  If[allVerticesPass,
   Style["✔ All vertices satisfy (N.0).",Darker[Green],Bold],
   Style["✘ Some vertices fail (N.0).",Red,Bold]]}]*)
uniqueCombos = {{1, 1, 1, 1}, {1, 1, 1, -1}, {1, 1, -1, -1}, {1, 1, -1, 1},
   {1, -1, 1, 1}, {1, -1, -1, 1}, {1, -1, 1, -1}, {1, -1, -1, -1}};
```

```
checkConditionN0Degrees[{α_, β_, γ_, δ_}] :=
  Module[{angles = {α, β, γ, δ}, results},
   results = Mod[uniqueCombos.angles, 360] // Chop;
   results];

(*apply to all vertices*)
resultsPerVertex = checkConditionN0Degrees /@ anglesDeg;

(*check pass/fail*)
conditionsN0 = Map[! MemberQ[#, 0] &, resultsPerVertex];
allVerticesPass = And @@ conditionsN0;

Column[{TextCell[Style["================= CONDITION (N.0) =================",
     Darker[Green], Bold, 16], "Text"], (*summary line*)If[allVerticesPass,
   Style["✔ All vertices satisfy (N.0).", Darker[Green], Bold],
   Style["✘ Some vertices fail (N.0).", Red, Bold]],
  (*detailed results for each vertex*)
  Grid[Prepend[Table[{"Vertex " <> ToString[i], resultsPerVertex[[i]],
      If[conditionsN0[[i]], "✔ Pass", "✘ Fail"]}, {i, Length[anglesDeg]}],
    {"Vertex", "Combinations (mod 360)", "Status"}], Frame → All]}]


(*=======================================================================
 =====*)
(*================================
   CONDITION (N.3)==================================*)
(*=======================================================================
 =====*)
Ms = FullSimplify[Times @@@ results];
allEqualQ = Simplify[Equal @@ Ms];

Column[{TextCell[Style["================= CONDITION (N.3) =================",
     Blue, Bold, 16], "Text"], If[allEqualQ,
   Row[{Style["✔ M1 = M2 = M3 = M4 = ", Bold], Ms[[1]]}],
   Style["✘ M_i are not all equal.", Red, Bold]]}]


(*=======================================================================
 =====*)
(*==========================CONDITION (N.4)==========================*)
(*=======================================================================
 =====*)
aList = results[[All, 1]];
cList = results[[All, 3]];
dList = results[[All, 4]];
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rList = FullSimplify /@ (aList * dList); {r1, r2, r3, r4} = rList;
sList = FullSimplify /@ (cList * dList); {s1, s2, s3, s4} = sList;

Column[{TextCell[Style["================= CONDITION (N.4) =================",
    Darker[Blue], Bold, 16], "Text"],
  If[r1 == r2 && r3 == r4 && s1 == s4 && s2 == s3, Column[
    {Row[{Style["✔ r1 = r2 = ", Bold], r1, Style["; ✔ r3 = r4 = ", Bold], r3}],
     Row[{Style["✔ s1 = s4 = ", Bold], s1, Style["; ✔ s2 = s3 = ", Bold],
       s2}]}], Style["✘ Condition (N.4) fails.", Red, Bold]]
 }]


(*======================================================================
 =====*)
(*================================
  CONDITION (N.5)================================*)
(*======================================================================
 =====*)
fList = FullSimplify /@ (aList * cList); {f1, f2, f3, f4} = fList;
M1 = Ms⟦1⟧;

m[M1_] := Piecewise[{{1 - M1, M1 < 1}, {(M1 - 1) / M1, M1 > 1}}] // N;
K = EllipticK[m[M1]] // N;
Kp = EllipticK[1 - m[M1]] // N;

computeTi[i_] :=
  Module[{sigma = sigmas⟦i⟧, r = rList⟦i⟧, s = sList⟦i⟧, f = fList⟦i⟧, base},
   base = I * Im[InverseJacobiDN[Piecewise[
         {{Sqrt[f], M1 < 1}, {1 / Sqrt[f], M1 > 1}}], m[M1]]] / Kp;
   Which[sigma < 180, Which[r > 1 && s > 1, base, (r < 1 && s > 1) || (r > 1 && s < 1),
     1 + base, r < 1 && s < 1, 2 + base], sigma > 180, Which[r > 1 && s > 1,
     2 + base, (r < 1 && s > 1) || (r > 1 && s < 1), 3 + base, r < 1 && s < 1, base]]];

tList = computeTi /@ Range[4];
RoundWithTolerance[x_, tol_ : 10 ^ (-15)] :=
  Module[{nearest}, nearest = Round[x];
   If[Abs[x - nearest] ≤ tol, nearest, x]];

checkValidCombination[M1_, ε_ : 10 ^ (-15)] :=
  Module[{i, combo, dotProd, rePart, imPart, n2, expr, foundQ = False},
   Print[Style["⚠ Approximate validation using ε-tolerance. For rigorous
       proof, see the referenced paper.", Darker@Orange, Italic]];
   Do[combo = uniqueCombos⟦i⟧;
    dotProd = tList.combo;
    rePart = Abs[Re[dotProd]];
    imPart = Abs[Im[dotProd]];
    If[M1 < 1,
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    If[Mod[RoundWithTolerance[rePart], 4] < ε,
     If[Mod[RoundWithTolerance[imPart], 2] < ε, expr =
       tList⟦1⟧ + combo⟦2⟧ × tList⟦2⟧ + combo⟦3⟧ × tList⟦3⟧ + combo⟦4⟧ × tList⟦4⟧;
      Print[Style["✅ Valid Combination Found (M < 1):",
        Darker[Green], Bold], "\n", Style["e1 = ", Bold], combo⟦2⟧,
       Style[", e2 = ", Bold], combo⟦3⟧, Style[", e3 = ", Bold], combo⟦4⟧,
       "\n", Style["t1 = ", Bold], Re[tList⟦1⟧], "K + ", Im[tList⟦1⟧], "iK'",
       "\n", Style["t2 = ", Bold], Re[tList⟦2⟧], "K + ", Im[tList⟦2⟧], "iK'",
       "\n", Style["t3 = ", Bold], Re[tList⟦3⟧], "K + ", Im[tList⟦3⟧], "iK'",
       "\n", Style["t4 = ", Bold], Re[tList⟦4⟧], "K + ", Im[tList⟦4⟧],
       "iK'", "\n", Style["t1 + e1*t2 + e2*t3 + e3*t4 = ", Bold],
       Re[expr], "K + ", Im[expr], "iK'"];
      foundQ = True;
      Break[]]]];
   If[M1 > 1,
    If[Mod[RoundWithTolerance[imPart], 2] < ε,
     n2 = Quotient[RoundWithTolerance[imPart], 2];
     If[Mod[RoundWithTolerance[Abs[rePart - 2 n2]], 4] < ε, expr =
       tList⟦1⟧ + combo⟦2⟧ × tList⟦2⟧ + combo⟦3⟧ × tList⟦3⟧ + combo⟦4⟧ × tList⟦4⟧;
      Print[Style["✅ Valid Combination Found (M > 1):",
        Darker[Green], Bold], "\n", Style["e1 = ", Bold], combo⟦2⟧,
       Style[", e2 = ", Bold], combo⟦3⟧, Style[", e3 = ", Bold], combo⟦4⟧,
       "\n", Style["t1 = ", Bold], Re[tList⟦1⟧], "K + ", Im[tList⟦1⟧], "iK'",
       "\n", Style["t2 = ", Bold], Re[tList⟦2⟧], "K + ", Im[tList⟦2⟧], "iK'",
       "\n", Style["t3 = ", Bold], Re[tList⟦3⟧], "K + ", Im[tList⟦3⟧], "iK'",
       "\n", Style["t4 = ", Bold], Re[tList⟦4⟧], "K + ", Im[tList⟦4⟧],
       "iK'", "\n", Style["t1 + e1*t2 + e2*t3 + e3*t4 = ", Bold],
       Re[expr], "K + ", Im[expr], "iK'"];
      foundQ = True;
      Break[]]]], {i, Length[uniqueCombos]}];
  If[! foundQ, Print[Style["❌ No valid combination found.", Red, Bold], "\n",
    Style["t1 = ", Bold], Re[tList⟦1⟧], "K + ", Im[tList⟦1⟧], "iK'", "\n",
    Style["t2 = ", Bold], Re[tList⟦2⟧], "K + ", Im[tList⟦2⟧], "iK'", "\n",
    Style["t3 = ", Bold], Re[tList⟦3⟧], "K + ", Im[tList⟦3⟧], "iK'", "\n",
    Style["t4 = ", Bold], Re[tList⟦4⟧], "K + ", Im[tList⟦4⟧], "iK'", "\n"]];];
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Column[{TextCell[Style["================= CONDITION (N.5) =================",
    Purple, Bold, 16], "Text"]}]
res = checkValidCombination[M1];
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(*======================================================================
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   OTHER PARAMETER================================*)
(*======================================================================
 =====*)
cosSigmaAlt[Mi_, fi_, ri_, si_] :=
  (Mi (fi + ri + si - 1) + ri fi si - ri fi - ri si - fi si) / (2 Sqrt[ri si fi] (1 - Mi));

Column[
 {TextCell[Style["================= OTHER PARAMETERS =================",
    Darker[Orange], Bold, 16], "Text"],
  Row[{Style["u = ", Bold], 1 - M1}],
  Row[{Style["σ1 = ", Bold], σ1 Degree, Style[", σ2 = ", Bold], σ2 Degree,
    Style[", σ3 = ", Bold], σ3 Degree, Style[", σ4 = ", Bold], σ4 Degree}],
  Row[{Style["σ1 ≈ ", Bold], N[σ1], Style["°", Bold], Style[", σ2 ≈ ", Bold],
    N[σ2], Style["°", Bold], Style[", σ3 ≈ ", Bold], N[σ3],
    Style["°", Bold], Style[", σ4 ≈ ", Bold], N[σ4], Style["°", Bold]}],
  Row[{Style["cosσ1 = ", Bold], Simplify@cosSigmaAlt[M1, f1, r1, s1],
    Style[", cosσ2 = ", Bold], Simplify@cosSigmaAlt[M1, f2, r2, s2],
    Style[", cosσ3 = ", Bold], Simplify@cosSigmaAlt[M1, f3, r3, s3],
    Style[", cosσ4 = ", Bold], Simplify@cosSigmaAlt[M1, f4, r4, s4]}],
  Row[{Style["f1 = ", Bold], f1, Style[", f2 = ", Bold],
    f2, Style[", f3 = ", Bold], f3, Style[", f4 = ", Bold], f4}],
  Row[{Style["z1 = ", Bold], FullSimplify[1 / (f1 - 1)], Style[", z2 = ", Bold],
    FullSimplify[1 / (f2 - 1)], Style[", z3 = ", Bold], FullSimplify[1 / (f3 - 1)],
    Style[", z4 = ", Bold], FullSimplify[1 / (f4 - 1)]}],
  Row[{Style["x1 = ", Bold], FullSimplify[1 / (r1 - 1)], Style[", x2 = ", Bold],
    FullSimplify[1 / (r2 - 1)], Style[", x3 = ", Bold], FullSimplify[1 / (r3 - 1)],
    Style[", x4 = ", Bold], FullSimplify[1 / (r4 - 1)]}],
  Row[{Style["y1 = ", Bold], FullSimplify[1 / (s1 - 1)], Style[", y2 = ", Bold],
    FullSimplify[1 / (s2 - 1)], Style[", y3 = ", Bold], FullSimplify[1 / (s3 - 1)],
    Style[", y4 = ", Bold], FullSimplify[1 / (s4 - 1)]}],
  Row[{Style["p1 = ", Bold], Simplify[Sqrt[r1 - 1]],
    Style[", p2 = ", Bold], Simplify[Sqrt[r2 - 1]], Style[", p3 = ", Bold],
    Simplify[Sqrt[r3 - 1]], Style[", p4 = ", Bold], Simplify[Sqrt[r4 - 1]]}],
  Row[{Style["q1 = ", Bold], Simplify[Sqrt[s1 - 1]],
    Style[", q2 = ", Bold], Simplify[Sqrt[s2 - 1]], Style[", q3 = ", Bold],
    Simplify[Sqrt[s3 - 1]], Style[", q4 = ", Bold], Simplify[Sqrt[s4 - 1]]}],
  Row[{Style["p1·q1 = ", Bold], Simplify[Sqrt[r1 - 1] Sqrt[s1 - 1]],
    Style[", p2·q2 = ", Bold], Simplify[Sqrt[r2 - 1] Sqrt[s2 - 1]],
    Style[", p3·q3 = ", Bold], Simplify[Sqrt[r3 - 1] Sqrt[s3 - 1]],
    Style[", p4·q4 = ", Bold], Simplify[Sqrt[r4 - 1] Sqrt[s4 - 1]]}]
 }]


(*======================================================================
 =====*)
(*==============================
  BRICARD's EQUATIONS==============================*)
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(*===============================================================================
 =====*)
(*Bricard's equation as defined in terms of angles and variables*)
BricardsEquation[{α_, β_, γ_, δ_}, σ_, x_, y_] := Module[
   {c22, c20, c02, c11, c00},
   c22 = Sin[σ - δ] Sin[σ - δ - β];
   c20 = Sin[σ - α] Sin[σ - α - β];
   c02 = Sin[σ - γ] Sin[σ - γ - β];
   c11 = -Sin[α] Sin[γ];
   c00 = Sin[σ] Sin[σ - β];
   c22 x^2 y^2 + c20 x^2 + c02 y^2 + 2 c11 x y + c00
  ];

(*Step 1: Bricard's system of equations*)
Column[{(*Header*)TextCell[
   Style["================ Bricard's System of Equations ================",
    Darker[Purple], Bold, 16], "Text"], (*Explanatory note*)
  Row[
   {TextCell[Style["We introduce new notation for the cotangents of half of
        the dihedral angles. Denote Z:= ", GrayLevel[0.3], 13],
     "Text"], TraditionalForm[cot[Subscript[θ, 1] / 2]], TextCell[
     Style[", W₂= ", GrayLevel[0.3], 13], "Text"],
    TraditionalForm[cot[Subscript[θ, 2] / 2]],
    TextCell[Style[", U:= ", GrayLevel[0.3], 13], "Text"],
    TraditionalForm[cot[Subscript[θ, 3] / 2]],
    TextCell[Style[", and W₁= ", GrayLevel[0.3], 13], "Text"],
    TraditionalForm[cot[Subscript[θ, 4] / 2]]
   }], Spacer[12],
  (*Traditional form results*)Row[{"P₁(Z, W₁) = ",
    TraditionalForm[Collect[FullSimplify[BricardsEquation[anglesDeg〚1〛 Degree,
        sigmas〚1〛 Degree, Z, W₁]], W₁]], " = 0"}], Spacer[6],
  Row[{"P₂(Z, W₂) = ",
    TraditionalForm[Collect[FullSimplify[BricardsEquation[anglesDeg〚2〛 Degree,
        sigmas〚2〛 Degree, Z, W₂]], W₂]], " = 0"}], Spacer[6],
  Row[{"P₃(U, W₂) = ",
    TraditionalForm[Collect[FullSimplify[BricardsEquation[anglesDeg〚3〛 Degree,
        sigmas〚3〛 Degree, U, W₂]], W₂]], " = 0"}], Spacer[6],
  Row[{"P₄(U, W₁) = ", TraditionalForm[Collect[FullSimplify[BricardsEquation[
        anglesDeg〚4〛 Degree, sigmas〚4〛 Degree, U, W₁]], W₁]], " = 0"}]
 }]
(*================================
 FLEXION 1================================*)
Z[t_] := t;

W1[t_] := (2.2689737907253456` (10.50393110877767` t - 2.542115267025096`
        Sqrt[(1 - 0.5721009760467486` t^2) (-1 + 7.594436687028424` t^2)])) /
```

```
    (14.662902846473798` + 7.594436687028424` t^2);
U[t_] := (0.20405148639750417` (1.1778865025892307` t + 10.102197432038`
          Sqrt[(1 - 0.5721009760467486` t^2) (-1 + 7.594436687028424` t^2)])) /
    (3.6622434357916323` - 2.099237700754583` t^2);
W2[t_] := (1.3011995890502461` (5.832687748284718` t - 1.6606733509905591`
          Sqrt[(1 - 0.5721009760467486` t^2) (-1 + 7.594436687028424` t^2)])) /
    (3.588495042139688` + 7.594436687028424` t^2);

(*==================================
 FLEXION 2==================================*)
Z2[t_] := t;
W12[t_] := (2.2689737907253456` (10.50393110877767` t + 2.542115267025096`
          Sqrt[(1 - 0.5721009760467486` t^2) (-1 + 7.594436687028424` t^2)])) /
    (14.662902846473798` + 7.594436687028424` t^2);
U2[t_] := (0.20405148639750417` (1.1778865025892307` t - 10.102197432038`
          Sqrt[(1 - 0.5721009760467486` t^2) (-1 + 7.594436687028424` t^2)])) /
    (3.6622434357916323` - 2.099237700754583` t^2);
W22[t_] := (1.3011995890502461` (5.832687748284718` t + 1.6606733509905591`
          Sqrt[(1 - 0.5721009760467486` t^2) (-1 + 7.594436687028424` t^2)])) /
    (3.588495042139688` + 7.594436687028424` t^2);


(*Step 2: Formulas for flexions*)
Column[
 {(*Header*)TextCell[Style["================= FLEXIONS =================",
    Red, Bold, 16], "Text"], (*Explanatory note*)TextCell[Style[
    "Solutions to Bricard's equations under a free parameter t := Z ∈ ℂ:",
    GrayLevel[0.3], 13], "Text"], Spacer[12],
  (*Heading for results*)TextCell[Style["Solution 1:", Bold, 14], "Text"],
  Spacer[6], (*Traditional form results*)
  Row[{"Z(t) = ", TraditionalForm[FullSimplify[Z[t]]]}], Spacer[6],
  Row[{"W₂(t) = ", TraditionalForm[FullSimplify[W2[t]]]}], Spacer[6],
  Row[{"U(t) = ", TraditionalForm[FullSimplify[U[t]]]}], Spacer[6],
  Row[{"W₁(t) = ", TraditionalForm[FullSimplify[W1[t]]]}], Spacer[12],
  (*Heading for results*)TextCell[Style["Solution 2:", Bold, 14], "Text"],
  Spacer[6], (*Traditional form results*)
  Row[{"Z(t) = ", TraditionalForm[FullSimplify[Z2[t]]]}], Spacer[6],
  Row[{"W₂(t) = ", TraditionalForm[FullSimplify[W22[t]]]}], Spacer[6],
  Row[{"U(t) = ", TraditionalForm[FullSimplify[U2[t]]]}], Spacer[6],
  Row[{"W₁(t) = ", TraditionalForm[FullSimplify[W12[t]]]}]
 }]
```

```
(*Step 3: Checking that W1(t) and W2(t) solves P1(t, W1) =
 0 and P2(t, W2) = 0 even when +- signs do NOT agree*)
(*t-range*)
tMin = 0.4;
tMax = 1.25;

Column[
 {(*Header*)TextCell[Style["================ Equations: P1(t, W1) = 0
      and P2(t, W2) = 0 ================",
    Orange, Bold, 16], "Text"], (*Explanatory note*)TextCell[
   Style["Let W1s1 and W1s2 be formulas for W1(t) from solutions
      1 and 2, respectively. Similarly, let W2s1 and W2s2 be
      the formulas for W2(t). We show that all four pairs -
      (W1s1, W2s1), (W1s1, W2s2), (W1s2, W2s1), and (W1s2, W2s2)
      - solve equations P1(t, W1) = 0 and P2(t, W2) = 0.",
    GrayLevel[0.3], 13], "Text"], Spacer[12],
  (*Heading for results*)

  TextCell[Style["Pair 1, (W1, W2) = (W1s1, W2s1):", Bold, 14], "Text"],
  Spacer[6], (*Traditional form results*)
  Row[{"P1(t, W1) = ", TraditionalForm[FullSimplify[BricardsEquation[
       anglesDeg[[1]] Degree, sigmas[[1]] Degree, Z[t], W1[t]]]]}], Spacer[6],
  Row[{"P2(t, W2) = ", TraditionalForm[FullSimplify[BricardsEquation[
       anglesDeg[[2]] Degree, sigmas[[2]] Degree, Z[t], W2[t]]]]}], Spacer[12],
  (*Verification Plots Module:This entire Module is now the last item inside
     the main Column.It will execute and place the resulting Panel here.*)
  Module[{PiExpr, labels}, (*1. Define the two polynomial expressions
    to be plotted*)PiExpr = {FullSimplify[BricardsEquation[
       anglesDeg[[1]] Degree, sigmas[[1]] Degree, Z[t], W1[t]]], FullSimplify[
      BricardsEquation[anglesDeg[[2]] Degree, sigmas[[2]] Degree, Z[t], W2[t]]]};
   (*2. Define the labels for each plot*)
   labels = {Row[{Subscript["P", 1], "(t, ", Subscript["W", 1], ")", " ≈ 0 ",
       Style["✓", Darker[Green], Bold]}], Row[{Subscript["P", 2], "(t, ",
       Subscript["W", 2], ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}]};
   (*3. Display the plots in the specified panel style*)
   Panel[GraphicsGrid[Partition[Table[Plot[PiExpr[[i]], {t, tMin, tMax},
        PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → {-10^-10, 10^-10},
        (*Zoom in to confirm zero*)AxesLabel → {"t", None}, ImageSize → 300],
       {i, Length[PiExpr]}], 2 (*Arrange plots in rows of 2*)],
     Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]],

  TextCell[Style["Pair 2, (W1, W2) = (W1s1, W2s2):", Bold, 14], "Text"],
  Spacer[6], (*Traditional form results*)
  Row[{"P1(t, W1) = ", TraditionalForm[FullSimplify[BricardsEquation[
       anglesDeg[[1]] Degree, sigmas[[1]] Degree, Z[t], W1[t]]]]}], Spacer[6],
  Row[{"P2(t, W2) = ", TraditionalForm[FullSimplify[BricardsEquation[
       anglesDeg[[2]] Degree, sigmas[[2]] Degree, Z[t], W22[t]]]]}], Spacer[12],
```

```
(*Verification Plots Module:This entire Module is now the last item inside
   the main Column.It will execute and place the resulting Panel here.*)
Module[{PiExpr, labels}, (*1. Define the two polynomial expressions
   to be plotted*)PiExpr = {FullSimplify[BricardsEquation[
      anglesDeg⟦1⟧ Degree, sigmas⟦1⟧ Degree, Z[t], W1[t]]], FullSimplify[
     BricardsEquation[anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z[t], W22[t]]]};
  (*2. Define the labels for each plot*)
  labels = {Row[{Subscript["P", 1], "(t, ", Subscript["W", 1], ")", " ≈ 0 ",
      Style["✓", Darker[Green], Bold]}], Row[{Subscript["P", 2], "(t, ",
      Subscript["W", 2], ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}]};
  (*3. Display the plots in the specified panel style*)
  Panel[GraphicsGrid[Partition[Table[Plot[PiExpr⟦i⟧, {t, tMin, tMax},
        PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → {-10^-10, 10^-10},
        (*Zoom in to confirm zero*)AxesLabel → {"t", None}, ImageSize → 300],
      {i, Length[PiExpr]}], 2 (*Arrange plots in rows of 2*)],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]],

TextCell[Style["Pair 3, (W_1, W_2) = (W_1s2, W_2s1):", Bold, 14], "Text"],
Spacer[6], (*Traditional form results*)
Row[{"P_1(t, W_1) = ", TraditionalForm[FullSimplify[BricardsEquation[
      anglesDeg⟦1⟧ Degree, sigmas⟦1⟧ Degree, Z[t], W12[t]]]]}], Spacer[6],
Row[{"P_2(t, W_2) = ", TraditionalForm[FullSimplify[BricardsEquation[
      anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z[t], W2[t]]]]}], Spacer[12],
(*Verification Plots Module:This entire Module is now the last item inside
   the main Column.It will execute and place the resulting Panel here.*)
Module[{PiExpr, labels}, (*1. Define the two polynomial expressions
   to be plotted*)PiExpr = {FullSimplify[BricardsEquation[
      anglesDeg⟦1⟧ Degree, sigmas⟦1⟧ Degree, Z[t], W12[t]]], FullSimplify[
     BricardsEquation[anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z[t], W2[t]]]};
  (*2. Define the labels for each plot*)
  labels = {Row[{Subscript["P", 1], "(t, ", Subscript["W", 1], ")", " ≈ 0 ",
      Style["✓", Darker[Green], Bold]}], Row[{Subscript["P", 2], "(t, ",
      Subscript["W", 2], ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}]};
  (*3. Display the plots in the specified panel style*)
  Panel[GraphicsGrid[Partition[Table[Plot[PiExpr⟦i⟧, {t, tMin, tMax},
        PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → {-10^-10, 10^-10},
        (*Zoom in to confirm zero*)AxesLabel → {"t", None}, ImageSize → 300],
      {i, Length[PiExpr]}], 2 (*Arrange plots in rows of 2*)],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]],

TextCell[Style["Pair 4, (W_1, W_2) = (W_1s2, W_2s2):", Bold, 14], "Text"],
Spacer[6], (*Traditional form results*)
Row[{"P_1(t, W_1) = ", TraditionalForm[FullSimplify[BricardsEquation[
      anglesDeg⟦1⟧ Degree, sigmas⟦1⟧ Degree, Z[t], W12[t]]]]}], Spacer[6],
Row[{"P_2(t, W_2) = ", TraditionalForm[FullSimplify[BricardsEquation[
      anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z[t], W22[t]]]]}], Spacer[12],
(*Verification Plots Module:This entire Module is now the last item inside
```

```
    the main Column.It will execute and place the resulting Panel here.*)
  Module[{PiExpr, labels}, (*1. Define the two polynomial expressions
     to be plotted*)PiExpr = {FullSimplify[BricardsEquation[
        anglesDeg⟦1⟧ Degree, sigmas⟦1⟧ Degree, Z[t], W12[t]]], FullSimplify[
       BricardsEquation[anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z[t], W22[t]]]};
    (*2. Define the labels for each plot*)
    labels = {Row[{Subscript["P", 1], "(t, ", Subscript["W", 1], ")", " ≈ 0 ",
        Style["✓", Darker[Green], Bold]}], Row[{Subscript["P", 2], "(t, ",
        Subscript["W", 2], ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}]};
    (*3. Display the plots in the specified panel style*)
    Panel[GraphicsGrid[Partition[Table[Plot[PiExpr⟦i⟧, {t, tMin, tMax},
          PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → {-10^-10, 10^-10},
          (*Zoom in to confirm zero*)AxesLabel → {"t", None}, ImageSize → 300],
        {i, Length[PiExpr]}], 2 (*Arrange plots in rows of 2*)],
      Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]]
 }]


(*Step 4: Checking that W1(t) and W2(t) does NOT satisfy W1(t)/W2(t) =
 c and W1(t)W2(t) = c where c is constant even when +- signs do NOT agree*)
expressions = {Z[t] * U[t], Z[t] / U[t], Z[t] * U2[t], Z[t] / U2[t],
   Z2[t] * U[t], Z2[t] / U[t], Z2[t] * U2[t], Z2[t] / U2[t]};
labels = {"W1s1 * W2s1", "W1s1 / W2s1", "W1s1 * W2s2", "W1s1 / W2s2",
   "W1s2 * W2s1", "W1s2 / W2s1", "W1s2 * W2s2", "W1s2 / W2s2"};
Column[
 {TextCell[Style["============ Plots of W1(t)W2(t) and W1(t)/W2(t) For All
      Pairs ============", Darker[Orange], Bold, 16], "Text"],

  (*Explanatory text*)
  TextCell[Style["Checking that all four pairs - (W1s1, W2s1), (W1s1, W2s2),
      (W1s2, W2s1), and (W1s2, W2s2) - does NOT satisfy W1(t)/W2(t)
      = const and W1(t)W2(t) = const.", GrayLevel[0.3]], "Text"],
  Spacer[12],

  (*Plots panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions⟦i⟧, {t, tMin, tMax},
        PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → All,
        AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
 }]


(*Step 5: Solving P3(U, W2) = 0 and P4(U, W1) = 0 for W1 and W2*)
(*Define the first Bricard equation P3(U, W2) = 0*)
P3 = BricardsEquation[anglesDeg⟦3⟧ Degree, sigmas⟦3⟧ Degree, U, Ww2];
(*Solve for W21,W22*)
solutionW2Expressions =
```

```
  (Ww2 /. FullSimplify[Solve[Rationalize[P3, 0] == 0, Ww2]]);
Ww21[U_] := solutionW2Expressions[[1]];
Ww22[U_] := solutionW2Expressions[[2]];

(*Define the fourth Bricard equation P4(U, W1) = 0*)
P4 = BricardsEquation[anglesDeg[[4]] Degree, sigmas[[4]] Degree, U, Ww1];
(*Solve for W11,W12*)
solutionW1Expressions =
  (Ww1 /. FullSimplify[Solve[Rationalize[P4, 0] == 0, Ww1]]);
Ww11[U_] := solutionW1Expressions[[1]];
Ww12[U_] := solutionW1Expressions[[2]];

Column[
 {(*Header*)TextCell[Style["================= Equations: P3(U, W2) = 0
      and P4(U, W1) = 0 =================",
    Darker[Cyan], Bold, 16], "Text"], (*Explanatory note*)
  TextCell[Style["We solve P3(U, W2) = 0 and P4(U, W1) = 0 for W1 and W2.",
    GrayLevel[0.3], 13], "Text"], Spacer[12],
  (*Heading for results*)Row[{"W1,1(U) = ", TraditionalForm[Ww11[U]]}],
  Row[{"W1,2(U) = ", TraditionalForm[Ww12[U]]}],
  Spacer[6], Row[{"W2,1(U) = ", TraditionalForm[Ww21[U]]}],
  Row[{"W2,2(U) = ", TraditionalForm[Ww22[U]]}]
 }]


(*Step 6: Checking that pairs (W1,1(U), W2,1(U)), (W1,1(U), W2,2(U)),
(W1,2(U), W2,1(U)), and (W1,2(U), W2,2(U)) does NOT satisfy W1(U)/W2(U) =
 c and W1(U)W2(U) = c where c is constant*)
UMin = 0;
UMax = 1;

expressions =
  {Ww11[U] * Ww21[U], Ww11[U] / Ww21[U], Ww11[U] * Ww22[U], Ww11[U] / Ww22[U],
   Ww12[U] * Ww21[U], Ww12[U] / Ww21[U], Ww12[U] * Ww22[U], Ww12[U] / Ww22[U]};
labels = {"W1,1 * W2,1", "W1,1 / W2,1", "W1,1 * W2,2",
   "W1,1 / W2,2", "W1,2 * W2,1", "W1,2 / W2,1", "W1,2 * W2,2", "W1,2 / W2,2"};
Column[
 {TextCell[Style["============ Plots of W1(U)W2(U) and W1(U)/W2(U) For All
      Pairs ============", Magenta, Bold, 16], "Text"],

  (*Explanatory text*)
  TextCell[Style["Checking that all four pairs - (W1,1(U),
      W2,1(U)), (W1,1(U), W2,2(U)), (W1,2(U), W2,1(U)), and
      (W1,2(U), W2,2(U)) - does NOT satisfy W1(U)/W2(U) = const
      and W1(U)W2(U) = const.", GrayLevel[0.3]], "Text"],
  Spacer[12],
```

```
  (*Plots panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions⟦i⟧, {U, UMin, UMax},
       PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → All,
       AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
 }]

(*Step 7: Compute and print all P_i for flexions 1 and 2*)
TextCell[
 Style["================ FLEXIBILITY (Double Checking) ================",
  Orange, Bold, 16], "Text"]
funcs = {{Z, W1}, {Z, W2}, {U, W2}, {U, W1}};
TextCell[Style["Solution 1:", Bold, 14], "Text"]
Do[angles = anglesDeg⟦i⟧ Degree;
  sigma = sigmas⟦i⟧ Degree;
  {α, β, γ, δ} = angles;
  poly = Which[i == 1, BricardsEquation[{α, β, γ, δ}, sigma, Z[t], W1[t]],
    i == 2, BricardsEquation[{α, β, γ, δ}, sigma, Z[t], W2[t]],
    i == 3, BricardsEquation[{α, β, γ, δ}, sigma, U[t], W2[t]],
    i == 4, BricardsEquation[{α, β, γ, δ}, sigma, U[t], W1[t]]];
  Print[Row[{Subscript["P", i], "(", funcs⟦i, 1⟧,
     ", ", funcs⟦i, 2⟧, ") = ", FullSimplify[poly]}]], {i, 1, 4}];

(*Build P_i(t) consistently with your printing logic*)
Clear[PiExpr, labels, funcs];
funcs = {{Z, W1}, {Z, W2}, {U, W2}, {U, W1}};

PiExpr = Table[Module[{angles = anglesDeg⟦i⟧ Degree,
      sigma = sigmas⟦i⟧ Degree, α, β, γ, δ, poly}, {α, β, γ, δ} = angles;
    poly = BricardsEquation[{α, β, γ, δ}, sigma, funcs⟦i, 1⟧[t], funcs⟦i, 2⟧[t]];
    FullSimplify[poly]], {i, 1, 4}];

labels = Table[Row[
    {Subscript["P", i], "(", ToString@funcs⟦i, 1⟧, ", ", ToString@funcs⟦i, 2⟧,
     ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}], {i, 1, 4}];

(*Pretty panel with plots like your style*)
Column[{TextCell[Style["================ FLEXION 1 ================",
    Darker[Cyan], Bold, 16], "Text"], TextCell[
   Style["Polynomials Pi(t) built from Bricard’s equations for flexion 1.",
    GrayLevel[0.3]], "Text"], Spacer[12],
  Panel[GraphicsGrid[Partition[Table[Plot[Evaluate@PiExpr⟦i⟧,
       {t, tMin, tMax}, PlotLabel → Style[labels⟦i⟧, Bold, 14],
       PlotRange → {-10^(-11), 10^(-11)}, AxesLabel → {"t", None},
       ImageSize → 250], {i, Length[PiExpr]}], 2], Spacings → {2, 2}],
   Background → Lighter[Gray, 0.95], FrameMargins → 15]}]
```

```
(*Compute and print all P_i for flexion 2*)
TextCell[Style["Solution 2:", Bold, 14], "Text"]
funcs = {{Z, W1}, {Z, W2}, {U, W2}, {U, W1}};
Do[angles = anglesDeg[[i]] Degree;
  sigma = sigmas[[i]] Degree;
  {α, β, γ, δ} = angles;
  poly = Which[i == 1, BricardsEquation[{α, β, γ, δ}, sigma, Z2[t], W12[t]],
    i == 2, BricardsEquation[{α, β, γ, δ}, sigma, Z2[t], W22[t]],
    i == 3, BricardsEquation[{α, β, γ, δ}, sigma, U2[t], W22[t]],
    i == 4, BricardsEquation[{α, β, γ, δ}, sigma, U2[t], W12[t]]];
  Print[Row[{Subscript["P", i], "(", funcs[[i, 1]],
     ", ", funcs[[i, 2]], ") = ", FullSimplify[poly]}]], {i, 1, 4}];

(*Build P_i(t) consistently with your printing logic*)
Clear[PiExpr, labels, funcs];
funcs = {{Z2, W12}, {Z2, W22}, {U2, W22}, {U2, W12}};

PiExpr = Table[Module[{angles = anglesDeg[[i]] Degree,
      sigma = sigmas[[i]] Degree, α, β, γ, δ, poly}, {α, β, γ, δ} = angles;
     poly = BricardsEquation[{α, β, γ, δ}, sigma, funcs[[i, 1]][t], funcs[[i, 2]][t]];
     FullSimplify[poly]], {i, 1, 4}];

labels = Table[Row[
     {Subscript["P", i], "(", ToString@funcs[[i, 1]], ", ", ToString@funcs[[i, 2]],
      ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}], {i, 1, 4}];

(*Pretty panel with plots like your style*)
Column[{TextCell[Style["================= FLEXION 2 =================",
     Magenta, Bold, 16], "Text"], TextCell[
    Style["Polynomials Pi(t) built from Bricard’s equations for flexion 2.",
     GrayLevel[0.3]], "Text"], Spacer[12],
  Panel[GraphicsGrid[Partition[Table[Plot[Evaluate@PiExpr[[i]],
        {t, tMin, tMax}, PlotLabel → Style[labels[[i]], Bold, 14],
        PlotRange → {-10^(-11), 10^(-11)}, AxesLabel → {"t", None},
        ImageSize → 250], {i, Length[PiExpr]}], 2], Spacings → {2, 2}],
   Background → Lighter[Gray, 0.95], FrameMargins → 15]}]


(*=========================================================================
 =====*)
(*===================NOT LINEAR COMPOUND==================*)
(*=========================================================================
 =====*)
(*{TraditionalForm[cot[Subscript[θ,1]/2]],
  TraditionalForm[cot[Subscript[θ,2]/2]],
  TraditionalForm[cot[Subscript[θ,3]/2]],
  TraditionalForm[cot[Subscript[θ,4]/2]]};*)
```

```
(*==================================
 FLEXION 1==================================*)
(*List of expressions& labels*)
expressions =
  {Z[t] * U[t], Z[t] / U[t], W1[t] * W2[t], W1[t] / W2[t], Z[t] * W2[t], Z[t] / W2[t],
   U[t] * W1[t], U[t] / W1[t], Z[t] * W1[t], Z[t] / W1[t], W2[t] * U[t], W2[t] / U[t]};
labels = {"Z * U", "Z / U", "W₁ * W₂", "W₁ / W₂", "Z * W₂", "Z / W₂",
   "U * W₁", "U / W₁", "Z * W₁", "Z / W₁", "W₂ * U", "W₂ / U"};

Column[{TextCell[Style["============= NOT LINEAR COMPOUND =============",
     Darker[Brown], Bold, 16], "Text"],
  (*Explanatory text*)TextCell[
   Style["Above we consider the first pair of equations (P₁(t, W₁) = 0 and
      P₂(t, W₂) = 0). Solving them as quadratic equations in W₁ and W₂,
      respectively we parametrize the solutions by the first two  and
      fourth expressions in Solutions 1 and 2 in a neighborhood of any
      point (W₁, t, W₂) such that the expression in the square root and
      denominators are not zero. Here, we choose any continuous branch
      of the square root in this neighborhood, and the signs in ± need
      Not agree (this means we consider all 4 pairs we describe above).
      We conclude that NO component of the solution set of the first
      pair of Bricard's equations satisfies W₁/W₂=const NOR W₁W₂=const.

Analogously, NO component of the solution set of the other pair
      of equations (P₃(U, W₂) = 0 and P₄(U, W₁) = 0) satisfies
      W₁/W₂=const NOR W₁W₂=const. As a result, NO component
      of the solution set of all four equations satisfies
      W₁/W₂=const NOR W₁W₂=const. So, our example does not belong
      to the linear compound class, even after switching the
boundary strips.", GrayLevel[0.3]], "Text"],
  Spacer[6],
  TextCell[Style[
    "Below, we also present the plots of functions ZU, Z/U, W₁W₂, W₁/W₂, ZW₂,
      Z/W₂, UW₁, U/W₁, ZW₁, Z/W₁, W₂U, W₂/U.", GrayLevel[0.3]], "Text"],
  Spacer[6],
  TextCell[Style["Solution 1:", Bold, 14], "Text"], Spacer[12],
  Panel[GraphicsGrid[Partition[Table[Plot[expressions[[i]], {t, tMin, tMax},
       PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → All,
       AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
 }]


(*==================================
 FLEXION 2==================================*)
(*List of expressions& labels*)
```

```
expressions = {Z2[t] * U2[t], Z2[t] / U2[t], W12[t] * W22[t], W12[t] / W22[t],
   Z2[t] * W22[t], Z2[t] / W22[t], U2[t] * W12[t], U2[t] / W12[t],
   Z2[t] * W12[t], Z2[t] / W12[t], W22[t] * U2[t], W22[t] / U2[t]};
labels = {"Z * U", "Z / U", "W₁ * W₂", "W₁ / W₂", "Z * W₂", "Z / W₂",
   "U * W₁", "U / W₁", "Z * W₁", "Z / W₁", "W₂ * U", "W₂ / U"};

Column[{ TextCell[Style["Solution 2:", Bold, 14], "Text"], Spacer[12],
  (*Plots panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions〚i〛, {t, tMin, tMax},
       PlotLabel → Style[labels〚i〛, Bold, 14], PlotRange → All,
       AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
 }]

(*===========================================================================
 =====*)
(*=================================
 NOT TRIVIAL=================================*)
(*===========================================================================
 =====*)
(*=================================
 FLEXION 1================================*)
(*List of expressions to plot*)
expressions = {Z[t], W2[t], U[t], W1[t]};
labels = {"Z", "W₂", "U", "W₁"};

Column[{TextCell[
   Style["================= NOT TRIVIAL (FLEXION 1) =================",
    Pink, Bold, 16], "Text"],

  (*Explanatory text*)
  TextCell[Style["This configuration does not belong to the trivial class –
      even after switching the boundary strips – since none of the
      functions Z, W2, U, or W1 is constant.", GrayLevel[0.3]], "Text"],
  Spacer[12],

  (*Plots in a light panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions〚i〛, {t, tMin, tMax},
       PlotLabel → Style[labels〚i〛, Bold, 14], PlotRange → All,
       AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]

 }]

(*=================================
 FLEXION 2================================*)
(*List of expressions to plot*)
```

```
expressions = {Z2[t], W22[t], U2[t], W12[t]};
labels = {"Z", "W₂", "U", "W₁"};

Column[{TextCell[
   Style["================= NOT TRIVIAL (FLEXION 2) =================",
    Darker[Pink], Bold, 16], "Text"],

  (*Explanatory text*)
  TextCell[Style["This configuration does not belong to the trivial class –
      even after switching the boundary strips – since none of the
      functions Z, W2, U, or W1 is constant.", GrayLevel[0.3]], "Text"],
  Spacer[12],

  (*Plots in a light panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions〚i〛, {t, tMin, tMax},
       PlotLabel → Style[labels〚i〛, Bold, 14], PlotRange → All,
       AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]

 }]



(*=========================================================================
 =====*)
(*==========================
 SWITCHING BOUNDARY STRIPS==========================*)
(*=========================================================================
 =====*)
SwitchingRightBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices:{row,column}={1,2},{1,3},{4,2},{4,3}*)
  modified〚1, 2〛 = 180 - anglesDeg〚1, 2〛; (*β1*)
  modified〚1, 3〛 = 180 - anglesDeg〚1, 3〛; (*γ1*)
  modified〚4, 2〛 = 180 - anglesDeg〚4, 2〛; (*β4*)
  modified〚4, 3〛 = 180 - anglesDeg〚4, 3〛; (*γ4*)
  modified]

SwitchingLeftBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices:{row,column}={2,2},{2,3},{3,2},{3,3}*)
  modified〚2, 2〛 = 180 - anglesDeg〚2, 2〛; (*β2*)
  modified〚2, 3〛 = 180 - anglesDeg〚2, 3〛; (*γ2*)
  modified〚3, 2〛 = 180 - anglesDeg〚3, 2〛; (*β3*)
  modified〚3, 3〛 = 180 - anglesDeg〚3, 3〛; (*γ3*)
  modified]

SwitchingLowerBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices:{row,column}={1,1},{1,2},{2,1},{2,2}*)
```

```
  modified⟦1, 1⟧ = 180 - anglesDeg⟦1, 1⟧; (*α1*)
  modified⟦1, 2⟧ = 180 - anglesDeg⟦1, 2⟧; (*β1*)
  modified⟦2, 1⟧ = 180 - anglesDeg⟦2, 1⟧; (*α2*)
  modified⟦2, 2⟧ = 180 - anglesDeg⟦2, 2⟧; (*β2*)
  modified]

SwitchingUpperBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices:{row,column}={3,1},{3,2},{4,1},{4,2}*)
  modified⟦3, 1⟧ = 180 - anglesDeg⟦3, 1⟧; (*α3*)
  modified⟦3, 2⟧ = 180 - anglesDeg⟦3, 2⟧; (*β3*)
  modified⟦4, 1⟧ = 180 - anglesDeg⟦4, 1⟧; (*α4*)
  modified⟦4, 2⟧ = 180 - anglesDeg⟦4, 2⟧; (*β4*)
  modified]


(*============================================================================
 =====*)
(*================================NOT CONIC & NOT CHIMERA & NOT LINEAR
     CONJUGATE & NOT ISOGONAL==================================*)
(*============================================================================
 =====*)
uniqueCombos = {{1, 1, 1, 1}, {1, 1, 1, -1}, {1, 1, -1, -1}, {1, 1, -1, 1},
   {1, -1, 1, 1}, {1, -1, -1, 1}, {1, -1, 1, -1}, {1, -1, -1, -1}};

checkConditionN0Degrees[{α_, β_, γ_, δ_}] :=
  Module[{angles = {α, β, γ, δ}, results},
   results = Mod[uniqueCombos.angles, 360] // Chop;
   ! MemberQ[results, 0]];

Column[{TextCell[Style[
    "================= NOT CONIC & NOT CHIMERA & NOT LINEAR CONJUGATE & NOT
      ISOGONAL=================", Darker[Magenta], Bold, 16], "Text"],
  TextCell[Style[
    "Condition (N.0) is satisfied for all i=1,...,4 ⇒ NOT equimodular-conic,
      NOT chimera, NOT isogonal and NOT linear conjugate.
      Applying any boundary-strip switch still preserves
      (N.0), so no conic, no chimera, no isogonal and no
      linear conjugate form emerges.", GrayLevel[0.3]], "Text"]
 }]

(*Now the exact same Module for checking all switch combinations…*)
Module[{angles = anglesDeg, switchers, combinations, results},
 (*Define switch functions*)
 switchers = <|"Right" → SwitchingRightBoundaryStrip,
   "Left" → SwitchingLeftBoundaryStrip, "Lower" → SwitchingLowerBoundaryStrip,
   "Upper" → SwitchingUpperBoundaryStrip|>;
 (*Generate all combinations of switches (from size 1 to 4)*)
```

```
combinations = Subsets[Keys[switchers], {1, Length[switchers]}];
(*Evaluate condition after each combination of switches*)results = Table[
  Module[{switched = angles, name, passQ}, name = StringRiffle[combo, " + "];
   Do[switched = switchers[sw][switched], {sw, combo}];
   passQ = And @@ (checkConditionN0Degrees /@ switched);
   (*Print the result after switching*)
   (*Print["\nSwitch combination: ",name];
   Print["Switched anglesDeg:"];
   Print[MatrixForm[switched]];*)
   {name, passQ}], {combo, combinations}];
(*Display results*)
Column[Prepend[Table[Module[{comboName = res⟦1⟧, passQ = res⟦2⟧},
    Row[{Style[comboName <> ": ", Bold],
      If[passQ,
       Style["Condition (N.0) is still satisfied.", Darker[Green]],
       Style["Condition (N.0) fails.", Red, Bold]
      ]
     }
    ]], {res, results}], TextCell[
   Style["CONDITION (N.0) AFTER SWITCHING BOUNDARY STRIPS", 14], "Text"]]]]


(*=============================================================================
 =====*)
(*================================
 NOT ORTHODIAGONAL================================*)
(*=============================================================================
 =====*)

(*Column[
  {TextCell[Style["================= NOT ORTHODIAGONAL =================",
     Darker[Blue],Bold,16],"Text"],
   TextCell[Style[
     "cos(αi)·cos(γi) ≠ cos(βi)·cos(δi) for each i = 1 ⇒ NOT orthodiagonal.
       Switching boundary strips does not
       correct this.",  GrayLevel[0.3]],"Text"]
  }]

Module[{angles=anglesDeg,switchers,combinations,results},
  (*Define switch functions*)switchers=<|"Right"→SwitchingRightBoundaryStrip,
    "Left"→SwitchingLeftBoundaryStrip,"Lower"→SwitchingLowerBoundaryStrip,
    "Upper"→SwitchingUpperBoundaryStrip|>;
  (*Helper function:compute and print difference only*)
  formatOrthodiagonalCheck[quad_List]:=
   Module[{vals},vals=Table[Module[{a,b,c,d,lhs,rhs,diff},{a,b,c,d}=quad⟦i⟧;
       lhs=FullSimplify[Cos[a Degree] Cos[c Degree]];
       rhs=FullSimplify[Cos[b Degree] Cos[d Degree]];
```

```
      diff=Chop[lhs-rhs];
      Style[Row[{"cos(α"<>ToString[i]<>")·cos(γ"<>ToString[i]<>") - ",
         "cos(β"<>ToString[i]<>")·cos(δ"<>ToString[i]<>") = ",
         NumberForm[diff,{5,3}]}],If[diff==0,Red,Black]]],{i,Length[quad]}];
    Column[vals]];
  (*Orthodiagonal check for anglesDeg before any switching*)
  Print[TextCell[Style["\nInitial anglesDeg (no switches):",Bold]]];
  Print[MatrixForm[angles]];
  Print[TextCell[Style[
     "Orthodiagonal check: cos(αi)·cos(γi) - cos(βi)·cos(δi) for i = 1..4",
     Italic]]];
  Print[formatOrthodiagonalCheck[angles]];
  (*Generate all combinations of switches (from size 1 to 4)*)
  combinations=Subsets[Keys[switchers],{1,Length[switchers]}];
  (*Evaluate condition after each combination of switches*)results=
   Table[Module[{switched=angles,name,passQ},name=StringRiffle[combo," + "];
     Do[switched=switchers[sw][switched],{sw,combo}];
     passQ=And@@(checkConditionN0Degrees/@switched);
     Print[Style["\nSwitch combination: ", Bold],name];
     Print[Style["Switched anglesDeg:", Italic]];
     Print[MatrixForm[switched]];
     Print[
      TextCell[Style["Orthodiagonal check: cos(αi)·cos(γi) - cos(βi)·cos(δi)
         for i = 1..4",Italic]]];
     Print[formatOrthodiagonalCheck[switched]];
     {name,passQ}],{combo,combinations}];]*)


Column[
 {TextCell[Style["================= ORTHOGONALITY CHECK =================",
    Brown, Bold, 16], "Text"],
  TextCell[Style["cos(αi)·cos(γi) ≠ cos(βi)·cos(δi) for at least
      one i = 1,..., 4 ⇒ NOT orthodiagonal. Switching boundary
      strips does not correct this.", GrayLevel[0.3]], "Text"]}]

(*Helper
 function:Returns True if at least one cosine product difference is non-
   zero.Returns False if all differences are zero.*)
isNotOrthodiagonal[quad_List] :=
  Or @@ Table[Module[{a, b, c, d, diff}, {a, b, c, d} = quad[[i]];
     diff = Chop[Cos[a Degree] Cos[c Degree] - Cos[b Degree] Cos[d Degree]];
     diff ≠ 0], {i, Length[quad]}];

(*First,check the initial,unswitched angles*)
Print[TextCell[Style["\nInitial anglesDeg (no switches):", Bold]]];
If[isNotOrthodiagonal[anglesDeg], Print[Style[
    " -> Condition met: At least one difference is non-zero.", Darker@Green]],
```

```
  Print[Style[" -> Condition NOT met: All differences are zero.", Red]]];

(*Now,use your desired module to check all switch combinations*)
Module[{angles = anglesDeg, switchers, combinations, results},
 (*Define switch functions*)
 switchers = <|"Right" → SwitchingRightBoundaryStrip,
   "Left" → SwitchingLeftBoundaryStrip, "Lower" → SwitchingLowerBoundaryStrip,
   "Upper" → SwitchingUpperBoundaryStrip|>;
 (*Generate all combinations of switches (from size 1 to 4)*)
 combinations = Subsets[Keys[switchers], {1, Length[switchers]}];
 (*Evaluate condition after each combination
  of switches and store in'results'*)results = Table[
   Module[{switched = angles, name, passQ}, name = StringRiffle[combo, " + "];
    Do[switched = switchers[sw][switched], {sw, combo}];
    (* ***THIS IS THE KEY CHANGE****)(*Set passQ using our
     new helper function*)passQ = isNotOrthodiagonal[switched];
    {name, passQ}], {combo, combinations}];
 (*Display results in the specified column format*)
 Column[Prepend[Table[Module[{comboName = res[[1]], passQ = res[[2]]},
     Row[{Style[comboName <> ": ", Bold], If[passQ,
        Style["Condition met (at least one difference is non-zero).", Darker[
          Green]], Style["Condition NOT met (all differences are zero).",
         Red, Bold]]}]], {res, results}], TextCell[
    Style["\nNON-ORTHOGONALITY CHECK AFTER SWITCHING BOUNDARY STRIPS", 14],
    "Text"]]]]



(*==========================================================================
 =====*)
(*============================
   NOT CONJUGATE-MODULAR=============================*)
(*==========================================================================
 =====*)
(*Column[{TextCell[
     Style["================= NOT CONJUGATE-MODULAR =================",
      Purple,Bold,16],"Text"],
    TextCell[Style["M1 = M2 = M3 = M4 = M
        and M ≠ 2 ⇒ NOT conjugate-modular. Boundary-strip
        switches preserve this.",GrayLevel[0.3]],"Text"]
   }]
  Ms=FullSimplify[Times@@@results];
allEqualQ=Simplify[Equal@@Ms];

Module[{angles=anglesDeg,switchers,combinations,results,
  computeConjugateModularInfo},(*Define switch functions*)
 switchers=<|"Right"→SwitchingRightBoundaryStrip,
```

```
    "Left"→SwitchingLeftBoundaryStrip,"Lower"→SwitchingLowerBoundaryStrip,
    "Upper"→SwitchingUpperBoundaryStrip|>;
  (*Computes Mᵢ and pᵢ and prints them,
  with classification*)computeConjugateModularInfo[quad_List]:=
   Module[{abcdList,Ms,summary},abcdList=computeABCD/@quad;
    Ms=FullSimplify[Times@@@abcdList];
    summary=If[Simplify[Equal@@Ms]&&Ms⟦1⟧=!=2,
      Style["M1 = M2 = M3 = M4 = M and M ≠ 2",Bold],
      Style["M1 = M2 = M3 = M4 = M and M = 2",Red,Bold]];
    Column[{Style["Mᵢ values:",Bold],Row[{"M1 = ",Ms⟦1⟧,
        ", M2 = ",Ms⟦2⟧,", M3 = ",Ms⟦3⟧,", M4 = ",Ms⟦4⟧}],summary}]];
  (*Original anglesDeg check*)
  Print[
   TextCell[Style["\nInitial configuration (no switches applied):",Bold]]];
  Print[MatrixForm[angles]];
  Print[computeConjugateModularInfo[angles]];
  (*Generate all switch combinations (from size 1 to 4)*)
  combinations=Subsets[Keys[switchers],{1,Length[switchers]}];
  (*Evaluate each switched configuration*)results=
   Table[Module[{switched=angles,name,passQ},name=StringRiffle[combo," + "];
     Do[switched=switchers[sw][switched],{sw,combo}];
     passQ=And@@(checkConditionN0Degrees/@switched);
     Print[Style["\nSwitch combination: ", Bold],name];
     Print[Style["Switched anglesDeg:", Italic]];
     Print[MatrixForm[switched]];
     Print[computeConjugateModularInfo[switched]];
     {name,passQ}],{combo,combinations}];]*)


Column[{TextCell[
    Style["================= CONJUGATE-MODULAR CHECK =================",
     Darker[Brown], Bold, 16], "Text"],
   TextCell[Style["M1 = M2 = M3 = M4 = M and M ≠ 2 ⇒ NOT conjugate-modular.
      Boundary-strip switches preserve this.", GrayLevel[0.3]], "Text"]}]

(*Helper Function:Returns True if all M_i values are equal
    AND their common value is not 2. Returns False otherwise.*)
isNotConjugateModular[quad_List] :=
   Module[{abcdList, Ms}, abcdList = computeABCD /@ quad;
    Ms = FullSimplify[Times @@@ abcdList];
    (*The condition is met if they are all equal AND the value isn't 2*)
    Simplify[Equal @@ Ms] && (Ms⟦1⟧ ≠ 2)];

(*First,check the initial,unswitched angles*)
Print[TextCell[Style["\nInitial anglesDeg (no switches):", Bold]]];
If[isNotConjugateModular[anglesDeg], Print[
    Style[" -> Condition met: All Mᵢ are equal and M ≠ 2.", Darker@Green]],
```

```
  Print[Style[" -> Condition NOT met.", Red]]];

(*Now,use the clean module to check all switch combinations*)
Module[{angles = anglesDeg, switchers, combinations, results},
 (*Define switch functions*)
 switchers = <|"Right" → SwitchingRightBoundaryStrip,
   "Left" → SwitchingLeftBoundaryStrip, "Lower" → SwitchingLowerBoundaryStrip,
   "Upper" → SwitchingUpperBoundaryStrip|>;
 (*Generate all combinations of switches (from size 1 to 4)*)
 combinations = Subsets[Keys[switchers], {1, Length[switchers]}];
 (*Evaluate condition after each combination
  of switches and store the result*)results = Table[
   Module[{switched = angles, name, passQ}, name = StringRiffle[combo, " + "];
    Do[switched = switchers[sw][switched], {sw, combo}];
    (*Set passQ using our new helper function for this check*)
    passQ = isNotConjugateModular[switched];
    {name, passQ}], {combo, combinations}];
 (*Display results in the specified column format*)
 Column[Prepend[Table[Module[{comboName = res⟦1⟧, passQ = res⟦2⟧},
     Row[{Style[comboName <> ": ", Bold], If[passQ,
        Style["Condition met (All Mᵢ are equal and M ≠ 2).", Darker[Green]],
        Style["Condition NOT met.", Red, Bold]]}]], {res, results}],
   TextCell[Style["\nCONJUGATE-MODULAR CHECK AFTER SWITCHING BOUNDARY STRIPS",
     14], "Text"]]]]
```

Out[248]=

================= CONDITION (N.0) =================

✔ All vertices satisfy (N.0).

Vertex	Combinations (mod 360)	Status
Vertex 1	{342.075, 102.075, 255.638, 135.638, 287.012, 80.5752, 47.0121, 200.575}	✔ Pass
Vertex 2	{334.867, 174.867, 315.305, 115.305, 276.196, 56.6343, 116.196, 256.634}	✔ Pass
Vertex 3	{319.476, 149.476, 330.241, 140.241, 92.1432, 272.908, 282.143, 102.908}	✔ Pass
Vertex 4	{332.626, 182.626, 299.113, 89.1129, 117.278, 233.765, 327.278, 83.7651}	✔ Pass

Out[251]=

================= CONDITION (N.3) =================

✔ **M1** = **M2** = **M3** = **M4** = 0.924668

Out[257]=

================= CONDITION (N.4) =================

✔ **r1** = **r2** = 1.13168; ✔ **r3** = **r4** = 0.68379

✔ **s1** = **s4** = 1.17093; ✔ **s2** = **s3** = 1.09802

Out[267]=

================= CONDITION (N.5) =================

⚠ Approximate validation using ε-tolerance. For rigorous proof, see the referenced paper.

✅ **Valid Combination Found (M < 1):**

e1 = −1, **e2** = 1, **e3** = −1

t1 = 0.K + 0.309024iK'

t2 = 0.K + 0.502198iK'

t3 = 1.K + 0.519082iK'

t4 = 1.K + 0.325908iK'

t1 + e1*t2 + e2*t3 + e3*t4 = 0.K + -3.33067×10^{-16}iK'

Out[270]=

================= OTHER PARAMETERS =================

u = 0.0753316

σ1 = 2.98516, **σ2** = 2.92226, **σ3** = 2.78795, **σ4** = 2.90271

σ1 ≈ 171.037°, **σ2** ≈ 167.433°, **σ3** ≈ 159.738°, **σ4** ≈ 166.313°

cosσ1 = −0.98779, **cosσ2** = −0.976043, **cosσ3** = −0.938119, **cosσ4** = −0.971603

f1 = 1.0682, **f2** = 1.27867, **f3** = 0.70414, **f4** = 0.857702

z1 = 14.6629, **z2** = 3.5885, **z3** = −3.37998, **z4** = −7.02752

x1 = 7.59444, **x2** = 7.59444, **x3** = −3.16246, **x4** = −3.16246

y1 = 5.85051, **y2** = 10.2019, **y3** = 10.2019, **y4** = 5.85051

p1 = 0.362871 , **p2** = 0.362871 , **p3** = 0. + 0.562325 i , **p4** = 0. + 0.562325 i

q1 = 0.413431, **q2** = 0.313084, **q3** = 0.313084, **q4** = 0.413431

p1·q1 = 0.150022 , **p2·q2** = 0.113609 , **p3·q3** = 0. + 0.176055 i , **p4·q4** = 0. + 0.232483 i

Out[272]=

================= Bricard's System of Equations =================

We introduce new notation for the cotangents of half of the dihedral angles. Denote Z:= $\cot\left(\frac{\theta_1}{2}\right)$, $W_2 = \cot\left(\frac{\theta_2}{2}\right)$, $U := \cot\left(\frac{\theta_3}{2}\right)$, and $W_1 = \cot\left(\frac{\theta_4}{2}\right)$

$$P_1(Z, W_1) = W_1^2\left(0.310125\, Z^2 + 0.598771\right) - 1.94649\, W_1 Z + 0.777254\, Z^2 + 0.0926553 = 0$$

$$P_2(Z, W_2) = W_2^2\left(0.848101\, Z^2 + 0.400742\right) - 1.6951\, W_2 Z + 0.298319\, Z^2 + 0.14531 = 0$$

$$P_3(U, W_2) = \left(-0.606188\, U^2 - 0.647883\right) W_2^2 + 0.200836\, U^2 - 1.03579\, U W_2 + 0.249417 = 0$$

$$P_4(U, W_1) = \left(-0.281613\, U^2 - 0.625792\right) W_1^2 + 0.338269\, U^2 - 0.803495\, U W_1 + 0.202049 = 0$$

Out[281]=

================= FLEXIONS =================

Solutions to Bricard's equations under a free parameter t := Z ∈ ℂ:

Solution 1:

$Z(t) = t$

$W_2(t) = \frac{0.999349\, t - 0.284533 \sqrt{-4.34478\, t^4 + 8.16654\, t^2 - 1}}{1.\, t^2 + 0.472516}$

$U(t) = \frac{0.98196 \sqrt{-4.34478\, t^4 + 8.16654\, t^2 - 1} + 0.114494\, t}{1.74456 - 1.\, t^2}$

$W_1(t) = \frac{3.13824\, t - 0.759502 \sqrt{-4.34478\, t^4 + 8.16654\, t^2 - 1}}{1.\, t^2 + 1.93074}$

Solution 2:

$Z(t) = t$

$W_2(t) = \frac{0.284533 \sqrt{-4.34478\, t^4 + 8.16654\, t^2 - 1} + 0.999349\, t}{1.\, t^2 + 0.472516}$

$U(t) = \frac{0.114494\, t - 0.98196 \sqrt{-4.34478\, t^4 + 8.16654\, t^2 - 1}}{1.74456 - 1.\, t^2}$

$W_1(t) = \frac{0.759502 \sqrt{-4.34478\, t^4 + 8.16654\, t^2 - 1} + 3.13824\, t}{1.\, t^2 + 1.93074}$

Out[284]=

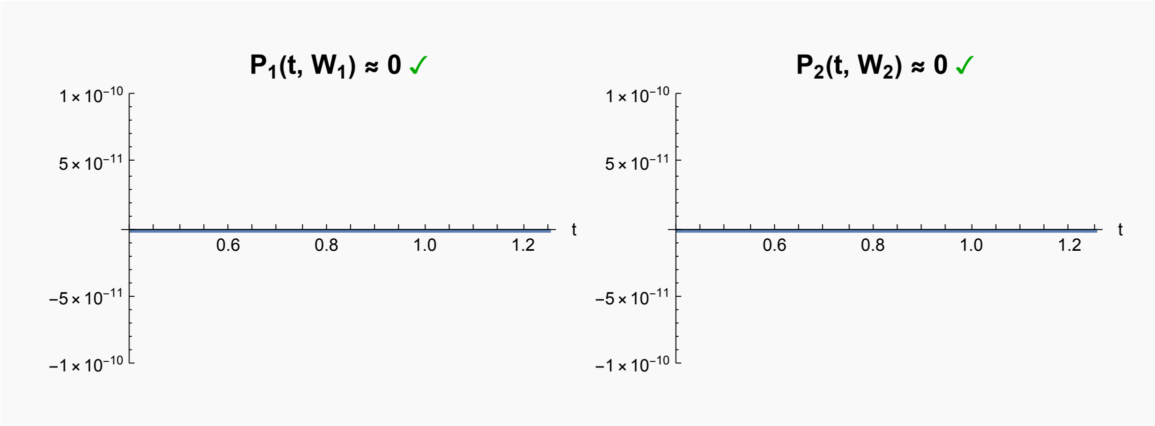
================= Equations: $P_1(t, W_1) = 0$ and $P_2(t, W_2) = 0$ =================

Let W_{1s1} and W_{1s2} be formulas for $W_1(t)$ from solutions 1 and 2, respectively. Similarly, let W_{2s1} and W_{2s2} be the formulas for $W_2(t)$. We show that all four pairs - (W_{1s1}, W_{2s1}), (W_{1s1}, W_{2s2}), (W_{1s2}, W_{2s1}), and (W_{1s2}, W_{2s2}) - solve equations $P_1(t, W_1) = 0$ and $P_2(t, W_2) = 0$.

Pair 1, $(W_1, W_2) = (W_{1s1}, W_{2s1})$:

$$P_1(t, W_1) = \frac{1}{(1.\, t^2 + 1.93074)^2} \Big(t \Big(t \Big(2.67507 \times 10^{-15}\, t^4 - 1.16063 \times 10^{-14}\, t^2 + 3.55271 \times 10^{-15} \sqrt{-4.34478\, t^4 + 8.16654\, t^2 - 1}\; t + 2.00199 \times 10^{-14} \Big) - 5.77316 \times 10^{-15} \sqrt{-4.34478\, t^4 + 8.16654\, t^2 - 1} \Big) - 2.41619 \times 10^{-15} \Big)$$

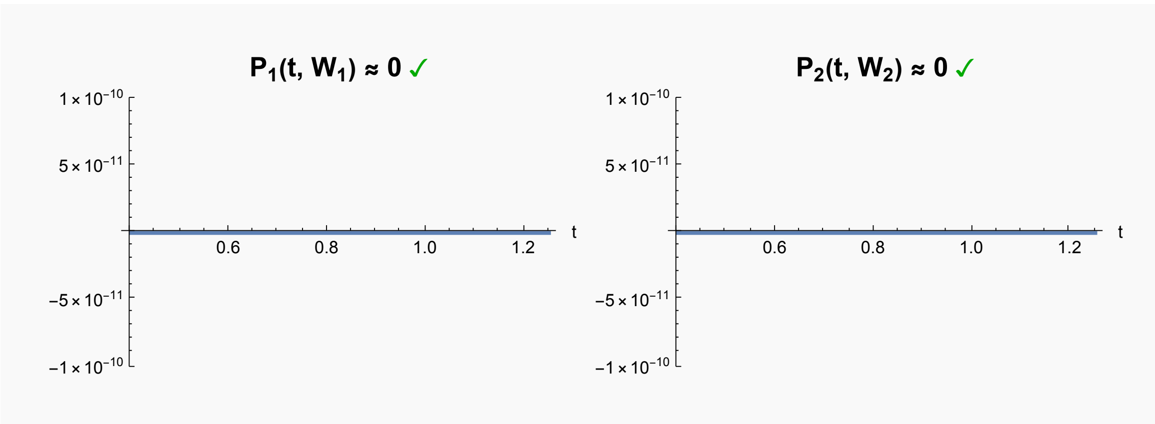
$$P_2(t, W_2) = \frac{1}{(1.\, t^2 + 0.472516)^2} \Big(t \Big(t \Big(1.3248 \times 10^{-16}\, t^4 - 6.62401 \times 10^{-17}\, t^2 + 3.88578 \times 10^{-16} \sqrt{-4.34478\, t^4 + 8.16654\, t^2 - 1}\; t - 2.18592 \times 10^{-15} \Big) + 8.88178 \times 10^{-16} \sqrt{-4.34478\, t^4 + 8.16654\, t^2 - 1} \Big) + 1.9458 \times 10^{-16} \Big)$$



Pair 2, $(W_1, W_2) = (W_{1s1}, W_{2s2})$:

$$P_1(t, W_1) = \frac{1}{(1.\ t^2+1.93074)^2}\left(t\left(t\left(2.67507\times10^{-15}\ t^4 - 1.16063\times10^{-14}\ t^2 + 3.55271\times10^{-15}\ \sqrt{-4.34478\ t^4+8.16654\ t^2-1}\ t + 2.00199\times10^{-14}\right) - 5.77316\times10^{-15}\ \sqrt{-4.34478\ t^4+8.16654\ t^2-1}\right) - 2.41619\times10^{-15}\right)$$

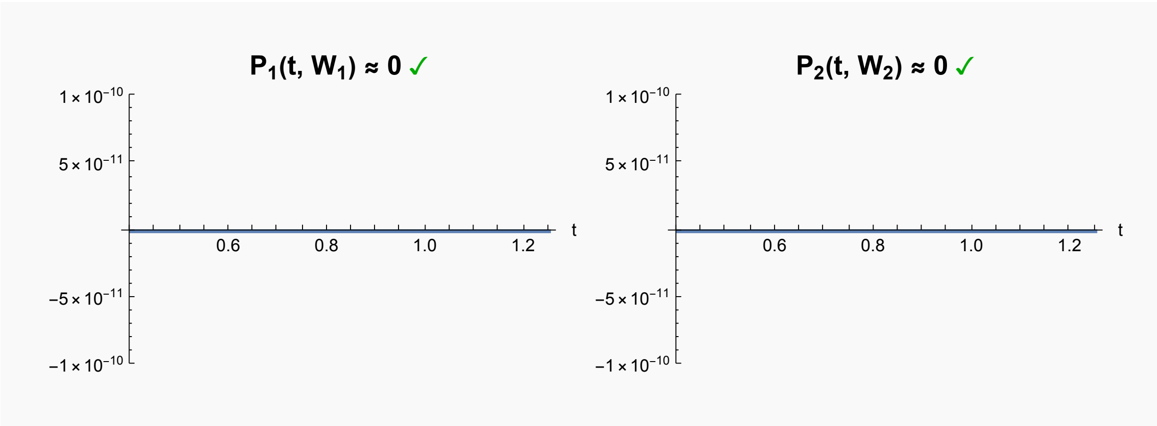
$$P_2(t, W_2) = \frac{1}{(1.\ t^2+0.472516)^2}\left(t\left(t\left(1.3248\times10^{-16}\ t^4 - 6.62401\times10^{-17}\ t^2 - 3.88578\times10^{-16}\ \sqrt{-4.34478\ t^4+8.16654\ t^2-1}\ t - 2.18592\times10^{-15}\right) - 8.88178\times10^{-16}\ \sqrt{-4.34478\ t^4+8.16654\ t^2-1}\right) + 1.9458\times10^{-16}\right)$$



Pair 3, $(W_1, W_2) = (W_{1s2}, W_{2s1})$:

$$P_1(t, W_1) = \frac{1}{(1.\ t^2+1.93074)^2}\left(t\left(t\left(2.67507\times10^{-15}\ t^4 - 1.16063\times10^{-14}\ t^2 - 3.55271\times10^{-15}\ \sqrt{-4.34478\ t^4+8.16654\ t^2-1}\ t + 2.00199\times10^{-14}\right) + 5.77316\times10^{-15}\ \sqrt{-4.34478\ t^4+8.16654\ t^2-1}\right) - 2.41619\times10^{-15}\right)$$

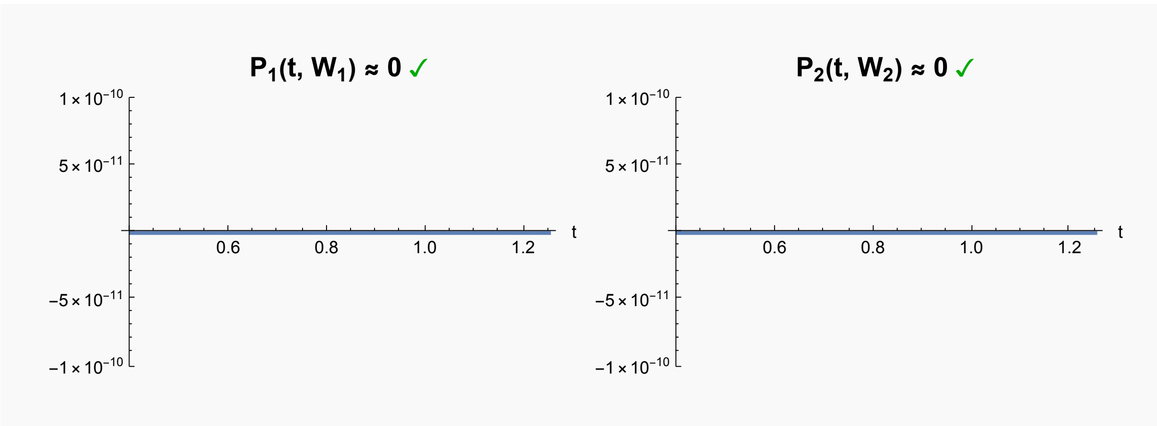
$$P_2(t, W_2) = \frac{1}{(1.\ t^2+0.472516)^2}\left(t\left(t\left(1.3248\times10^{-16}\ t^4 - 6.62401\times10^{-17}\ t^2 + 3.88578\times10^{-16}\ \sqrt{-4.34478\ t^4+8.16654\ t^2-1}\ t - 2.18592\times10^{-15}\right) + 8.88178\times10^{-16}\ \sqrt{-4.34478\ t^4+8.16654\ t^2-1}\right) + 1.9458\times10^{-16}\right)$$



Pair 4, $(W_1, W_2) = (W_{1s2}, W_{2s2})$:

$$P_1(t, W_1) = \frac{1}{(1.\ t^2 + 1.93074)^2}\left(t\left(t\left(2.67507\times10^{-15}\ t^4 - 1.16063\times10^{-14}\ t^2 - 3.55271\times10^{-15}\ \sqrt{-4.34478\ t^4 + 8.16654\ t^2 - 1}\ t + 2.00199\times10^{-14}\right) + 5.77316\times10^{-15}\ \sqrt{-4.34478\ t^4 + 8.16654\ t^2 - 1}\right) - 2.41619\times10^{-15}\right)$$

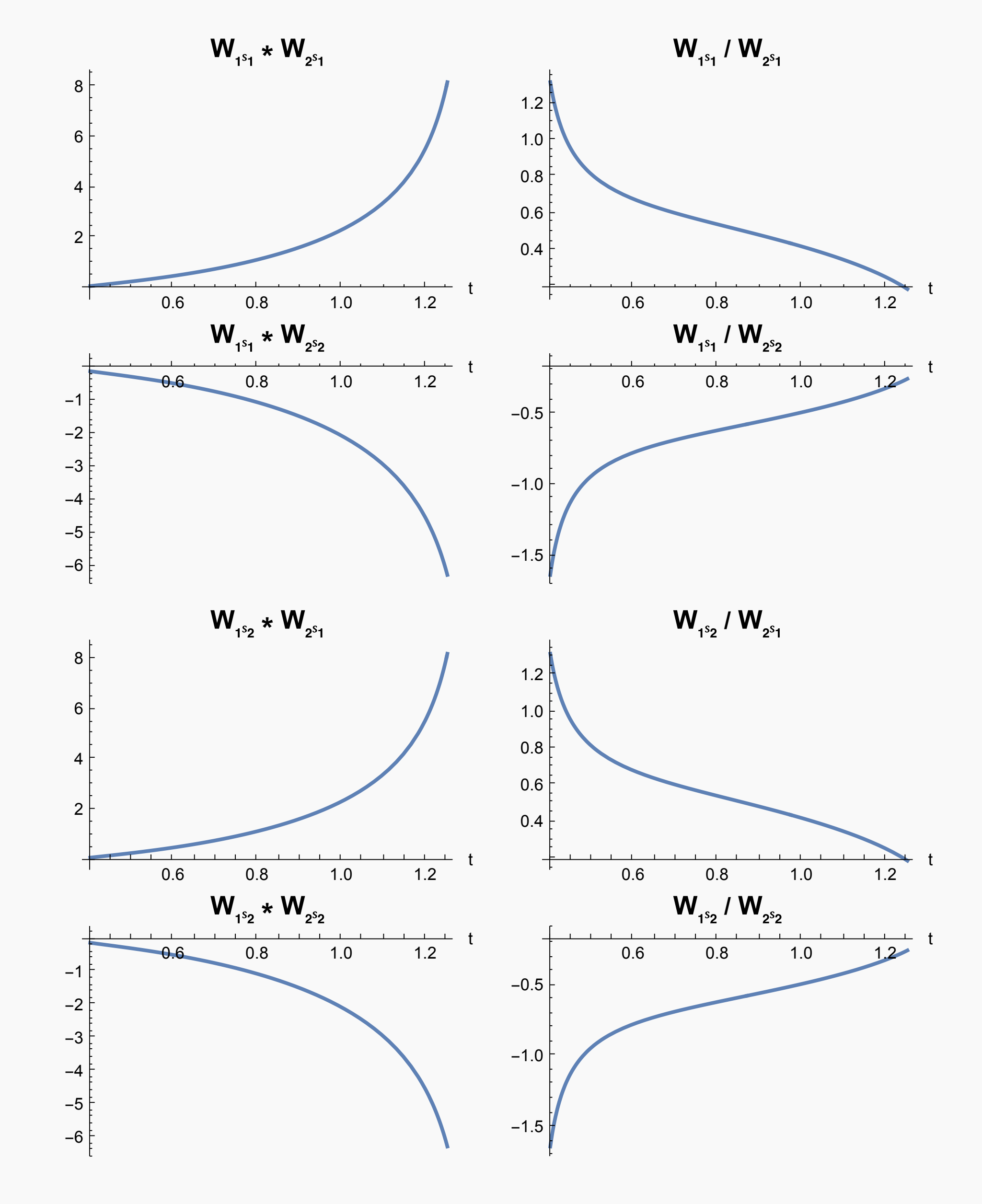
$$P_2(t, W_2) = \frac{1}{(1.\ t^2 + 0.472516)^2}\left(t\left(t\left(1.3248\times10^{-16}\ t^4 - 6.62401\times10^{-17}\ t^2 - 3.88578\times10^{-16}\ \sqrt{-4.34478\ t^4 + 8.16654\ t^2 - 1}\ t - 2.18592\times10^{-15}\right) - 8.88178\times10^{-16}\ \sqrt{-4.34478\ t^4 + 8.16654\ t^2 - 1}\right) + 1.9458\times10^{-16}\right)$$



Out[287]=

============= Plots of $W_1(t) W_2(t)$ and $W_1(t)/W_2(t)$ For All Pairs =============

Checking that all four pairs – (W_{1s1}, W_{2s1}), (W_{1s1}, W_{2s2}), (W_{1s2}, W_{2s1}), and (W_{1s2}, W_{2s2}) – does NOT satisfy $W_1(t)/W_2(t)$ = const and $W_1(t) W_2(t)$ = const.



Out[296]=

================= Equations: $P_3(U, W_2) = 0$ and $P_4(U, W_1) = 0$ =================

We solve $P_3(U, W_2) = 0$ and $P_4(U, W_1) = 0$ for W_1 and W_2.

$$W_{1,1}(U) = \left(785\,075\,686\,532\,519 \left(-\sqrt{\left(\frac{496\,280\,591\,367\,232\,664\,614\,220\,778\,988\,068\, U^4}{3\,103\,731\,454\,578\,013} + \frac{68\,067\,333\,362\,384\,661\,407\,133\,160\,086\,126\,094\,655\,740\,418\,273\, U^2}{94\,309\,449\,855\,884\,820\,125\,100\,815\,987} + \frac{45\,142\,470\,930\,042\,455\,801\,905\,312\,527\,302}{212\,700\,795\,365\,993}\right)} - 520\,496\,100\, U\right)\right) \Big/ \left(215\,929\,997\, \left(1\,326\,526\,088\,509\,749\, U^2 + 2\,947\,765\,589\,535\,578\right)\right)$$

$$W_{1,2}(U) = \left(785\,075\,686\,532\,519 \left(\sqrt{\left(\frac{496\,280\,591\,367\,232\,664\,614\,220\,778\,988\,068\, U^4}{3\,103\,731\,454\,578\,013} + \frac{68\,067\,333\,362\,384\,661\,407\,133\,160\,086\,126\,094\,655\,740\,418\,273\, U^2}{94\,309\,449\,855\,884\,820\,125\,100\,815\,987} + \frac{45\,142\,470\,930\,042\,455\,801\,905\,312\,527\,302}{212\,700\,795\,365\,993}\right)} - 520\,496\,100\, U\right)\right) \Big/ \left(215\,929\,997\, \left(1\,326\,526\,088\,509\,749\, U^2 + 2\,947\,765\,589\,535\,578\right)\right)$$

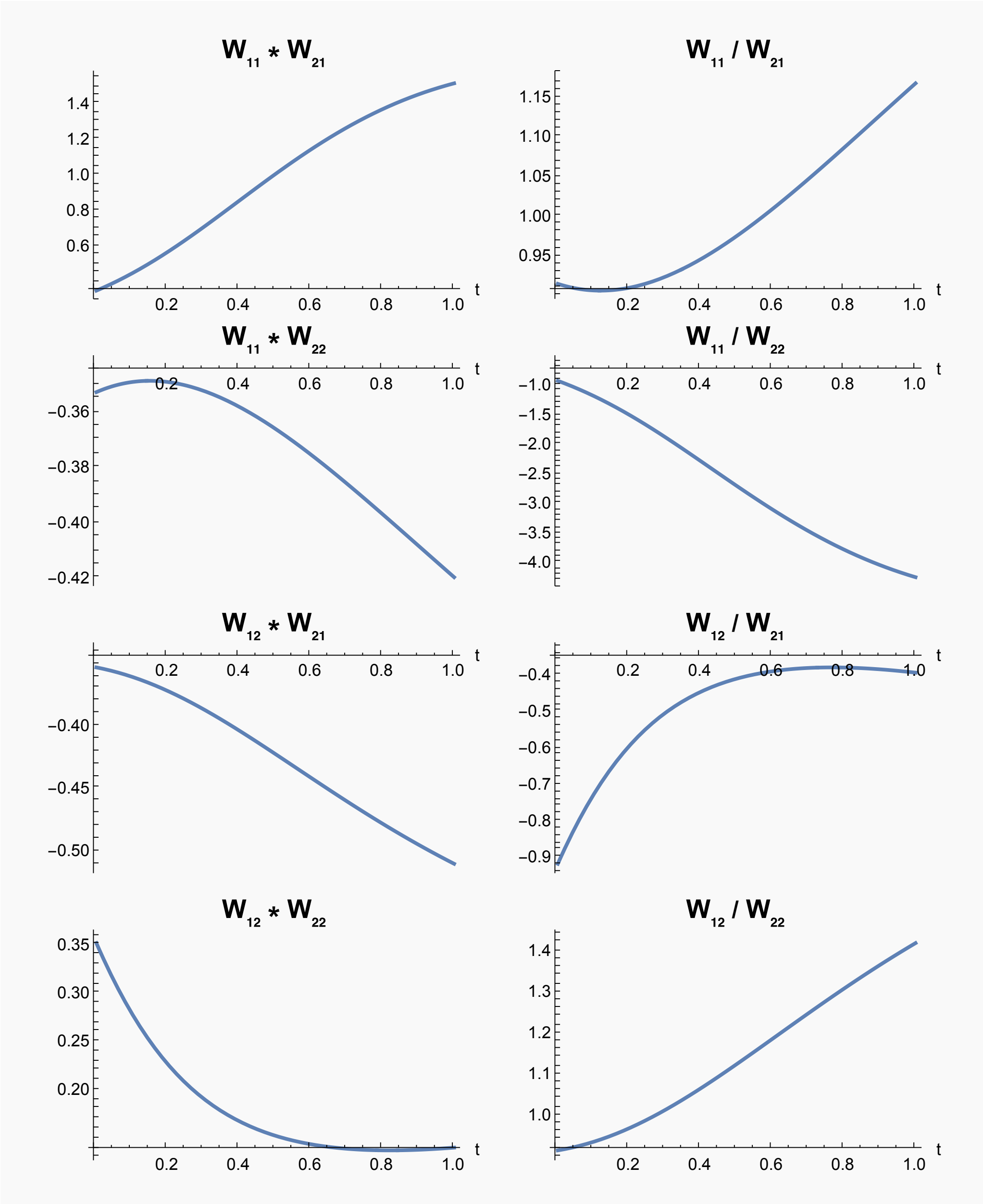
$$W_{2,1}(U) = \left(12\,632\,129\,231\,006\,053 \left(-\frac{1}{\sqrt{366\,633\,289\,649\,269\,544\,689\,138\,024\,494\,489}} \left(\sqrt{\left(1\,422\,696\,506\,085\,811\,215\,664\,299\,703\,025\,802\,263\,771\,973\,484\,032\, U^4 + 6\,421\,738\,798\,276\,188\,584\,442\,497\,071\,245\,567\,749\,166\,671\,004\,433\, U^2 + 1\,888\,362\,313\,921\,430\,846\,177\,320\,630\,986\,380\,830\,579\,012\,874\,896\right)}\right) - 92\,460\,837\, U\right)\right) \Big/ \left(714\,126\,584\, \left(1\,914\,362\,650\,630\,586\, U^2 + 2\,046\,034\,380\,138\,707\right)\right)$$

$$W_{2,2}(U) = \left(12\,632\,129\,231\,006\,053 \left(\frac{1}{\sqrt{366\,633\,289\,649\,269\,544\,689\,138\,024\,494\,489}} \left(\sqrt{\left(1\,422\,696\,506\,085\,811\,215\,664\,299\,703\,025\,802\,263\,771\,973\,484\,032\, U^4 + 6\,421\,738\,798\,276\,188\,584\,442\,497\,071\,245\,567\,749\,166\,671\,004\,433\, U^2 + 1\,888\,362\,313\,921\,430\,846\,177\,320\,630\,986\,380\,830\,579\,012\,874\,896\right)}\right) - 92\,460\,837\, U\right)\right) \Big/ \left(714\,126\,584\, \left(1\,914\,362\,650\,630\,586\, U^2 + 2\,046\,034\,380\,138\,707\right)\right)$$

Out[301]=

============= Plots of $W_1(U) W_2(U)$ and $W_1(U)/W_2(U)$ For All Pairs =============

Checking that all four pairs – $(W_{11}(U), W_{21}(U))$, $(W_{11}(U), W_{22}(U))$, $(W_{12}(U), W_{21}(U))$, and $(W_{12}(U), W_{22}(U))$ – does NOT satisfy $W_1(U)/W_2(U)$ = const and $W_1(U) W_2(U)$ = const.



Out[302]=

================= FLEXIBILITY (Double Checking) =================

Out[304]=

Solution 1:

$$P_1(Z, W_1) = \frac{1}{(1.93074 + 1.\, t^2)^2} \Big(-2.41619\times10^{-15} + t\Big(-5.77316\times10^{-15}\sqrt{-1 + 8.16654\, t^2 - 4.34478\, t^4} + t\Big(2.00199\times10^{-14} - 1.16063\times10^{-14}\, t^2 + 2.67507\times10^{-15}\, t^4 + 3.55271\times10^{-15}\, t\sqrt{-1 + 8.16654\, t^2 - 4.34478\, t^4}\Big)\Big)\Big)$$

$$P_2(Z, W_2) = \frac{1}{(0.472516 + 1.\, t^2)^2} \Big(1.9458\times10^{-16} + t\Big(8.88178\times10^{-16}\sqrt{-1 + 8.16654\, t^2 - 4.34478\, t^4} + t\Big(-2.18592\times10^{-15} - 6.62401\times10^{-17}\, t^2 + 1.3248\times10^{-16}\, t^4 + 3.88578\times10^{-16}\, t\sqrt{-1 + 8.16654\, t^2 - 4.34478\, t^4}\Big)\Big)\Big)$$

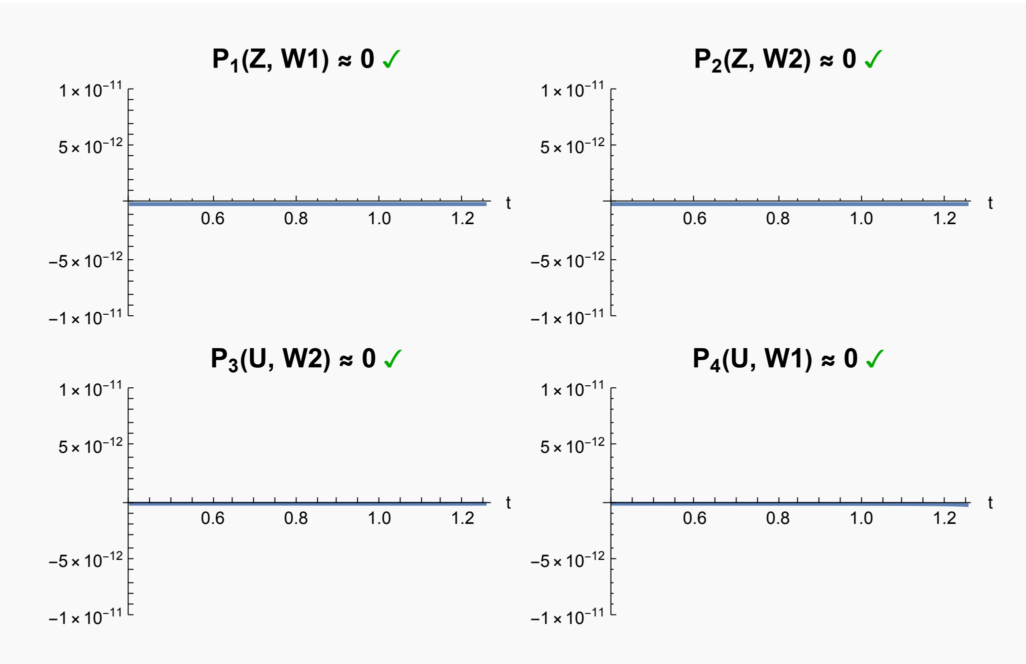
$$P_3(U, W_2) = \frac{1}{(0.824332 + 1.27204\, t^2 - 1.\, t^4)^2} \Big(1.68121\times10^{-16} + t\Big(-3.78271\times10^{-15}\sqrt{-1 + 8.16654\, t^2 - 4.34478\, t^4} + t\Big(1.1096\times10^{-14} + t\Big(-5.37986\times10^{-15}\sqrt{-1 + 8.16654\, t^2 - 4.34478\, t^4} + t\Big(8.74227\times10^{-15} - 2.03426\times10^{-14}\, t^2 + 6.85092\times10^{-15}\, t^4 + 4.37114\times10^{-15}\, t\sqrt{-1 + 8.16654\, t^2 - 4.34478\, t^4}\Big)\Big)\Big)\Big)\Big)$$

$$P_4(U, W_1) = \frac{1}{(3.36829 - 0.186184\, t^2 - 1.\, t^4)^2} \Big(2.78246\times10^{-15} + t\Big(-4.47977\times10^{-14}\sqrt{-1 + 8.16654\, t^2 - 4.34478\, t^4} + t\Big(4.89714\times10^{-14} + t\Big(2.22597\times10^{-15}\sqrt{-1 + 8.16654\, t^2 - 4.34478\, t^4} + t\Big(-7.5683\times10^{-14} - 6.67791\times10^{-15}\, t^2 + 1.81556\times10^{-14}\, t^4 + 1.00169\times10^{-14}\, t\sqrt{-1 + 8.16654\, t^2 - 4.34478\, t^4}\Big)\Big)\Big)\Big)\Big)$$

Out[310]=

================= FLEXION 1 =================

Polynomials $P_i(t)$ built from Bricard's equations for flexion 1.



Out[311]=

Solution 2:

$$P_1(Z,\ W_1) = \frac{1}{(1.93074 + 1.\,t^2)^2}\Big(-2.41619\times10^{-15} + t\Big(5.77316\times10^{-15}\sqrt{-1+8.16654\,t^2-4.34478\,t^4} + t\Big(2.00199\times10^{-14} - 1.16063\times10^{-14}\,t^2 + 2.67507\times10^{-15}\,t^4 - 3.55271\times10^{-15}\,t\sqrt{-1+8.16654\,t^2-4.34478\,t^4}\Big)\Big)\Big)$$

$$P_2(Z,\ W_2) = \frac{1}{(0.472516 + 1.\,t^2)^2}\Big(1.9458\times10^{-16} + t\Big(-8.88178\times10^{-16}\sqrt{-1+8.16654\,t^2-4.34478\,t^4} + t\Big(-2.18592\times10^{-15} - 6.62401\times10^{-17}\,t^2 + 1.3248\times10^{-16}\,t^4 - 3.88578\times10^{-16}\,t\sqrt{-1+8.16654\,t^2-4.34478\,t^4}\Big)\Big)\Big)$$

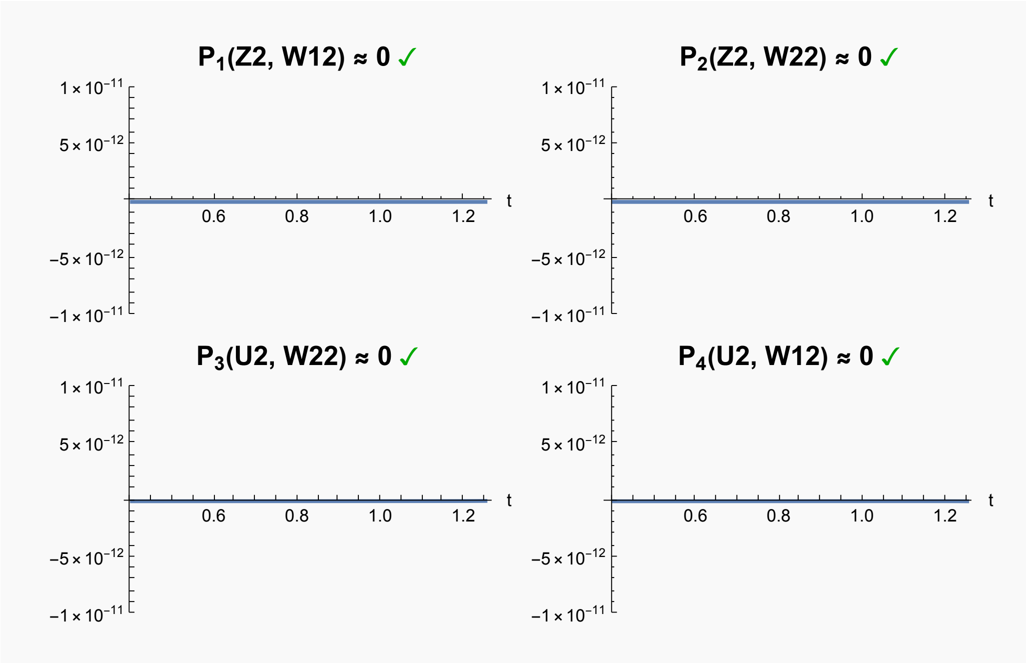
$$P_3(U,\ W_2) = \frac{1}{(0.824332 + 1.27204\,t^2 - 1.\,t^4)^2}\Big(1.68121\times10^{-16} + t\Big(3.78271\times10^{-15}\sqrt{-1+8.16654\,t^2-4.34478\,t^4} + t\Big(1.1096\times10^{-14} + t\Big(5.37986\times10^{-15}\sqrt{-1+8.16654\,t^2-4.34478\,t^4} + t\Big(8.74227\times10^{-15} - 2.03426\times10^{-14}\,t^2 + 6.85092\times10^{-15}\,t^4 - 4.37114\times10^{-15}\,t\sqrt{-1+8.16654\,t^2-4.34478\,t^4}\Big)\Big)\Big)\Big)\Big)$$

$$P_4(U,\ W_1) = \frac{1}{(3.36829 - 0.186184\,t^2 - 1.\,t^4)^2}\Big(2.78246\times10^{-15} + t\Big(4.47977\times10^{-14}\sqrt{-1+8.16654\,t^2-4.34478\,t^4} + t\Big(4.89714\times10^{-14} + t\Big(-2.22597\times10^{-15}\sqrt{-1+8.16654\,t^2-4.34478\,t^4} + t\Big(-7.5683\times10^{-14} - 6.67791\times10^{-15}\,t^2 + 1.81556\times10^{-14}\,t^4 - 1.00169\times10^{-14}\,t\sqrt{-1+8.16654\,t^2-4.34478\,t^4}\Big)\Big)\Big)\Big)\Big)$$

Out[318]=

================= FLEXION 2 =================

Polynomials $P_i(t)$ built from Bricard's equations for flexion 2.



Out[321]=

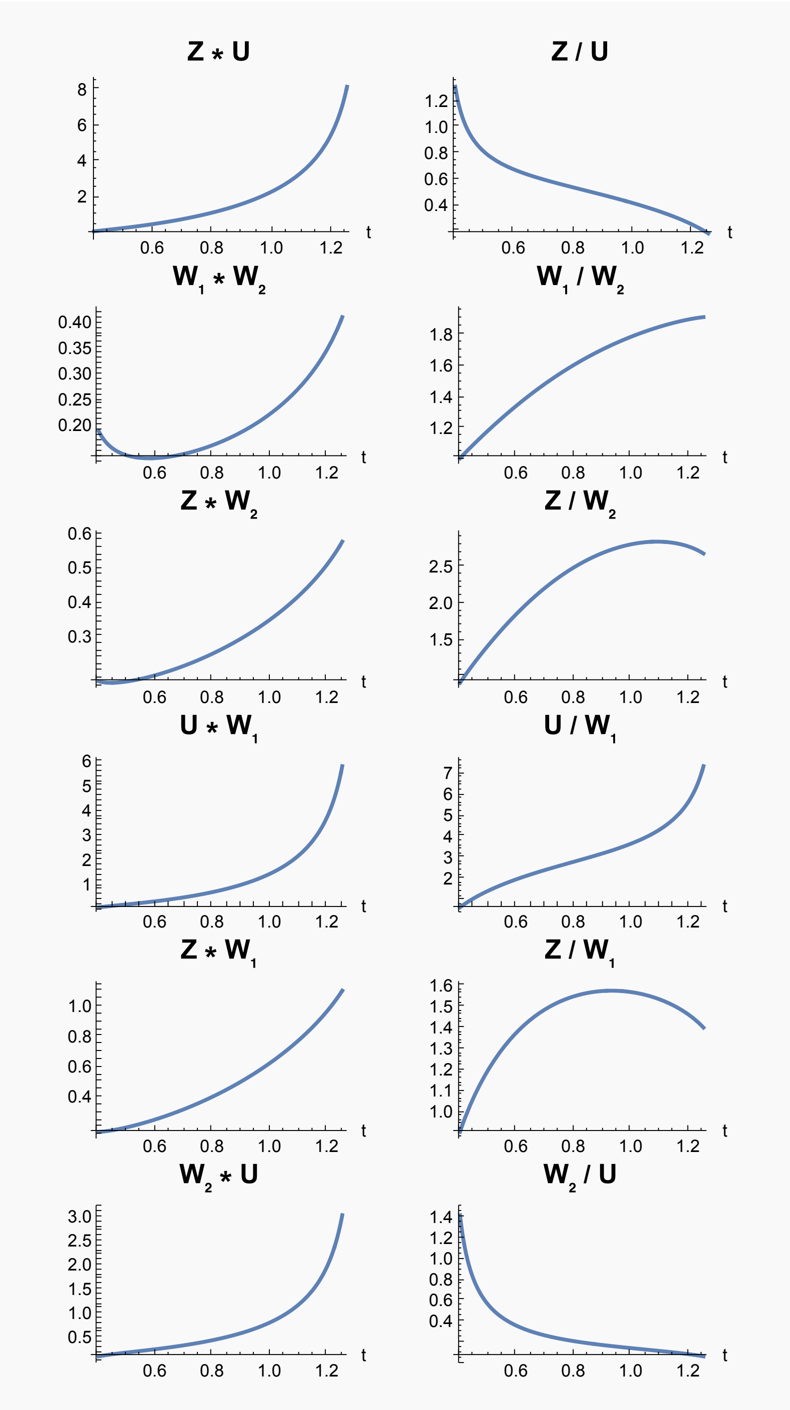
============= NOT LINEAR COMPOUND =============

Above we consider the first pair of equations ($P_1(t, W_1) = 0$ and $P_2(t, W_2) = 0$). Solving them as quadratic equations in W_1 and W_2, respectively we parametrize the solutions by the first two and fourth expressions in Solutions 1 and 2 in a neighborhood of any point (W_1, t, W_2) such that the expression in the square root and denominators are not zero. Here, we choose any continuous branch of the square root in this neighborhood, and the signs in ± need Not agree (this means we consider all 4 pairs we describe above). We conclude that NO component of the solution set of the first pair of Bricard's equations satisfies W_1/W_2=const NOR W_1W_2=const.

Analogously, NO component of the solution set of the other pair of equations ($P_3(U, W_2) = 0$ and $P_4(U, W_1) = 0$) satisfies W_1/W_2=const NOR W_1W_2=const. As a result, NO component of the solution set of all four equations satisfies W_1/W_2=const NOR W_1W_2=const. So, our example does not belong to the linear compound class, even after switching the boundary strips.

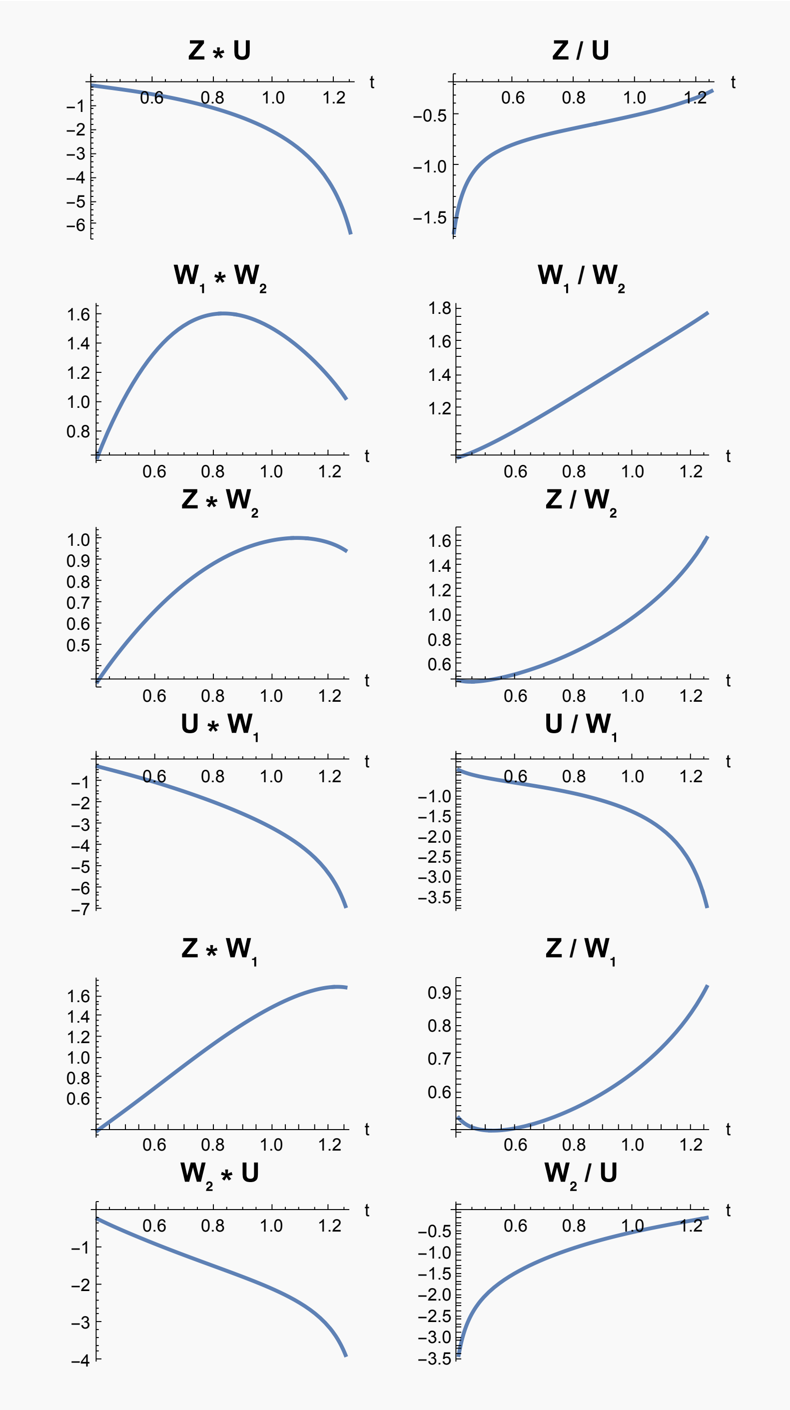
Below, we also present the plots of functions ZU, Z/U, W_1W_2, W_1/W_2, ZW_2, Z/W_2, UW_1, U/W_1, ZW_1, Z/W_1, W_2U, W_2/U.

Solution 1:



Out[324]=

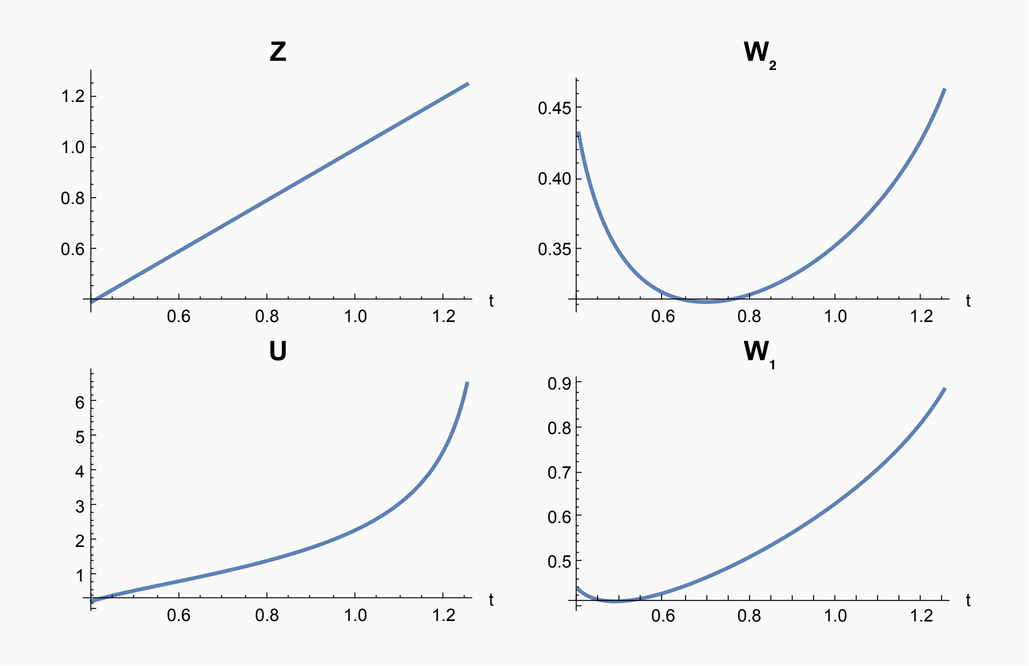
Solution 2:



Out[327]=

================= NOT TRIVIAL (FLEXION 1) =================

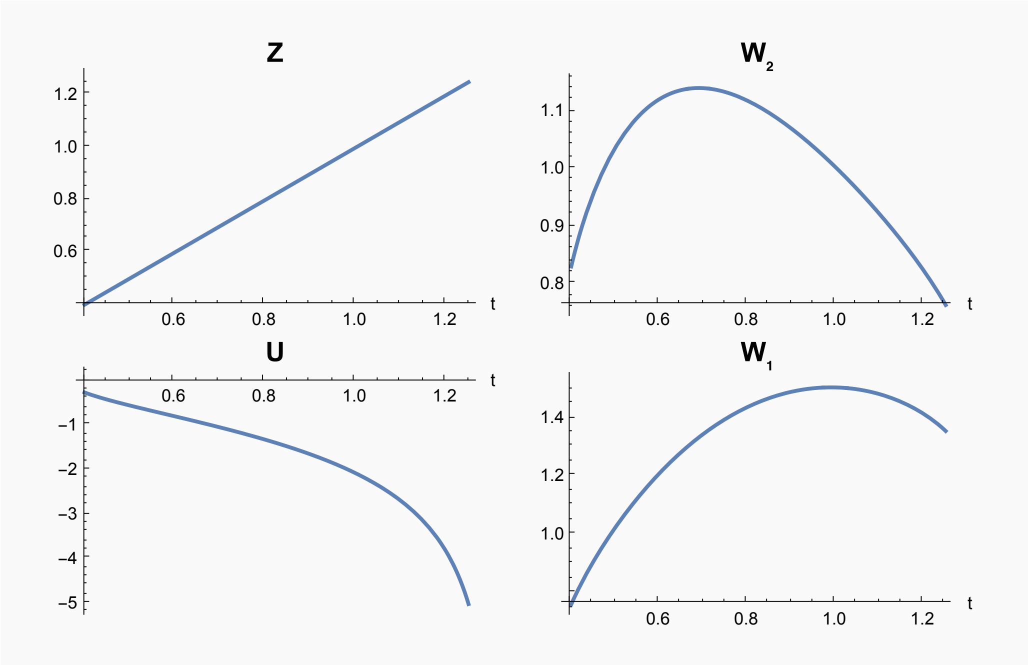
This configuration does not belong to the trivial
class — even after switching the boundary strips —
since none of the functions Z, W2, U, or W1 is constant.



Out[330]=

================= NOT TRIVIAL (FLEXION 2) =================

This configuration does not belong to the trivial class – even after switching the boundary strips – since none of the functions Z, W2, U, or W1 is constant.



Out[337]=

================= NOT CONIC & NOT CHIMERA & NOT LINEAR CONJUGATE & NOT ISOGONAL=================

Condition (N.0) is satisfied for all i=1,...,4 ⇒ NOT equimodular-conic, NOT chimera, NOT isogonal and NOT linear conjugate. Applying any boundary-strip switch still preserves (N.0), so no conic, no chimera, no isogonal and no linear conjugate form emerges.

Out[338]=

CONDITION (N.0) AFTER SWITCHING BOUNDARY STRIPS
Right: Condition (N.0) is still satisfied.
Left: Condition (N.0) is still satisfied.
Lower: Condition (N.0) is still satisfied.
Upper: Condition (N.0) is still satisfied.
Right + Left: Condition (N.0) is still satisfied.
Right + Lower: Condition (N.0) is still satisfied.
Right + Upper: Condition (N.0) is still satisfied.
Left + Lower: Condition (N.0) is still satisfied.
Left + Upper: Condition (N.0) is still satisfied.
Lower + Upper: Condition (N.0) is still satisfied.
Right + Left + Lower: Condition (N.0) is still satisfied.
Right + Left + Upper: Condition (N.0) is still satisfied.
Right + Lower + Upper: Condition (N.0) is still satisfied.
Left + Lower + Upper: Condition (N.0) is still satisfied.
Right + Left + Lower + Upper: Condition (N.0) is still satisfied.

Out[339]=

================== **ORTHOGONALITY CHECK** ==================

$\cos(\alpha_i)\cdot\cos(\gamma_i) \neq \cos(\beta_i)\cdot\cos(\delta_i)$ for
at least one i = 1,..., 4 ⇒ NOT orthodiagonal.
Switching boundary strips does not correct this.

Initial anglesDeg (no switches):

-> Condition met: At least one difference is non-zero.

Out[343]=

NON-ORTHOGONALITY CHECK AFTER SWITCHING BOUNDARY STRIPS
Right: Condition met (at least one difference is non-zero).
Left: Condition met (at least one difference is non-zero).
Lower: Condition met (at least one difference is non-zero).
Upper: Condition met (at least one difference is non-zero).
Right + Left: Condition met (at least one difference is non-zero).
Right + Lower: Condition met (at least one difference is non-zero).
Right + Upper: Condition met (at least one difference is non-zero).
Left + Lower: Condition met (at least one difference is non-zero).
Left + Upper: Condition met (at least one difference is non-zero).
Lower + Upper: Condition met (at least one difference is non-zero).
Right + Left + Lower: Condition met (at least one difference is non-zero).
Right + Left + Upper: Condition met (at least one difference is non-zero).
Right + Lower + Upper: Condition met (at least one difference is non-zero).
Left + Lower + Upper: Condition met (at least one difference is non-zero).
Right + Left + Lower + Upper:
Condition met (at least one difference is non-zero).

Out[344]=

================== **CONJUGATE-MODULAR CHECK** ==================

M1 = M2 = M3 = M4 = M and M ≠ 2 ⇒ NOT
conjugate-modular. Boundary-strip switches preserve this.

Initial anglesDeg (no switches):

-> Condition met: All M_i are equal and $M \neq 2$.

Out[348]=

CONJUGATE-MODULAR CHECK AFTER SWITCHING BOUNDARY STRIPS

Right: Condition met (All M_i are equal and $M \neq 2$).
Left: Condition met (All M_i are equal and $M \neq 2$).
Lower: Condition met (All M_i are equal and $M \neq 2$).
Upper: Condition met (All M_i are equal and $M \neq 2$).
Right + Left: Condition met (All M_i are equal and $M \neq 2$).
Right + Lower: Condition met (All M_i are equal and $M \neq 2$).
Right + Upper: Condition met (All M_i are equal and $M \neq 2$).
Left + Lower: Condition met (All M_i are equal and $M \neq 2$).
Left + Upper: Condition met (All M_i are equal and $M \neq 2$).
Lower + Upper: Condition met (All M_i are equal and $M \neq 2$).
Right + Left + Lower: Condition met (All M_i are equal and $M \neq 2$).
Right + Left + Upper: Condition met (All M_i are equal and $M \neq 2$).
Right + Lower + Upper: Condition met (All M_i are equal and $M \neq 2$).
Left + Lower + Upper: Condition met (All M_i are equal and $M \neq 2$).
Right + Left + Lower + Upper: Condition met (All M_i are equal and $M \neq 2$).