

Computational Companion to “Quasi-Symmetric Nets: A Constructive Approach to the Equimodular Elliptic Type of Kokotsakis Polyhedra” — Example 4 Helper

A. Nurmatov, M. Skopenkov, F. Rist, J. Klein, D. L. Michels
Tested on: Mathematica 14.0

In[5790]:=

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(*=====*)
=====*)
(*=====*)
=====*)
(*=====*)
=====*)
(*Quit*)
(*All angle sets in degrees*)
anglesDeg = {
  {91.32487959870187, 27.53122212644406,
   103.21844931187813, 120}, (*Vertex 1*)
  {115.75047063536742,
   29.335366103921295, 109.7807695499394, 80}, (*Vertex 2*)
  {31.19200181228523,
   113.66642350596918, 89.61760260813426, 85}, (*Vertex 3*)
  {28.19551791700786, 107.67391515450669,
   121.75654544610931, 75} (*Vertex 4*)};

(*-----*)
(*Function to compute sigma from 4 angles*)
computeSigma[{ $\alpha$ _,  $\beta$ _,  $\gamma$ _,  $\delta$ _}] := ( $\alpha$  +  $\beta$  +  $\gamma$  +  $\delta$ ) / 2;
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(*-----*)
(*Function to compute a,b,c,d from angles*)
computeABCD[{ $\alpha$ _,  $\beta$ _,  $\gamma$ _,  $\delta$ _}] :=
Module[{alpha =  $\alpha$  Degree, beta =  $\beta$  Degree, gamma =  $\gamma$  Degree,
  delta =  $\delta$  Degree, sigma}, sigma = computeSigma[{ $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ ] Degree];
{Sin[alpha] / Sin[sigma - alpha], Sin[beta] / Sin[sigma - beta],
  Sin[gamma] / Sin[sigma - gamma], Sin[delta] / Sin[sigma - delta]};

(*-----*)
(*Compute all  $\sigma$  values*)
sigmas = computeSigma /@ anglesDeg;

(*Compute all {a,b,c,d} values*)
results = computeABCD /@ anglesDeg;

(*Optional:extract individual values*)
{a1, b1, c1, d1} = results[[1]];
{a2, b2, c2, d2} = results[[2]];
{a3, b3, c3, d3} = results[[3]];
{a4, b4, c4, d4} = results[[4]];
{ $\sigma$ 1,  $\sigma$ 2,  $\sigma$ 3,  $\sigma$ 4} = FullSimplify[sigmas];

(*=====
====*)
(*=====
CONDITION (N.0) =====*)
(*=====
====*)
(*uniqueCombos={{1,1,1,1},{1,1,1,-1},{1,1,-1,-1},
  {1,1,-1,1},{1,-1,1,1},{1,-1,-1,1},{1,-1,1,-1},{1,-1,-1,-1}};

checkConditionN0Degrees[{ $\alpha$ _, $\beta$ _, $\gamma$ _, $\delta$ _}]:=Module[
  {angles={ $\alpha$ , $\beta$ , $\gamma$ , $\delta$ },results},results=Mod[uniqueCombos.angles,360]//Chop;
  !MemberQ[results,0]];

conditionsN0=checkConditionN0Degrees/@anglesDeg;
allVerticesPass=And@@conditionsN0;

Column[{TextCell[Style["===== CONDITION (N.0) =====",
  Darker[Green],Bold, 16],"Text"],
  If[allVerticesPass,
    Style["✓ All vertices satisfy (N.0).",Darker[Green],Bold],
    Style["✗ Some vertices fail (N.0).",Red,Bold]]]*),
uniqueCombos = {{1, 1, 1, 1}, {1, 1, 1, -1}, {1, 1, -1, -1}, {1, 1, -1, 1},
  {1, -1, 1, 1}, {1, -1, -1, 1}, {1, -1, 1, -1}, {1, -1, -1, -1}};

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checkConditionN0Degrees[{ $\alpha$ _,  $\beta$ _,  $\gamma$ _,  $\delta$ _}] :=
Module[{angles = { $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ }, results},
  results = Mod[uniqueCombos.angles, 360] // Chop;
  results];

(*apply to all vertices*)
resultsPerVertex = checkConditionN0Degrees /@ anglesDeg;

(*check pass/fail*)
conditionsN0 = Map[! MemberQ[#, 0] &, resultsPerVertex];
allVerticesPass = And@@conditionsN0;

Column[{TextCell[Style["===== CONDITION (N.0) =====",
  Darker[Green], Bold, 16], "Text"], (*summary line*)If[allVerticesPass,
  Style["✓ All vertices satisfy (N.0).", Darker[Green], Bold],
  Style["✗ Some vertices fail (N.0).", Red, Bold]],
  (*detailed results for each vertex*)
  Grid[Prepend[Table[{"Vertex " <> ToString[i], resultsPerVertex[[i],
    If[conditionsN0[[i], "✓ Pass", "✗ Fail"]], {i, Length[anglesDeg]}],
    {"Vertex", "Combinations (mod 360)", "Status"}], Frame → All]]]

(*=====*)
(*=====*)
  CONDITION (N.3)=====*)
(*=====*)
  =====*)
Ms = FullSimplify[Times@@@results];
(*Set precision explicitly*)
Ms = SetPrecision[Ms, 17];
allEqualQ = Simplify[Equal@@Ms];

Column[
  {TextCell[Style["===== CONDITION (N.3) =====", Blue,
    Bold, 16], "Text"],
    (*Spacer[{0,10}],(*Adds some vertical space*)*)
    (* ===PART 2:Always display the calculated values===*)
    Style["Calculated Values:", Bold, 12],
    Grid[Table[{Style["M" <> ToString[i], Bold], " = ", NumberForm[Ms[[i], 16]}],
      {i, Length[Ms]}], Alignment → {Left, Center}],
    (*Spacer[{0,15}],(*Adds more vertical space*)*)
    (* ===PART 3:Check for equality and print the result===*)
    TextCell[Style["Equality Check:", Bold, 12]], If[allEqualQ,
      Style["✓ All values are equal up to 15 digits after the decimal point.",
        Darker[Green], Bold, 14],

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    Style["× The values are not all equal.", Red, Bold, 14]]
  ]]

(*=====
====*)
(*=====CONDITION (N.4)=====*)
(*=====
====*)
aList = results[[All, 1]];
cList = results[[All, 3]];
dList = results[[All, 4]];

rList = FullSimplify /@ (aList * dList);
sList = FullSimplify /@ (cList * dList);

(*Set precision explicitly for each value*)
rList = SetPrecision[rList, 17];
sList = SetPrecision[sList, 17];

{r1, r2, r3, r4} = rList;
{s1, s2, s3, s4} = sList;

condition4Passed = (r1 == r2 && r3 == r4 && s1 == s4 && s2 == s3);

Column[{TextCell[Style["===== CONDITION (N.4) =====",
  Darker[Blue], Bold, 16], "Text"],
  Style["Calculated Values:", Bold, 12],
  Grid[{{Style["r1", Bold], " = ", NumberForm[r1, 17],
    Style["; ", Bold], Style["s1", Bold], " = ", NumberForm[s1, 17]},
    {Style["r2", Bold], " = ", NumberForm[r2, 17], Style["; ", Bold],
    Style["s2", Bold], " = ", NumberForm[s2, 17]},
    {Style["r3", Bold], " = ", NumberForm[r3, 16], Style["; ", Bold],
    Style["s3", Bold], " = ", NumberForm[s3, 17]}, {Style["r4", Bold],
    " = ", NumberForm[r4, 16], Style["; ", Bold], Style["s4", Bold],
    " = ", NumberForm[s4, 17]}}, Alignment → {Left, Center}],

  TextCell[Style["Equality Checks:", Bold, 12]],
  If[condition4Passed,
    Style["✓ All conditions are met.", Darker[Green], Bold, 14],
    Style["× Condition (N.4) fails.", Red, Bold]],
  Column[
    {If[r1 == r2, Style["✓ r1 = r2 up to 15 digits after the decimal point.",
      Darker[Green], Bold], Style["× r1 ≠ r2", Red, Bold]],
    If[r3 == r4, Style["✓ r3 = r4 up to 14 digits after the decimal point.",
      Darker[Green], Bold], Style["× r3 ≠ r4", Red, Bold]],

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If[s1 == s4, Style["✓ s1 = s4 up to 12 digits after the decimal point.",
  Darker[Green], Bold], Style["✗ s1 ≠ s4", Red, Bold]],
If[s2 == s3, Style["✓ s2 = s3 up to 16 digits after the decimal point.",
  Darker[Green], Bold], Style["✗ s2 = s3", Red, Bold]]], Spacings → 0.5]
}]

(*=====
====*)
(*=====
CONDITION (N.5)=====*)
(*=====
====*)
fList = FullSimplify /@ (aList * cList);
(*Set precision explicitly*)
fList = SetPrecision[fList, 15];
{f1, f2, f3, f4} = fList;
M1 = Ms[[1]];
M1 = N[Floor[M1 * 10^15] / 10^15];

m[M1_] := Piecewise[{{1 - M1, M1 < 1}, {(M1 - 1) / M1, M1 > 1}}] // N;
K = EllipticK[m[M1]] // N;
K = SetPrecision[K, 15];
Kp = EllipticK[1 - m[M1]] // N;
Kp = SetPrecision[Kp, 15];

computeTi[i_] :=
Module[{sigma = sigmas[[i]], r = rList[[i]], s = sList[[i]], f = fList[[i]], base},
  base = I * Im[InverseJacobiDN[Piecewise[
    {{Sqrt[f], M1 < 1}, {1 / Sqrt[f], M1 > 1}}], m[M1]]] / Kp;
  Which[sigma < 180, Which[r > 1 && s > 1, base, (r < 1 && s > 1) || (r > 1 && s < 1),
    1 + base, r < 1 && s < 1, 2 + base], sigma > 180, Which[r > 1 && s > 1,
    2 + base, (r < 1 && s > 1) || (r > 1 && s < 1), 3 + base, r < 1 && s < 1, base]]];

tList = computeTi /@ Range[4];
tList = SetPrecision[tList, 15];
RoundWithTolerance[x_, tol_ : 10^(-15)] :=
Module[{nearest}, nearest = Round[x];
  If[Abs[x - nearest] ≤ tol, nearest, x]];

checkValidCombination[M1_, ε_ : 10^(-15)] :=
Module[{i, combo, dotProd, rePart, imPart, n2, expr, foundQ = False},
  Print[Style["△ Approximate validation using ε-tolerance. For rigorous
    proof, see the referenced paper.", Darker@Orange, Italic]];
  Do[combo = uniqueCombos[[i]];
    dotProd = tList.combo;
    rePart = Abs[Re[dotProd]]];

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imPart = Abs[Im[dotProd]];
If[M1 < 1,
  If[Mod[RoundWithTolerance[rePart], 4] < ε,
    If[Mod[RoundWithTolerance[imPart], 2] < ε, expr =
      tList[[1]] + combo[[2]] tList[[2]] + combo[[3]] tList[[3]] + combo[[4]] tList[[4]];
    Print[Style["✔ Valid Combination Found (M < 1):",
      Darker[Green], Bold], "\n", Style["e1 = ", Bold], combo[[2]],
      Style["", e2 = ", Bold], combo[[3]], Style["", e3 = ", Bold], combo[[4]],
      "\n", Style["t1 = ", Bold], Re[tList[[1]], "K + ", Im[tList[[1]], "iK'",
      "\n", Style["t2 = ", Bold], Re[tList[[2]], "K + ", Im[tList[[2]], "iK'",
      "\n", Style["t3 = ", Bold], Re[tList[[3]], "K + ", Im[tList[[3]], "iK'",
      "\n", Style["t4 = ", Bold], Re[tList[[4]], "K + ", Im[tList[[4]],
      "iK'", "\n", Style["t1 + e1*t2 + e2*t3 + e3*t4 = ", Bold],
      Re[expr], "K + ", Im[expr], "iK'"];
    foundQ = True;
    Break[]]]];
If[M1 > 1,
  If[Mod[RoundWithTolerance[imPart], 2] < ε,
    n2 = Quotient[RoundWithTolerance[imPart], 2];
    If[Mod[RoundWithTolerance[Abs[rePart - 2 n2]], 4] < ε, expr =
      tList[[1]] + combo[[2]] tList[[2]] + combo[[3]] tList[[3]] + combo[[4]] tList[[4]];
    Print[Style["✔ Valid Combination Found (M > 1):",
      Darker[Green], Bold], "\n", Style["e1 = ", Bold], combo[[2]],
      Style["", e2 = ", Bold], combo[[3]], Style["", e3 = ", Bold], combo[[4]],
      "\n", Style["t1 = ", Bold], Re[tList[[1]], "K + ", Im[tList[[1]], "iK'",
      "\n", Style["t2 = ", Bold], Re[tList[[2]], "K + ", Im[tList[[2]], "iK'",
      "\n", Style["t3 = ", Bold], Re[tList[[3]], "K + ", Im[tList[[3]], "iK'",
      "\n", Style["t4 = ", Bold], Re[tList[[4]], "K + ", Im[tList[[4]],
      "iK'", "\n", Style["t1 + e1*t2 + e2*t3 + e3*t4 = ", Bold],
      Re[expr], "K + ", Im[expr], "iK'"];
    foundQ = True;
    Break[]]]], {i, Length[uniqueCombos]}}];
If[! foundQ, Print[Style["✘ No valid combination found.", Red, Bold], "\n",
  Style["t1 = ", Bold], Re[tList[[1]], "K + ", Im[tList[[1]], "iK'", "\n",
  Style["t2 = ", Bold], Re[tList[[2]], "K + ", Im[tList[[2]], "iK'", "\n",
  Style["t3 = ", Bold], Re[tList[[3]], "K + ", Im[tList[[3]], "iK'", "\n",
  Style["t4 = ", Bold], Re[tList[[4]], "K + ", Im[tList[[4]], "iK'", "\n"]];];

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Column[{TextCell[Style["===== CONDITION (N.5) =====",
  Purple, Bold, 16], "Text"]}]}
res = checkValidCombination[M1];

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(*=====

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====*)
(*=====
    OTHER PARAMETER=====*)
(*=====
====*)
cosSigmaAlt[Mi_, fi_, ri_, si_] :=
    (Mi (fi + ri + si - 1) + ri fi si - ri fi - ri si - fi si) / (2 Sqrt[ri si fi] (1 - Mi));

Column[
{TextCell[Style["===== OTHER PARAMETERS =====",
    Darker[Orange], Bold, 16], "Text"],
Row[{Style["u = ", Bold], 1 - M1}],
Row[{Style["σ1 = ", Bold], σ1 Degree, Style["", σ2 = ", Bold], σ2 Degree,
    Style["", σ3 = ", Bold], σ3 Degree, Style["", σ4 = ", Bold], σ4 Degree}],
Row[{Style["σ1 ≈ ", Bold], N[σ1], Style["", σ2 ≈ ", Bold],
    N[σ2], Style["", σ3 ≈ ", Bold], N[σ3],
    Style["", σ4 ≈ ", Bold], N[σ4], Style["", σ4 ≈ ", Bold]}],
Row[{Style["cosσ1 = ", Bold], Simplify@cosSigmaAlt[M1, f1, r1, s1],
    Style["", cosσ2 = ", Bold], Simplify@cosSigmaAlt[M1, f2, r2, s2],
    Style["", cosσ3 = ", Bold], Simplify@cosSigmaAlt[M1, f3, r3, s3],
    Style["", cosσ4 = ", Bold], Simplify@cosSigmaAlt[M1, f4, r4, s4]}],
Row[{Style["f1 = ", Bold], f1, Style["", f2 = ", Bold],
    f2, Style["", f3 = ", Bold], f3, Style["", f4 = ", Bold], f4}],
Row[{Style["z1 = ", Bold], FullSimplify[1 / (f1 - 1)], Style["", z2 = ", Bold],
    FullSimplify[1 / (f2 - 1)], Style["", z3 = ", Bold], FullSimplify[1 / (f3 - 1)],
    Style["", z4 = ", Bold], FullSimplify[1 / (f4 - 1)]}],
Row[{Style["x1 = ", Bold], FullSimplify[1 / (r1 - 1)], Style["", x2 = ", Bold],
    FullSimplify[1 / (r2 - 1)], Style["", x3 = ", Bold], FullSimplify[1 / (r3 - 1)],
    Style["", x4 = ", Bold], FullSimplify[1 / (r4 - 1)]}],
Row[{Style["y1 = ", Bold], FullSimplify[1 / (s1 - 1)], Style["", y2 = ", Bold],
    FullSimplify[1 / (s2 - 1)], Style["", y3 = ", Bold], FullSimplify[1 / (s3 - 1)],
    Style["", y4 = ", Bold], FullSimplify[1 / (s4 - 1)]}],
Row[{Style["p1 = ", Bold], Simplify[Sqrt[r1 - 1]],
    Style["", p2 = ", Bold], Simplify[Sqrt[r2 - 1]], Style["", p3 = ", Bold],
    Simplify[Sqrt[r3 - 1]], Style["", p4 = ", Bold], Simplify[Sqrt[r4 - 1]]}],
Row[{Style["q1 = ", Bold], Simplify[Sqrt[s1 - 1]],
    Style["", q2 = ", Bold], Simplify[Sqrt[s2 - 1]], Style["", q3 = ", Bold],
    Simplify[Sqrt[s3 - 1]], Style["", q4 = ", Bold], Simplify[Sqrt[s4 - 1]]}],
Row[{Style["p1.q1 = ", Bold], Simplify[Sqrt[r1 - 1] Sqrt[s1 - 1]],
    Style["", p2.q2 = ", Bold], Simplify[Sqrt[r2 - 1] Sqrt[s2 - 1]],
    Style["", p3.q3 = ", Bold], Simplify[Sqrt[r3 - 1] Sqrt[s3 - 1]],
    Style["", p4.q4 = ", Bold], Simplify[Sqrt[r4 - 1] Sqrt[s4 - 1]]}]]
}]

(*=====
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(*=====
BRICARD'S EQUATIONS=====*)
(*=====
=====*)
(*Bricard's equation as defined in terms of angles and variables*)
BricardsEquation[{α_, β_, γ_, δ_}, σ_, x_, y_] := Module[
  {c22, c20, c02, c11, c00},
  c22 = Sin[σ - δ] Sin[σ - δ - β];
  c20 = Sin[σ - α] Sin[σ - α - β];
  c02 = Sin[σ - γ] Sin[σ - γ - β];
  c11 = -Sin[α] Sin[γ];
  c00 = Sin[σ] Sin[σ - β];
  c22 x^2 y^2 + c20 x^2 + c02 y^2 + 2 c11 x y + c00
];

(*Step 1: Bricard's system of equations*)
Column[{(*Header*)TextCell[
  Style["===== Bricard's System of Equations =====",
    Darker[Purple], Bold, 16], "Text"], (*Explanatory note*)
Row[
  {TextCell[Style["We introduce new notation for the cotangents of half of
    the dihedral angles. Denote Z:= ", GrayLevel[0.3], 13],
    "Text"], TraditionalForm[cot[Subscript[θ, 1] / 2]], TextCell[
    Style["", W2= ", GrayLevel[0.3], 13], "Text"],
    TraditionalForm[cot[Subscript[θ, 2] / 2]],
    TextCell[Style["", U:= ", GrayLevel[0.3], 13], "Text"],
    TraditionalForm[cot[Subscript[θ, 3] / 2]],
    TextCell[Style["", and W1= ", GrayLevel[0.3], 13], "Text"],
    TraditionalForm[cot[Subscript[θ, 4] / 2]]
  ]], Spacer[12],
(*Traditional form results*)Row[{"P1(Z, W1) = ",
  TraditionalForm[Collect[FullSimplify[BricardsEquation[anglesDeg[[1]] Degree,
    sigmas[[1]] Degree, Z, W1]], W1]], " = 0"]], Spacer[6],
Row[{"P2(Z, W2) = ",
  TraditionalForm[Collect[FullSimplify[BricardsEquation[anglesDeg[[2]] Degree,
    sigmas[[2]] Degree, Z, W2]], W2]], " = 0"]], Spacer[6],
Row[{"P3(U, W2) = ",
  TraditionalForm[Collect[FullSimplify[BricardsEquation[anglesDeg[[3]] Degree,
    sigmas[[3]] Degree, U, W2]], W2]], " = 0"]], Spacer[6],
Row[{"P4(U, W1) = ", TraditionalForm[Collect[FullSimplify[BricardsEquation[
  anglesDeg[[4]] Degree, sigmas[[4]] Degree, U, W1]], W1]], " = 0"]]}
];
(*=====
FLEXION 1=====*)
Z[t_] := t;

W1[t_] := (2.2689737907253456` (10.50393110877767` t - 2.542115267025096`

```



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$$\frac{\sqrt{(1 - 0.5721009760467486 t^2) (-1 + 7.594436687028424 t^2)}}{(14.662902846473798 + 7.594436687028424 t^2)};$$

U[t_] :=  $\frac{0.20405148639750417 (1.1778865025892307 t + 10.102197432038 \sqrt{(1 - 0.5721009760467486 t^2) (-1 + 7.594436687028424 t^2)})}{(3.6622434357916323 - 2.099237700754583 t^2)};$ 
W2[t_] :=  $\frac{1.3011995890502461 (5.832687748284718 t - 1.6606733509905591 \sqrt{(1 - 0.5721009760467486 t^2) (-1 + 7.594436687028424 t^2)})}{(3.588495042139688 + 7.594436687028424 t^2)};$ 

(*=====
FLEXION 2=====*)
Z2[t_] := t;
W12[t_] :=  $\frac{2.2689737907253456 (10.50393110877767 t + 2.542115267025096 \sqrt{(1 - 0.5721009760467486 t^2) (-1 + 7.594436687028424 t^2)})}{(14.662902846473798 + 7.594436687028424 t^2)};$ 
U2[t_] :=  $\frac{0.20405148639750417 (1.1778865025892307 t - 10.102197432038 \sqrt{(1 - 0.5721009760467486 t^2) (-1 + 7.594436687028424 t^2)})}{(3.6622434357916323 - 2.099237700754583 t^2)};$ 
W22[t_] :=  $\frac{1.3011995890502461 (5.832687748284718 t + 1.6606733509905591 \sqrt{(1 - 0.5721009760467486 t^2) (-1 + 7.594436687028424 t^2)})}{(3.588495042139688 + 7.594436687028424 t^2)};$ 

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(*Step 2: Formulas for flexions*)

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Column[
{(*Header*)TextCell[Style["===== FLEXIONS =====",
  Red, Bold, 16], "Text"], (*Explanatory note*)TextCell[Style[
  "Solutions to Bricard's equations under a free parameter t := Z ∈ ℂ:",
  GrayLevel[0.3], 13], "Text"], Spacer[12],
(*Heading for results*)TextCell[Style["Solution 1:", Bold, 14], "Text"],
Spacer[6], (*Traditional form results*)
Row[{"Z(t) = ", TraditionalForm[FullSimplify[Z[t]]}], Spacer[6],
Row[{"W2(t) = ", TraditionalForm[FullSimplify[W2[t]]}], Spacer[6],
Row[{"U(t) = ", TraditionalForm[FullSimplify[U[t]]}], Spacer[6],
Row[{"W1(t) = ", TraditionalForm[FullSimplify[W1[t]]}], Spacer[12],
(*Heading for results*)TextCell[Style["Solution 2:", Bold, 14], "Text"],
Spacer[6], (*Traditional form results*)
Row[{"Z(t) = ", TraditionalForm[FullSimplify[Z2[t]]}], Spacer[6],
Row[{"W2(t) = ", TraditionalForm[FullSimplify[W22[t]]}], Spacer[6],
Row[{"U(t) = ", TraditionalForm[FullSimplify[U2[t]]}], Spacer[6],
Row[{"W1(t) = ", TraditionalForm[FullSimplify[W12[t]]]}]
}]

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(*Step 3: Checking that  $W_1(t)$  and  $W_2(t)$  solves  $P_1(t, W_1) = 0$ 
and  $P_2(t, W_2) = 0$  even when +- signs do NOT agree*)
(*t-range*)
tMin = 0.4;
tMax = 1.25;

Column[
{(*Header*)TextCell[Style["===== Equations:  $P_1(t, W_1) = 0$ 
and  $P_2(t, W_2) = 0$  =====",
Orange, Bold, 16], "Text"], (*Explanatory note*)TextCell[
Style["Let  $W_{1s1}$  and  $W_{1s2}$  be formulas for  $W_1(t)$  from solutions
1 and 2, respectively. Similarly, let  $W_{2s1}$  and  $W_{2s2}$  be
the formulas for  $W_2(t)$ . We show that all four pairs -
 $(W_{1s1}, W_{2s1})$ ,  $(W_{1s1}, W_{2s2})$ ,  $(W_{1s2}, W_{2s1})$ , and  $(W_{1s2}, W_{2s2})$ 
- solve equations  $P_1(t, W_1) = 0$  and  $P_2(t, W_2) = 0$ .",
GrayLevel[0.3], 13], "Text"], Spacer[12],
(*Heading for results*)

TextCell[Style["Pair 1,  $(W_1, W_2) = (W_{1s1}, W_{2s1})$ :", Bold, 14], "Text"],
Spacer[6], (*Traditional form results*)
Row[{" $P_1(t, W_1) =$ ", TraditionalForm[FullSimplify[BricardsEquation[
anglesDeg[[1] Degree, sigmas[[1] Degree, Z[t], W1[t]]]]]], Spacer[6],
Row[{" $P_2(t, W_2) =$ ", TraditionalForm[FullSimplify[BricardsEquation[
anglesDeg[[2] Degree, sigmas[[2] Degree, Z[t], W2[t]]]]]], Spacer[12],
(*Verification Plots Module: This entire Module is now the last item inside
the main Column. It will execute and place the resulting Panel here. *)
Module[{PiExpr, labels}, (*1. Define the two polynomial expressions
to be plotted*) PiExpr = {FullSimplify[BricardsEquation[
anglesDeg[[1] Degree, sigmas[[1] Degree, Z[t], W1[t]]], FullSimplify[
BricardsEquation[anglesDeg[[2] Degree, sigmas[[2] Degree, Z[t], W2[t]]]]];
(*2. Define the labels for each plot*)
labels = {Row[{Subscript["P", 1], "(t, ", Subscript["W", 1], ")", "  $\approx 0$  ",
Style["✓", Darker[Green], Bold]}], Row[{Subscript["P", 2], "(t, ",
Subscript["W", 2], ")", "  $\approx 0$  ", Style["✓", Darker[Green], Bold]}]};
(*3. Display the plots in the specified panel style*)
Panel[GraphicsGrid[Partition[Table[Plot[PiExpr[[i]], {t, tMin, tMax},
PlotLabel  $\rightarrow$  Style[labels[[i]], Bold, 14], PlotRange  $\rightarrow$  {-1010 - 1010},
(*Zoom in to confirm zero*) AxesLabel  $\rightarrow$  {"t", None}, ImageSize  $\rightarrow$  300],
{i, Length[PiExpr]}], 2 (*Arrange plots in rows of 2*)],
Spacings  $\rightarrow$  {2, 2}], Background  $\rightarrow$  Lighter[Gray, 0.95], FrameMargins  $\rightarrow$  15]],

TextCell[Style["Pair 2,  $(W_1, W_2) = (W_{1s1}, W_{2s2})$ :", Bold, 14], "Text"],
Spacer[6], (*Traditional form results*)
Row[{" $P_1(t, W_1) =$ ", TraditionalForm[FullSimplify[BricardsEquation[
anglesDeg[[1] Degree, sigmas[[1] Degree, Z[t], W1[t]]]]]], Spacer[6],
Row[{" $P_2(t, W_2) =$ ", TraditionalForm[FullSimplify[BricardsEquation[

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    anglesDeg[[2]] Degree, sigmas[[2]] Degree, Z[t], W22[t]]]]], Spacer[12],
(*Verification Plots Module:This entire Module is now the last item inside
the main Column.It will execute and place the resulting Panel here.*)
Module[{PiExpr, labels}, (*1. Define the two polynomial expressions
to be plotted*)PiExpr = {FullSimplify[BricardsEquation[
    anglesDeg[[1]] Degree, sigmas[[1]] Degree, Z[t], W1[t]], FullSimplify[
    BricardsEquation[anglesDeg[[2]] Degree, sigmas[[2]] Degree, Z[t], W22[t]]]];
(*2. Define the labels for each plot*)
labels = {Row[{Subscript["P", 1], "(t, ", Subscript["W", 1], ")", " ≈ 0 ",
    Style["✓", Darker[Green], Bold]}], Row[{Subscript["P", 2], "(t, ",
    Subscript["W", 2], ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}]};
(*3. Display the plots in the specified panel style*)
Panel[GraphicsGrid[Partition[Table[Plot[PiExpr[[i]], {t, tMin, tMax},
    PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → {-10^-10, 10^-10},
    (*Zoom in to confirm zero*)AxesLabel → {"t", None}, ImageSize → 300],
    {i, Length[PiExpr]}], 2 (*Arrange plots in rows of 2*)],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]],

TextCell[Style["Pair 3, (W1, W2) = (W1;2, W2;1):", Bold, 14], "Text"],
Spacer[6], (*Traditional form results*)
Row[{"P1(t, W1) = ", TraditionalForm[FullSimplify[BricardsEquation[
    anglesDeg[[1]] Degree, sigmas[[1]] Degree, Z[t], W12[t]]]]], Spacer[6],
Row[{"P2(t, W2) = ", TraditionalForm[FullSimplify[BricardsEquation[
    anglesDeg[[2]] Degree, sigmas[[2]] Degree, Z[t], W2[t]]]]], Spacer[12],
(*Verification Plots Module:This entire Module is now the last item inside
the main Column.It will execute and place the resulting Panel here.*)
Module[{PiExpr, labels}, (*1. Define the two polynomial expressions
to be plotted*)PiExpr = {FullSimplify[BricardsEquation[
    anglesDeg[[1]] Degree, sigmas[[1]] Degree, Z[t], W12[t]], FullSimplify[
    BricardsEquation[anglesDeg[[2]] Degree, sigmas[[2]] Degree, Z[t], W2[t]]]];
(*2. Define the labels for each plot*)
labels = {Row[{Subscript["P", 1], "(t, ", Subscript["W", 1], ")", " ≈ 0 ",
    Style["✓", Darker[Green], Bold]}], Row[{Subscript["P", 2], "(t, ",
    Subscript["W", 2], ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}]};
(*3. Display the plots in the specified panel style*)
Panel[GraphicsGrid[Partition[Table[Plot[PiExpr[[i]], {t, tMin, tMax},
    PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → {-10^-10, 10^-10},
    (*Zoom in to confirm zero*)AxesLabel → {"t", None}, ImageSize → 300],
    {i, Length[PiExpr]}], 2 (*Arrange plots in rows of 2*)],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]],

TextCell[Style["Pair 4, (W1, W2) = (W1;2, W2;2):", Bold, 14], "Text"],
Spacer[6], (*Traditional form results*)
Row[{"P1(t, W1) = ", TraditionalForm[FullSimplify[BricardsEquation[
    anglesDeg[[1]] Degree, sigmas[[1]] Degree, Z[t], W12[t]]]]], Spacer[6],
Row[{"P2(t, W2) = ", TraditionalForm[FullSimplify[BricardsEquation[
    anglesDeg[[2]] Degree, sigmas[[2]] Degree, Z[t], W22[t]]]]], Spacer[12],

```

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(*Verification Plots Module:This entire Module is now the last item inside
the main Column.It will execute and place the resulting Panel here.*)
Module[{PiExpr, labels}, (*1. Define the two polynomial expressions
to be plotted*)PiExpr = {FullSimplify[BricardsEquation[
  anglesDeg[[1]] Degree, sigmas[[1]] Degree, Z[t], W12[t]], FullSimplify[
  BricardsEquation[anglesDeg[[2]] Degree, sigmas[[2]] Degree, Z[t], W22[t]]];
(*2. Define the labels for each plot*)
labels = {Row[{Subscript["P", 1], "(t, ", Subscript["W", 1], ") ", " ≈ 0 ",
  Style["✓", Darker[Green], Bold]}], Row[{Subscript["P", 2], "(t, ",
  Subscript["W", 2], ") ", " ≈ 0 ", Style["✓", Darker[Green], Bold]}]};
(*3. Display the plots in the specified panel style*)
Panel[GraphicsGrid[Partition[Table[Plot[PiExpr[[i]], {t, tMin, tMax},
  PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → {-10^-10, 10^-10},
  (*Zoom in to confirm zero*)AxesLabel → {"t", None}, ImageSize → 300],
  {i, Length[PiExpr]}], 2 (*Arrange plots in rows of 2*),
  Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]]
]]

(*Step 4: Checking that  $W_1(t)$  and  $W_2(t)$  does NOT satisfy  $W_1(t)/W_2(t) =$ 
c and  $W_1(t)W_2(t) = c$  where c is constant even when +- signs do NOT agree*)
expressions = {Z[t] * U[t], Z[t] / U[t], Z[t] * U2[t], Z[t] / U2[t],
  Z2[t] * U[t], Z2[t] / U[t], Z2[t] * U2[t], Z2[t] / U2[t]};
labels = {"W1s1 * W2s1", "W1s1 / W2s1", "W1s1 * W2s2", "W1s1 / W2s2",
  "W1s2 * W2s1", "W1s2 / W2s1", "W1s2 * W2s2", "W1s2 / W2s2"};
Column[
  {TextCell[Style["===== Plots of  $W_1(t)W_2(t)$  and  $W_1(t)/W_2(t)$  For All
    Pairs =====", Darker[Orange], Bold, 16], "Text"],

  (*Explanatory text*)
  TextCell[Style["Checking that all four pairs - ( $W_{1s1}, W_{2s1}$ ), ( $W_{1s1}, W_{2s2}$ ),
    ( $W_{1s2}, W_{2s1}$ ), and ( $W_{1s2}, W_{2s2}$ ) - does NOT satisfy  $W_1(t)/W_2(t)$ 
    = const and  $W_1(t)W_2(t) = \text{const.}$ ", GrayLevel[0.3]], "Text"],
  Spacer[12],

  (*Plots panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions[[i]], {t, tMin, tMax},
    PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → All,
    AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]]
]]

(*Step 5: Solving  $P_3(U, W_2) = 0$  and  $P_4(U, W_1) = 0$  for  $W_1$  and  $W_2$ *)
(*Define the first Bricard equation  $P_3(U, W_2) = 0$ *)
P3 = BricardsEquation[anglesDeg[[3]] Degree, sigmas[[3]] Degree, U, Ww2];
(*Solve for W21,W22*)

```

```

solutionW2Expressions =
  (Ww2 /. FullSimplify[Solve[Rationalize[P3, 0] == 0, Ww2]]);
Ww21[U_] := Simplify[solutionW2Expressions[[1]] // N];
Ww22[U_] := Simplify[solutionW2Expressions[[2]] // N];

(*Define the fourth Bricard equation  $P_4(U, W_1) = 0$ *)
P4 = BricardsEquation[anglesDeg[[4]] Degree, sigmas[[4]] Degree, U, Ww1];
(*Solve for W11,W12*)
solutionW1Expressions =
  (Ww1 /. FullSimplify[Solve[Rationalize[P4, 0] == 0, Ww1]]);
Ww11[U_] := Simplify[solutionW1Expressions[[1]] // N];
Ww12[U_] := Simplify[solutionW1Expressions[[2]] // N];

Column[
  {(*Header*)TextCell[Style["===== Equations:  $P_3(U, W_2) = 0$ 
    and  $P_4(U, W_1) = 0$  =====",
    Darker[Cyan], Bold, 16], "Text"), (*Explanatory note*)
    TextCell[Style["We solve  $P_3(U, W_2) = 0$  and  $P_4(U, W_1) = 0$  for  $W_1$  and  $W_2$ .",
    GrayLevel[0.3], 13], "Text"), Spacer[12],
    (*Heading for results*)Row[{" $W_{1,1}(U) =$ ", TraditionalForm[Ww11[U]]}],
    Row[{" $W_{1,2}(U) =$ ", TraditionalForm[Ww12[U]]}],
    Spacer[6], Row[{" $W_{2,1}(U) =$ ", TraditionalForm[Ww21[U]]}],
    Row[{" $W_{2,2}(U) =$ ", TraditionalForm[Ww22[U]]}]
  ]
}

(*Step 6: Checking that pairs  $(W_{1,1}(U), W_{2,1}(U))$ ,  $(W_{1,1}(U), W_{2,2}(U))$ ,
 $(W_{1,2}(U), W_{2,1}(U))$ , and  $(W_{1,2}(U), W_{2,2}(U))$  does NOT satisfy  $W_1(U)/W_2(U) =$ 
c and  $W_1(U)W_2(U) = c$  where c is constant*)
UMin = 0;
UMax = 1;

expressions =
  {Ww11[U] * Ww21[U], Ww11[U] / Ww21[U], Ww11[U] * Ww22[U], Ww11[U] / Ww22[U],
    Ww12[U] * Ww21[U], Ww12[U] / Ww21[U], Ww12[U] * Ww22[U], Ww12[U] / Ww22[U]};
labels = {" $W_{1,1} * W_{2,1}$ ", " $W_{1,1} / W_{2,1}$ ", " $W_{1,1} * W_{2,2}$ ",
  " $W_{1,1} / W_{2,2}$ ", " $W_{1,2} * W_{2,1}$ ", " $W_{1,2} / W_{2,1}$ ", " $W_{1,2} * W_{2,2}$ ", " $W_{1,2} / W_{2,2}$ "};
Column[
  {TextCell[Style["===== Plots of  $W_1(U)W_2(U)$  and  $W_1(U)/W_2(U)$  For All
    Pairs =====", Magenta, Bold, 16], "Text"],

    (*Explanatory text*)
    TextCell[Style["Checking that all four pairs -  $(W_{1,1}(U),$ 
       $W_{2,1}(U))$ ,  $(W_{1,1}(U), W_{2,2}(U))$ ,  $(W_{1,2}(U), W_{2,1}(U))$ , and
       $(W_{1,2}(U), W_{2,2}(U))$  - does NOT satisfy  $W_1(U)/W_2(U) = \text{const}$ 
      and  $W_1(U)W_2(U) = \text{const}$ .", GrayLevel[0.3]], "Text"],
    Spacer[12],

```

```

(*Plots panel*)
Panel[GraphicsGrid[Partition[Table[Plot[expressions[[i]], {U, UMin, UMax},
  PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → All,
  AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
  Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
}]

(*Step 7: Compute and print all P_i for flexions 1 and 2*)
TextCell[
  Style["===== FLEXIBILITY (Double Checking) =====",
    Orange, Bold, 16], "Text"]
funcs = {{Z, W1}, {Z, W2}, {U, W2}, {U, W1}};
TextCell[Style["Solution 1:", Bold, 14], "Text"]
Do[angles = anglesDeg[[i]] Degree;
  sigma = sigmas[[i]] Degree;
  {α, β, γ, δ} = angles;
  poly = Which[i == 1, BricardsEquation[{α, β, γ, δ}, sigma, Z[t], W1[t]],
    i == 2, BricardsEquation[{α, β, γ, δ}, sigma, Z[t], W2[t]],
    i == 3, BricardsEquation[{α, β, γ, δ}, sigma, U[t], W2[t]],
    i == 4, BricardsEquation[{α, β, γ, δ}, sigma, U[t], W1[t]]];
  Print[Row[{Subscript["P", i], "("}, funcs[[i, 1],
    ", ", funcs[[i, 2]], ") = ", FullSimplify[poly]]], {i, 1, 4}];

(*Build P_i(t) consistently with your printing logic*)
Clear[PiExpr, labels, funcs];
funcs = {{Z, W1}, {Z, W2}, {U, W2}, {U, W1}};

PiExpr = Table[Module[{angles = anglesDeg[[i]] Degree,
  sigma = sigmas[[i]] Degree, α, β, γ, δ, poly}, {α, β, γ, δ} = angles;
  poly = BricardsEquation[{α, β, γ, δ}, sigma, funcs[[i, 1]][t], funcs[[i, 2]][t]];
  FullSimplify[poly]], {i, 1, 4}];

labels = Table[Row[
  {Subscript["P", i], "("}, ToString@funcs[[i, 1], ", ", ToString@funcs[[i, 2],
  ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]]], {i, 1, 4}];

(*Pretty panel with plots like your style*)
Column[{TextCell[Style["===== FLEXION 1 =====",
  Darker[Cyan], Bold, 16], "Text"], TextCell[
  Style["Polynomials P_i(t) built from Bricard's equations for flexion 1.",
  GrayLevel[0.3]], "Text"], Spacer[12],
  Panel[GraphicsGrid[Partition[Table[Plot[Evaluate@PiExpr[[i]],
    {t, tMin, tMax}, PlotLabel → Style[labels[[i]], Bold, 14],
    PlotRange → {-10^(-11), 10^(-11)}, AxesLabel → {"t", None},
    ImageSize → 250], {i, Length[PiExpr]}], 2], Spacings → {2, 2}],
  Background → Lighter[Gray, 0.95], FrameMargins → 15]]]

```

```

(*Compute and print all P_i for flexion 2*)
TextCell[Style["Solution 2:", Bold, 14], "Text"]
funcs = {{Z, W1}, {Z, W2}, {U, W2}, {U, W1}};
Do[angles = anglesDeg[[i]] Degree;
  sigma = sigmas[[i]] Degree;
  {α, β, γ, δ} = angles;
  poly = Which[i == 1, BricardsEquation[{α, β, γ, δ}, sigma, Z2[t], W12[t]],
    i == 2, BricardsEquation[{α, β, γ, δ}, sigma, Z2[t], W22[t]],
    i == 3, BricardsEquation[{α, β, γ, δ}, sigma, U2[t], W22[t]],
    i == 4, BricardsEquation[{α, β, γ, δ}, sigma, U2[t], W12[t]]];
  Print[Row[{Subscript["P", i], "(", funcs[[i, 1],
    " ", " ", funcs[[i, 2]], ") = ", FullSimplify[poly]]}], {i, 1, 4}];

(*Build P_i(t) consistently with your printing logic*)
Clear[PiExpr, labels, funcs];
funcs = {{Z2, W12}, {Z2, W22}, {U2, W22}, {U2, W12}};

PiExpr = Table[Module[{angles = anglesDeg[[i]] Degree,
  sigma = sigmas[[i]] Degree, α, β, γ, δ, poly}, {α, β, γ, δ} = angles;
  poly = BricardsEquation[{α, β, γ, δ}, sigma, funcs[[i, 1]][t], funcs[[i, 2]][t]];
  FullSimplify[poly]], {i, 1, 4}];

labels = Table[Row[
  {Subscript["P", i], "(", ToString@funcs[[i, 1], " ", " ", ToString@funcs[[i, 2],
    ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}], {i, 1, 4}];

(*Pretty panel with plots like your style*)
Column[{TextCell[Style["===== FLEXION 2 =====",
  Magenta, Bold, 16], "Text"], TextCell[
  Style["Polynomials Pi(t) built from Bricard's equations for flexion 2.",
  GrayLevel[0.3]], "Text"], Spacer[12],
  Panel[GraphicsGrid[Partition[Table[Plot[Evaluate@PiExpr[[i]],
    {t, tMin, tMax}, PlotLabel → Style[labels[[i]], Bold, 14],
    PlotRange → {-10-11, 10-11}, AxesLabel → {"t", None},
    ImageSize → 250], {i, Length[PiExpr]}], 2], Spacings → {2, 2}],
  Background → Lighter[Gray, 0.95], FrameMargins → 15]]]

(*=====
====*)
(*=====NOT LINEAR COMPOUND=====*)
(*=====
====*)
(*{TraditionalForm[cot[Subscript[θ,1]/2]],
  TraditionalForm[cot[Subscript[θ,2]/2]],
  TraditionalForm[cot[Subscript[θ,3]/2]],

```

```

TraditionalForm[cot[Subscript[ $\theta$ ,4]/2]]];*)

(*=====
FLEXION 1=====*)
(*List of expressions& labels*)
expressions =
{Z[t] * U[t], Z[t] / U[t], W1[t] * W2[t], W1[t] / W2[t], Z[t] * W2[t], Z[t] / W2[t],
U[t] * W1[t], U[t] / W1[t], Z[t] * W1[t], Z[t] / W1[t], W2[t] * U[t], W2[t] / U[t]};
labels = {"Z * U", "Z / U", "W1 * W2", "W1 / W2", "Z * W2", "Z / W2",
"U * W1", "U / W1", "Z * W1", "Z / W1", "W2 * U", "W2 / U"};

Column[{TextCell[Style["===== NOT LINEAR COMPOUND =====",
Darker[Brown], Bold, 16], "Text"],
(*Explanatory text*)TextCell[
Style["Above we consider the first pair of equations ( $P_1(t, W_1) = 0$  and
 $P_2(t, W_2) = 0$ ). Solving them as quadratic equations in  $W_1$  and  $W_2$ ,
respectively we parametrize the solutions by the first two and
fourth expressions in Solutions 1 and 2 in a neighborhood of any
point  $(W_1, t, W_2)$  such that the expression in the square root and
denominators are not zero. Here, we choose any continuous branch
of the square root in this neighborhood, and the signs in  $\pm$  need
Not agree (this means we consider all 4 pairs we describe above).
We conclude that NO component of the solution set of the first
pair of Bricard's equations satisfies  $W_1/W_2=\text{const}$  NOR  $W_1W_2=\text{const}$ .

Analogously, NO component of the solution set of the other pair
of equations ( $P_3(U, W_2) = 0$  and  $P_4(U, W_1) = 0$ ) satisfies
 $W_1/W_2=\text{const}$  NOR  $W_1W_2=\text{const}$ . As a result, NO component
of the solution set of all four equations satisfies
 $W_1/W_2=\text{const}$  NOR  $W_1W_2=\text{const}$ . So, our example does not belong
to the linear compound class, even after switching the
boundary strips.", GrayLevel[0.3]], "Text"],
Spacer[6],
TextCell[Style[
"Below, we also present the plots of functions  $ZU, Z/U, W_1W_2, W_1/W_2, ZW_2,$ 
 $Z/W_2, UW_1, U/W_1, ZW_1, Z/W_1, W_2U, W_2/U$ .", GrayLevel[0.3]], "Text"],
Spacer[6],
TextCell[Style["Solution 1:", Bold, 14], "Text"], Spacer[12],
Panel[GraphicsGrid[Partition[Table[Plot[expressions[[i]], {t, tMin, tMax},
PlotLabel  $\rightarrow$  Style[labels[[i]], Bold, 14], PlotRange  $\rightarrow$  All,
AxesLabel  $\rightarrow$  {"t", None}, ImageSize  $\rightarrow$  250], {i, Length[expressions]}], 2],
Spacings  $\rightarrow$  {2, 2}], Background  $\rightarrow$  Lighter[Gray, 0.95], FrameMargins  $\rightarrow$  15]
]]

(*=====
FLEXION 2=====*)

```



```

(*List of expressions& labels*)
expressions = {Z2[t] * U2[t], Z2[t] / U2[t], W12[t] * W22[t], W12[t] / W22[t],
  Z2[t] * W22[t], Z2[t] / W22[t], U2[t] * W12[t], U2[t] / W12[t],
  Z2[t] * W12[t], Z2[t] / W12[t], W22[t] * U2[t], W22[t] / U2[t]};
labels = {"Z * U", "Z / U", "W1 * W2", "W1 / W2", "Z * W2", "Z / W2",
  "U * W1", "U / W1", "Z * W1", "Z / W1", "W2 * U", "W2 / U"};

Column[{ TextCell[Style["Solution 2:", Bold, 14], "Text"], Spacer[12],
  (*Plots panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions[[i]], {t, tMin, tMax},
    PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → All,
    AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
  ]],

(*=====*)
(*=====*)
NOT TRIVIAL=====*)
(*=====*)
(*=====*)
FLEXION 1=====*)
(*List of expressions to plot*)
expressions = {Z[t], W2[t], U[t], W1[t]};
labels = {"Z", "W2", "U", "W1"};

Column[{TextCell[
  Style["===== NOT TRIVIAL (FLEXION 1) =====",
    Pink, Bold, 16], "Text"],

  (*Explanatory text*)
  TextCell[Style["This configuration does not belong to the trivial class –
    even after switching the boundary strips – since none of the
    functions Z, W2, U, or W1 is constant.", GrayLevel[0.3]], "Text"],
  Spacer[12],

  (*Plots in a light panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions[[i]], {t, tMin, tMax},
    PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → All,
    AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
  ]],

(*=====*)
FLEXION 2=====*)

```

```

(*List of expressions to plot*)
expressions = {Z2[t], W22[t], U2[t], W12[t]};
labels = {"Z", "W2", "U", "W1"};

Column[{TextCell[
  Style["===== NOT TRIVIAL (FLEXION 2) =====",
    Darker[Pink], Bold, 16], "Text"],

  (*Explanatory text*)
  TextCell[Style["This configuration does not belong to the trivial class –
    even after switching the boundary strips – since none of the
    functions Z, W2, U, or W1 is constant.", GrayLevel[0.3]], "Text"],
  Spacer[12],

  (*Plots in a light panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions[[i]], {t, tMin, tMax},
    PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → All,
    AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]

  ]}]

(*=====*)
(*=====
SWITCHING BOUNDARY STRIPS=====*)
(*=====*)

SwitchingRightBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices: {row, column} = {1, 2}, {1, 3}, {4, 2}, {4, 3}*)
  modified[[1, 2]] = 180 - anglesDeg[[1, 2]]; (*β1*)
  modified[[1, 3]] = 180 - anglesDeg[[1, 3]]; (*γ1*)
  modified[[4, 2]] = 180 - anglesDeg[[4, 2]]; (*β4*)
  modified[[4, 3]] = 180 - anglesDeg[[4, 3]]; (*γ4*)
  modified]

SwitchingLeftBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices: {row, column} = {2, 2}, {2, 3}, {3, 2}, {3, 3}*)
  modified[[2, 2]] = 180 - anglesDeg[[2, 2]]; (*β2*)
  modified[[2, 3]] = 180 - anglesDeg[[2, 3]]; (*γ2*)
  modified[[3, 2]] = 180 - anglesDeg[[3, 2]]; (*β3*)
  modified[[3, 3]] = 180 - anglesDeg[[3, 3]]; (*γ3*)
  modified]

SwitchingLowerBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},

```

```

(*Indices:{row,column}={1,1},{1,2},{2,1},{2,2}*)
modified[[1, 1]] = 180 - anglesDeg[[1, 1]]; (* $\alpha_1$ *)
modified[[1, 2]] = 180 - anglesDeg[[1, 2]]; (* $\beta_1$ *)
modified[[2, 1]] = 180 - anglesDeg[[2, 1]]; (* $\alpha_2$ *)
modified[[2, 2]] = 180 - anglesDeg[[2, 2]]; (* $\beta_2$ *)
modified]

SwitchingUpperBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices:{row,column}={3,1},{3,2},{4,1},{4,2}*)
  modified[[3, 1]] = 180 - anglesDeg[[3, 1]]; (* $\alpha_3$ *)
  modified[[3, 2]] = 180 - anglesDeg[[3, 2]]; (* $\beta_3$ *)
  modified[[4, 1]] = 180 - anglesDeg[[4, 1]]; (* $\alpha_4$ *)
  modified[[4, 2]] = 180 - anglesDeg[[4, 2]]; (* $\beta_4$ *)
  modified]

(*=====
====*)
(*=====NOT CONIC & NOT CHIMERA & NOT LINEAR
CONJUGATE & NOT ISOGONAL=====*)
(*=====
====*)
(*Uncomment this part for the detailed print*)
(*uniqueCombos={{1,1,1,1},{1,1,1,-1},{1,1,-1,-1},
{1,1,-1,1},{1,-1,1,1},{1,-1,-1,1},{1,-1,1,-1},{1,-1,-1,-1}};

(*checkConditionN0Degrees[{ $\alpha$ _, $\beta$ _, $\gamma$ _, $\delta$ _}]:=Module[
  {angles={ $\alpha$ , $\beta$ , $\gamma$ , $\delta$ },results},results=Mod[uniqueCombos.angles,360]//Chop;
  !MemberQ[results,0]]];*)

checkConditionN0Degrees[{ $\alpha$ _, $\beta$ _, $\gamma$ _, $\delta$ _}]:=Module[
  {angles={ $\alpha$ , $\beta$ , $\gamma$ , $\delta$ },results},results=Mod[uniqueCombos.angles,360]//Chop;
  results];

Column[{TextCell[Style[
  "===== NOT CONIC & NOT CHIMERA & NOT LINEAR CONJUGATE & NOT
  ISOGONAL=====",Darker[Magenta],Bold,16],"Text"],
TextCell[Style["Condition (N.0) is satisfied for all i=1,...,4  $\Rightarrow$  NOT
  equimodular-conic, NOT chimera, NOT isogonal and NOT
  linear conjugate. Applying any boundary-strip switch still
  preserves (N.0), so no conic, no chimera, no isogonal and
  no linear conjugate form emerges.", GrayLevel[0.3]],"Text"]
}],

(*Now the exact same Module for checking all switch combinations...*)
Module[{angles=anglesDeg,switchers,combinations,results},
  (*Define switch functions*)switchers=<|"Right"→SwitchingRightBoundaryStrip,
```

```

"Left"→SwitchingLeftBoundaryStrip,"Lower"→SwitchingLowerBoundaryStrip,
"Upper"→SwitchingUpperBoundaryStrip|>;
(*Generate all combinations of switches (from size 1 to 4)*)
combinations=Subsets[Keys[switchers],{1,Length[switchers]}];
(*Evaluate condition after each combination of switches*)
results=Table[Module[{switched=angles,name,passQ,
  resultsPerVertex,conditionsN0},name=StringRiffle[combo," + "];
Do[switched=switchers[sw][switched],{sw,combo}];
resultsPerVertex=checkConditionN0Degrees/@switched;
conditionsN0=Map[!MemberQ[#,0]&,resultsPerVertex];
passQ=And@@conditionsN0;
(*Print the result after switching*)
(*Print["\nSwitch combination: ",name];
Print["Switched anglesDeg:"];
Print[MatrixForm[switched]]];*)
{name,passQ, resultsPerVertex,conditionsN0}},{combo,combinations}];
(*Display results*)Column[Prepend[
Table[Module[{comboName=res[[1]],passQ=res[[2]], resultsPerVertex=res[[3]],
conditionsN0=res[[4]]},Column[{Style[comboName<>": ",Bold],
If[passQ,
Style["Condition (N.0) is still satisfied.",Darker[Green]],
Style["Condition (N.0) fails.",Red,Bold]
],
Grid[Prepend[Table[{"Vertex "<>ToString[i],resultsPerVertex[[i]],
If[conditionsN0[[i]],"✓ Pass","✗ Fail"]},{i,Length[anglesDeg]}],
{"Vertex","Combinations (mod 360)","Status"}],Frame→All]
}
]],{res,results}],TextCell[Style[
"CONDITION (N.0) AFTER SWITCHING BOUNDARY STRIPS",14],"Text"]]]]*)

(*Comment this part during the detailed print*)
uniqueCombos = {{1, 1, 1, 1}, {1, 1, 1, -1}, {1, 1, -1, -1}, {1, 1, -1, 1},
{1, -1, 1, 1}, {1, -1, -1, 1}, {1, -1, 1, -1}, {1, -1, -1, -1}};

checkConditionN0Degrees[{ $\alpha$ _,  $\beta$ _,  $\gamma$ _,  $\delta$ _}] :=
Module[{angles = { $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ }, results},
results = Mod[uniqueCombos.angles, 360] // Chop;
!MemberQ[results, 0]];

Column[{TextCell[Style[
"===== NOT CONIC & NOT CHIMERA & NOT LINEAR CONJUGATE & NOT
ISOGONAL=====", Darker[Magenta], Bold, 16], "Text"],
TextCell[Style[
"Condition (N.0) is satisfied for all i=1,...,4  $\Rightarrow$  NOT equimodular-conic,
NOT chimera, NOT isogonal and NOT linear conjugate.
Applying any boundary-strip switch still preserves

```

```

(N.0), so no conic, no chimera, no isogonal and no
linear conjugate form emerges.", GrayLevel[0.3]], "Text"]
}]

(*Now the exact same Module for checking all switch combinations...*)
Module[{angles = anglesDeg, switchers, combinations, results},
  (*Define switch functions*)
  switchers = <|"Right" → SwitchingRightBoundaryStrip,
    "Left" → SwitchingLeftBoundaryStrip, "Lower" → SwitchingLowerBoundaryStrip,
    "Upper" → SwitchingUpperBoundaryStrip|>;
  (*Generate all combinations of switches (from size 1 to 4)*)
  combinations = Subsets[Keys[switchers], {1, Length[switchers]}];
  (*Evaluate condition after each combination of switches*) results = Table[
    Module[{switched = angles, name, passQ}, name = StringRiffle[combo, " + "];
    Do[switched = switchers[sw][switched], {sw, combo}];
    passQ = And@@ (checkConditionN0Degrees /@ switched);
    (*Print the result after switching*)
    (*Print["\nSwitch combination: ", name];
    Print["Switched anglesDeg:"];
    Print[MatrixForm[switched]]];*)
    {name, passQ}], {combo, combinations}];
  (*Display results*)
  Column[Prepend[Table[Module[{comboName = res[[1]], passQ = res[[2]]},
    Row[{Style[comboName <> ": ", Bold],
      If[passQ,
        Style["Condition (N.0) is still satisfied.", Darker[Green]],
        Style["Condition (N.0) fails.", Red, Bold]
      ]
    }, {res, results}], TextCell[
    Style["CONDITION (N.0) AFTER SWITCHING BOUNDARY STRIPS", 14], "Text"]]]]

(*=====
====*)
(*=====
NOT ORTHODIAGONAL=====*)
(*=====
====*)

(*Column[
  {TextCell[Style["===== NOT ORTHODIAGONAL =====",
    Darker[Blue], Bold, 16], "Text"],
  TextCell[Style[
    "cos(αi) · cos(γi) ≠ cos(βi) · cos(δi) for each i = 1 ⇒ NOT orthodiagonal.
    Switching boundary strips does not
    correct this.", GrayLevel[0.3]], "Text"]

```

```

}]

Module[{angles=anglesDeg,switchers,combinations,results},
  (*Define switch functions*)switchers=<|"Right"→SwitchingRightBoundaryStrip,
    "Left"→SwitchingLeftBoundaryStrip,"Lower"→SwitchingLowerBoundaryStrip,
    "Upper"→SwitchingUpperBoundaryStrip|>;
  (*Helper function:compute and print difference only*)
  formatOrthodiagonalCheck[quad_List]:=
    Module[{vals},vals=Table[Module[{a,b,c,d,lhs,rhs,diff},{a,b,c,d}=quad[[i]];
      lhs=FullSimplify[Cos[a Degree] Cos[c Degree]];
      rhs=FullSimplify[Cos[b Degree] Cos[d Degree]];
      diff=Chop[lhs-rhs];
      Style[Row[{"cos(α">ToString[i]<>")·cos(γ">ToString[i]<>") - ",
        "cos(β">ToString[i]<>")·cos(δ">ToString[i]<>") = ",
        NumberForm[diff,{5,3}]]],If[diff==0,Red,Black]]],{i,Length[quad]}};
    Column[vals]];
  (*Orthodiagonal check for anglesDeg before any switching*)
  Print[TextCell[Style["\nInitial anglesDeg (no switches):",Bold]]];
  Print[MatrixForm[angles]];
  Print[TextCell[Style[
    "Orthodiagonal check: cos(αi)·cos(γi) - cos(βi)·cos(δi) for i = 1..4",
    Italic]]];
  Print[formatOrthodiagonalCheck[angles]];
  (*Generate all combinations of switches (from size 1 to 4)*)
  combinations=Subsets[Keys[switchers],{1,Length[switchers]}];
  (*Evaluate condition after each combination of switches*)results=
    Table[Module[{switched=angles,name,passQ},name=StringRiffle[combo," + "];
      Do[switched=switchers[sw][switched],{sw,combo}];
      passQ=And@@(checkConditionN0Degrees/@switched);
      Print[Style["\nSwitch combination: ", Bold],name];
      Print[Style["Switched anglesDeg:", Italic]];
      Print[MatrixForm[switched]];
      Print[
        TextCell[Style["Orthodiagonal check: cos(αi)·cos(γi) - cos(βi)·cos(δi)
          for i = 1..4",Italic]]];
      Print[formatOrthodiagonalCheck[switched]];
      {name,passQ}],{combo,combinations}];]*)

Column[
  {TextCell[Style["===== ORTHOGONALITY CHECK =====",
    Brown, Bold, 16], "Text"],
    TextCell[Style["cos(αi)·cos(γi) ≠ cos(βi)·cos(δi) for at least
      one i = 1,..., 4 ⇒ NOT orthodiagonal. Switching boundary
      strips does not correct this.", GrayLevel[0.3]], "Text"]}]

(*Helper

```

```

function>Returns True if at least one cosine product difference is non-
zero.Returns False if all differences are zero.
isNotOrthodiagonal[quad_List] :=
  Or@@Table[Module[{a, b, c, d, diff}, {a, b, c, d} = quad[[i]];
    diff = Chop[Cos[a Degree] Cos[c Degree] - Cos[b Degree] Cos[d Degree]];
    diff ≠ 0], {i, Length[quad]}];

(*First,check the initial,unswitched angles*)
Print[TextCell[Style["\nInitial anglesDeg (no switches):", Bold]]];
If[isNotOrthodiagonal[anglesDeg], Print[Style[
  " -> Condition met: At least one difference is non-zero.", Darker@Green]],
  Print[Style[" -> Condition NOT met: All differences are zero.", Red]]];

(*Now,use your desired module to check all switch combinations*)
Module[{angles = anglesDeg, switchers, combinations, results},
  (*Define switch functions*)
  switchers = <|"Right" → SwitchingRightBoundaryStrip,
    "Left" → SwitchingLeftBoundaryStrip, "Lower" → SwitchingLowerBoundaryStrip,
    "Upper" → SwitchingUpperBoundaryStrip|>;
  (*Generate all combinations of switches (from size 1 to 4)*)
  combinations = Subsets[Keys[switchers], {1, Length[switchers]}];
  (*Evaluate condition after each combination
  of switches and store in'results'*)results = Table[
    Module[{switched = angles, name, passQ}, name = StringRiffle[combo, " + "];
    Do[switched = switchers[sw][switched], {sw, combo}];
    (* ***THIS IS THE KEY CHANGE*****) (*Set passQ using our
    new helper function*)passQ = isNotOrthodiagonal[switched];
    {name, passQ}], {combo, combinations}];
  (*Display results in the specified column format*)
  Column[Prepend[Table[Module[{comboName = res[[1]], passQ = res[[2]]},
    Row[{Style[comboName <> ": ", Bold], If[passQ,
      Style["Condition met (at least one difference is non-zero).", Darker[
        Green]], Style["Condition NOT met (all differences are zero).",
        Red, Bold]}]]], {res, results}], TextCell[
    Style["\nNON-ORTHOGONALITY CHECK AFTER SWITCHING BOUNDARY STRIPS", 14],
    "Text"]]]];

(*=====
====*)
(*=====
NOT CONJUGATE-MODULAR=====*)
(*=====
====*)
(*Column[{TextCell[
  Style["===== NOT CONJUGATE-MODULAR =====",

```

```

    Purple,Bold,16], "Text"],
    TextCell[Style["M1 = M2 = M3 = M4 = M
    and M  $\neq$  2  $\Rightarrow$  NOT conjugate-modular. Boundary-strip
    switches preserve this.", GrayLevel[0.3]], "Text"]
  ]]
  Ms=FullSimplify[Times@@@results];
  allEqualQ=Simplify[Equal@@Ms];

Module[{angles=anglesDeg,switchers,combinations,results,
  computeConjugateModularInfo},(*Define switch functions*)
  switchers=<|"Right"→SwitchingRightBoundaryStrip,
    "Left"→SwitchingLeftBoundaryStrip,"Lower"→SwitchingLowerBoundaryStrip,
    "Upper"→SwitchingUpperBoundaryStrip|>;
  (*Computes Mi and pi and prints them,
  with classification*)computeConjugateModularInfo[quad_List]:=
  Module[{abcdList,Ms,summary},abcdList=computeABCD/@quad;
  Ms=FullSimplify[Times@@@abcdList];
  summary=If[Simplify[Equal@@Ms]&&Ms[[1]]!=2,
    Style["M1 = M2 = M3 = M4 = M and M  $\neq$  2",Bold],
    Style["M1 = M2 = M3 = M4 = M and M = 2",Red,Bold]];
  Column[{Style["Mi values:",Bold],Row[{"M1 = ",Ms[[1]],
    ", M2 = ",Ms[[2]],", M3 = ",Ms[[3]],", M4 = ",Ms[[4]]},summary]}}];
  (*Original anglesDeg check*)
  Print[
    TextCell[Style["\nInitial configuration (no switches applied):",Bold]]];
  Print[MatrixForm[angles]];
  Print[computeConjugateModularInfo[angles]];
  (*Generate all switch combinations (from size 1 to 4)*)
  combinations=Subsets[Keys[switchers],{1,Length[switchers]}];
  (*Evaluate each switched configuration*)results=
  Table[Module[{switched=angles,name,passQ},name=StringRiffle[combo," + "];
    Do[switched=switchers[sw][switched],{sw,combo}];
    passQ=And@@(checkConditionN0Degrees/@switched);
    Print[Style["\nSwitch combination: ", Bold],name];
    Print[Style["Switched anglesDeg:", Italic]];
    Print[MatrixForm[switched]];
    Print[computeConjugateModularInfo[switched]];
    {name,passQ}},{combo,combinations}];];*)

Column[{TextCell[
  Style["===== CONJUGATE-MODULAR CHECK =====",
    Darker[Brown], Bold, 16], "Text"],
  TextCell[Style["M1 = M2 = M3 = M4 = M and M  $\neq$  2  $\Rightarrow$  NOT conjugate-modular.
    Boundary-strip switches preserve this.", GrayLevel[0.3]], "Text"]}

(*Helper Function>Returns True if all Mi values are equal

```



```

    AND their common value is not 2. Returns False otherwise. *)
isNotConjugateModular[quad_List] :=
Module[{abcdList, Ms}, abcdList = computeABCD /@ quad;
  Ms = FullSimplify[Times@@@abcdList];
  (*The condition is met if they are all equal AND the value isn't 2*)
  Simplify[Equal@@Ms] && (Ms[[1]] ≠ 2)];

(*First, check the initial, unswitched angles*)
Print[TextCell[Style["\nInitial anglesDeg (no switches):", Bold]]];
If[isNotConjugateModular[anglesDeg], Print[
  Style[" -> Condition met: All  $M_i$  are equal and  $M \neq 2$ .", Darker@Green]],
  Print[Style[" -> Condition NOT met.", Red]]];

(*Now, use the clean module to check all switch combinations*)
Module[{angles = anglesDeg, switchers, combinations, results},
  (*Define switch functions*)
  switchers = <|"Right" → SwitchingRightBoundaryStrip,
    "Left" → SwitchingLeftBoundaryStrip, "Lower" → SwitchingLowerBoundaryStrip,
    "Upper" → SwitchingUpperBoundaryStrip|>;
  (*Generate all combinations of switches (from size 1 to 4)*)
  combinations = Subsets[Keys[switchers], {1, Length[switchers]}];
  (*Evaluate condition after each combination
  of switches and store the result*) results = Table[
    Module[{switched = angles, name, passQ}, name = StringRiffle[combo, " + "];
    Do[switched = switchers[sw][switched], {sw, combo}];
    (*Set passQ using our new helper function for this check*)
    passQ = isNotConjugateModular[switched];
    {name, passQ}], {combo, combinations}];
  (*Display results in the specified column format*)
  Column[Prepend[Table[Module[{comboName = res[[1]], passQ = res[[2]]},
    Row[{Style[comboName <> ": ", Bold], If[passQ,
      Style["Condition met (All  $M_i$  are equal and  $M \neq 2$ ).", Darker@Green]],
      Style["Condition NOT met.", Red, Bold]]}], {res, results}],
    TextCell[Style["\nCONJUGATE-MODULAR CHECK AFTER SWITCHING BOUNDARY STRIPS",
      14, "Text"]]]]

```

Out[5805]=

===== CONDITION (N.0) =====
 ✓ All vertices satisfy (N.0).

Vertex	Combinations (mod 360)	Status
Vertex 1	{342.075, 102.075, 255.638, 135.638, 287.012, 80.5752, 47.0121, 200.575}	✓ Pass
Vertex 2	{334.867, 174.867, 315.305, 115.305, 276.196, 56.6343, 116.196, 256.634}	✓ Pass
Vertex 3	{319.476, 149.476, 330.241, 140.241, 92.1432, 272.908, 282.143, 102.908}	✓ Pass
Vertex 4	{332.626, 182.626, 299.113, 89.1129, 117.278, 233.765, 327.278, 83.7651}	✓ Pass

Out[5809]=

===== CONDITION (N.3) =====

Calculated Values:

M1 = 0.9246684119410834

M2 = 0.9246684119410838

M3 = 0.9246684119410838

M4 = 0.9246684119410834

Equality Check:

✓ All values are equal up to 15 digits after the decimal point.

Out[5820]=

===== CONDITION (N.4) =====

Calculated Values:

r1 = 1.1316753356714444 ; s1 = 1.1709253989194002

r2 = 1.1316753356714440 ; s2 = 1.0980214314247647

r3 = 0.6837903705449951 ; s3 = 1.0980214314247647

r4 = 0.6837903705449949 ; s4 = 1.1709253989193993

Equality Checks:

✓ All conditions are met.

✓ r1 = r2 up to 15 digits after the decimal point.

✓ r3 = r4 up to 14 digits after the decimal point.

✓ s1 = s4 up to 12 digits after the decimal point.

✓ s2 = s3 up to 16 digits after the decimal point.

Out[5836]=

===== CONDITION (N.5) =====

△ Approximate validation using

ε -tolerance. For rigorous proof, see the referenced paper.

✓ Valid Combination Found (M < 1):

e1 = -1, e2 = 1, e3 = -1

t1 = 0K + 0.309023709800244iK'

t2 = 0K + 0.502198011562077iK'

t3 = 1.000000000000000K + 0.519081863076006iK'

t4 = 1.000000000000000K + 0.325907561314174iK'

t1 + e1*t2 + e2*t3 + e3*t4 = 0. × 10⁻¹⁵K + 0. × 10⁻¹⁵iK'

Out[5839]=

```

===== OTHER PARAMETERS =====
u = 0.0753316
σ1 = 2.98516, σ2 = 2.92226, σ3 = 2.78795, σ4 = 2.90271
σ1 ≈ 171.037°, σ2 ≈ 167.433°, σ3 ≈ 159.738°, σ4 ≈ 166.313°
cosσ1 = -0.98779, cosσ2 = -0.976043, cosσ3 = -0.938119, cosσ4 = -0.971603
f1 = 1.06819931977115, f2 = 1.27866835212450
, f3 = 0.704139915597430, f4 = 0.857702279855926
z1 = 14.662902846474, z2 = 3.5884950421397
, z3 = -3.37997605192095, z4 = -7.0275194780880
x1 = 7.594436687028425, x2 = 7.594436687028450
, x3 = -3.1624590361891408, x4 = -3.1624590361891385
y1 = 5.850505579171120, y2 = 10.20185061026720
, y3 = 10.20185061026720, y4 = 5.850505579171150
p1 = 0.3628709628386438, p2 = 0.3628709628386432
, p3 = 0.56232519902188703 i, p4 = 0.56232519902188723 i
q1 = 0.4134312505355639, q2 = 0.3130837450663396
, q3 = 0.3130837450663396, q4 = 0.4134312505355628
p1·q1 = 0.1500221959494246, p2·q2 = 0.1136090000213509
, p3·q3 = 0.1760548792549471 i, p4·q4 = 0.2324828102392781 i

```

Out[5841]=

```

===== Bricard's
System of Equations =====

```

We introduce new notation for the

cotangents of half of the dihedral angles. Denote $Z :=$

$\cot\left(\frac{\theta_1}{2}\right)$, $W_2 = \cot\left(\frac{\theta_2}{2}\right)$, $U := \cot\left(\frac{\theta_3}{2}\right)$, and $W_1 = \cot\left(\frac{\theta_4}{2}\right)$

$$P_1(Z, W_1) = W_1^2 (0.310125 Z^2 + 0.598771) - 1.94649 W_1 Z + 0.777254 Z^2 + 0.0926553 = 0$$

$$P_2(Z, W_2) = W_2^2 (0.848101 Z^2 + 0.400742) - 1.6951 W_2 Z + 0.298319 Z^2 + 0.14531 = 0$$

$$P_3(U, W_2) = (-0.606188 U^2 - 0.647883) W_2^2 + 0.200836 U^2 - 1.03579 U W_2 + 0.249417 = 0$$

$$P_4(U, W_1) = (-0.281613 U^2 - 0.625792) W_1^2 + 0.338269 U^2 - 0.803495 U W_1 + 0.202049 = 0$$

Out[5850]=

===== FLEXIONS =====

Solutions to Bricard's equations under a free parameter $t := Z \in \mathbb{C}$:**Solution 1:**

$$Z(t) = t$$

$$W_2(t) = \frac{0.999349 t - 0.284533 \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1}}{1. t^2 + 0.472516}$$

$$U(t) = \frac{0.98196 \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} + 0.114494 t}{1.74456 - 1. t^2}$$

$$W_1(t) = \frac{3.13824 t - 0.759502 \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1}}{1. t^2 + 1.93074}$$

Solution 2:

$$Z(t) = t$$

$$W_2(t) = \frac{0.284533 \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} + 0.999349 t}{1. t^2 + 0.472516}$$

$$U(t) = \frac{0.114494 t - 0.98196 \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1}}{1.74456 - 1. t^2}$$

$$W_1(t) = \frac{0.759502 \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} + 3.13824 t}{1. t^2 + 1.93074}$$

Out[5853]=

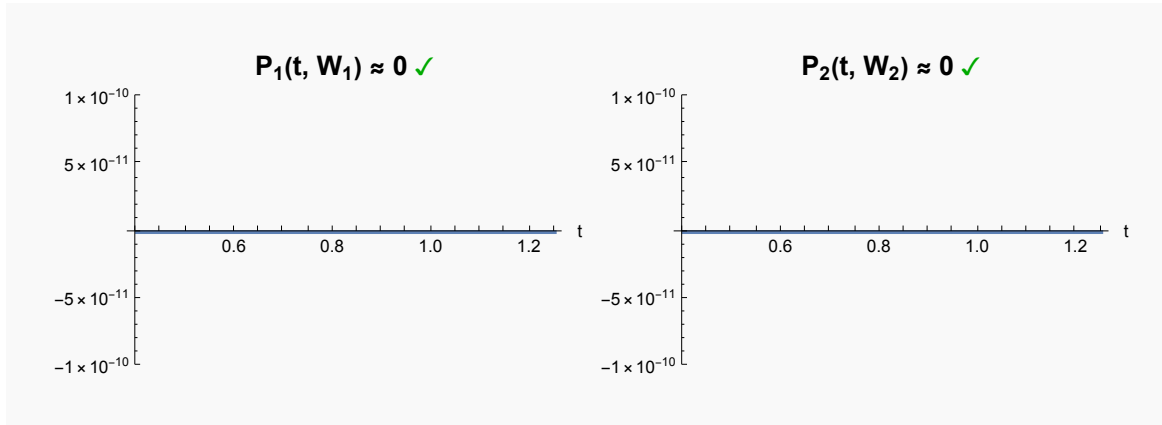
===== Equations: $P_1(t, W_1) = 0$ and $P_2(t, W_2) = 0$ =====Let W_{1s1} and W_{1s2} be formulas for $W_1(t)$ from solutions 1 and 2, respectively.

Similarly, let W_{2s1} and W_{2s2} be the formulas for $W_2(t)$. We show that all four pairs - (W_{1s1}, W_{2s1}) , (W_{1s1}, W_{2s2}) , (W_{1s2}, W_{2s1}) , and (W_{1s2}, W_{2s2}) - solve equations $P_1(t, W_1) = 0$ and $P_2(t, W_2) = 0$.

Pair 1, $(W_1, W_2) = (W_{1s1}, W_{2s1})$:

$$P_1(t, W_1) = \frac{1}{(1. t^2 + 1.93074)^2} \left(t \left(t \left(2.67507 \times 10^{-15} t^4 - 1.16063 \times 10^{-14} t^2 + 3.55271 \times 10^{-15} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} t + 2.00199 \times 10^{-14} \right) - 5.77316 \times 10^{-15} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} \right) - 2.41619 \times 10^{-15} \right)$$

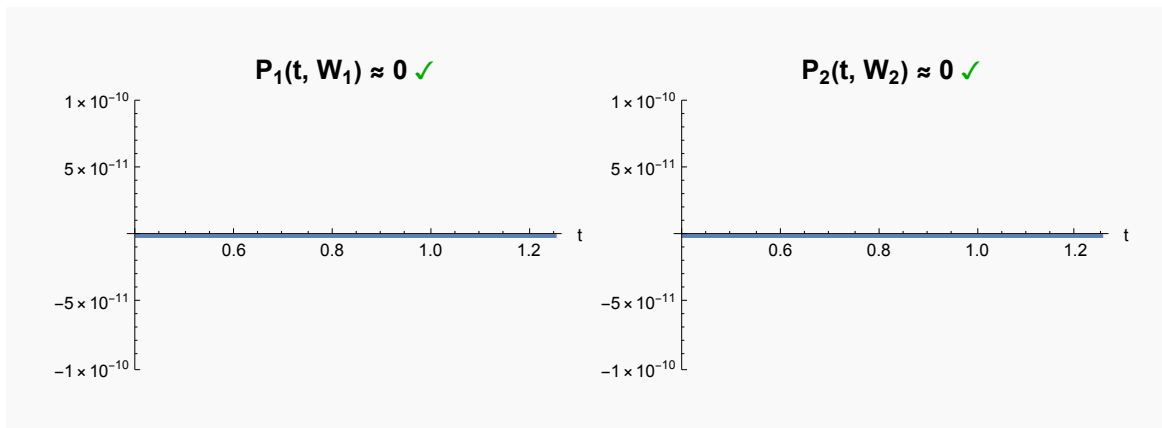
$$P_2(t, W_2) = \frac{1}{(1. t^2 + 0.472516)^2} \left(t \left(t \left(1.3248 \times 10^{-16} t^4 - 6.62401 \times 10^{-17} t^2 + 3.88578 \times 10^{-16} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} t - 2.18592 \times 10^{-15} \right) + 8.88178 \times 10^{-16} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} \right) + 1.9458 \times 10^{-16} \right)$$



Pair 2, $(W_1, W_2) = (W_{1s1}, W_{2s2})$:

$$P_1(t, W_1) = \frac{1}{(1. t^2 + 1.93074)^2} \left(t \left(t \left(2.67507 \times 10^{-15} t^4 - 1.16063 \times 10^{-14} t^2 + 3.55271 \times 10^{-15} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} t + 2.00199 \times 10^{-14} \right) - 5.77316 \times 10^{-15} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} \right) - 2.41619 \times 10^{-15} \right)$$

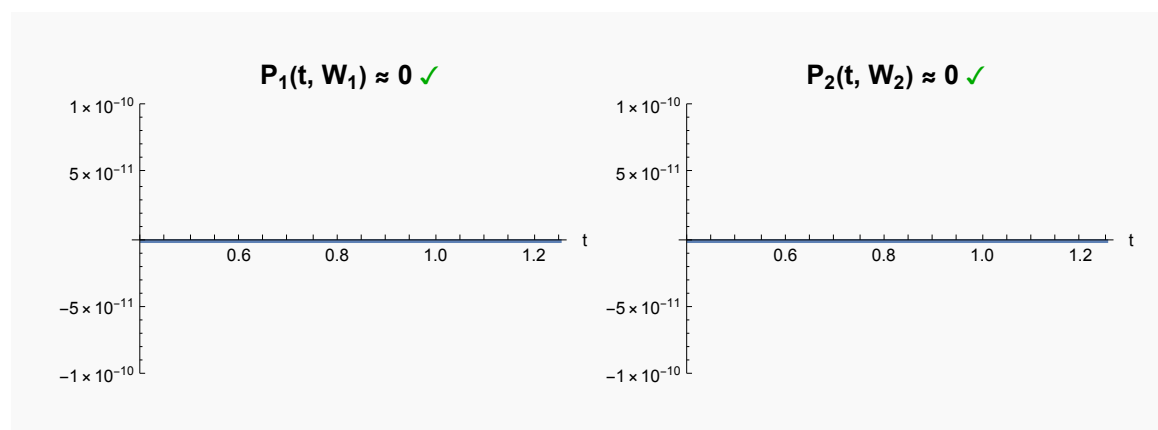
$$P_2(t, W_2) = \frac{1}{(1. t^2 + 0.472516)^2} \left(t \left(t \left(1.3248 \times 10^{-16} t^4 - 6.62401 \times 10^{-17} t^2 - 3.88578 \times 10^{-16} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} t - 2.18592 \times 10^{-15} \right) - 8.88178 \times 10^{-16} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} \right) + 1.9458 \times 10^{-16} \right)$$



Pair 3, $(W_1, W_2) = (W_{1s2}, W_{2s1})$:

$$P_1(t, W_1) = \frac{1}{(1. t^2 + 1.93074)^2} \left(t \left(t \left(2.67507 \times 10^{-15} t^4 - 1.16063 \times 10^{-14} t^2 - 3.55271 \times 10^{-15} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} t + 2.00199 \times 10^{-14} \right) + 5.77316 \times 10^{-15} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} \right) - 2.41619 \times 10^{-15} \right)$$

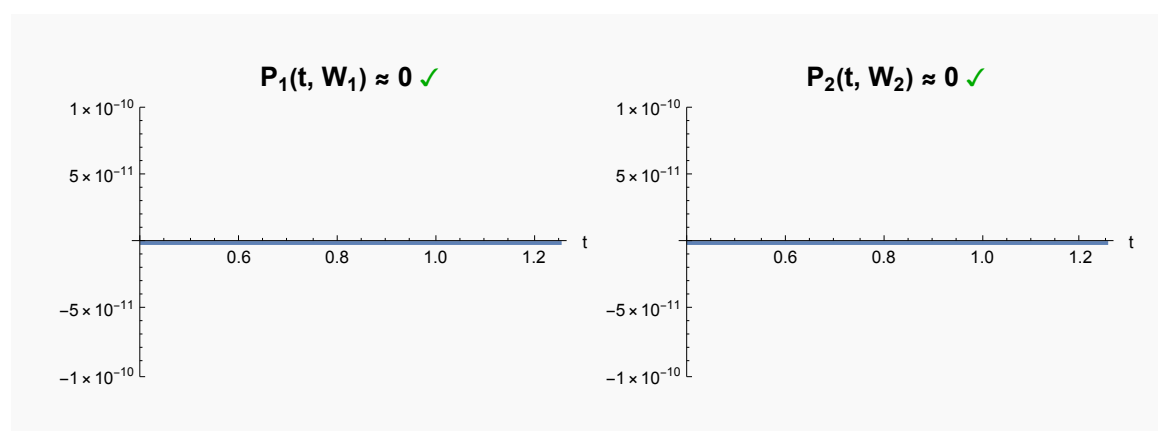
$$P_2(t, W_2) = \frac{1}{(1. t^2 + 0.472516)^2} \left(t \left(t \left(1.3248 \times 10^{-16} t^4 - 6.62401 \times 10^{-17} t^2 + 3.88578 \times 10^{-16} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} t - 2.18592 \times 10^{-15} \right) + 8.88178 \times 10^{-16} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} \right) + 1.9458 \times 10^{-16} \right)$$



Pair 4, $(W_1, W_2) = (W_{1s2}, W_{2s2})$:

$$P_1(t, W_1) = \frac{1}{(1. t^2 + 1.93074)^2} \left(t \left(t \left(2.67507 \times 10^{-15} t^4 - 1.16063 \times 10^{-14} t^2 - 3.55271 \times 10^{-15} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} t + 2.00199 \times 10^{-14} \right) + 5.77316 \times 10^{-15} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} \right) - 2.41619 \times 10^{-15} \right)$$

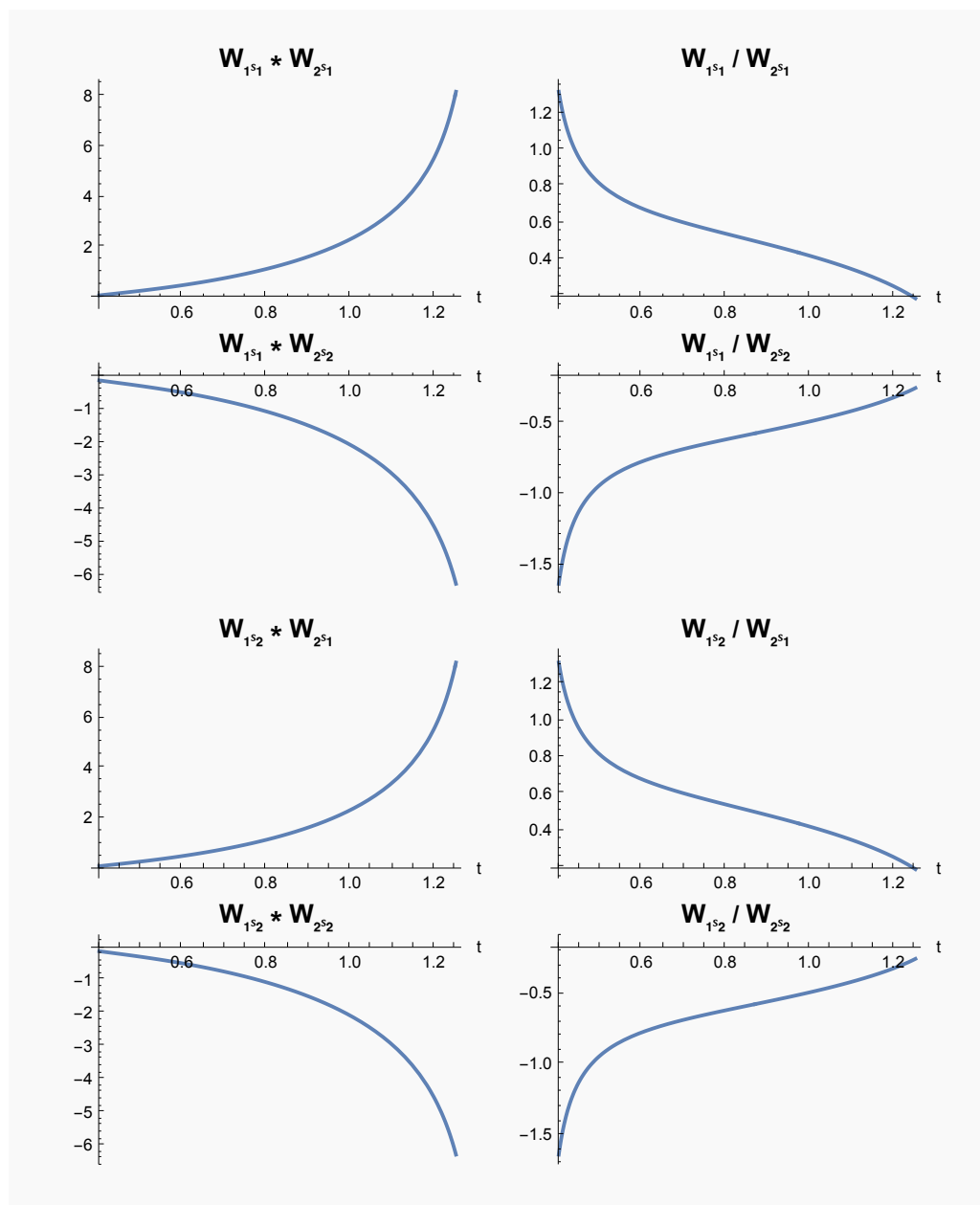
$$P_2(t, W_2) = \frac{1}{(1. t^2 + 0.472516)^2} \left(t \left(t \left(1.3248 \times 10^{-16} t^4 - 6.62401 \times 10^{-17} t^2 - 3.88578 \times 10^{-16} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} t - 2.18592 \times 10^{-15} \right) - 8.88178 \times 10^{-16} \sqrt{-4.34478 t^4 + 8.16654 t^2 - 1} \right) + 1.9458 \times 10^{-16} \right)$$



Out[5856]=

**===== Plots of $W_1(t)W_2(t)$
and $W_1(t)/W_2(t)$ For All Pairs =====**

Checking that all four pairs - (W_{1s1}, W_{2s1}) ,
 (W_{1s1}, W_{2s2}) , (W_{1s2}, W_{2s1}) , and (W_{1s2}, W_{2s2}) - does NOT
satisfy $W_1(t)/W_2(t) = \text{const}$ and $W_1(t)W_2(t) = \text{const}$.



Out[5865]=

===== Equations: $P_3(U,$
 $W_2) = 0$ and $P_4(U, W_1) = 0$ =====

We solve $P_3(U, W_2) = 0$ and $P_4(U, W_1) = 0$ for W_1 and W_2 .

$$W_{1,1}(U) = \frac{-3.63579 \times 10^6 \sqrt{1.59898 \times 10^{17} U^4 + 7.21745 \times 10^{17} U^2 + 2.12235 \times 10^{17}} - 1.89241 \times 10^{15} U}{1.32653 \times 10^{15} U^2 + 2.94777 \times 10^{15}}$$

$$W_{1,2}(U) = \frac{3.63579 \times 10^6 \left(\sqrt{1.59898 \times 10^{17} U^4 + 7.21745 \times 10^{17} U^2 + 2.12235 \times 10^{17}} - 5.20496 \times 10^8 U \right)}{1.32653 \times 10^{15} U^2 + 2.94777 \times 10^{15}}$$

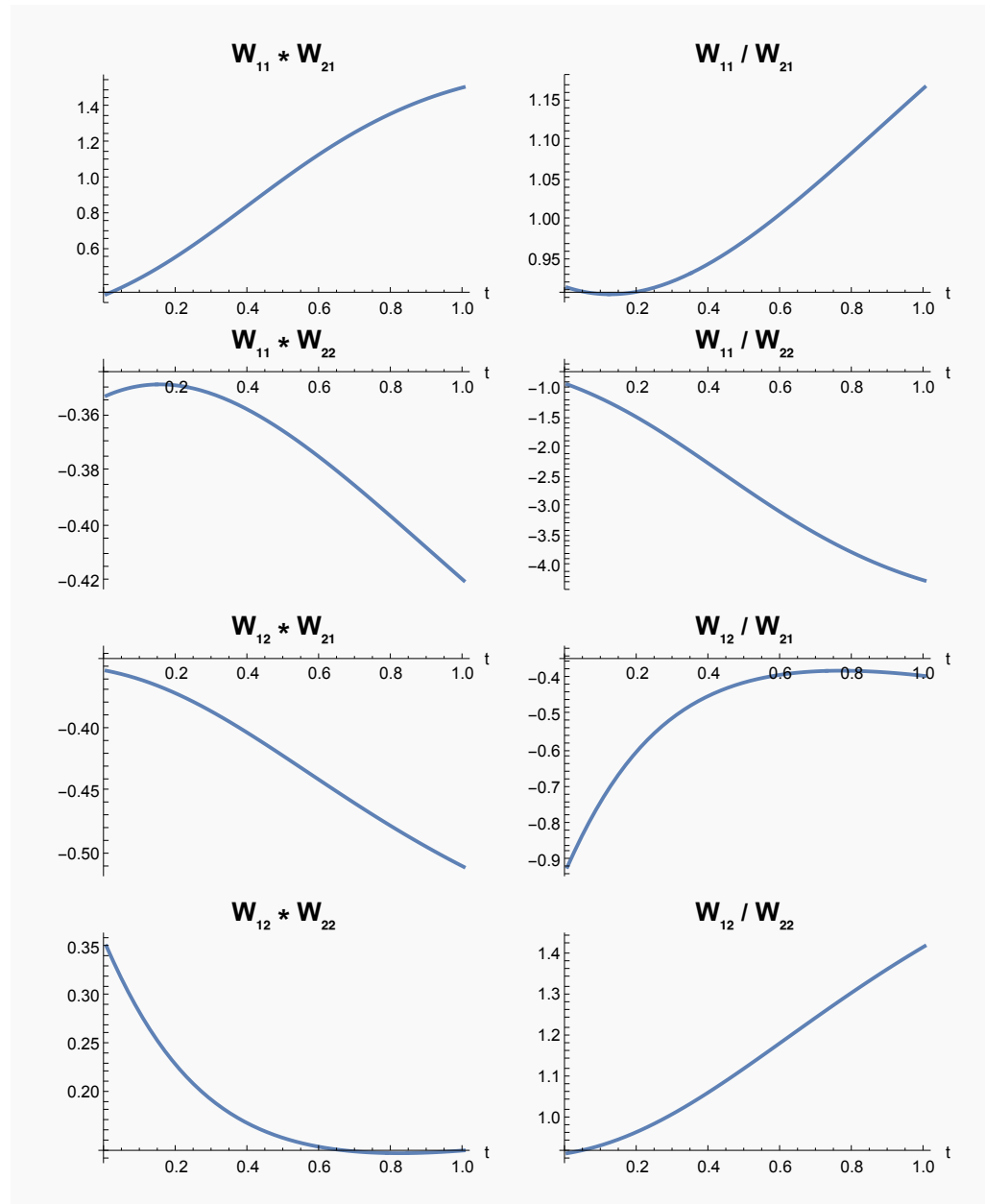
$$W_{2,1}(U) = \frac{-9.23816 \times 10^{-10} \sqrt{1.4227 \times 10^{48} U^4 + 6.42174 \times 10^{48} U^2 + 1.88836 \times 10^{48}} - 1.63553 \times 10^{15} U}{1.91436 \times 10^{15} U^2 + 2.04603 \times 10^{15}}$$

$$W_{2,2}(U) = \frac{9.23816 \times 10^{-10} \sqrt{1.4227 \times 10^{48} U^4 + 6.42174 \times 10^{48} U^2 + 1.88836 \times 10^{48}} - 1.63553 \times 10^{15} U}{1.91436 \times 10^{15} U^2 + 2.04603 \times 10^{15}}$$

Out[5870]=

===== Plots of $W_1(U)W_2(U)$
and $W_1(U)/W_2(U)$ For All Pairs =====

Checking that all four pairs – $(W_{11}(U), W_{21}(U))$, $(W_{11}(U), W_{22}(U))$, $(W_{12}(U), W_{21}(U))$, and $(W_{12}(U), W_{22}(U))$ – does NOT satisfy $W_1(U)/W_2(U) = \text{const}$ and $W_1(U)W_2(U) = \text{const}$.



Out[5871]=

===== FLEXIBILITY
(Double Checking) =====

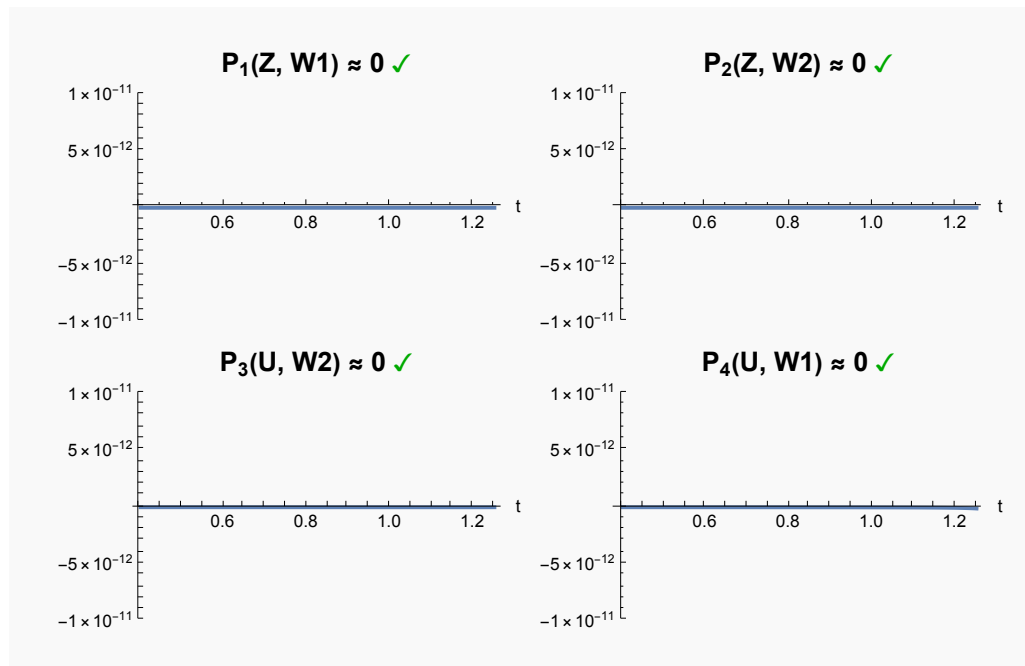
Out[5873]=

Solution 1:

$$\begin{aligned}
P_1(Z, W_1) &= \frac{1}{(1.93074 + 1. t^2)^2} \left(-2.41619 \times 10^{-15} + \right. \\
&\quad \left. t \left(-5.77316 \times 10^{-15} \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} + t \left(2.00199 \times 10^{-14} - 1.16063 \times 10^{-14} t^2 + \right. \right. \right. \\
&\quad \left. \left. 2.67507 \times 10^{-15} t^4 + 3.55271 \times 10^{-15} t \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} \right) \right) \left. \right) \\
P_2(Z, W_2) &= \frac{1}{(0.472516 + 1. t^2)^2} \left(1.9458 \times 10^{-16} + \right. \\
&\quad \left. t \left(8.88178 \times 10^{-16} \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} + t \left(-2.18592 \times 10^{-15} - 6.62401 \times 10^{-17} t^2 + \right. \right. \right. \\
&\quad \left. \left. 1.3248 \times 10^{-16} t^4 + 3.88578 \times 10^{-16} t \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} \right) \right) \left. \right) \\
P_3(U, W_2) &= \frac{1}{(0.824332 + 1.27204 t^2 - 1. t^4)^2} \\
&\quad \left(1.68121 \times 10^{-16} + t \left(-3.78271 \times 10^{-15} \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} + \right. \right. \\
&\quad \left. t \left(1.1096 \times 10^{-14} + t \left(-5.37986 \times 10^{-15} \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} + \right. \right. \right. \\
&\quad \left. \left. t \left(8.74227 \times 10^{-15} - 2.03426 \times 10^{-14} t^2 + 6.85092 \times 10^{-15} t^4 + \right. \right. \right. \\
&\quad \left. \left. \left. 4.37114 \times 10^{-15} t \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} \right) \right) \right) \left. \right) \left. \right) \\
P_4(U, W_1) &= \frac{1}{(3.36829 - 0.186184 t^2 - 1. t^4)^2} \\
&\quad \left(2.78246 \times 10^{-15} + t \left(-4.47977 \times 10^{-14} \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} + t \left(4.89714 \times 10^{-14} + \right. \right. \right. \\
&\quad \left. t \left(2.22597 \times 10^{-15} \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} + t \left(-7.5683 \times 10^{-14} - 6.67791 \times 10^{-15} \right. \right. \right. \\
&\quad \left. \left. \left. t^2 + 1.81556 \times 10^{-14} t^4 + 1.00169 \times 10^{-14} t \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} \right) \right) \right) \left. \right) \left. \right)
\end{aligned}$$

Out[5879]=

===== FLEXION 1 =====

Polynomials $P_i(t)$ built from Bricard's equations for flexion 1.

Out[5880]=

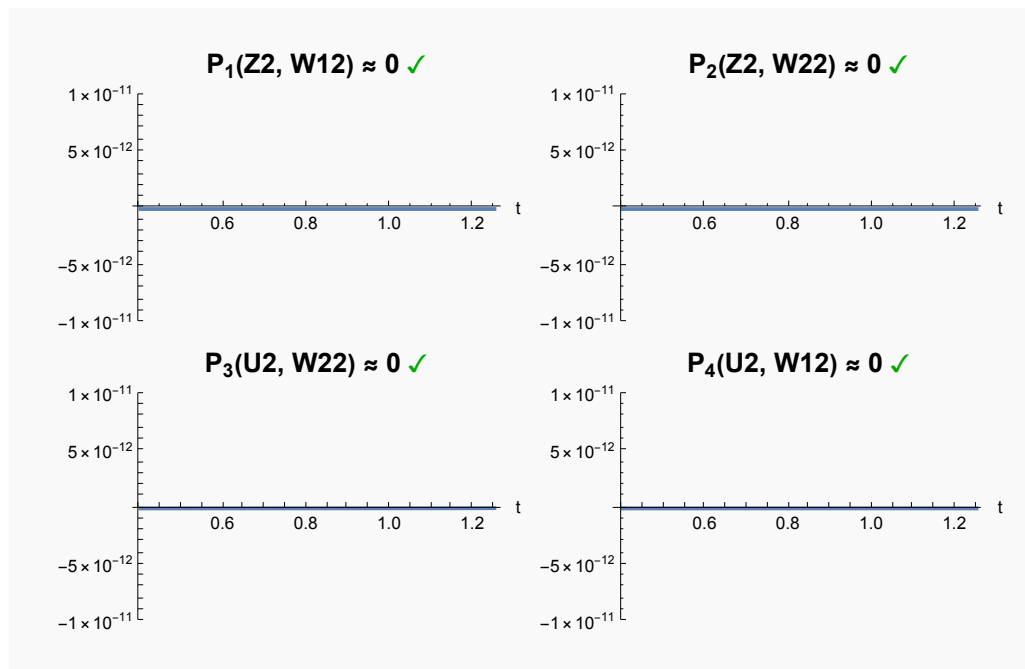
Solution 2:

$$\begin{aligned}
P_1(Z, W_1) &= \frac{1}{(1.93074 + 1. t^2)^2} \left(-2.41619 \times 10^{-15} + \right. \\
&\quad t \left(5.77316 \times 10^{-15} \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} + t \left(2.00199 \times 10^{-14} - 1.16063 \times 10^{-14} t^2 + \right. \right. \\
&\quad \left. \left. 2.67507 \times 10^{-15} t^4 - 3.55271 \times 10^{-15} t \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} \right) \right) \left. \right) \\
P_2(Z, W_2) &= \frac{1}{(0.472516 + 1. t^2)^2} \left(1.9458 \times 10^{-16} + \right. \\
&\quad t \left(-8.88178 \times 10^{-16} \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} + t \left(-2.18592 \times 10^{-15} - 6.62401 \times 10^{-17} t^2 + \right. \right. \\
&\quad \left. \left. 1.3248 \times 10^{-16} t^4 - 3.88578 \times 10^{-16} t \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} \right) \right) \left. \right) \\
P_3(U, W_2) &= \frac{1}{(0.824332 + 1.27204 t^2 - 1. t^4)^2} \\
&\quad \left(1.68121 \times 10^{-16} + t \left(3.78271 \times 10^{-15} \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} + t \left(1.1096 \times 10^{-14} + \right. \right. \right. \\
&\quad \left. \left. t \left(5.37986 \times 10^{-15} \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} + t \left(8.74227 \times 10^{-15} - 2.03426 \times 10^{-14} \right. \right. \right. \right. \\
&\quad \left. \left. \left. t^2 + 6.85092 \times 10^{-15} t^4 - 4.37114 \times 10^{-15} t \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} \right) \right) \right) \left. \right) \left. \right) \\
P_4(U, W_1) &= \frac{1}{(3.36829 - 0.186184 t^2 - 1. t^4)^2} \\
&\quad \left(2.78246 \times 10^{-15} + t \left(4.47977 \times 10^{-14} \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} + \right. \right. \\
&\quad \left. \left. t \left(4.89714 \times 10^{-14} + t \left(-2.22597 \times 10^{-15} \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} + \right. \right. \right. \\
&\quad \left. \left. \left. t \left(-7.5683 \times 10^{-14} - 6.67791 \times 10^{-15} t^2 + 1.81556 \times 10^{-14} t^4 - \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. 1.00169 \times 10^{-14} t \sqrt{-1 + 8.16654 t^2 - 4.34478 t^4} \right) \right) \right) \right) \left. \right) \left. \right) \left. \right)
\end{aligned}$$

Out[5887]=

===== FLEXION 2 =====

Polynomials $P_i(t)$ built from Bricard's equations for flexion 2.



Out[5890]=

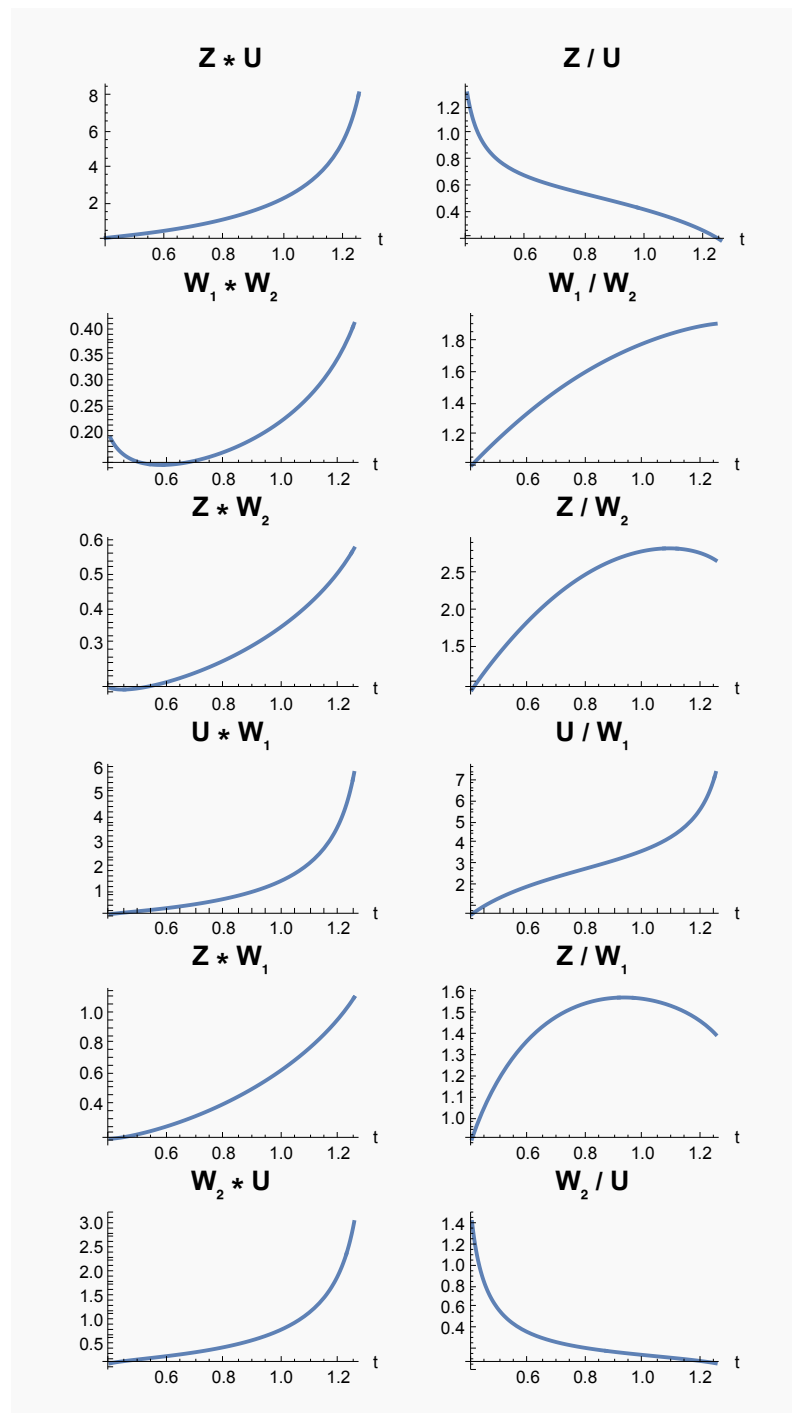
===== NOT LINEAR COMPOUND =====

Above we consider the first pair of equations ($P_1(t, W_1) = 0$ and $P_2(t, W_2) = 0$). Solving them as quadratic equations in W_1 and W_2 , respectively we parametrize the solutions by the first two and fourth expressions in Solutions 1 and 2 in a neighborhood of any point (W_1, t, W_2) such that the expression in the square root and denominators are not zero. Here, we choose any continuous branch of the square root in this neighborhood, and the signs in \pm need Not agree (this means we consider all 4 pairs we describe above). We conclude that NO component of the solution set of the first pair of Bricard's equations satisfies $W_1/W_2 = \text{const}$ NOR $W_1 W_2 = \text{const}$.

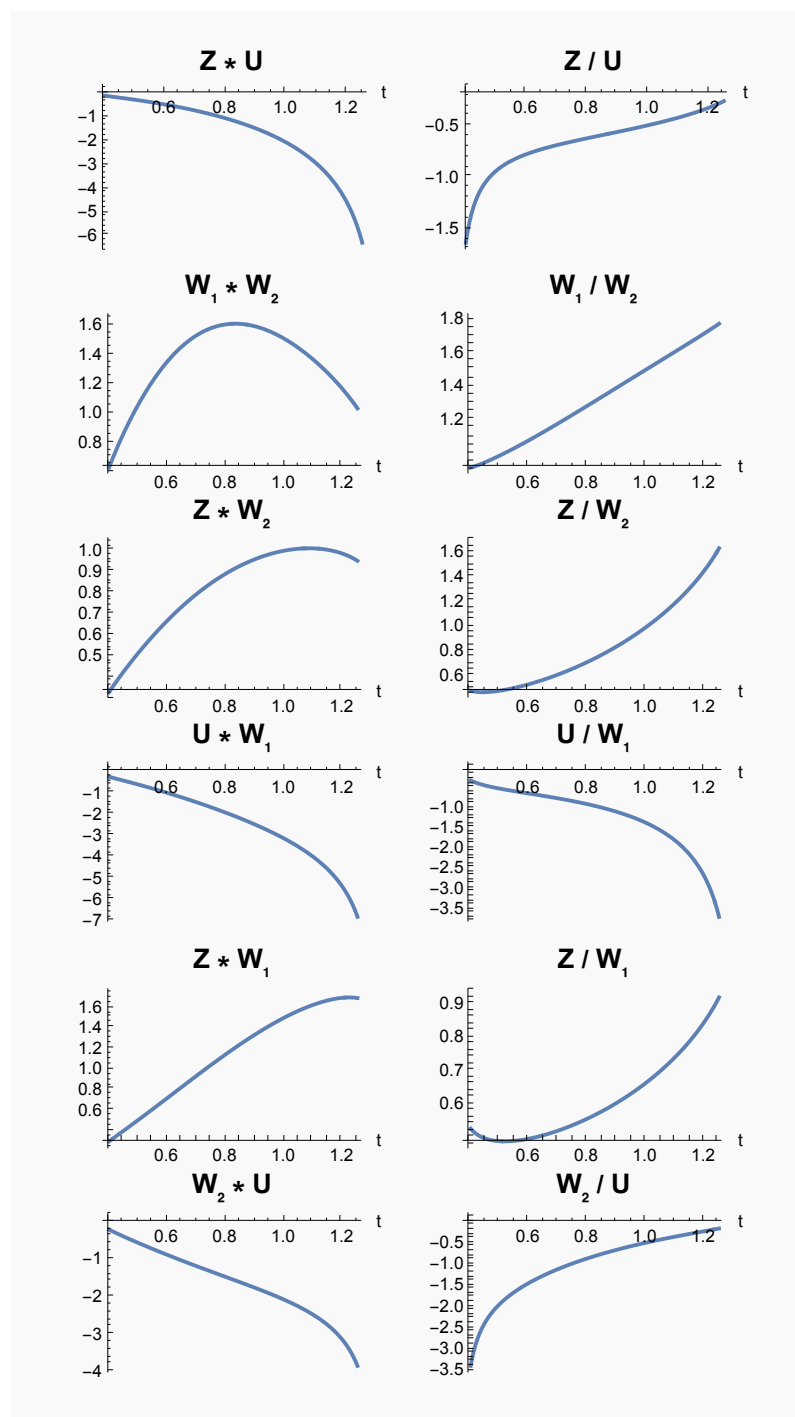
Analogously, NO component of the solution set of the other pair of equations ($P_3(U, W_2) = 0$ and $P_4(U, W_1) = 0$) satisfies $W_1/W_2 = \text{const}$ NOR $W_1 W_2 = \text{const}$. As a result, NO component of the solution set of all four equations satisfies $W_1/W_2 = \text{const}$ NOR $W_1 W_2 = \text{const}$. So, our example does not belong to the linear compound class, even after switching the boundary strips.

Below, we also present the plots of functions ZU , Z/U , W_1W_2 , W_1/W_2 , ZW_2 , Z/W_2 , UW_1 , U/W_1 , ZW_1 , Z/W_1 , W_2U , W_2/U .

Solution 1:



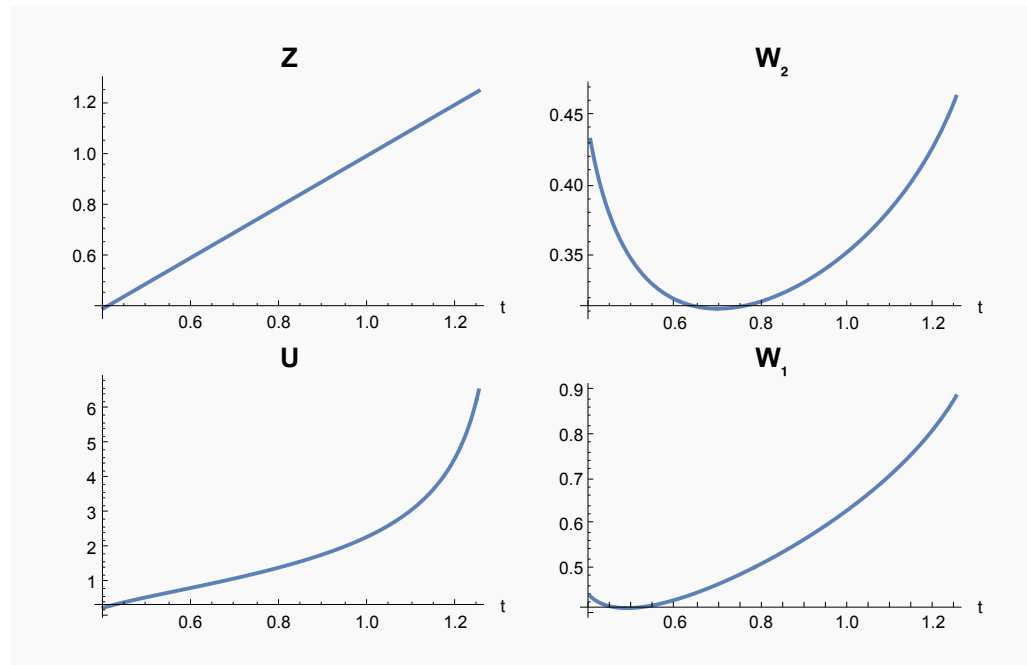
Out[5893]=

Solution 2:

Out[5896]=

===== NOT TRIVIAL (FLEXION 1) =====

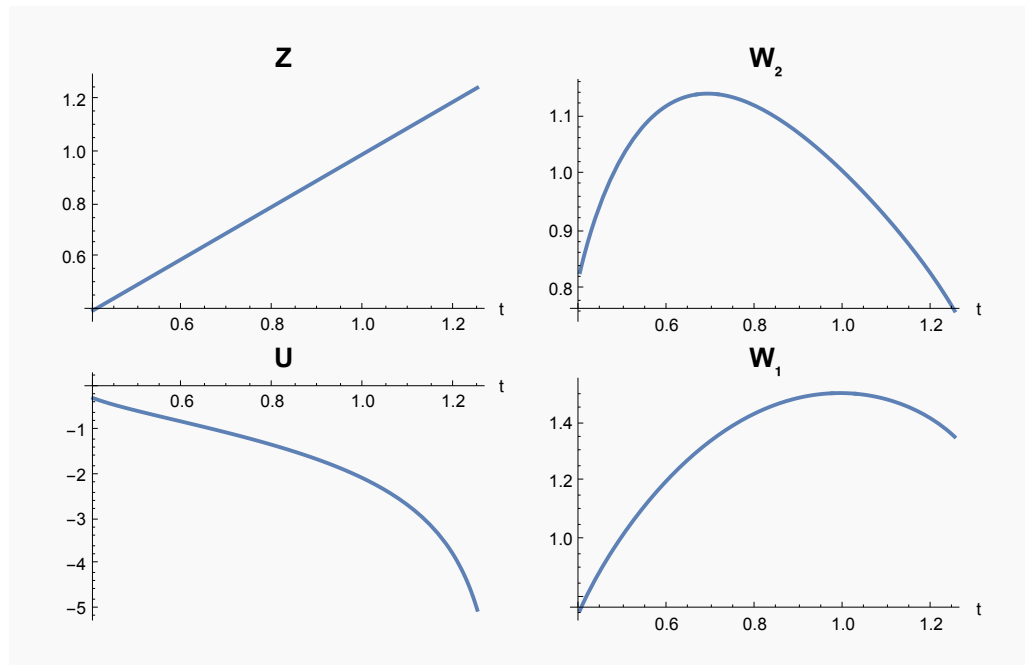
This configuration does not belong to the trivial class – even after switching the boundary strips – since none of the functions Z , W_2 , U , or W_1 is constant.



Out[5899]=

===== NOT TRIVIAL (FLEXION 2) =====

This configuration does not belong to the trivial class – even after switching the boundary strips – since none of the functions Z , W_2 , U , or W_1 is constant.



Out[5906]=

===== NOT CONIC & NOT CHIMERA & NOT
LINEAR CONJUGATE & NOT ISOGONAL=====

Condition (N.0) is satisfied for all $i=1,\dots,4$

\Rightarrow NOT equimodular-conic, NOT chimera, NOT isogonal and NOT linear conjugate. Applying any boundary-strip switch still preserves (N.0), so no conic, no chimera, no isogonal and no linear conjugate form emerges.

Out[5907]=

CONDITION (N.0) AFTER SWITCHING BOUNDARY STRIPS

Right: Condition (N.0) is still satisfied.**Left:** Condition (N.0) is still satisfied.**Lower:** Condition (N.0) is still satisfied.**Upper:** Condition (N.0) is still satisfied.**Right + Left:** Condition (N.0) is still satisfied.**Right + Lower:** Condition (N.0) is still satisfied.**Right + Upper:** Condition (N.0) is still satisfied.**Left + Lower:** Condition (N.0) is still satisfied.**Left + Upper:** Condition (N.0) is still satisfied.**Lower + Upper:** Condition (N.0) is still satisfied.**Right + Left + Lower:** Condition (N.0) is still satisfied.**Right + Left + Upper:** Condition (N.0) is still satisfied.**Right + Lower + Upper:** Condition (N.0) is still satisfied.**Left + Lower + Upper:** Condition (N.0) is still satisfied.**Right + Left + Lower + Upper:** Condition (N.0) is still satisfied.

Out[5908]=

===== ORTHOGONALITY CHECK =====

 $\cos(\alpha_i) \cdot \cos(\gamma_i) \neq \cos(\beta_i) \cdot \cos(\delta_i)$ forat least one $i = 1, \dots, 4 \Rightarrow$ NOT orthodiagonal.

Switching boundary strips does not correct this.

Initial anglesDeg (no switches):

-> Condition met: At least one difference is non-zero.

Out[5912]=

NON-ORTHOGONALITY CHECK AFTER SWITCHING BOUNDARY STRIPS

Right: Condition met (at least one difference is non-zero).**Left:** Condition met (at least one difference is non-zero).**Lower:** Condition met (at least one difference is non-zero).**Upper:** Condition met (at least one difference is non-zero).**Right + Left:** Condition met (at least one difference is non-zero).**Right + Lower:** Condition met (at least one difference is non-zero).**Right + Upper:** Condition met (at least one difference is non-zero).**Left + Lower:** Condition met (at least one difference is non-zero).**Left + Upper:** Condition met (at least one difference is non-zero).**Lower + Upper:** Condition met (at least one difference is non-zero).**Right + Left + Lower:** Condition met (at least one difference is non-zero).**Right + Left + Upper:** Condition met (at least one difference is non-zero).**Right + Lower + Upper:** Condition met (at least one difference is non-zero).**Left + Lower + Upper:** Condition met (at least one difference is non-zero).**Right + Left + Lower + Upper:**

Condition met (at least one difference is non-zero).

Out[5913]=

===== CONJUGATE-MODULAR CHECK =====

 $M_1 = M_2 = M_3 = M_4 = M$ and $M \neq 2 \Rightarrow$ NOT

conjugate-modular. Boundary-strip switches preserve this.

Initial anglesDeg (no switches):

-> Condition met: All M_i are equal and $M \neq 2$.

Out[5917]=

CONJUGATE-MODULAR CHECK AFTER SWITCHING BOUNDARY STRIPS

Right: Condition met (All M_i are equal and $M \neq 2$).

Left: Condition met (All M_i are equal and $M \neq 2$).

Lower: Condition met (All M_i are equal and $M \neq 2$).

Upper: Condition met (All M_i are equal and $M \neq 2$).

Right + Left: Condition met (All M_i are equal and $M \neq 2$).

Right + Lower: Condition met (All M_i are equal and $M \neq 2$).

Right + Upper: Condition met (All M_i are equal and $M \neq 2$).

Left + Lower: Condition met (All M_i are equal and $M \neq 2$).

Left + Upper: Condition met (All M_i are equal and $M \neq 2$).

Lower + Upper: Condition met (All M_i are equal and $M \neq 2$).

Right + Left + Lower: Condition met (All M_i are equal and $M \neq 2$).

Right + Left + Upper: Condition met (All M_i are equal and $M \neq 2$).

Right + Lower + Upper: Condition met (All M_i are equal and $M \neq 2$).

Left + Lower + Upper: Condition met (All M_i are equal and $M \neq 2$).

Right + Left + Lower + Upper: Condition met (All M_i are equal and $M \neq 2$).