

Computational Companion to “Quasi-Symmetric Nets: A Constructive Approach to the Equimodular Elliptic Type of Kokotsakis Polyhedra” — Criterion Helper

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Tested on: Mathematica 14.0

Section 0

Results of Section 0 is used in Theorem 1 (see paper)

```
In[ ]:= (*Define the initial parameters*)
 $\delta_1 = \text{Pi} / 2;$ 

(*Angles for the vertices*)
anglesRad = {
  { $\alpha_1, \beta_1, \gamma_1, \delta_1$ }, (*Vertex 1*)
  { $\delta_1, \gamma_1, \beta_1, \alpha_1$ }, (*Vertex 2*)
  { $\text{Pi} - \delta_1, \text{Pi} - \gamma_1, \text{Pi} - \beta_1, \text{Pi} - \alpha_1$ }, (*Vertex 3*)
  { $\alpha_1, \beta_1, \gamma_1, \delta_1$ } (*Vertex 4*)
};

(*Compute sigma values for each vertex*)
computeSigma[{ $\alpha_$ ,  $\beta_$ ,  $\gamma_$ ,  $\delta_$ }] := ( $\alpha + \beta + \gamma + \delta$ ) / 2;
sigmas = computeSigma /@ anglesRad;

(*Compute the differences for convenience*)
 $\sigma_1 = \text{sigmas}[[1]];$ 
```

```

 $\bar{\alpha}1 = \sigma1 - \alpha1;$ 
 $\bar{\beta}1 = \sigma1 - \beta1;$ 
 $\bar{\gamma}1 = \sigma1 - \gamma1;$ 
 $\bar{\delta}1 = \sigma1 - \delta1;$ 

(*Bricard's equation as defined in terms of angles and variables*)
BricardsEquation[{ $\alpha_$ ,  $\beta_$ ,  $\gamma_$ ,  $\delta_$ },  $\sigma_$ ,  $x_$ ,  $y_$ ] := Module[
  {c22, c20, c02, c11, c00},
  c22 = Sin[ $\sigma - \delta$ ] Sin[ $\sigma - \delta - \beta$ ];
  c20 = Sin[ $\sigma - \alpha$ ] Sin[ $\sigma - \alpha - \beta$ ];
  c02 = Sin[ $\sigma - \gamma$ ] Sin[ $\sigma - \gamma - \beta$ ];
  c11 = -Sin[ $\alpha$ ] Sin[ $\gamma$ ];
  c00 = Sin[ $\sigma$ ] Sin[ $\sigma - \beta$ ];
  c22 x^2 y^2 + c20 x^2 + c02 y^2 + 2 c11 x y + c00
];

(*Flexion 1*)
e0 = +1;
Z[t_] :=
  (-t * Sin[ $\beta1$ ] + e0 * Sqrt[(t^2 * Sin[ $\bar{\alpha}1$ ] * Sin[ $\bar{\beta}1$ ] + Sin[ $\sigma1$ ] * Sin[ $\bar{\alpha}1 - \beta1$ ]) *
    (Sin[ $\bar{\gamma}1$ ] * Sin[ $\bar{\delta}1$ ] + t^2 * Sin[ $\bar{\gamma}1 - \beta1$ ] * Sin[ $\bar{\delta}1 - \beta1$ ]))] /
  (Sin[ $\bar{\delta}1$ ] * Sin[ $\bar{\alpha}1 - \beta1$ ] + t^2 * Sin[ $\bar{\alpha}1$ ] * Sin[ $\bar{\delta}1 - \beta1$ ]);

W2[t_] := t;

U[t_] :=
  (t * Sin[ $\beta1$ ] + e0 * Sqrt[(t^2 * Sin[ $\bar{\alpha}1$ ] * Sin[ $\bar{\beta}1$ ] + Sin[ $\sigma1$ ] * Sin[ $\bar{\alpha}1 - \beta1$ ]) *
    (Sin[ $\bar{\gamma}1$ ] * Sin[ $\bar{\delta}1$ ] + t^2 * Sin[ $\bar{\gamma}1 - \beta1$ ] * Sin[ $\bar{\delta}1 - \beta1$ ]))] /
  (Sin[ $\bar{\delta}1$ ] * Sin[ $\bar{\alpha}1 - \beta1$ ] + t^2 * Sin[ $\bar{\alpha}1$ ] * Sin[ $\bar{\delta}1 - \beta1$ ]);

W1[t_] := e0 * Sqrt[(t^2 * Sin[ $\bar{\alpha}1$ ] * Sin[ $\bar{\beta}1$ ] + Sin[ $\sigma1$ ] * Sin[ $\bar{\alpha}1 - \beta1$ ]) *
  (Sin[ $\bar{\gamma}1$ ] * Sin[ $\bar{\delta}1$ ] + t^2 * Sin[ $\bar{\gamma}1 - \beta1$ ] * Sin[ $\bar{\delta}1 - \beta1$ ])) /
  (Sin[ $\bar{\gamma}1$ ] * Sin[ $\bar{\delta}1$ ] + t^2 * Sin[ $\bar{\gamma}1 - \beta1$ ] * Sin[ $\bar{\delta}1 - \beta1$ ]);

(*Compute the expressions for Bricard's equations*)
P1 = BricardsEquation[anglesRad[[1]], sigmas[[1]], Z[t], W1[t]];
P2 = BricardsEquation[anglesRad[[2]], sigmas[[2]], Z[t], W2[t]];
P3 = BricardsEquation[anglesRad[[3]], sigmas[[3]], U[t], W2[t]];
P4 = BricardsEquation[anglesRad[[4]], sigmas[[4]], U[t], W1[t]];

(*Full simplification of the equations*)
simplifiedP1 = FullSimplify[P1];
simplifiedP2 = FullSimplify[P2];
simplifiedP3 = FullSimplify[P3];
simplifiedP4 = FullSimplify[P4];

```

```

(*LHS and RHS for the table*)
lhs = {"P1(Z, W1)", "P2(Z, W2)", "P3(U, W2)", "P4(U, W1)"};
rhs = {simplifiedP1, simplifiedP2, simplifiedP3, simplifiedP4};

(*Check if the expression simplifies to zero*)
isZeroQ[expr_] := TrueQ[FullSimplify[expr] === 0];

(*Create the table with a title manually inserted*)
title = "Flexion 1";
(*Create the table*)
table = TableForm[Table[{lhs[[i]], rhs[[i]], If[isZeroQ[rhs[[i]]], "✓", "x"]},
  {i, Length[lhs]}], TableHeadings -> {None, {"LHS", "RHS", "LHS = RHS?"}}];

(*Combine the title and the table*)
Column[{title, table}]

(*Flexion 2*)
ClearAll[Z, W2, U, W1, e0]
e0 = -1;
Z[t_] :=
  (-t * Sin[β1] + e0 * Sqrt[(t^2 * Sin[α1] * Sin[β1] + Sin[σ1] * Sin[α1 - β1]) *
    (Sin[γ1] * Sin[δ1] + t^2 * Sin[γ1 - β1] * Sin[δ1 - β1])]) /
  (Sin[δ1] * Sin[α1 - β1] + t^2 * Sin[α1] * Sin[δ1 - β1]);

W2[t_] := t;

U[t_] :=
  (t * Sin[β1] + e0 * Sqrt[(t^2 * Sin[α1] * Sin[β1] + Sin[σ1] * Sin[α1 - β1]) *
    (Sin[γ1] * Sin[δ1] + t^2 * Sin[γ1 - β1] * Sin[δ1 - β1])]) /
  (Sin[δ1] * Sin[α1 - β1] + t^2 * Sin[α1] * Sin[δ1 - β1]);

W1[t_] := e0 * Sqrt[(t^2 * Sin[α1] * Sin[β1] + Sin[σ1] * Sin[α1 - β1]) *
  (Sin[γ1] * Sin[δ1] + t^2 * Sin[γ1 - β1] * Sin[δ1 - β1])] /
  (Sin[γ1] * Sin[δ1] + t^2 * Sin[γ1 - β1] * Sin[δ1 - β1]);

(*Compute the expressions for Bricard's equations*)
P1 = BricardsEquation[anglesRad[[1]], sigmas[[1]], Z[t], W1[t]];
P2 = BricardsEquation[anglesRad[[2]], sigmas[[2]], Z[t], W2[t]];
P3 = BricardsEquation[anglesRad[[3]], sigmas[[3]], U[t], W2[t]];
P4 = BricardsEquation[anglesRad[[4]], sigmas[[4]], U[t], W1[t]];

(*Full simplification of the equations*)
simplifiedP1 = FullSimplify[P1];
simplifiedP2 = FullSimplify[P2];
simplifiedP3 = FullSimplify[P3];

```

```

simplifiedP4 = FullSimplify[P4];

(*LHS and RHS for the table*)
lhs = {"P1(Z, W1)", "P2(Z, W2)", "P3(U, W2)", "P4(U, W1)"};
rhs = {simplifiedP1, simplifiedP2, simplifiedP3, simplifiedP4};

(*Check if the expression simplifies to zero*)
isZeroQ[expr_] := TrueQ[FullSimplify[expr] == 0];

(*Create the table with a title manually inserted*)
title = "Flexion 2";
(*Create the table*)
table = TableForm[Table[{lhs[[i]], rhs[[i]], If[isZeroQ[rhs[[i]]], "✓", "x"]},
  {i, Length[lhs]}], TableHeadings → {None, {"LHS", "RHS", "LHS = RHS?"}}];

(*Combine the title and the table*)
Column[{title, table}]

```

Out[]=

Flexion 1		
LHS	RHS	LHS = RHS?
$P_1(Z, W_1)$	0	✓
$P_2(Z, W_2)$	0	✓
$P_3(U, W_2)$	0	✓
$P_4(U, W_1)$	0	✓

Out[]=

Flexion 2		
LHS	RHS	LHS = RHS?
$P_1(Z, W_1)$	0	✓
$P_2(Z, W_2)$	0	✓
$P_3(U, W_2)$	0	✓
$P_4(U, W_1)$	0	✓

Section 1

Results of Section 1 is used in Lemma 1 (see paper) and in Sections 2 and 3 of this file

```

(*Clear all previous definitions*)
ClearAll[ $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ ,  $\sigma$ ,  $\bar{\alpha}$ ,  $\bar{\beta}$ ,  $\bar{\gamma}$ ,  $\bar{\delta}$ , a, b, c, d, M]

(*Define the angles and variables*)
 $\sigma = (\alpha + \beta + \gamma + \delta) / 2$ ;
 $\bar{\alpha} = \sigma - \alpha$ ;
 $\bar{\beta} = \sigma - \beta$ ;
 $\bar{\gamma} = \sigma - \gamma$ ;
 $\bar{\delta} = \sigma - \delta$ ;

a = Sin[ $\alpha$ ] / Sin[ $\bar{\alpha}$ ];
b = Sin[ $\beta$ ] / Sin[ $\bar{\beta}$ ];
c = Sin[ $\gamma$ ] / Sin[ $\bar{\gamma}$ ];
d = Sin[ $\delta$ ] / Sin[ $\bar{\delta}$ ];
M = a * b * c * d;

(*Define the 7 identities*)
expr1 = 1 - a * b == Sin[ $\sigma$ ] * Sin[ $\bar{\alpha} - \beta$ ] / (Sin[ $\bar{\alpha}$ ] * Sin[ $\bar{\beta}$ ]);
expr2 = 1 - b * c == Sin[ $\sigma$ ] * Sin[ $\bar{\gamma} - \beta$ ] / (Sin[ $\bar{\gamma}$ ] * Sin[ $\bar{\beta}$ ]);
expr3 = 1 - b * d == Sin[ $\sigma$ ] * Sin[ $\bar{\delta} - \beta$ ] / (Sin[ $\bar{\delta}$ ] * Sin[ $\bar{\beta}$ ]);
expr4 = c * d - 1 == Sin[ $\sigma$ ] * Sin[ $\bar{\alpha} - \beta$ ] / (Sin[ $\bar{\gamma}$ ] * Sin[ $\bar{\delta}$ ]);
expr5 = a * d - 1 == Sin[ $\sigma$ ] * Sin[ $\bar{\gamma} - \beta$ ] / (Sin[ $\bar{\alpha}$ ] * Sin[ $\bar{\delta}$ ]);
expr6 = a * c - 1 == Sin[ $\sigma$ ] * Sin[ $\bar{\delta} - \beta$ ] / (Sin[ $\bar{\alpha}$ ] * Sin[ $\bar{\gamma}$ ]);
expr7 = 1 - M ==
  Sin[ $\sigma$ ] * Sin[ $\bar{\alpha} - \beta$ ] * Sin[ $\bar{\gamma} - \beta$ ] * Sin[ $\bar{\delta} - \beta$ ] / (Sin[ $\bar{\alpha}$ ] * Sin[ $\bar{\beta}$ ] * Sin[ $\bar{\gamma}$ ] * Sin[ $\bar{\delta}$ ]);

(*Store the expressions and labels*)
lhs = {"1 - ab", "1 - bc", "1 - bd", "cd - 1", "ad - 1", "ac - 1", "1 - M"};
rhs =
  {"sin( $\sigma$ )sin( $\bar{\alpha} - \beta$ ) / (sin( $\bar{\alpha}$ )sin( $\bar{\beta}$ ))", "sin( $\sigma$ )sin( $\bar{\gamma} - \beta$ ) / (sin( $\bar{\beta}$ )sin( $\bar{\gamma}$ ))",
   "sin( $\sigma$ )sin( $\bar{\delta} - \beta$ ) / (sin( $\bar{\beta}$ )sin( $\bar{\delta}$ ))", "sin( $\sigma$ )sin( $\bar{\alpha} - \beta$ ) / (sin( $\bar{\gamma}$ )sin( $\bar{\delta}$ ))",
   "sin( $\sigma$ )sin( $\bar{\gamma} - \beta$ ) / (sin( $\bar{\alpha}$ )sin( $\bar{\delta}$ ))", "sin( $\sigma$ )sin( $\bar{\delta} - \beta$ ) / (sin( $\bar{\alpha}$ )sin( $\bar{\gamma}$ ))",
   "sin( $\sigma$ )sin( $\bar{\alpha} - \beta$ )sin( $\bar{\gamma} - \beta$ )sin( $\bar{\delta} - \beta$ ) / (sin( $\bar{\alpha}$ )sin( $\bar{\beta}$ )sin( $\bar{\gamma}$ )sin( $\bar{\delta}$ ))"};

expressions = {expr1, expr2, expr3, expr4, expr5, expr6, expr7};

(*Function to test equality*)
isTrueQ[expr_] := TrueQ[FullSimplify[expr]]

(*Build the result table*)
TableForm[Table[{lhs[[i]], rhs[[i]], If[isTrueQ[expressions[[i]]], "✓", "X"]}],
  {i, Length[expressions]}],
  TableHeadings → {None, {"LHS", "RHS", "LHS = RHS?"}}]

```

Out[]//TableForm=

LHS	RHS	LI
1 - ab	$\sin(\sigma) \sin(\bar{\alpha} - \beta) / (\sin(\bar{\alpha}) \sin(\bar{\beta}))$	✓
1 - bc	$\sin(\sigma) \sin(\bar{\gamma} - \beta) / (\sin(\bar{\beta}) \sin(\bar{\gamma}))$	✓
1 - bd	$\sin(\sigma) \sin(\bar{\delta} - \beta) / (\sin(\bar{\beta}) \sin(\bar{\delta}))$	✓
cd - 1	$\sin(\sigma) \sin(\bar{\alpha} - \beta) / (\sin(\bar{\gamma}) \sin(\bar{\delta}))$	✓
ad - 1	$\sin(\sigma) \sin(\bar{\gamma} - \beta) / (\sin(\bar{\alpha}) \sin(\bar{\delta}))$	✓
ac - 1	$\sin(\sigma) \sin(\bar{\delta} - \beta) / (\sin(\bar{\alpha}) \sin(\bar{\gamma}))$	✓
1 - M	$\sin(\sigma) \sin(\bar{\alpha} - \beta) \sin(\bar{\gamma} - \beta) \sin(\bar{\delta} - \beta) / (\sin(\bar{\alpha}) \sin(\bar{\beta}) \sin(\bar{\gamma}) \sin(\bar{\delta}))$	✓

Section 2

Results of Section 2 is used in the proof of Lemma 6 (see paper)

```
(*Clear all previous definitions*)
ClearAll[α, β, γ, δ, σ, ᾱ, β̄, γ̄, δ̄, u, x, y, z, d1, d2, d3, d4, d5,
  rhsCosAlpha, rhsCosBeta, rhsCosGamma, rhsCosDelta, rhsCosSigma]

(*Define assumptions for the angles*)
assumptions = {0 < α < Pi, 0 < β < Pi, 0 < γ < Pi,
  0 < δ < Pi, 0 < ᾱ < Pi, 0 < β̄ < Pi, 0 < γ̄ < Pi, 0 < δ̄ < Pi};

(*Define the angles and intermediate variables*)
σ = (α + β + γ + δ) / 2;
ᾱ = σ - α;
β̄ = σ - β;
γ̄ = σ - γ;
δ̄ = σ - δ;
ε = Abs[Sin[σ]] / Sin[σ];

(*Define Section 1 results*)
OneMinusM =
  Sin[σ] * Sin[ᾱ - β] * Sin[γ̄ - β] * Sin[δ̄ - β] / (Sin[ᾱ] * Sin[β̄] * Sin[γ̄] * Sin[δ̄]);
rMinusOne = Sin[σ] * Sin[γ̄ - β] / (Sin[ᾱ] * Sin[δ̄]);
sMinusOne = Sin[σ] * Sin[ᾱ - β] / (Sin[γ̄] * Sin[δ̄]);
fMinusOne = Sin[σ] * Sin[δ̄ - β] / (Sin[ᾱ] * Sin[γ̄]);

u = OneMinusM;
x = 1 / rMinusOne;
y = 1 / sMinusOne;
z = 1 / fMinusOne;

(*Define expressions in denominators*)
d1 = x * z * u * (1 + y) * (1 + u * y);
d2 = y * z * u * (1 + x) * (1 + u * x);
```

```

d3 = x * y * u * (1 + z) * (1 + u * z);
d4 = u * (1 + x) * (1 + y) * (1 + z) * (1 + u * x) * (1 + u * y) * (1 + u * z);
d5 = x * y * z * u^2 * (1 + x) * (1 + y) * (1 + z);

(*Define denominators*)
denAlpha = 2 * FullSimplify[Sqrt[d1], assumptions];
denBeta = 2 * FullSimplify[Sqrt[d4], assumptions];
denGamma = 2 * FullSimplify[Sqrt[d2], assumptions];
denDelta = 2 * FullSimplify[Sqrt[d3], assumptions];
denSigma = 2 * FullSimplify[Sqrt[d5], assumptions];

(*Define RHS expressions*)
rhsCosAlpha = FullSimplify[ε * (1 - y * z * u + x * z * u - x * y * u) / denAlpha] /.
  Cos[β] - Cos[α + γ + δ] ⇒ 2 Sin[(α + β + γ + δ) / 2] Sin[(α - β + γ + δ) / 2];

rhsCosBeta =
  FullSimplify[ε * (u * (1 + x) * (1 + y) * (1 + z) + (1 + u * x) * (1 + u * y) * (1 + u * z) -
    u * x * y * z * (u - 1)^2) / denBeta] /.
  Cos[α] - Cos[β + γ + δ] ⇒ 2 Sin[(α + β + γ + δ) / 2] Sin[(-α + β + γ + δ) / 2];

rhsCosGamma = FullSimplify[ε * (1 + y * z * u - x * z * u - x * y * u) / denGamma] /.
  Cos[β] - Cos[α + γ + δ] ⇒ 2 Sin[(α + β + γ + δ) / 2] Sin[(α - β + γ + δ) / 2];

rhsCosDelta = FullSimplify[ε * (1 - y * z * u - x * z * u + x * y * u) / denDelta] /.
  Cos[β] - Cos[α + γ + δ] ⇒ 2 Sin[(α + β + γ + δ) / 2] Sin[(α - β + γ + δ) / 2];

rhsCosSigma = (1 - u * (x * y + x * z + y * z + 2 * x * y * z)) / denSigma;

(*Define equations to check*)
exprAlpha = Cos[α] == FullSimplify[rhsCosAlpha, assumptions];
exprBeta = Cos[β] == FullSimplify[rhsCosBeta, assumptions];
exprGamma = Cos[γ] == FullSimplify[rhsCosGamma, assumptions];
exprDelta = Cos[δ] == FullSimplify[rhsCosDelta, assumptions];
exprSigma = Cos[σ] == FullSimplify[rhsCosSigma, assumptions];

(*Build labeled table*)
lhs = {"cos(α)", "cos(β)", "cos(γ)", "cos(δ)", "cos(σ)"};
rhs = {
  "ε(1 - yzu + xzu - xyu) / (2√(xzu(1 + y)(1 + uy)))",
  "ε(u(1 + x)(1 + y)(1 + z) + (1 + ux)(1 + uy)(1 + uz) - uxyz(u - 1)^2) / (2√(u(1 + x)(1 + y)(1 + z)(1 + ux)(1 + uy)(1 + uz)))",
  "ε(1 + yzu - xzu - xyu) / (2√(yzu(1 + x)(1 + ux)))",
  "ε(1 - yzu - xzu + xyu) / (2√(xyu(1 + z)(1 + uz)))",
  "(1 - u(xy + xz + yz + 2xyz)) / (2√(xyzu^2(1 + x)(1 + y)(1 + z)))"};
expressions = {exprAlpha, exprBeta, exprGamma, exprDelta, exprSigma};

```



```
isTrueQ[expr_] := TrueQ[FullSimplify[expr, assumptions]];
```

```
TableForm[Table[{lhs[[i]], rhs[[i]], If[isTrueQ[expressions[[i]]], "✓", "X"]},
  {i, Length[expressions]}],
  TableHeadings → {None, {"LHS", "RHS", "LHS = RHS?"}}]
```

```
(*(*old version:check equalities*)
```

```
(*cos(α)*)
```

```
FullSimplify[
```

```
(ε (Sin[σ - β] Sin[σ] + Sin[σ - α] Sin[σ - α - β] - Sin[σ - γ] Sin[σ - γ -
  β] - Sin[σ - δ] Sin[σ - δ - β])) / (2 Sin[γ] Sin[δ]) == ε Cos[α]]
```

```
(*cos(β)*)
```

```
FullSimplify[
```

```
(ε (Sin[σ - α] Sin[σ] + Sin[σ - β] Sin[σ - α - β] + Sin[σ - δ] Sin[σ - γ -
  β] + Sin[σ - γ] Sin[σ - δ - β])) / (2 Sin[γ] Sin[δ]) == ε Cos[β]]
```

```
(*cos(γ)*)
```

```
FullSimplify[
```

```
(ε (Sin[σ - β] Sin[σ] - Sin[σ - α] Sin[σ - α - β] + Sin[σ - γ] Sin[σ - γ -
  β] - Sin[σ - δ] Sin[σ - δ - β])) / (2 Sin[α] Sin[δ]) == ε Cos[γ]]
```

```
(*cos(δ)*)
```

```
FullSimplify[
```

```
(ε (Sin[σ - β] Sin[σ] - Sin[σ - α] Sin[σ - α - β] - Sin[σ - γ] Sin[σ - γ -
  β] + Sin[σ - δ] Sin[σ - δ - β])) / (2 Sin[α] Sin[γ]) == ε Cos[δ]]
```

```
(*cos(σ)*)
```

```
FullSimplify[(Sin[σ - β] Sin[σ] ^2 - Sin[σ - α] Sin[σ - α - β] Sin[σ] -
  Sin[σ - γ] Sin[σ - γ - β] Sin[σ] - Sin[σ - δ] Sin[σ - δ - β] Sin[σ] -
  2 Sin[σ - α] Sin[σ - γ] Sin[σ - δ]) /
  (2 Sin[α] Sin[γ] Sin[δ]) == Cos[σ]]*)
```

```
Out[*]//TableForm=
```

LHS	RHS
cos(α)	$\varepsilon (1 - yzu + xzu - xyu) / (2\sqrt{xzu(1+y)(1+uy)})$
cos(β)	$\varepsilon (u(1+x)(1+y)(1+z) + (1+ux)(1+uy)(1+uz) - uxyz(u-1)^2) /$
cos(γ)	$\varepsilon (1 + yzu - xzu - xyu) / (2\sqrt{yzu(1+x)(1+ux)})$
cos(δ)	$\varepsilon (1 - yzu - xzu + xyu) / (2\sqrt{xyu(1+z)(1+uz)})$
cos(σ)	$(1 - u(xy + xz + yz + 2xyz)) / (2\sqrt{xyz u^2(1+x)(1+y)(1+z)})$

Section 3

Results of Section 3 is used in the proof of Lemma 7 (see paper)

```
In[*]:= (*Clear previous definitions*)
```

```
ClearAll[α, β, γ, δ, σ, ā, β̄, γ̄, δ̄, ε, θ, A, ξ, η, lhs, rhs, expr, thetaIndex];
```

```

(*Define assumptions for the angles*)
assumptions = {0 <  $\alpha$  < Pi, 0 <  $\beta$  < Pi, 0 <  $\gamma$  < Pi,
  0 <  $\delta$  < Pi, 0 <  $\bar{\alpha}$  < Pi, 0 <  $\bar{\beta}$  < Pi, 0 <  $\bar{\gamma}$  < Pi, 0 <  $\bar{\delta}$  < Pi};

(*---Definitions---*)
 $\sigma = (\alpha + \beta + \gamma + \delta) / 2$ ;
 $\bar{\alpha} = \sigma - \alpha$ ;
 $\bar{\beta} = \sigma - \beta$ ;
 $\bar{\gamma} = \sigma - \gamma$ ;
 $\bar{\delta} = \sigma - \delta$ ;
 $\varepsilon = \text{Abs}[\text{Sin}[\sigma]] / \text{Sin}[\sigma]$ ;

(*Define Section 1 results*)
OneMinusM =
  Sin[ $\sigma$ ] * Sin[ $\bar{\alpha} - \beta$ ] * Sin[ $\bar{\gamma} - \beta$ ] * Sin[ $\bar{\delta} - \beta$ ] / (Sin[ $\bar{\alpha}$ ] * Sin[ $\bar{\beta}$ ] * Sin[ $\bar{\gamma}$ ] * Sin[ $\bar{\delta}$ ]);
rMinusOne = Sin[ $\sigma$ ] * Sin[ $\bar{\gamma} - \beta$ ] / (Sin[ $\bar{\alpha}$ ] * Sin[ $\bar{\delta}$ ]);
sMinusOne = Sin[ $\sigma$ ] * Sin[ $\bar{\alpha} - \beta$ ] / (Sin[ $\bar{\gamma}$ ] * Sin[ $\bar{\delta}$ ]);
fMinusOne = Sin[ $\sigma$ ] * Sin[ $\bar{\delta} - \beta$ ] / (Sin[ $\bar{\alpha}$ ] * Sin[ $\bar{\gamma}$ ]);

u = OneMinusM;
x = 1 / rMinusOne;
y = 1 / sMinusOne;
z = 1 / fMinusOne;

(*Define expressions in denominators*)
dexp = x * y * u * (1 + z) * (1 + u * z);

(*Define denominators*)
den = 2 * FullSimplify[Sqrt[dexp], assumptions];

(*Cyclic index function*)
thetaIndex[k_] := Mod[k - 1, 4] + 1;

(*Definition of A[i,j]*)
A[i_, j_] /; 1 ≤ i ≤ 4 && 1 ≤ j ≤ 4 :=
  4 Cos[ $\theta[\text{thetaIndex}[i]] / 2 + (\text{Pi} / 4) i j (j - 1) + (\text{Pi} / 2) j j^2 * \text{Cos}[\theta[\text{thetaIndex}[i - 1]] / 2 + (\text{Pi} / 4) (i - 1) j (j - 1) + (\text{Pi} / 2) j j^2$ ;
(*Table[A[i,j],{i,1,4},{j,1,4}];*)

(*Define  $\xi[i]$  and  $\eta[i]$ *)
 $\xi[i\_? \text{IntegerQ}] /; 1 \leq i \leq 4 :=$ 
  If[OddQ[i],  $\theta[\text{thetaIndex}[i]]$ ,  $\theta[\text{thetaIndex}[i - 1]]$ ];
 $\eta[i\_? \text{IntegerQ}] /; 1 \leq i \leq 4 :=$  If[OddQ[i],  $\theta[\text{thetaIndex}[i - 1]]$ ,  $\theta[\text{thetaIndex}[i]]$ ];
(*Table[{i, $\xi[i]$ , $\eta[i]$ },{i,1,4}];*)

```

```
(*Define left-hand side and right-hand side expressions*)
lhs[i_Integer] := FullSimplify[
  (A[i, 1] + A[i, 2] * y * z * u + A[i, 3] * x * z * u + A[i, 4] * x * y * u) / den] /.
  Cos[β] - Cos[α + γ + δ] → 2 Sin[(α + β + γ + δ) / 2] Sin[(α - β + γ + δ) / 2];

rhs[i_Integer] :=
  ε (Cos[β] - Cos[γ] (Cos[α] Cos[δ] + Cos[ξ[i]] Sin[α] Sin[δ]) - Cos[η[i]]
    Sin[γ] (Cos[α] Sin[δ] - Cos[ξ[i]] Sin[α] Cos[δ])) / (Sin[α] Sin[γ]);

expr[i_Integer] := FullSimplify[Equal[lhs[i], rhs[i]], assumptions];

(*---Build labeled result table---*)
lhsList =
  {"(A1,1 + A1,2y1z1u1 + A1,3x1z1u1 + A1,4x1y1u1) / (2√(x1y1u1(1 + z1)(1
    + u1z1)))", "(A2,1 + A2,2y2z2u2 + A2,3x2z2u2 +
    A2,4x2y2u2) / (2√(x2y2u2(1 + z2)(1 + u2z2)))",
  "(A3,1 + A3,2y3z3u3 + A3,3x3z3u3 + A3,4x3y3u3) / (2√(x3y3u3(1 +
    z3)(1 + u3z3)))", "(A4,1 + A4,2y4z4u4 + A4,3x4z4u4
    + A4,4x4y4u4) / (2√(x4y4u4(1 + z4)(1 + u4z4)))"};

rhsList = {"ε1((cos(β1) - cos(γ1)(cos(α1)cos(δ1) +
  cos(θ1)sin(α1)sin(δ1)) - cos(θ4)sin(γ1)(cos(α1)sin(δ1)
  - cos(θ1)sin(α1)cos(δ1))) / (sin(α1)sin(γ1))",
  "ε2((cos(β2) - cos(γ2)(cos(α2)cos(δ2) + cos(θ1)sin(α2)sin(δ2))
  - cos(θ2)sin(γ2)(cos(α2)sin(δ2) -
  cos(θ1)sin(α2)cos(δ2))) / (sin(α2)sin(γ2))",
  "ε3((cos(β3) - cos(γ3)(cos(α3)cos(δ3) + cos(θ3)sin(α3)sin(δ3))
  - cos(θ2)sin(γ3)(cos(α3)sin(δ3) -
  cos(θ3)sin(α3)cos(δ3))) / (sin(α3)sin(γ3))",
  "ε4((cos(β4) - cos(γ4)(cos(α4)cos(δ4) + cos(θ3)sin(α4)sin(δ4))
  - cos(θ4)sin(γ4)(cos(α4)sin(δ4) -
  cos(θ3)sin(α4)cos(δ4))) / (sin(α4)sin(γ4))"};

results = Table[If[TrueQ[expr[i]], "✓", "✗"], {i, 1, 4}];

TableForm[Table[{i, lhsList[[i]], rhsList[[i]], results[[i]]}, {i, 1, 4}],
  TableHeadings → {None, {"i", "LHS", "RHS", "LHS = RHS?"}},
  TableAlignments → Left]
```

Out[]//TableForm=

i	LHS
1	$(A_{1,1} + A_{1,2}y_1z_1u_1 + A_{1,3}x_1z_1u_1 + A_{1,4}x_1y_1u_1) / (2\sqrt{x_1y_1u_1(1 + z_1)(1 + u_1z_1)})$
2	$(A_{2,1} + A_{2,2}y_2z_2u_2 + A_{2,3}x_2z_2u_2 + A_{2,4}x_2y_2u_2) / (2\sqrt{x_2y_2u_2(1 + z_2)(1 + u_2z_2)})$
3	$(A_{3,1} + A_{3,2}y_3z_3u_3 + A_{3,3}x_3z_3u_3 + A_{3,4}x_3y_3u_3) / (2\sqrt{x_3y_3u_3(1 + z_3)(1 + u_3z_3)})$
4	$(A_{4,1} + A_{4,2}y_4z_4u_4 + A_{4,3}x_4z_4u_4 + A_{4,4}x_4y_4u_4) / (2\sqrt{x_4y_4u_4(1 + z_4)(1 + u_4z_4)})$