# Computational Companion to “Quasi-Symmetric Nets: A Constructive Approach to the Equimodular Elliptic Type of Kokotsakis Polyhedra” — Example 5 Helper

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Tested on: Mathematica 14.0

In[233]:=

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(*==========================================================================
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(*Quit*)
(*All angle sets in degrees*)
anglesDeg = {
   {26.20863403213998, 82.2407675648952, 21.949109994264898, 60}, (*Vertex 1*)
   {16.166237389600262,
    130.87095233025335, 18.85247535405415, 115}, (*Vertex 2*)
   {134.65533802039442,
    34.44439013740831, 145.3694664686027, 80}, (*Vertex 3*)
   {117.95117201340666,
    49.52829397349284, 149.0275482144225, 105} (*Vertex 4*)};


(*------------------------------------------*)
(*Function to compute sigma from 4 angles*)
computeSigma[{α_, β_, γ_, δ_}] := (α + β + γ + δ) / 2;
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(*------------------------------------------*)
(*Function to compute a,b,c,d from angles*)
computeABCD[{α_, β_, γ_, δ_}] :=
  Module[{alpha = α Degree, beta = β Degree, gamma = γ Degree,
    delta = δ Degree, sigma}, sigma = computeSigma[{α, β, γ, δ} Degree];
   {Sin[alpha] / Sin[sigma - alpha], Sin[beta] / Sin[sigma - beta],
    Sin[gamma] / Sin[sigma - gamma], Sin[delta] / Sin[sigma - delta]}];

(*------------------------------------------*)
(*Compute all σ values*)
sigmas = computeSigma /@ anglesDeg;

(*Compute all {a,b,c,d} values*)
results = computeABCD /@ anglesDeg;

(*Optional:extract individual values*)
{a1, b1, c1, d1} = results[[1]];
{a2, b2, c2, d2} = results[[2]];
{a3, b3, c3, d3} = results[[3]];
{a4, b4, c4, d4} = results[[4]];
{σ1, σ2, σ3, σ4} = FullSimplify[sigmas];


(*==========================================================================
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(*================================
   CONDITION (N.0)===================================*)
(*==========================================================================
 =====*)
(*uniqueCombos={{1,1,1,1},{1,1,1,-1},{1,1,-1,-1},
  {1,1,-1,1},{1,-1,1,1},{1,-1,-1,1},{1,-1,1,-1},{1,-1,-1,-1}};

checkConditionN0Degrees[{α_,β_,γ_,δ_}]:=Module[
  {angles={α,β,γ,δ},results},results=Mod[uniqueCombos.angles,360]//Chop;
  !MemberQ[results,0]];

conditionsN0=checkConditionN0Degrees/@anglesDeg;
allVerticesPass=And@@conditionsN0;

Column[{TextCell[Style["================= CONDITION (N.0) =================",
    Darker[Green],Bold, 16],"Text"],
  If[allVerticesPass,
   Style["✓ All vertices satisfy (N.0).",Darker[Green],Bold],
   Style["✘ Some vertices fail (N.0).",Red,Bold]]}]*)
uniqueCombos = {{1, 1, 1, 1}, {1, 1, 1, -1}, {1, 1, -1, -1}, {1, 1, -1, 1},
   {1, -1, 1, 1}, {1, -1, -1, 1}, {1, -1, 1, -1}, {1, -1, -1, -1}};
```

```
checkConditionN0Degrees[{α_, β_, γ_, δ_}] :=
  Module[{angles = {α, β, γ, δ}, results},
   results = Mod[uniqueCombos.angles, 360] // Chop;
   results];

(*apply to all vertices*)
resultsPerVertex = checkConditionN0Degrees /@ anglesDeg;

(*check pass/fail*)
conditionsN0 = Map[! MemberQ[#, 0] &, resultsPerVertex];
allVerticesPass = And @@ conditionsN0;

Column[{TextCell[Style["================= CONDITION (N.0) =================",
    Darker[Green], Bold, 16], "Text"], (*summary line*)If[allVerticesPass,
   Style["✔ All vertices satisfy (N.0).", Darker[Green], Bold],
   Style["✘ Some vertices fail (N.0).", Red, Bold]],
  (*detailed results for each vertex*)
  Grid[Prepend[Table[{"Vertex " <> ToString[i], resultsPerVertex[[i]],
      If[conditionsN0[[i]], "✔ Pass", "✘ Fail"]}, {i, Length[anglesDeg]}],
    {"Vertex", "Combinations (mod 360)", "Status"}], Frame → All]}]


(*======================================================================
 =====*)
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   CONDITION (N.3)================================*)
(*======================================================================
 =====*)
Ms = FullSimplify[Times @@@ results];
allEqualQ = Simplify[Equal @@ Ms];

Column[{TextCell[Style["================= CONDITION (N.3) =================",
    Darker[Blue], Bold, 16], "Text"], If[allEqualQ,
   Row[{Style["✔ M1 = M2 = M3 = M4 = ", Bold], Ms[[1]]}],
   Style["✘ M_i are not all equal.", Red, Bold]]}]


(*======================================================================
 =====*)
(*=========================CONDITION (N.4)=========================*)
(*======================================================================
 =====*)
aList = results[[All, 1]];
cList = results[[All, 3]];
dList = results[[All, 4]];

rList = FullSimplify /@ (aList * dList); {r1, r2, r3, r4} = rList;
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sList = FullSimplify /@ (cList * dList); {s1, s2, s3, s4} = sList;

Column[{TextCell[Style["================= CONDITION (N.4) =================",
    Darker[Purple], Bold, 16], "Text"],
  If[r1 == r2 && r3 == r4 && s1 == s4 && s2 == s3, Column[
    {Row[{Style["✓ r1 = r2 = ", Bold], r1, Style["; ✓ r3 = r4 = ", Bold], r3}],
     Row[{Style["✓ s1 = s4 = ", Bold], s1, Style["; ✓ s2 = s3 = ", Bold],
       s2}]}], Style["✘ Condition (N.4) fails.", Red, Bold]]
 }]


(*============================================================================
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  CONDITION (N.5)===============================*)
(*============================================================================
 =====*)
fList = FullSimplify /@ (aList * cList); {f1, f2, f3, f4} = fList;
M1 = Ms[[1]];

m[M1_] := Piecewise[{{1 - M1, M1 < 1}, {(M1 - 1) / M1, M1 > 1}}] // N;
K = EllipticK[m[M1]] // N;
Kp = EllipticK[1 - m[M1]] // N;

computeTi[i_] :=
  Module[{sigma = sigmas[[i]], r = rList[[i]], s = sList[[i]], f = fList[[i]], base},
   base = I * Im[InverseJacobiDN[Piecewise[
         {{Sqrt[f], M1 < 1}, {1 / Sqrt[f], M1 > 1}}], m[M1]]] / Kp;
   Which[sigma < 180, Which[r > 1 && s > 1, base, (r < 1 && s > 1) || (r > 1 && s < 1),
     1 + base, r < 1 && s < 1, 2 + base], sigma > 180, Which[r > 1 && s > 1,
     2 + base, (r < 1 && s > 1) || (r > 1 && s < 1), 3 + base, r < 1 && s < 1, base]]];

tList = computeTi /@ Range[4];
RoundWithTolerance[x_, tol_ : 10^(-14)] :=
  Module[{nearest}, nearest = Round[x];
   If[Abs[x - nearest] ≤ tol, nearest, x]];

checkValidCombination[M1_, ε_ : 10^(-14)] :=
  Module[{i, combo, dotProd, rePart, imPart, n2, expr, foundQ = False},
   Print[Style["⚠ Approximate validation using ε-tolerance. For rigorous
       proof, see the referenced paper.", Darker@Orange, Italic]];
   Do[combo = uniqueCombos[[i]];
    dotProd = tList.combo;
    rePart = Abs[Re[dotProd]];
    imPart = Abs[Im[dotProd]];
    If[M1 < 1,
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    If[Mod[RoundWithTolerance[rePart], 4] < ε,
     If[Mod[RoundWithTolerance[imPart], 2] < ε, expr =
       tList⟦1⟧ + combo⟦2⟧ × tList⟦2⟧ + combo⟦3⟧ × tList⟦3⟧ + combo⟦4⟧ × tList⟦4⟧;
      Print[Style["✅ Valid Combination Found (M < 1):",
         Darker[Green], Bold], "\n", Style["e1 = ", Bold], combo⟦2⟧,
       Style[", e2 = ", Bold], combo⟦3⟧, Style[", e3 = ", Bold], combo⟦4⟧,
       "\n", Style["t1 = ", Bold], Re[tList⟦1⟧], "K + ", Im[tList⟦1⟧], "iK'",
       "\n", Style["t2 = ", Bold], Re[tList⟦2⟧], "K + ", Im[tList⟦2⟧], "iK'",
       "\n", Style["t3 = ", Bold], Re[tList⟦3⟧], "K + ", Im[tList⟦3⟧], "iK'",
       "\n", Style["t4 = ", Bold], Re[tList⟦4⟧], "K + ", Im[tList⟦4⟧],
       "iK'", "\n", Style["t1 + e1*t2 + e2*t3 + e3*t4 = ", Bold],
       Re[expr], "K + ", Im[expr], "iK'"];
      foundQ = True;
      Break[]]];
   If[M1 > 1,
    If[Mod[RoundWithTolerance[imPart], 2] < ε,
     n2 = Quotient[RoundWithTolerance[imPart], 2];
     If[Mod[RoundWithTolerance[Abs[rePart - 2 n2]], 4] < ε, expr =
       tList⟦1⟧ + combo⟦2⟧ × tList⟦2⟧ + combo⟦3⟧ × tList⟦3⟧ + combo⟦4⟧ × tList⟦4⟧;
      Print[Style["✅ Valid Combination Found (M > 1):",
         Darker[Green], Bold], "\n", Style["e1 = ", Bold], combo⟦2⟧,
       Style[", e2 = ", Bold], combo⟦3⟧, Style[", e3 = ", Bold], combo⟦4⟧,
       "\n", Style["t1 = ", Bold], Re[tList⟦1⟧], "K + ", Im[tList⟦1⟧], "iK'",
       "\n", Style["t2 = ", Bold], Re[tList⟦2⟧], "K + ", Im[tList⟦2⟧], "iK'",
       "\n", Style["t3 = ", Bold], Re[tList⟦3⟧], "K + ", Im[tList⟦3⟧], "iK'",
       "\n", Style["t4 = ", Bold], Re[tList⟦4⟧], "K + ", Im[tList⟦4⟧],
       "iK'", "\n", Style["t1 + e1*t2 + e2*t3 + e3*t4 = ", Bold],
       Re[expr], "K + ", Im[expr], "iK'"];
      foundQ = True;
      Break[]]]], {i, Length[uniqueCombos]}];
  If[! foundQ, Print[Style["❌ No valid combination found.", Red, Bold], "\n",
    Style["t1 = ", Bold], Re[tList⟦1⟧], "K + ", Im[tList⟦1⟧], "iK'", "\n",
    Style["t2 = ", Bold], Re[tList⟦2⟧], "K + ", Im[tList⟦2⟧], "iK'", "\n",
    Style["t3 = ", Bold], Re[tList⟦3⟧], "K + ", Im[tList⟦3⟧], "iK'", "\n",
    Style["t4 = ", Bold], Re[tList⟦4⟧], "K + ", Im[tList⟦4⟧], "iK'", "\n"]];];


Column[{TextCell[Style["================= CONDITION (N.5) =================",
    Darker[Red], Bold, 16], "Text"]}]
res = checkValidCombination[M1];


(*========================================================================
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   OTHER PARAMETER================================*)
(*======================================================================
 =====*)
cosSigmaAlt[Mi_, fi_, ri_, si_] :=
  (Mi (fi + ri + si - 1) + ri fi si - ri fi - ri si - fi si) / (2 Sqrt[ri si fi] (1 - Mi));

Column[
 {TextCell[Style["================= OTHER PARAMETERS =================",
    Darker[Orange], Bold, 16], "Text"],
  Row[{Style["u = ", Bold], 1 - M1}],
  Row[{Style["σ1 = ", Bold], σ1 Degree, Style[", σ2 = ", Bold], σ2 Degree,
    Style[", σ3 = ", Bold], σ3 Degree, Style[", σ4 = ", Bold], σ4 Degree}],
  Row[{Style["σ1 ≈ ", Bold], N[σ1], Style["°", Bold], Style[", σ2 ≈ ", Bold],
    N[σ2], Style["°", Bold], Style[", σ3 ≈ ", Bold], N[σ3],
    Style["°", Bold], Style[", σ4 ≈ ", Bold], N[σ4], Style["°", Bold]}],
  Row[{Style["cosσ1 = ", Bold], Simplify@cosSigmaAlt[M1, f1, r1, s1],
    Style[", cosσ2 = ", Bold], Simplify@cosSigmaAlt[M1, f2, r2, s2],
    Style[", cosσ3 = ", Bold], Simplify@cosSigmaAlt[M1, f3, r3, s3],
    Style[", cosσ4 = ", Bold], Simplify@cosSigmaAlt[M1, f4, r4, s4]}],
  Row[{Style["f1 = ", Bold], f1, Style[", f2 = ", Bold],
    f2, Style[", f3 = ", Bold], f3, Style[", f4 = ", Bold], f4}],
  Row[{Style["z1 = ", Bold], FullSimplify[1 / (f1 - 1)], Style[", z2 = ", Bold],
    FullSimplify[1 / (f2 - 1)], Style[", z3 = ", Bold], FullSimplify[1 / (f3 - 1)],
    Style[", z4 = ", Bold], FullSimplify[1 / (f4 - 1)]}],
  Row[{Style["x1 = ", Bold], FullSimplify[1 / (r1 - 1)], Style[", x2 = ", Bold],
    FullSimplify[1 / (r2 - 1)], Style[", x3 = ", Bold], FullSimplify[1 / (r3 - 1)],
    Style[", x4 = ", Bold], FullSimplify[1 / (r4 - 1)]}],
  Row[{Style["y1 = ", Bold], FullSimplify[1 / (s1 - 1)], Style[", y2 = ", Bold],
    FullSimplify[1 / (s2 - 1)], Style[", y3 = ", Bold], FullSimplify[1 / (s3 - 1)],
    Style[", y4 = ", Bold], FullSimplify[1 / (s4 - 1)]}],
  Row[{Style["p1 = ", Bold], Simplify[Sqrt[r1 - 1]],
    Style[", p2 = ", Bold], Simplify[Sqrt[r2 - 1]], Style[", p3 = ", Bold],
    Simplify[Sqrt[r3 - 1]], Style[", p4 = ", Bold], Simplify[Sqrt[r4 - 1]]}],
  Row[{Style["q1 = ", Bold], Simplify[Sqrt[s1 - 1]],
    Style[", q2 = ", Bold], Simplify[Sqrt[s2 - 1]], Style[", q3 = ", Bold],
    Simplify[Sqrt[s3 - 1]], Style[", q4 = ", Bold], Simplify[Sqrt[s4 - 1]]}],
  Row[{Style["p1·q1 = ", Bold], Simplify[Sqrt[r1 - 1] Sqrt[s1 - 1]],
    Style[", p2·q2 = ", Bold], Simplify[Sqrt[r2 - 1] Sqrt[s2 - 1]],
    Style[", p3·q3 = ", Bold], Simplify[Sqrt[r3 - 1] Sqrt[s3 - 1]],
    Style[", p4·q4 = ", Bold], Simplify[Sqrt[r4 - 1] Sqrt[s4 - 1]]}]
 }]


(*======================================================================
 =====*)
(*==============================
  BRICARD's EQUATIONS==============================*)
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(*==========================================================================
 =====*)
(*Bricard's equation as defined in terms of angles and variables*)
BricardsEquation[{α_, β_, γ_, δ_}, σ_, x_, y_] := Module[
   {c22, c20, c02, c11, c00},
   c22 = Sin[σ - δ] Sin[σ - δ - β];
   c20 = Sin[σ - α] Sin[σ - α - β];
   c02 = Sin[σ - γ] Sin[σ - γ - β];
   c11 = -Sin[α] Sin[γ];
   c00 = Sin[σ] Sin[σ - β];
   c22 x^2 y^2 + c20 x^2 + c02 y^2 + 2 c11 x y + c00
  ];

(*Step 1: Bricard's system of equations*)
Column[{(*Header*)TextCell[
   Style["================= Bricard's System of Equations =================",
    Darker[Purple], Bold, 16], "Text"], (*Explanatory note*)
  Row[
   {TextCell[Style["We introduce new notation for the cotangents of half of
        the dihedral angles. Denote Z:= ", GrayLevel[0.3], 13],
     "Text"], TraditionalForm[cot[Subscript[θ, 1] / 2]], TextCell[
     Style[", W2= ", GrayLevel[0.3], 13], "Text"],
    TraditionalForm[cot[Subscript[θ, 2] / 2]],
    TextCell[Style[", U:= ", GrayLevel[0.3], 13], "Text"],
    TraditionalForm[cot[Subscript[θ, 3] / 2]],
    TextCell[Style[", and W1= ", GrayLevel[0.3], 13], "Text"],
    TraditionalForm[cot[Subscript[θ, 4] / 2]]
   }], Spacer[12],
  (*Traditional form results*)Row[{"P1(Z, W1) = ",
    TraditionalForm[Collect[FullSimplify[BricardsEquation[anglesDeg[[1]] Degree,
        sigmas[[1]] Degree, Z, W1]], W1]], " = 0"}], Spacer[6],
  Row[{"P2(Z, W2) = ",
    TraditionalForm[Collect[FullSimplify[BricardsEquation[anglesDeg[[2]] Degree,
        sigmas[[2]] Degree, Z, W2]], W2]], " = 0"}], Spacer[6],
  Row[{"P3(U, W2) = ",
    TraditionalForm[Collect[FullSimplify[BricardsEquation[anglesDeg[[3]] Degree,
        sigmas[[3]] Degree, U, W2]], W2]], " = 0"}], Spacer[6],
  Row[{"P4(U, W1) = ", TraditionalForm[Collect[FullSimplify[BricardsEquation[
        anglesDeg[[4]] Degree, sigmas[[4]] Degree, U, W1]], W1]], " = 0"}]
 }]
(*==================================
 FLEXION 1==================================*)

Z[t_] := t;
W1[t_] := (1.8303883744906646` (0.739190870110122` t - 0.8185802872931142`
        Sqrt[(1 + 3.4575776313801847` t^2) (1 - 0.7811714739558353` t^2)])) /
```

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    (-1.2264950862699229` - 3.4575776313801847` t^2);
U[t_] := (0.18029302872898165` (11.610011024543208` t + 6.663793331850769`
          Sqrt[(1 + 3.4575776313801847` t^2) (1 - 0.7811714739558353` t^2)])) /
    (2.6091661366212175` + 3.845171738795376` t^2);
W2[t_] := (0.8842187622039149` (0.8494336559689466` t - 1.1387226496890441`
          Sqrt[(1 + 3.4575776313801847` t^2) (1 - 0.7811714739558353` t^2)])) /
    (-1.1465569838598522` - 3.4575776313801847` t^2);

(*=================================
 FLEXION 2=================================*)
Z2[t_] := t;
W12[t_] := (1.8303883744906646` (0.739190870110122` t + 0.8185802872931142`
          Sqrt[(1 + 3.4575776313801847` t^2) (1 - 0.7811714739558353` t^2)])) /
    (-1.2264950862699229` - 3.4575776313801847` t^2);
U2[t_] := (0.18029302872898165` (11.610011024543208` t - 6.663793331850769`
          Sqrt[(1 + 3.4575776313801847` t^2) (1 - 0.7811714739558353` t^2)])) /
    (2.6091661366212175` + 3.845171738795376` t^2);
W22[t_] := (0.8842187622039149` (0.8494336559689466` t + 1.1387226496890441`
          Sqrt[(1 + 3.4575776313801847` t^2) (1 - 0.7811714739558353` t^2)])) /
    (-1.1465569838598522` - 3.4575776313801847` t^2);

(*Step 2: Formulas for flexions*)
Column[
 {(*Header*)TextCell[Style["================= FLEXIONS =================",
    Red, Bold, 16], "Text"], (*Explanatory note*)TextCell[Style[
    "Solutions to Bricard's equations under a free parameter t := Z ∈ ℂ:",
    GrayLevel[0.3], 13], "Text"], Spacer[12],
  (*Heading for results*)TextCell[Style["Solution 1:", Bold, 14], "Text"],
  Spacer[6], (*Traditional form results*)
  Row[{"Z(t) = ", TraditionalForm[FullSimplify[Z[t]]]}], Spacer[6],
  Row[{"W₂(t) = ", TraditionalForm[FullSimplify[W2[t]]]}], Spacer[6],
  Row[{"U(t) = ", TraditionalForm[FullSimplify[U[t]]]}], Spacer[6],
  Row[{"W₁(t) = ", TraditionalForm[FullSimplify[W1[t]]]}],  Spacer[12],
  (*Heading for results*)TextCell[Style["Solution 2:", Bold, 14], "Text"],
  Spacer[6], (*Traditional form results*)
  Row[{"Z(t) = ", TraditionalForm[FullSimplify[Z2[t]]]}], Spacer[6],
  Row[{"W₂(t) = ", TraditionalForm[FullSimplify[W22[t]]]}], Spacer[6],
  Row[{"U(t) = ", TraditionalForm[FullSimplify[U2[t]]]}], Spacer[6],
  Row[{"W₁(t) = ", TraditionalForm[FullSimplify[W12[t]]]}]
 }]

(*Step 3: Checking that W₁(t) and W₂(t) solves P₁(t, W₁) =
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 0 and P₂(t, W₂) = 0 even when +- signs do NOT agree*)
(*t-range*)
tMin = 0;
tMax = 0.6;

Column[
 {(*Header*)TextCell[Style["================= Equations: P₁(t, W₁) = 0
      and P₂(t, W₂) = 0 =================",
    Orange, Bold, 16], "Text"], (*Explanatory note*)TextCell[
   Style["Let W₁ₛ₁ and W₁ₛ₂ be formulas for W₁(t) from solutions
      1 and 2, respectively. Similarly, let W₂ₛ₁ and W₂ₛ₂ be
      the formulas for W₂(t). We show that all four pairs -
      (W₁ₛ₁, W₂ₛ₁), (W₁ₛ₁, W₂ₛ₂), (W₁ₛ₂, W₂ₛ₁), and (W₁ₛ₂, W₂ₛ₂)
      - solve equations P₁(t, W₁) = 0 and P₂(t, W₂) = 0.",
    GrayLevel[0.3], 13], "Text"], Spacer[12],
  (*Heading for results*)

  TextCell[Style["Pair 1, (W₁, W₂) = (W₁ₛ₁, W₂ₛ₁):", Bold, 14], "Text"],
  Spacer[6], (*Traditional form results*)
  Row[{"P₁(t, W₁) = ", TraditionalForm[FullSimplify[BricardsEquation[
       anglesDeg⟦1⟧ Degree, sigmas⟦1⟧ Degree, Z[t], W1[t]]]]}], Spacer[6],
  Row[{"P₂(t, W₂) = ", TraditionalForm[FullSimplify[BricardsEquation[
       anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z[t], W2[t]]]]}], Spacer[12],
  (*Verification Plots Module:This entire Module is now the last item inside
     the main Column.It will execute and place the resulting Panel here.*)
  Module[{PiExpr, labels}, (*1. Define the two polynomial expressions
    to be plotted*)PiExpr = {FullSimplify[BricardsEquation[
       anglesDeg⟦1⟧ Degree, sigmas⟦1⟧ Degree, Z[t], W1[t]]], FullSimplify[
      BricardsEquation[anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z[t], W2[t]]]};
   (*2. Define the labels for each plot*)
   labels = {Row[{Subscript["P", 1], "(t, ", Subscript["W", 1], ")", " ≈ 0 ",
       Style["✓", Darker[Green], Bold]}], Row[{Subscript["P", 2], "(t, ",
       Subscript["W", 2], ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}]};
   (*3. Display the plots in the specified panel style*)
   Panel[GraphicsGrid[Partition[Table[Plot[PiExpr⟦i⟧, {t, tMin, tMax},
        PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → {-10^-10, 10^-10},
        (*Zoom in to confirm zero*)AxesLabel → {"t", None}, ImageSize → 300],
       {i, Length[PiExpr]}], 2 (*Arrange plots in rows of 2*)],
     Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]],

  TextCell[Style["Pair 2, (W₁, W₂) = (W₁ₛ₁, W₂ₛ₂):", Bold, 14], "Text"],
  Spacer[6], (*Traditional form results*)
  Row[{"P₁(t, W₁) = ", TraditionalForm[FullSimplify[BricardsEquation[
       anglesDeg⟦1⟧ Degree, sigmas⟦1⟧ Degree, Z[t], W1[t]]]]}], Spacer[6],
  Row[{"P₂(t, W₂) = ", TraditionalForm[FullSimplify[BricardsEquation[
       anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z[t], W22[t]]]]}], Spacer[12],
  (*Verification Plots Module:This entire Module is now the last item inside
```

```
   the main Column.It will execute and place the resulting Panel here.*)
Module[{PiExpr, labels}, (*1. Define the two polynomial expressions
  to be plotted*)PiExpr = {FullSimplify[BricardsEquation[
     anglesDeg⟦1⟧ Degree, sigmas⟦1⟧ Degree, Z[t], W1[t]]], FullSimplify[
    BricardsEquation[anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z[t], W22[t]]]};
 (*2. Define the labels for each plot*)
 labels = {Row[{Subscript["P", 1], "(t, ", Subscript["W", 1], ")", " ≈ 0 ",
     Style["✓", Darker[Green], Bold]}], Row[{Subscript["P", 2], "(t, ",
     Subscript["W", 2], ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}]};
 (*3. Display the plots in the specified panel style*)
 Panel[GraphicsGrid[Partition[Table[Plot[PiExpr⟦i⟧, {t, tMin, tMax},
      PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → {-10^-10, 10^-10},
      (*Zoom in to confirm zero*)AxesLabel → {"t", None}, ImageSize → 300],
     {i, Length[PiExpr]}], 2 (*Arrange plots in rows of 2*)],
   Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]],

TextCell[Style["Pair 3, (W₁, W₂) = (W₁ₛ₂, W₂ₛ₁):", Bold, 14], "Text"],
Spacer[6], (*Traditional form results*)
Row[{"P₁(t, W₁) = ", TraditionalForm[FullSimplify[BricardsEquation[
     anglesDeg⟦1⟧ Degree, sigmas⟦1⟧ Degree, Z[t], W12[t]]]]}], Spacer[6],
Row[{"P₂(t, W₂) = ", TraditionalForm[FullSimplify[BricardsEquation[
     anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z[t], W2[t]]]]}], Spacer[12],
(*Verification Plots Module:This entire Module is now the last item inside
   the main Column.It will execute and place the resulting Panel here.*)
Module[{PiExpr, labels}, (*1. Define the two polynomial expressions
  to be plotted*)PiExpr = {FullSimplify[BricardsEquation[
     anglesDeg⟦1⟧ Degree, sigmas⟦1⟧ Degree, Z[t], W12[t]]], FullSimplify[
    BricardsEquation[anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z[t], W2[t]]]};
 (*2. Define the labels for each plot*)
 labels = {Row[{Subscript["P", 1], "(t, ", Subscript["W", 1], ")", " ≈ 0 ",
     Style["✓", Darker[Green], Bold]}], Row[{Subscript["P", 2], "(t, ",
     Subscript["W", 2], ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}]};
 (*3. Display the plots in the specified panel style*)
 Panel[GraphicsGrid[Partition[Table[Plot[PiExpr⟦i⟧, {t, tMin, tMax},
      PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → {-10^-10, 10^-10},
      (*Zoom in to confirm zero*)AxesLabel → {"t", None}, ImageSize → 300],
     {i, Length[PiExpr]}], 2 (*Arrange plots in rows of 2*)],
   Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]],

TextCell[Style["Pair 4, (W₁, W₂) = (W₁ₛ₂, W₂ₛ₂):", Bold, 14], "Text"],
Spacer[6], (*Traditional form results*)
Row[{"P₁(t, W₁) = ", TraditionalForm[FullSimplify[BricardsEquation[
     anglesDeg⟦1⟧ Degree, sigmas⟦1⟧ Degree, Z[t], W12[t]]]]}], Spacer[6],
Row[{"P₂(t, W₂) = ", TraditionalForm[FullSimplify[BricardsEquation[
     anglesDeg⟦2⟧ Degree, sigmas⟦2⟧ Degree, Z[t], W22[t]]]]}], Spacer[12],
(*Verification Plots Module:This entire Module is now the last item inside
   the main Column.It will execute and place the resulting Panel here.*)
```

```
 Module[{PiExpr, labels}, (*1. Define the two polynomial expressions
   to be plotted*)PiExpr = {FullSimplify[BricardsEquation[
      anglesDeg[[1]] Degree, sigmas[[1]] Degree, Z[t], W12[t]]], FullSimplify[
     BricardsEquation[anglesDeg[[2]] Degree, sigmas[[2]] Degree, Z[t], W22[t]]]};
  (*2. Define the labels for each plot*)
  labels = {Row[{Subscript["P", 1], "(t, ", Subscript["W", 1], ")", " ≈ 0 ",
      Style["✓", Darker[Green], Bold]}], Row[{Subscript["P", 2], "(t, ",
      Subscript["W", 2], ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}]};
  (*3. Display the plots in the specified panel style*)
  Panel[GraphicsGrid[Partition[Table[Plot[PiExpr[[i]], {t, tMin, tMax},
       PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → {-10^-10, 10^-10},
       (*Zoom in to confirm zero*)AxesLabel → {"t", None}, ImageSize → 300],
      {i, Length[PiExpr]}], 2 (*Arrange plots in rows of 2*)],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]]
 }]


(*Step 4: Checking that W1(t) and W2(t) does NOT satisfy W1(t)/W2(t) =
 c and W1(t)W2(t) = c where c is constant even when +- signs do NOT agree*)
expressions = {Z[t] * U[t], Z[t] / U[t], Z[t] * U2[t], Z[t] / U2[t],
   Z2[t] * U[t], Z2[t] / U[t], Z2[t] * U2[t], Z2[t] / U2[t]};
labels = {"W1s1 * W2s1", "W1s1 / W2s1", "W1s1 * W2s2", "W1s1 / W2s2",
   "W1s2 * W2s1", "W1s2 / W2s1", "W1s2 * W2s2", "W1s2 / W2s2"};
Column[
 {TextCell[Style["============= Plots of W1(t)W2(t) and W1(t)/W2(t) For All
      Pairs =============", Darker[Orange], Bold, 16], "Text"],

  (*Explanatory text*)
  TextCell[Style["Checking that all four pairs - (W1s1, W2s1), (W1s1, W2s2),
     (W1s2, W2s1), and (W1s2, W2s2) - does NOT satisfy W1(t)/W2(t)
     = const and W1(t)W2(t) = const.", GrayLevel[0.3]], "Text"],
  Spacer[12],

  (*Plots panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions[[i]], {t, tMin, tMax},
      PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → All,
      AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
 }]


(*Step 5: Solving P3(U, W2) = 0 and P4(U, W1) = 0 for W1 and W2*)
(*Define the first Bricard equation P3(U, W2) = 0*)
P3 = BricardsEquation[anglesDeg[[3]] Degree, sigmas[[3]] Degree, U, Ww2];
(*Solve for W21,W22*)
solutionW2Expressions =
  (Ww2 /. FullSimplify[Solve[Rationalize[P3, 0] == 0, Ww2]]);
```

```
Ww21[U_] := Simplify[solutionW2Expressions[[1]] // N];
Ww22[U_] := Simplify[solutionW2Expressions[[2]] // N];

(*Define the fourth Bricard equation P_4(U, W_1) = 0*)
P4 = BricardsEquation[anglesDeg[[4]] Degree, sigmas[[4]] Degree, U, Ww1];
(*Solve for W11,W12*)
solutionW1Expressions =
  (Ww1 /. FullSimplify[Solve[Rationalize[P4, 0] == 0, Ww1]]);
Ww11[U_] := Simplify[solutionW1Expressions[[1]] // N];
Ww12[U_] := Simplify[solutionW1Expressions[[2]] // N];

Column[
 {(*Header*)TextCell[Style["================= Equations: P_3(U, W_2) = 0
      and P_4(U, W_1) = 0 =================",
    Darker[Cyan], Bold, 16], "Text"], (*Explanatory note*)
  TextCell[Style["We solve P_3(U, W_2) = 0 and P_4(U, W_1) = 0 for W_1 and W_2.",
    GrayLevel[0.3], 13], "Text"], Spacer[12],
  (*Heading for results*)Row[{"W_1,1(U) = ", TraditionalForm[Ww11[U]]}],
  Row[{"W_1,2(U) = ", TraditionalForm[Ww12[U]]}],
  Spacer[6], Row[{"W_2,1(U) = ", TraditionalForm[Ww21[U]]}],
  Row[{"W_2,2(U) = ", TraditionalForm[Ww22[U]]}]
 }]


(*Step 6: Checking that pairs (W_1,1(U), W_2,1(U)), (W_1,1(U), W_2,2(U)),
(W_1,2(U), W_2,1(U)), and (W_1,2(U), W_2,2(U)) does NOT satisfy W_1(U)/W_2(U) =
 c and W_1(U)W_2(U) = c where c is constant*)
UMin = 0;
UMax = 0.4;

expressions =
  {Ww11[U] * Ww21[U], Ww11[U] / Ww21[U], Ww11[U] * Ww22[U], Ww11[U] / Ww22[U],
   Ww12[U] * Ww21[U], Ww12[U] / Ww21[U], Ww12[U] * Ww22[U], Ww12[U] / Ww22[U]};
labels = {"W_1,1 * W_2,1", "W_1,1 / W_2,1", "W_1,1 * W_2,2",
   "W_1,1 / W_2,2", "W_1,2 * W_2,1", "W_1,2 / W_2,1", "W_1,2 * W_2,2", "W_1,2 / W_2,2"};
Column[
 {TextCell[Style["============= Plots of W_1(U)W_2(U) and W_1(U)/W_2(U) For All
      Pairs =============", Magenta, Bold, 16], "Text"],

  (*Explanatory text*)
  TextCell[Style["Checking that all four pairs - (W_1,1(U),
     W_2,1(U)), (W_1,1(U), W_2,2(U)), (W_1,2(U), W_2,1(U)), and
     (W_1,2(U), W_2,2(U)) - does NOT satisfy W_1(U)/W_2(U) = const
     and W_1(U)W_2(U) = const.", GrayLevel[0.3]], "Text"],
  Spacer[12],

  (*Plots panel*)
```

```
  Panel[GraphicsGrid[Partition[Table[Plot[expressions[[i]], {U, UMin, UMax},
       PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → All,
       AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
 }]

(*Step 7: Compute and print all P_i for flexions 1 and 2*)
TextCell[
 Style["================= FLEXIBILITY (Double Checking) =================",
  Orange, Bold, 16], "Text"]
funcs = {{Z, W1}, {Z, W2}, {U, W2}, {U, W1}};
TextCell[Style["Solution 1:", Bold, 14], "Text"]
Do[angles = anglesDeg[[i]] Degree;
  sigma = sigmas[[i]] Degree;
  {α, β, γ, δ} = angles;
  poly = Which[i == 1, BricardsEquation[{α, β, γ, δ}, sigma, Z[t], W1[t]],
    i == 2, BricardsEquation[{α, β, γ, δ}, sigma, Z[t], W2[t]],
    i == 3, BricardsEquation[{α, β, γ, δ}, sigma, U[t], W2[t]],
    i == 4, BricardsEquation[{α, β, γ, δ}, sigma, U[t], W1[t]]];
  Print[Row[{Subscript["P", i], "(", funcs[[i, 1]],
     ", ", funcs[[i, 2]], ") = ", FullSimplify[poly]}]], {i, 1, 4}];

(*Build P_i(t) consistently with your printing logic*)
Clear[PiExpr, labels, funcs];
funcs = {{Z, W1}, {Z, W2}, {U, W2}, {U, W1}};

PiExpr = Table[Module[{angles = anglesDeg[[i]] Degree,
     sigma = sigmas[[i]] Degree, α, β, γ, δ, poly}, {α, β, γ, δ} = angles;
    poly = BricardsEquation[{α, β, γ, δ}, sigma, funcs[[i, 1]][t], funcs[[i, 2]][t]];
    FullSimplify[poly]], {i, 1, 4}];

labels = Table[Row[
    {Subscript["P", i], "(", ToString@funcs[[i, 1]], ", ", ToString@funcs[[i, 2]],
     ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}], {i, 1, 4}];

(*Pretty panel with plots like your style*)
Column[{TextCell[Style["================= FLEXION 1 =================",
    Darker[Cyan], Bold, 16], "Text"], TextCell[
   Style["Polynomials Pi(t) built from Bricard's equations for flexion 1.",
    GrayLevel[0.3]], "Text"], Spacer[12],
  Panel[GraphicsGrid[Partition[Table[Plot[Evaluate@PiExpr[[i]],
       {t, tMin, tMax}, PlotLabel → Style[labels[[i]], Bold, 14],
       PlotRange → {-10^(-11), 10^(-11)}, AxesLabel → {"t", None},
       ImageSize → 250], {i, Length[PiExpr]}], 2], Spacings → {2, 2}],
   Background → Lighter[Gray, 0.95], FrameMargins → 15]}]

(*Compute and print all P_i for flexion 2*)
```

```
TextCell[Style["Solution 2:", Bold, 14], "Text"]
funcs = {{Z, W₁}, {Z, W₂}, {U, W₂}, {U, W₁}};
Do[angles = anglesDeg⟦i⟧ Degree;
  sigma = sigmas⟦i⟧ Degree;
  {α, β, γ, δ} = angles;
  poly = Which[i == 1, BricardsEquation[{α, β, γ, δ}, sigma, Z2[t], W12[t]],
    i == 2, BricardsEquation[{α, β, γ, δ}, sigma, Z2[t], W22[t]],
    i == 3, BricardsEquation[{α, β, γ, δ}, sigma, U2[t], W22[t]],
    i == 4, BricardsEquation[{α, β, γ, δ}, sigma, U2[t], W12[t]]];
  Print[Row[{Subscript["P", i], "(", funcs⟦i, 1⟧,
     ", ", funcs⟦i, 2⟧, ") = ", FullSimplify[poly]}]], {i, 1, 4}];

(*Build P_i(t) consistently with your printing logic*)
Clear[PiExpr, labels, funcs];
funcs = {{Z2, W12}, {Z2, W22}, {U2, W22}, {U2, W12}};

PiExpr = Table[Module[{angles = anglesDeg⟦i⟧ Degree,
      sigma = sigmas⟦i⟧ Degree, α, β, γ, δ, poly}, {α, β, γ, δ} = angles;
     poly = BricardsEquation[{α, β, γ, δ}, sigma, funcs⟦i, 1⟧[t], funcs⟦i, 2⟧[t]];
     FullSimplify[poly]], {i, 1, 4}];

labels = Table[Row[
     {Subscript["P", i], "(", ToString@funcs⟦i, 1⟧, ", ", ToString@funcs⟦i, 2⟧,
      ")", " ≈ 0 ", Style["✓", Darker[Green], Bold]}], {i, 1, 4}];

(*Pretty panel with plots like your style*)
Column[{TextCell[Style["================= FLEXION 2 =================",
     Magenta, Bold, 16], "Text"], TextCell[
    Style["Polynomials Pᵢ(t) built from Bricard’s equations for flexion 2.",
     GrayLevel[0.3]], "Text"], Spacer[12],
  Panel[GraphicsGrid[Partition[Table[Plot[Evaluate@PiExpr⟦i⟧,
        {t, tMin, tMax}, PlotLabel → Style[labels⟦i⟧, Bold, 14],
        PlotRange → {-10^(-11), 10^(-11)}, AxesLabel → {"t", None},
        ImageSize → 250], {i, Length[PiExpr]}], 2], Spacings → {2, 2}],
   Background → Lighter[Gray, 0.95], FrameMargins → 15]}]

(*======================================================================
 =====*)
(*==================NOT LINEAR COMPOUND==================*)
(*======================================================================
 =====*)
(*{TraditionalForm[cot[Subscript[θ,1]/2]],
  TraditionalForm[cot[Subscript[θ,2]/2]],
  TraditionalForm[cot[Subscript[θ,3]/2]],
  TraditionalForm[cot[Subscript[θ,4]/2]]};*)

(*==================================
```

```
 FLEXION 1=================================*)
(*List of expressions& labels*)
expressions =
  {Z[t] * U[t], Z[t] / U[t], W1[t] * W2[t], W1[t] / W2[t], Z[t] * W2[t], Z[t] / W2[t],
   U[t] * W1[t], U[t] / W1[t], Z[t] * W1[t], Z[t] / W1[t], W2[t] * U[t], W2[t] / U[t]};
labels = {"Z * U", "Z / U", "W1 * W2", "W1 / W2", "Z * W2", "Z / W2",
   "U * W1", "U / W1", "Z * W1", "Z / W1", "W2 * U", "W2 / U"};

Column[{TextCell[Style["============ NOT LINEAR COMPOUND ============",
     Darker[Brown], Bold, 16], "Text"],
  (*Explanatory text*)TextCell[
   Style["Above we consider the first pair of equations (P1(t, W1) = 0 and
      P2(t, W2) = 0). Solving them as quadratic equations in W1 and W2,
      respectively we parametrize the solutions by the first two  and
      fourth expressions in Solutions 1 and 2 in a neighborhood of any
      point (W1, t, W2) such that the expression in the square root and
      denominators are not zero. Here, we choose any continuous branch
      of the square root in this neighborhood, and the signs in ± need
      Not agree (this means we consider all 4 pairs we describe above).
      We conclude that NO component of the solution set of the first
      pair of Bricard's equations satisfies W1/W2=const NOR W1W2=const.

Analogously, NO component of the solution set of the other pair
      of equations (P3(U, W2) = 0 and P4(U, W1) = 0) satisfies
      W1/W2=const NOR W1W2=const. As a result, NO component
      of the solution set of all four equations satisfies
      W1/W2=const NOR W1W2=const. So, our example does not belong
      to the linear compound class, even after switching the
boundary strips.", GrayLevel[0.3]], "Text"],
  Spacer[6],
  TextCell[Style[
    "Below, we also present the plots of functions ZU, Z/U, W1W2, W1/W2, ZW2,
      Z/W2, UW1, U/W1, ZW1, Z/W1, W2U, W2/U.", GrayLevel[0.3]], "Text"],
  Spacer[6],
  TextCell[Style["Solution 1:", Bold, 14], "Text"], Spacer[12],
  Panel[GraphicsGrid[Partition[Table[Plot[expressions⟦i⟧, {t, tMin, tMax},
       PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → All,
       AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
 }]


(*=================================
 FLEXION 2=================================*)
(*List of expressions& labels*)
expressions = {Z2[t] * U2[t], Z2[t] / U2[t], W12[t] * W22[t], W12[t] / W22[t],
   Z2[t] * W22[t], Z2[t] / W22[t], U2[t] * W12[t], U2[t] / W12[t],
```

```
   Z2[t] * W12[t], Z2[t] / W12[t], W22[t] * U2[t], W22[t] / U2[t]};
labels = {"Z * U", "Z / U", "W₁ * W₂", "W₁ / W₂", "Z * W₂", "Z / W₂",
   "U * W₁", "U / W₁", "Z * W₁", "Z / W₁", "W₂ * U", "W₂ / U"};

Column[{ TextCell[Style["Solution 2:", Bold, 14], "Text"], Spacer[12],
  (*Plots panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions[[i]], {t, tMin, tMax},
      PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → All,
      AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]
 }]

(*============================================================================
 =====*)
(*===================================
 NOT TRIVIAL===================================*)
(*============================================================================
 =====*)
(*===================================
 FLEXION 1=================================*)
(*List of expressions to plot*)
expressions = {Z[t], W2[t], U[t], W1[t]};
labels = {"Z", "W₂", "U", "W₁"};

Column[{TextCell[
   Style["================= NOT TRIVIAL (FLEXION 1) =================",
    Pink, Bold, 16], "Text"],

  (*Explanatory text*)
  TextCell[Style["This configuration does not belong to the trivial class —
     even after switching the boundary strips — since none of the
     functions Z, W2, U, or W1 is constant.", GrayLevel[0.3]], "Text"],
  Spacer[12],

  (*Plots in a light panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions[[i]], {t, tMin, tMax},
      PlotLabel → Style[labels[[i]], Bold, 14], PlotRange → All,
      AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]

 }]

(*===================================
 FLEXION 2=================================*)
(*List of expressions to plot*)
expressions = {Z2[t], W22[t], U2[t], W12[t]};
labels = {"Z", "W₂", "U", "W₁"};
```

```
Column[{TextCell[
   Style["================= NOT TRIVIAL (FLEXION 2) =================",
    Darker[Pink], Bold, 16], "Text"],

  (*Explanatory text*)
  TextCell[Style["This configuration does not belong to the trivial class –
      even after switching the boundary strips – since none of the
      functions Z, W2, U, or W1 is constant.", GrayLevel[0.3]], "Text"],
  Spacer[12],

  (*Plots in a light panel*)
  Panel[GraphicsGrid[Partition[Table[Plot[expressions⟦i⟧, {t, tMin, tMax},
       PlotLabel → Style[labels⟦i⟧, Bold, 14], PlotRange → All,
       AxesLabel → {"t", None}, ImageSize → 250], {i, Length[expressions]}], 2],
    Spacings → {2, 2}], Background → Lighter[Gray, 0.95], FrameMargins → 15]

 }]



(*=========================================================================
 =====*)
(*===========================
 SWITCHING BOUNDARY STRIPS===========================*)
(*=========================================================================
 =====*)
SwitchingRightBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices:{row,column}={1,2},{1,3},{4,2},{4,3}*)
  modified⟦1, 2⟧ = 180 - anglesDeg⟦1, 2⟧; (*β1*)
  modified⟦1, 3⟧ = 180 - anglesDeg⟦1, 3⟧; (*γ1*)
  modified⟦4, 2⟧ = 180 - anglesDeg⟦4, 2⟧; (*β4*)
  modified⟦4, 3⟧ = 180 - anglesDeg⟦4, 3⟧; (*γ4*)
  modified]

SwitchingLeftBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices:{row,column}={2,2},{2,3},{3,2},{3,3}*)
  modified⟦2, 2⟧ = 180 - anglesDeg⟦2, 2⟧; (*β2*)
  modified⟦2, 3⟧ = 180 - anglesDeg⟦2, 3⟧; (*γ2*)
  modified⟦3, 2⟧ = 180 - anglesDeg⟦3, 2⟧; (*β3*)
  modified⟦3, 3⟧ = 180 - anglesDeg⟦3, 3⟧; (*γ3*)
  modified]

SwitchingLowerBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices:{row,column}={1,1},{1,2},{2,1},{2,2}*)
  modified⟦1, 1⟧ = 180 - anglesDeg⟦1, 1⟧; (*α1*)
  modified⟦1, 2⟧ = 180 - anglesDeg⟦1, 2⟧; (*β1*)
```

```
  modified⟦2, 1⟧ = 180 - anglesDeg⟦2, 1⟧; (*α2*)
  modified⟦2, 2⟧ = 180 - anglesDeg⟦2, 2⟧; (*β2*)
  modified]

SwitchingUpperBoundaryStrip[anglesDeg_List] := Module[{modified = anglesDeg},
  (*Indices:{row,column}={3,1},{3,2},{4,1},{4,2}*)
  modified⟦3, 1⟧ = 180 - anglesDeg⟦3, 1⟧; (*α3*)
  modified⟦3, 2⟧ = 180 - anglesDeg⟦3, 2⟧; (*β3*)
  modified⟦4, 1⟧ = 180 - anglesDeg⟦4, 1⟧; (*α4*)
  modified⟦4, 2⟧ = 180 - anglesDeg⟦4, 2⟧; (*β4*)
  modified]


(*=========================================================================
 =====*)
(*==================================NOT CONIC & NOT CHIMERA & NOT LINEAR
     CONJUGATE & NOT ISOGONAL===================================*)
(*=========================================================================
 =====*)
uniqueCombos = {{1, 1, 1, 1}, {1, 1, 1, -1}, {1, 1, -1, -1}, {1, 1, -1, 1},
   {1, -1, 1, 1}, {1, -1, -1, 1}, {1, -1, 1, -1}, {1, -1, -1, -1}};

checkConditionN0Degrees[{α_, β_, γ_, δ_}] :=
  Module[{angles = {α, β, γ, δ}, results},
   results = Mod[uniqueCombos.angles, 360] // Chop;
   ! MemberQ[results, 0]];

Column[{TextCell[Style[
    "================= NOT CONIC & NOT CHIMERA & NOT LINEAR CONJUGATE & NOT
      ISOGONAL=================", Darker[Magenta], Bold, 16], "Text"],
  TextCell[Style[
    "Condition (N.0) is satisfied for all i=1,...,4 ⇒ NOT equimodular-conic,
      NOT chimera, NOT isogonal and NOT linear conjugate.
      Applying any boundary-strip switch still preserves
      (N.0), so no conic, no chimera, no isogonal and no
      linear conjugate form emerges.", GrayLevel[0.3]], "Text"]
 }]

(*Now the exact same Module for checking all switch combinations…*)
Module[{angles = anglesDeg, switchers, combinations, results},
 (*Define switch functions*)
 switchers = <|"Right" → SwitchingRightBoundaryStrip,
   "Left" → SwitchingLeftBoundaryStrip, "Lower" → SwitchingLowerBoundaryStrip,
   "Upper" → SwitchingUpperBoundaryStrip|>;
 (*Generate all combinations of switches (from size 1 to 4)*)
 combinations = Subsets[Keys[switchers], {1, Length[switchers]}];
 (*Evaluate condition after each combination of switches*)results = Table[
```

```
   Module[{switched = angles, name, passQ}, name = StringRiffle[combo, " + "];
    Do[switched = switchers[sw][switched], {sw, combo}];
    passQ = And @@ (checkConditionN0Degrees /@ switched);
    (*Print the result after switching*)
    (*Print["\nSwitch combination: ",name];
    Print["Switched anglesDeg:"];
    Print[MatrixForm[switched]];*)
    {name, passQ}], {combo, combinations}];
 (*Display results*)
 Column[Prepend[Table[Module[{comboName = res⟦1⟧, passQ = res⟦2⟧},
     Row[{Style[comboName <> ": ", Bold],
       If[passQ,
        Style["Condition (N.0) is still satisfied.", Darker[Green]],
        Style["Condition (N.0) fails.", Red, Bold]
       ]
      }
     ]], {res, results}], TextCell[
    Style["CONDITION (N.0) AFTER SWITCHING BOUNDARY STRIPS", 14], "Text"]]]]


(*============================================================================
 =====*)
(*===============================
 NOT ORTHODIAGONAL===============================*)
(*============================================================================
 =====*)

(*Column[
  {TextCell[Style["================= NOT ORTHODIAGONAL =================",
     Darker[Blue],Bold,16],"Text"],
   TextCell[Style[
     "cos(αᵢ)·cos(γᵢ) ≠ cos(βᵢ)·cos(δᵢ) for each i = 1 ⇒ NOT orthodiagonal.
       Switching boundary strips does not
       correct this.",  GrayLevel[0.3]],"Text"]
  }]

Module[{angles=anglesDeg,switchers,combinations,results},
  (*Define switch functions*)switchers=<|"Right"→SwitchingRightBoundaryStrip,
    "Left"→SwitchingLeftBoundaryStrip,"Lower"→SwitchingLowerBoundaryStrip,
    "Upper"→SwitchingUpperBoundaryStrip|>;
  (*Helper function:compute and print difference only*)
  formatOrthodiagonalCheck[quad_List]:=
   Module[{vals},vals=Table[Module[{a,b,c,d,lhs,rhs,diff},{a,b,c,d}=quad⟦i⟧;
       lhs=FullSimplify[Cos[a Degree] Cos[c Degree]];
       rhs=FullSimplify[Cos[b Degree] Cos[d Degree]];
       diff=Chop[lhs-rhs];
       Style[Row[{"cos(α"<>ToString[i]<>")·cos(γ"<>ToString[i]<>") - ",
```

```
          "cos(β"<>ToString[i]<>")·cos(δ"<>ToString[i]<>") = ",
          NumberForm[diff,{5,3}]}],If[diff==0,Red,Black]]],{i,Length[quad]}];
    Column[vals]];
  (*Orthodiagonal check for anglesDeg before any switching*)
  Print[TextCell[Style["\nInitial anglesDeg (no switches):",Bold]]];
  Print[MatrixForm[angles]];
  Print[TextCell[Style[
     "Orthodiagonal check: cos(αi)·cos(γi) - cos(βi)·cos(δi) for i = 1..4",
     Italic]]];
  Print[formatOrthodiagonalCheck[angles]];
  (*Generate all combinations of switches (from size 1 to 4)*)
  combinations=Subsets[Keys[switchers],{1,Length[switchers]}];
  (*Evaluate condition after each combination of switches*)results=
   Table[Module[{switched=angles,name,passQ},name=StringRiffle[combo," + "];
     Do[switched=switchers[sw][switched],{sw,combo}];
     passQ=And@@(checkConditionN0Degrees/@switched);
     Print[Style["\nSwitch combination: ", Bold],name];
     Print[Style["Switched anglesDeg:", Italic]];
     Print[MatrixForm[switched]];
     Print[
      TextCell[Style["Orthodiagonal check: cos(αi)·cos(γi) - cos(βi)·cos(δi)
          for i = 1..4",Italic]]];
     Print[formatOrthodiagonalCheck[switched]];
     {name,passQ}],{combo,combinations}];]*)


Column[
 {TextCell[Style["================= ORTHOGONALITY CHECK =================",
    Brown, Bold, 16], "Text"],
  TextCell[Style["cos(αi)·cos(γi) ≠ cos(βi)·cos(δi) for at least
      one i = 1,..., 4 ⇒ NOT orthodiagonal. Switching boundary
      strips does not correct this.", GrayLevel[0.3]], "Text"]}]

(*Helper
 function:Returns True if at least one cosine product difference is non-
   zero.Returns False if all differences are zero.*)
isNotOrthodiagonal[quad_List] :=
  Or @@ Table[Module[{a, b, c, d, diff}, {a, b, c, d} = quad[[i]];
     diff = Chop[Cos[a Degree] Cos[c Degree] - Cos[b Degree] Cos[d Degree]];
     diff ≠ 0], {i, Length[quad]}];

(*First,check the initial,unswitched angles*)
Print[TextCell[Style["\nInitial anglesDeg (no switches):", Bold]]];
If[isNotOrthodiagonal[anglesDeg], Print[Style[
    " -> Condition met: At least one difference is non-zero.", Darker@Green]],
  Print[Style[" -> Condition NOT met: All differences are zero.", Red]]];
```

```
(*Now,use your desired module to check all switch combinations*)
Module[{angles = anglesDeg, switchers, combinations, results},
 (*Define switch functions*)
 switchers = <|"Right" → SwitchingRightBoundaryStrip,
   "Left" → SwitchingLeftBoundaryStrip, "Lower" → SwitchingLowerBoundaryStrip,
   "Upper" → SwitchingUpperBoundaryStrip|>;
 (*Generate all combinations of switches (from size 1 to 4)*)
 combinations = Subsets[Keys[switchers], {1, Length[switchers]}];
 (*Evaluate condition after each combination
  of switches and store in'results'*)results = Table[
   Module[{switched = angles, name, passQ}, name = StringRiffle[combo, " + "];
    Do[switched = switchers[sw][switched], {sw, combo}];
    (* ***THIS IS THE KEY CHANGE****)(*Set passQ using our
     new helper function*)passQ = isNotOrthodiagonal[switched];
    {name, passQ}], {combo, combinations}];
 (*Display results in the specified column format*)
 Column[Prepend[Table[Module[{comboName = res[[1]], passQ = res[[2]]},
     Row[{Style[comboName <> ": ", Bold], If[passQ,
        Style["Condition met (at least one difference is non-zero).", Darker[
          Green]], Style["Condition NOT met (all differences are zero).",
         Red, Bold]]}]], {res, results}], TextCell[
    Style["\nNON-ORTHOGONALITY CHECK AFTER SWITCHING BOUNDARY STRIPS", 14],
    "Text"]]]]
```

```
(*=============================================================================
 =====*)
(*============================
   NOT CONJUGATE-MODULAR=============================*)
(*=============================================================================
 =====*)
(*Column[{TextCell[
     Style["================= NOT CONJUGATE-MODULAR =================",
      Purple,Bold,16],"Text"],
    TextCell[Style["M1 = M2 = M3 = M4 = M
        and M ≠ 2 ⇒ NOT conjugate-modular. Boundary-strip
        switches preserve this.",GrayLevel[0.3]],"Text"]
   }]
  Ms=FullSimplify[Times@@@results];
allEqualQ=Simplify[Equal@@Ms];

Module[{angles=anglesDeg,switchers,combinations,results,
  computeConjugateModularInfo},(*Define switch functions*)
 switchers=<|"Right"→SwitchingRightBoundaryStrip,
   "Left"→SwitchingLeftBoundaryStrip,"Lower"→SwitchingLowerBoundaryStrip,
   "Upper"→SwitchingUpperBoundaryStrip|>;
```

```
 (*Computes Mi and pi and prints them,
 with classification*)computeConjugateModularInfo[quad_List]:=
  Module[{abcdList,Ms,summary},abcdList=computeABCD/@quad;
   Ms=FullSimplify[Times@@@abcdList];
   summary=If[Simplify[Equal@@Ms]&&Ms[[1]]=!=2,
     Style["M1 = M2 = M3 = M4 = M and M ≠ 2",Bold],
     Style["M1 = M2 = M3 = M4 = M and M = 2",Red,Bold]];
   Column[{Style["Mi values:",Bold],Row[{"M1 = ",Ms[[1]],
       ", M2 = ",Ms[[2]],", M3 = ",Ms[[3]],", M4 = ",Ms[[4]]}],summary}]];
 (*Original anglesDeg check*)
 Print[
  TextCell[Style["\nInitial configuration (no switches applied):",Bold]]];
 Print[MatrixForm[angles]];
 Print[computeConjugateModularInfo[angles]];
 (*Generate all switch combinations (from size 1 to 4)*)
 combinations=Subsets[Keys[switchers],{1,Length[switchers]}];
 (*Evaluate each switched configuration*)results=
  Table[Module[{switched=angles,name,passQ},name=StringRiffle[combo," + "];
    Do[switched=switchers[sw][switched],{sw,combo}];
    passQ=And@@(checkConditionN0Degrees/@switched);
    Print[Style["\nSwitch combination: ", Bold],name];
    Print[Style["Switched anglesDeg:", Italic]];
    Print[MatrixForm[switched]];
    Print[computeConjugateModularInfo[switched]];
    {name,passQ}],{combo,combinations}];]*)


Column[{TextCell[
   Style["================ CONJUGATE-MODULAR CHECK ================",
    Darker[Brown], Bold, 16], "Text"],
  TextCell[Style["M1 = M2 = M3 = M4 = M and M ≠ 2 ⇒ NOT conjugate-modular.
      Boundary-strip switches preserve this.", GrayLevel[0.3]], "Text"]}]

(*Helper Function:Returns True if all M_i values are equal
   AND their common value is not 2. Returns False otherwise.*)
isNotConjugateModular[quad_List] :=
  Module[{abcdList, Ms}, abcdList = computeABCD /@ quad;
   Ms = FullSimplify[Times @@@ abcdList];
   (*The condition is met if they are all equal AND the value isn't 2*)
   Simplify[Equal @@ Ms] && (Ms[[1]] ≠ 2)];

(*First,check the initial,unswitched angles*)
Print[TextCell[Style["\nInitial anglesDeg (no switches):", Bold]]];
If[isNotConjugateModular[anglesDeg], Print[
   Style[" -> Condition met: All Mi are equal and M ≠ 2.", Darker@Green]],
  Print[Style[" -> Condition NOT met.", Red]]];
```

```
(*Now,use the clean module to check all switch combinations*)
Module[{angles = anglesDeg, switchers, combinations, results},
 (*Define switch functions*)
 switchers = <|"Right" → SwitchingRightBoundaryStrip,
   "Left" → SwitchingLeftBoundaryStrip, "Lower" → SwitchingLowerBoundaryStrip,
   "Upper" → SwitchingUpperBoundaryStrip|>;
 (*Generate all combinations of switches (from size 1 to 4)*)
 combinations = Subsets[Keys[switchers], {1, Length[switchers]}];
 (*Evaluate condition after each combination
  of switches and store the result*)results = Table[
   Module[{switched = angles, name, passQ}, name = StringRiffle[combo, " + "];
    Do[switched = switchers[sw][switched], {sw, combo}];
    (*Set passQ using our new helper function for this check*)
    passQ = isNotConjugateModular[switched];
    {name, passQ}], {combo, combinations}];
 (*Display results in the specified column format*)
 Column[Prepend[Table[Module[{comboName = res[[1]], passQ = res[[2]]},
     Row[{Style[comboName <> ": ", Bold], If[passQ,
        Style["Condition met (All Mᵢ are equal and M ≠ 2).", Darker[Green]],
        Style["Condition NOT met.", Red, Bold]]}]], {res, results}],
   TextCell[Style["\nCONJUGATE-MODULAR CHECK AFTER SWITCHING BOUNDARY STRIPS",
     14], "Text"]]]]
```

Out[248]=

**================= CONDITION (N.0) =================**

**✔ All vertices satisfy (N.0).**

| Vertex | Combinations (mod 360) | Status |
|---|---|---|
| Vertex 1 | {190.399, 70.3985, 26.5003, 146.5, 25.917, 342.019, 265.917, 222.019} | ✔ Pass |
| Vertex 2 | {280.89, 50.8897, 13.1847, 243.185, 19.1478, 341.443, 149.148, 111.443} | ✔ Pass |
| Vertex 3 | {34.4692, 234.469, 303.73, 103.73, 325.58, 34.8415, 165.58, 234.841} | ✔ Pass |
| Vertex 4 | {61.507, 211.507, 273.452, 123.452, 322.45, 24.3953, 112.45, 174.395} | ✔ Pass |

Out[251]=

**================= CONDITION (N.3) =================**

✔ **M1** = **M2** = **M3** = **M4** = 1.22593

Out[257]=

**================= CONDITION (N.4) =================**

✔ **r1** = **r2** = 0.71078; ✔ **r3** = **r4** = 0.887609

✔ **s1** = **s4** = 0.58646; ✔ **s2** = **s3** = 0.800228

Out[267]=

**================= CONDITION (N.5) =================**

⚠ *Approximate validation using ε-tolerance. For rigorous proof, see the referenced paper.*

✅ **Valid Combination Found (M > 1):**

**e1** = −1, **e2** = −1, **e3** = 1

**t1** = 2.K + 0.801037iK'

**t2** = 2.K + 0.837691iK'

**t3** = 0.K + 0.579573iK'

**t4** = 0.K + 0.616227iK'

**t1** + **e1*t2** + **e2*t3** + **e3*t4** = 0.K + $2.33147 \times 10^{-15}$iK'

Out[270]=

**================= OTHER PARAMETERS =================**

**u** = −0.22593

**σ1** = 1.66154, **σ2** = 2.45122, **σ3** = 3.44239, **σ4** = 3.67834

**σ1** ≈ 95.1993°, **σ2** ≈ 140.445°, **σ3** ≈ 197.235°, **σ4** ≈ 210.754°

**cosσ1** = −0.0906196, **cosσ2** = −0.771012, **cosσ3** = −0.9551, **cosσ4** = −0.859375

**f1** = 0.184669, **f2** = 0.127824, **f3** = 0.578993, **f4** = 0.516796

**z1** = −1.2265, **z2** = −1.14656, **z3** = −2.37526, **z4** = −2.06952

**x1** = −3.45758, **x2** = −3.45758, **x3** = −8.89754, **x4** = −8.89754

**y1** = −2.41814, **y2** = −5.00571, **y3** = −5.00571, **y4** = −2.41814

**p1** = 0. + 0.537792 i , **p2** = 0. + 0.537792 i , **p3** = 0. + 0.335247 i , **p4** = 0. + 0.335247 i

**q1** = 0. + 0.643071 i , **q2** = 0. + 0.446958 i , **q3** = 0. + 0.446958 i , **q4** = 0. + 0.643071 i

**p1·q1** = −0.345838 + 0. i, **p2·q2** = −0.24037 + 0. i , **p3·q3** = −0.149841 + 0. i, **p4·q4** = −0.215588 + 0. i

Out[272]=

**================= Bricard's System of Equations =================**

We introduce new notation for the cotangents of half of the dihedral angles. Denote Z:= $\cot\left(\frac{\theta_1}{2}\right)$, $W_2 = \cot\left(\frac{\theta_2}{2}\right)$, U:= $\cot\left(\frac{\theta_3}{2}\right)$, and $W_1 = \cot\left(\frac{\theta_4}{2}\right)$

$$P_1(Z, W_1) = W_1^2\left(-0.421853\,Z^2 - 0.149642\right) - 0.330156\,W_1\,Z - 0.213966\,Z^2 + 0.223322 = 0$$

$$P_2(Z, W_2) = W_2^2\left(-0.414164\,Z^2 - 0.13734\right) - 0.179936\,W_2\,Z - 0.0948641\,Z^2 + 0.105916 = 0$$

$$P_3(U, W_2) = \left(0.88211\,U^2 + 0.235485\right) W_2^2 + 0.41857\,U^2 - 0.808492\,U\,W_2 - 0.0876622 = 0$$

$$P_4(U, W_1) = \left(0.800007\,U^2 + 0.186077\right) W_1^2 + 0.684669\,U^2 - 0.909187\,U\,W_1 - 0.164576 = 0$$

Out[281]=

================= **FLEXIONS** =================

Solutions to Bricard's equations under a free parameter t := Z ∈ ℂ:

**Solution 1:**

$Z(t) = t$

$$W_2(t) = \frac{0.29121\sqrt{-2.70096\,t^4+2.67641\,t^2+1}-0.217229\,t}{1.\,t^2+0.331607}$$

$$U(t) = \frac{0.312453\sqrt{-2.70096\,t^4+2.67641\,t^2+1}+0.544372\,t}{1.\,t^2+0.678556}$$

$$W_1(t) = \frac{0.433344\sqrt{-2.70096\,t^4+2.67641\,t^2+1}-0.391316\,t}{1.\,t^2+0.354727}$$

**Solution 2:**

$Z(t) = t$

$$W_2(t) = \frac{-0.29121\sqrt{-2.70096\,t^4+2.67641\,t^2+1}-0.217229\,t}{1.\,t^2+0.331607}$$

$$U(t) = \frac{0.544372\,t-0.312453\sqrt{-2.70096\,t^4+2.67641\,t^2+1}}{1.\,t^2+0.678556}$$

$$W_1(t) = \frac{-0.433344\sqrt{-2.70096\,t^4+2.67641\,t^2+1}-0.391316\,t}{1.\,t^2+0.354727}$$

Out[284]=

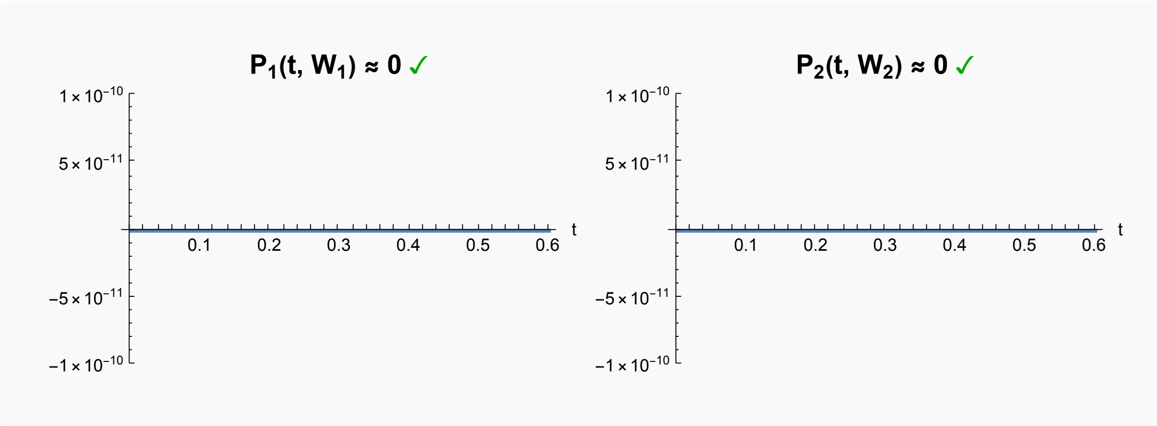
================= **Equations: $P_1(t, W_1) = 0$ and $P_2(t, W_2) = 0$** =================

Let $W_{1s1}$ and $W_{1s2}$ be formulas for $W_1(t)$ from solutions 1 and 2, respectively. Similarly, let $W_{2s1}$ and $W_{2s2}$ be the formulas for $W_2(t)$. We show that all four pairs – $(W_{1s1}, W_{2s1})$, $(W_{1s1}, W_{2s2})$, $(W_{1s2}, W_{2s1})$, and $(W_{1s2}, W_{2s2})$ – solve equations $P_1(t, W_1) = 0$ and $P_2(t, W_2) = 0$.

**Pair 1, $(W_1, W_2) = (W_{1s1}, W_{2s1})$:**

$$P_1(t, W_1) = \frac{1}{(1.\,t^2+0.354727)^2}\Big(t\,\Big(t\,\big(1.9004\times10^{-16}\,t^4-1.52032\times10^{-15}\,t^2+3.88578\times10^{-16}\sqrt{-2.70096\,t^4+2.67641\,t^2+1}\;t-8.55179\times10^{-16}\big)+7.63278\times10^{-17}\sqrt{-2.70096\,t^4+2.67641\,t^2+1}\Big)-1.12836\times10^{-16}\Big)$$

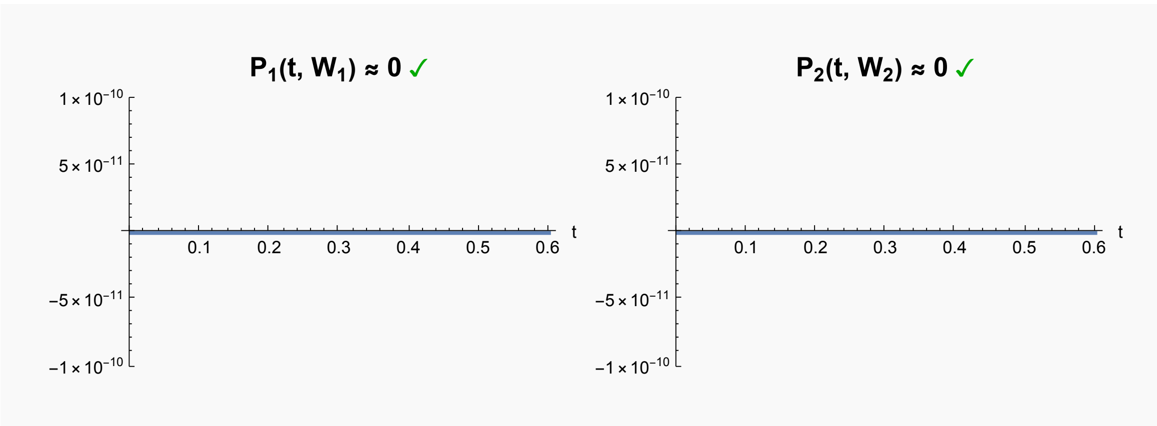
$$P_2(t, W_2) = \frac{1}{(1.\,t^2+0.331607)^2}\Big(t\,\Big(t\,\big(-1.0532\times10^{-16}\,t^4-5.6873\times10^{-16}\,t^2+7.63278\times10^{-17}\sqrt{-2.70096\,t^4+2.67641\,t^2+1}\;t-1.89577\times10^{-16}\big)-2.08167\times10^{-17}\sqrt{-2.70096\,t^4+2.67641\,t^2+1}\Big)-1.0532\times10^{-17}\Big)$$




**Pair 2, $(W_1, W_2) = (W_{1s1}, W_{2s2})$:**

$$P_1(t, W_1) = \frac{1}{(1.\ t^2+0.354727)^2}\left(t\left(t\left(1.9004\times10^{-16}\ t^4 - 1.52032\times10^{-15}\ t^2 + 3.88578\times10^{-16}\ \sqrt{-2.70096\ t^4+2.67641\ t^2+1}\ t - 8.55179\times10^{-16}\right) + 7.63278\times10^{-17}\ \sqrt{-2.70096\ t^4+2.67641\ t^2+1}\right) - 1.12836\times10^{-16}\right)$$

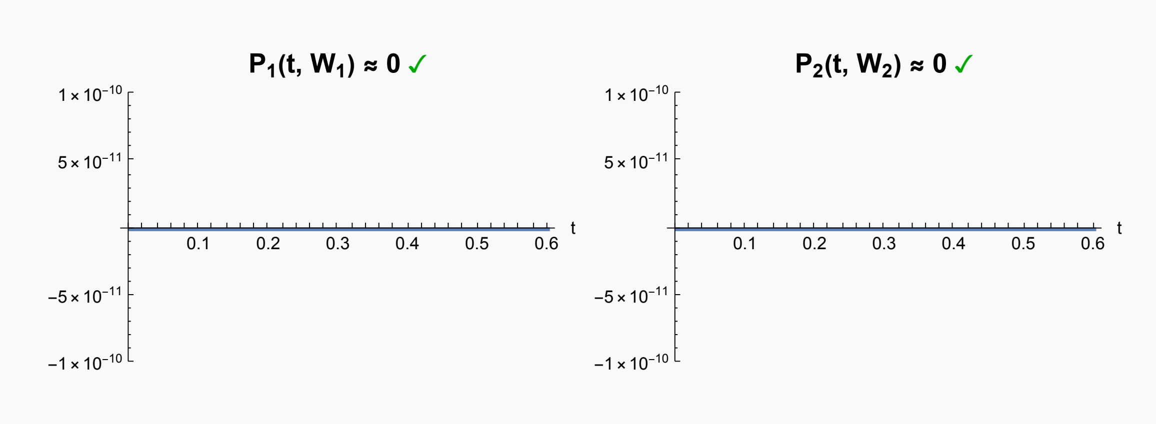
$$P_2(t, W_2) = \frac{1}{(1.\ t^2+0.331607)^2}\left(t\left(t\left(-1.0532\times10^{-16}\ t^4 - 5.6873\times10^{-16}\ t^2 - 7.63278\times10^{-17}\ \sqrt{-2.70096\ t^4+2.67641\ t^2+1}\ t - 1.89577\times10^{-16}\right) + 2.08167\times10^{-17}\ \sqrt{-2.70096\ t^4+2.67641\ t^2+1}\right) - 1.0532\times10^{-17}\right)$$




**Pair 3, $(W_1, W_2) = (W_{1s2}, W_{2s1})$:**

$$P_1(t, W_1) = \frac{1}{(1.\ t^2+0.354727)^2}\left(t\left(t\left(1.9004\times10^{-16}\ t^4 - 1.52032\times10^{-15}\ t^2 - 3.88578\times10^{-16}\ \sqrt{-2.70096\ t^4+2.67641\ t^2+1}\ t - 8.55179\times10^{-16}\right) - 7.63278\times10^{-17}\ \sqrt{-2.70096\ t^4+2.67641\ t^2+1}\right) - 1.12836\times10^{-16}\right)$$

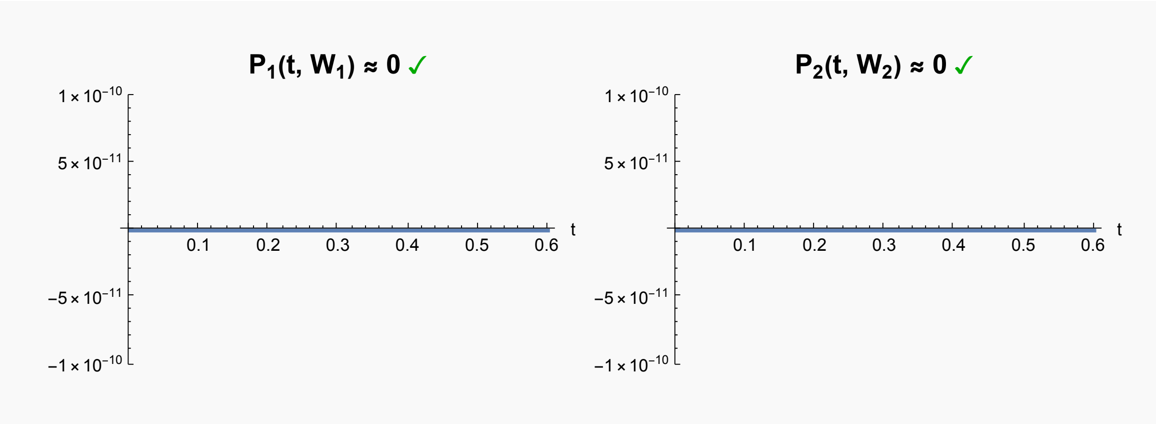
$$P_2(t, W_2) = \frac{1}{(1.\ t^2+0.331607)^2}\left(t\left(t\left(-1.0532\times10^{-16}\ t^4 - 5.6873\times10^{-16}\ t^2 + 7.63278\times10^{-17}\ \sqrt{-2.70096\ t^4+2.67641\ t^2+1}\ t - 1.89577\times10^{-16}\right) - 2.08167\times10^{-17}\ \sqrt{-2.70096\ t^4+2.67641\ t^2+1}\right) - 1.0532\times10^{-17}\right)$$




**Pair 4, $(W_1, W_2) = (W_{1s2}, W_{2s2})$:**

$$P_1(t, W_1) = \frac{1}{(1.\ t^2+0.354727)^2}\left(t\left(t\left(1.9004\times10^{-16}\ t^4-1.52032\times10^{-15}\ t^2-3.88578\times10^{-16}\ \sqrt{-2.70096\ t^4+2.67641\ t^2+1}\ t-8.55179\times10^{-16}\right)-7.63278\times10^{-17}\ \sqrt{-2.70096\ t^4+2.67641\ t^2+1}\right)-1.12836\times10^{-16}\right)$$

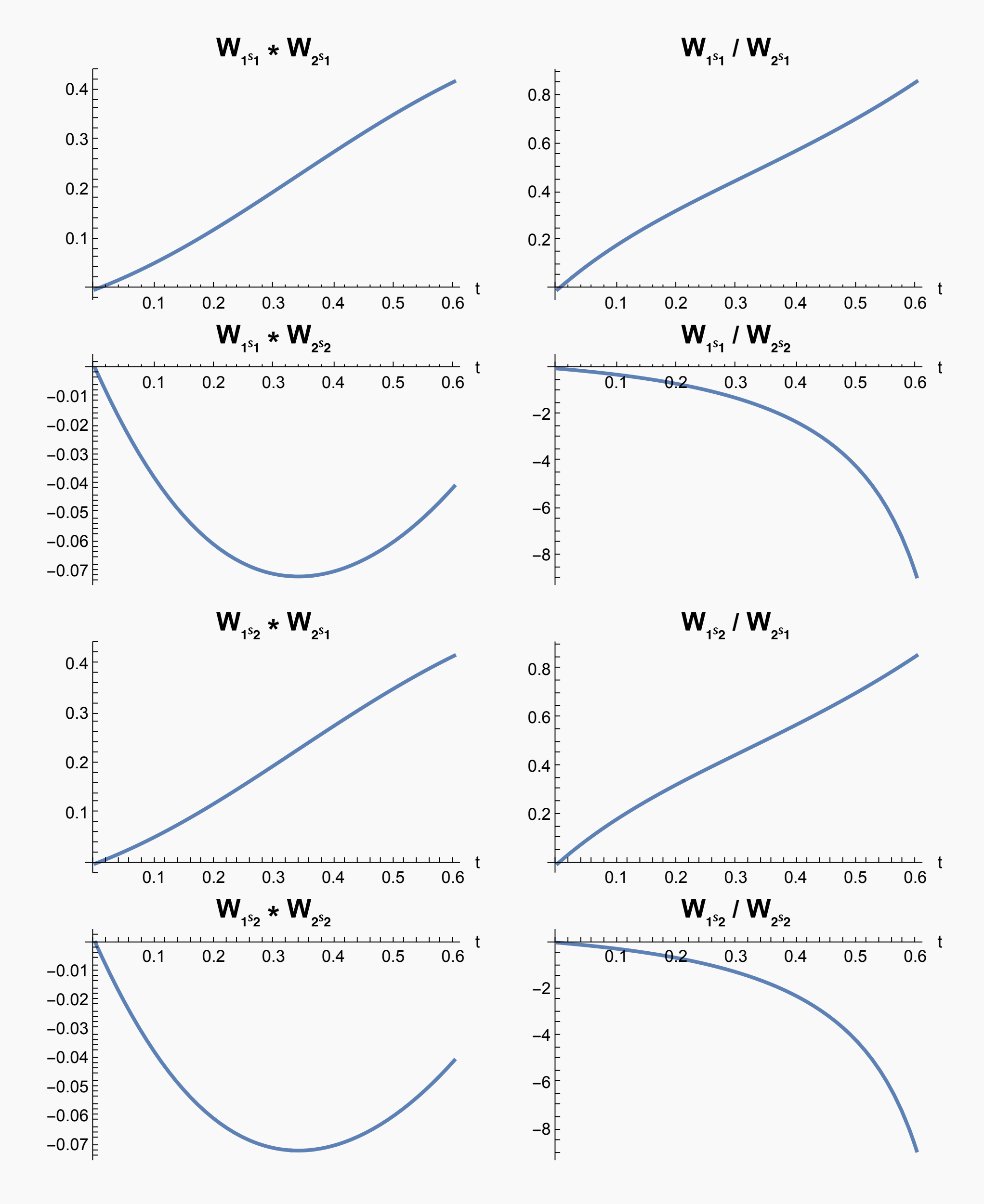
$$P_2(t, W_2) = \frac{1}{(1.\ t^2+0.331607)^2}\left(t\left(t\left(-1.0532\times10^{-16}\ t^4-5.6873\times10^{-16}\ t^2-7.63278\times10^{-17}\ \sqrt{-2.70096\ t^4+2.67641\ t^2+1}\ t-1.89577\times10^{-16}\right)+2.08167\times10^{-17}\ \sqrt{-2.70096\ t^4+2.67641\ t^2+1}\right)-1.0532\times10^{-17}\right)$$



Out[287]=

**============= Plots of $W_1(t)W_2(t)$ and $W_1(t)/W_2(t)$ For All Pairs =============**

Checking that all four pairs – $(W_{1s1}, W_{2s1})$, $(W_{1s1}, W_{2s2})$, $(W_{1s2}, W_{2s1})$, and $(W_{1s2}, W_{2s2})$ – does NOT satisfy $W_1(t)/W_2(t)$ = const and $W_1(t)W_2(t)$ = const.



Out[296]=

## ================= Equations: $P_3(U, W_2) = 0$ and $P_4(U, W_1) = 0$ =================

We solve $P_3(U, W_2) = 0$ and $P_4(U, W_1) = 0$ for $W_1$ and $W_2$.

$$W_{1\,1}(U) = \frac{1.97005\times10^{16}\,U - 1.95903\times10^{-8}\,\sqrt{-2.6804\times10^{48}\,U^4 + 1.03213\times10^{48}\,U^2 + 1.49859\times10^{47}}}{3.46695\times10^{16}\,U^2 + 8.06392\times10^{15}}$$

$$W_{1\,2}(U) = \frac{1.95903\times10^{-8}\,\sqrt{-2.6804\times10^{48}\,U^4 + 1.03213\times10^{48}\,U^2 + 1.49859\times10^{47}} + 1.97005\times10^{16}\,U}{3.46695\times10^{16}\,U^2 + 8.06392\times10^{15}}$$

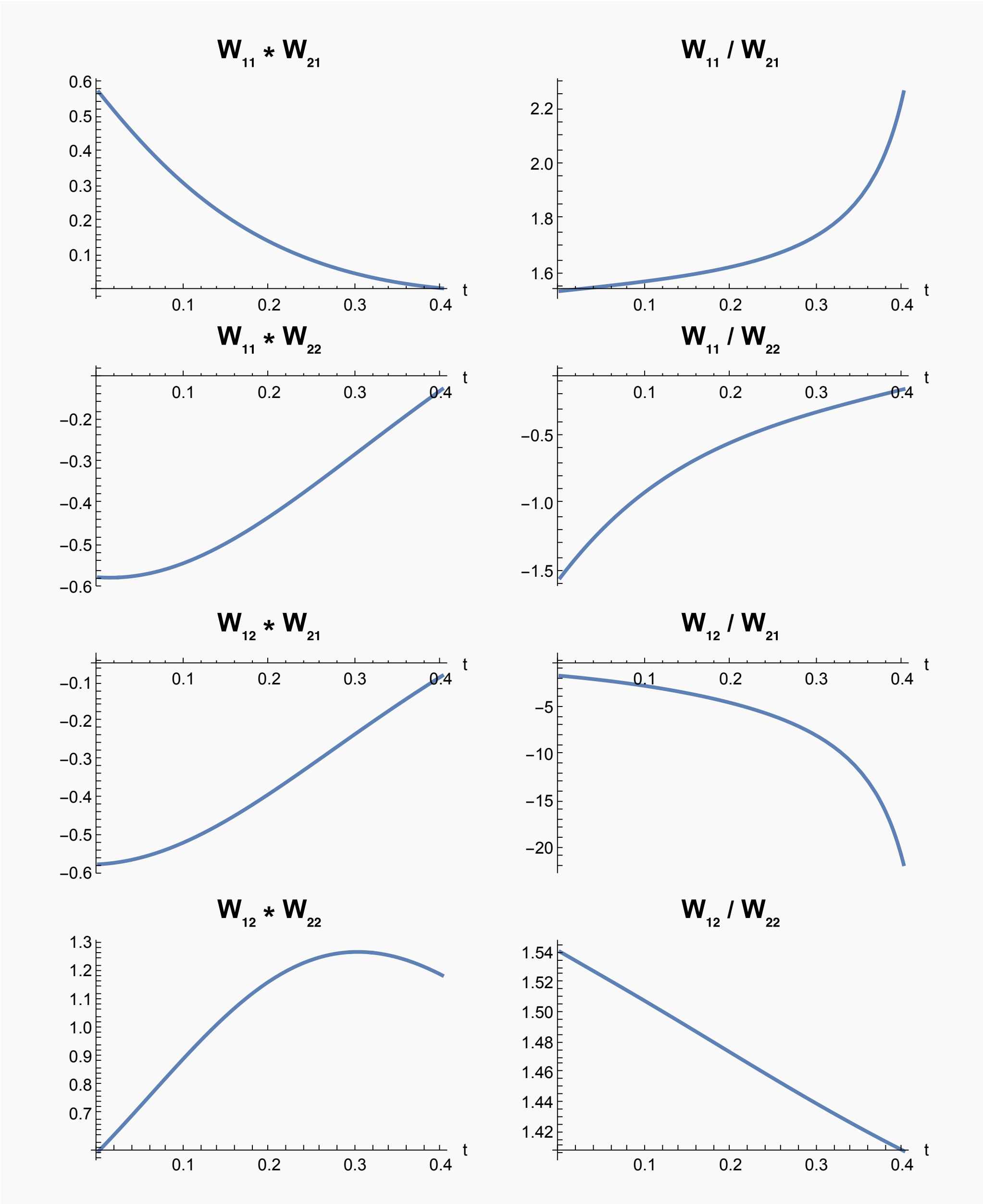
$$W_{2\,1}(U) = \frac{1.48108\times10^{17}\,U - 1.70137\times10^{-8}\,\sqrt{-1.71222\times10^{50}\,U^4 + 6.59317\times10^{49}\,U^2 + 9.57292\times10^{48}}}{3.23189\times10^{17}\,U^2 + 8.62774\times10^{16}}$$

$$W_{2\,2}(U) = \frac{1.70137\times10^{-8}\,\sqrt{-1.71222\times10^{50}\,U^4 + 6.59317\times10^{49}\,U^2 + 9.57292\times10^{48}} + 1.48108\times10^{17}\,U}{3.23189\times10^{17}\,U^2 + 8.62774\times10^{16}}$$

Out[301]=

**============= Plots of $W_1(U)W_2(U)$ and $W_1(U)/W_2(U)$ For All Pairs =============**

Checking that all four pairs – $(W_{11}(U), W_{21}(U))$, $(W_{11}(U), W_{22}(U))$, $(W_{12}(U), W_{21}(U))$, and $(W_{12}(U), W_{22}(U))$ – does NOT satisfy $W_1(U)/W_2(U)$ = const and $W_1(U)W_2(U)$ = const.




Out[302]=

**================= FLEXIBILITY (Double Checking) =================**

Out[304]=

**Solution 1:**

$$P_1(Z, W_1) = \frac{1}{(0.354727 + 1.\,t^2)^2}\left(-1.12836\times10^{-16} + t\left(7.63278\times10^{-17}\sqrt{1+2.67641\,t^2-2.70096\,t^4} + t\left(-8.55179\times10^{-16} - 1.52032\times10^{-15}\,t^2 + 1.9004\times10^{-16}\,t^4 + 3.88578\times10^{-16}\,t\sqrt{1+2.67641\,t^2-2.70096\,t^4}\right)\right)\right)$$

$$P_2(Z, W_2) = \frac{1}{(0.331607 + 1.\,t^2)^2}\left(-1.0532\times10^{-17} + t\left(-2.08167\times10^{-17}\sqrt{1+2.67641\,t^2-2.70096\,t^4} + t\left(-1.89577\times10^{-16} - 5.6873\times10^{-16}\,t^2 - 1.0532\times10^{-16}\,t^4 + 7.63278\times10^{-17}\,t\sqrt{1+2.67641\,t^2-2.70096\,t^4}\right)\right)\right)$$

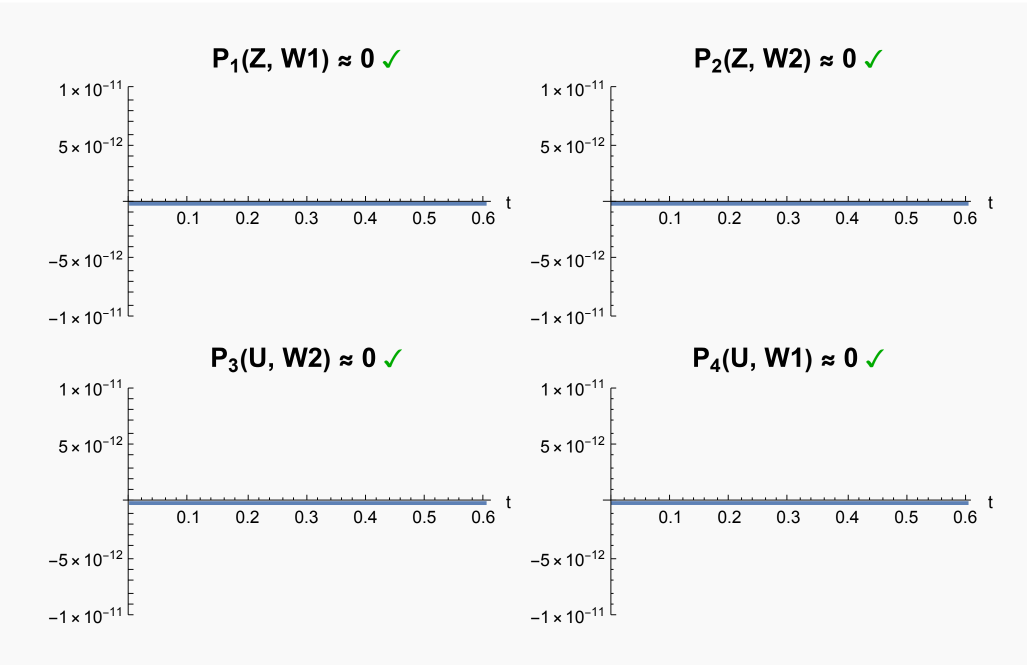
$$P_3(U, W_2) = \frac{1}{(0.225014 + 1.01016\,t^2 + 1.\,t^4)^2}\left(8.02692\times10^{-17} + t\left(9.7296\times10^{-17}\sqrt{1+2.67641\,t^2-2.70096\,t^4} + t\left(6.42255\times10^{-16} + t\left(4.21616\times10^{-16}\sqrt{1+2.67641\,t^2-2.70096\,t^4} + t\left(1.82917\times10^{-15} + 1.66701\times10^{-15}\,t^2 + 2.20538\times10^{-16}\,t^4 + 4.54048\times10^{-16}\,t\sqrt{1+2.67641\,t^2-2.70096\,t^4}\right)\right)\right)\right)\right)$$

$$P_4(U, W_1) = \frac{1}{(0.240702 + 1.03328\,t^2 + 1.\,t^4)^2}\left(2.31221\times10^{-16} + t\left(4.95008\times10^{-16}\sqrt{1+2.67641\,t^2-2.70096\,t^4} + t\left(1.00956\times10^{-15} + t\left(1.30265\times10^{-15}\sqrt{1+2.67641\,t^2-2.70096\,t^4} + t\left(9.90016\times10^{-16} - 1.0812\times10^{-15}\,t^2 - 2.60531\times10^{-16}\,t^4 + 4.81982\times10^{-16}\,t\sqrt{1+2.67641\,t^2-2.70096\,t^4}\right)\right)\right)\right)\right)$$

Out[310]=

**================= FLEXION 1 =================**

Polynomials $P_i(t)$ built from Bricard's equations for flexion 1.




Out[311]=

**Solution 2:**

$$P_1(Z, W_1) = \frac{1}{(0.354727 + 1.\,t^2)^2}\left(-1.12836\times10^{-16} + t\left(-7.63278\times10^{-17}\sqrt{1+2.67641\,t^2-2.70096\,t^4} + t\left(-8.55179\times10^{-16} - 1.52032\times10^{-15}\,t^2 + 1.9004\times10^{-16}\,t^4 - 3.88578\times10^{-16}\,t\sqrt{1+2.67641\,t^2-2.70096\,t^4}\right)\right)\right)$$

$$P_2(Z, W_2) = \frac{1}{(0.331607 + 1.\,t^2)^2}\left(-1.0532\times10^{-17} + t\left(2.08167\times10^{-17}\sqrt{1+2.67641\,t^2-2.70096\,t^4} + t\left(-1.89577\times10^{-16} - 5.6873\times10^{-16}\,t^2 - 1.0532\times10^{-16}\,t^4 - 7.63278\times10^{-17}\,t\sqrt{1+2.67641\,t^2-2.70096\,t^4}\right)\right)\right)$$

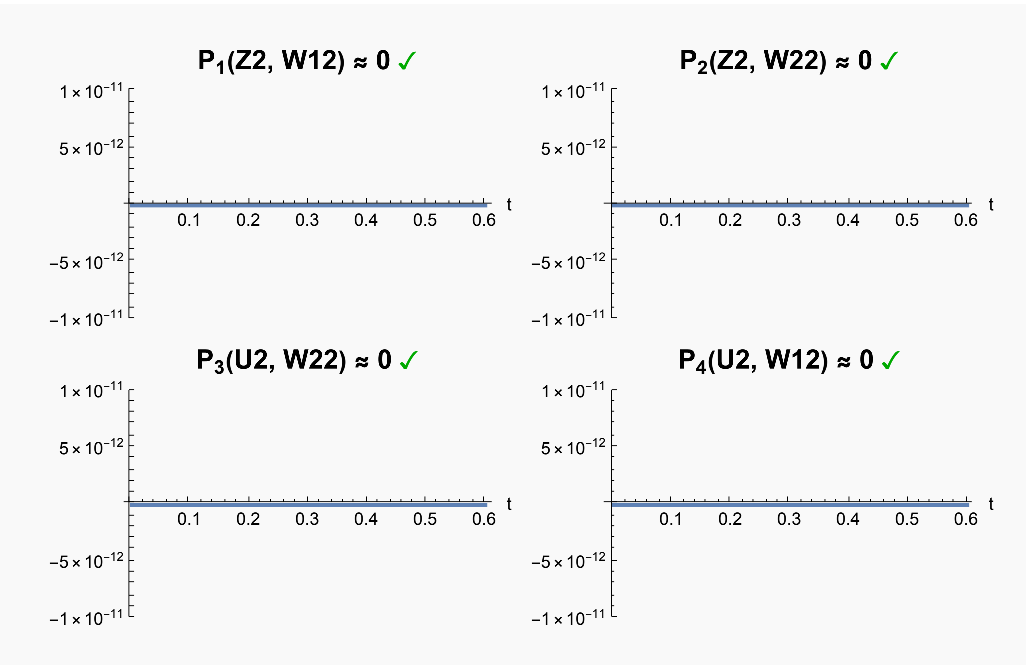
$$P_3(U, W_2) = \frac{1}{(0.225014 + 1.01016\,t^2 + 1.\,t^4)^2}\left(8.02692\times10^{-17} + t\left(-9.7296\times10^{-17}\sqrt{1+2.67641\,t^2-2.70096\,t^4} + t\left(6.42255\times10^{-16} + t\left(-4.21616\times10^{-16}\sqrt{1+2.67641\,t^2-2.70096\,t^4} + t\left(1.82917\times10^{-15} + 1.66701\times10^{-15}\,t^2 + 2.20538\times10^{-16}\,t^4 - 4.54048\times10^{-16}\,t\sqrt{1+2.67641\,t^2-2.70096\,t^4}\right)\right)\right)\right)\right)$$

$$P_4(U, W_1) = \frac{1}{(0.240702 + 1.03328\,t^2 + 1.\,t^4)^2}\left(2.31221\times10^{-16} + t\left(-4.95008\times10^{-16}\sqrt{1+2.67641\,t^2-2.70096\,t^4} + t\left(1.00956\times10^{-15} + t\left(-1.30265\times10^{-15}\sqrt{1+2.67641\,t^2-2.70096\,t^4} + t\left(9.90016\times10^{-16} - 1.0812\times10^{-15}\,t^2 - 2.60531\times10^{-16}\,t^4 - 4.81982\times10^{-16}\,t\sqrt{1+2.67641\,t^2-2.70096\,t^4}\right)\right)\right)\right)\right)$$

Out[318]=

**================= FLEXION 2 =================**

Polynomials $P_i(t)$ built from Bricard's equations for flexion 2.




Out[321]=

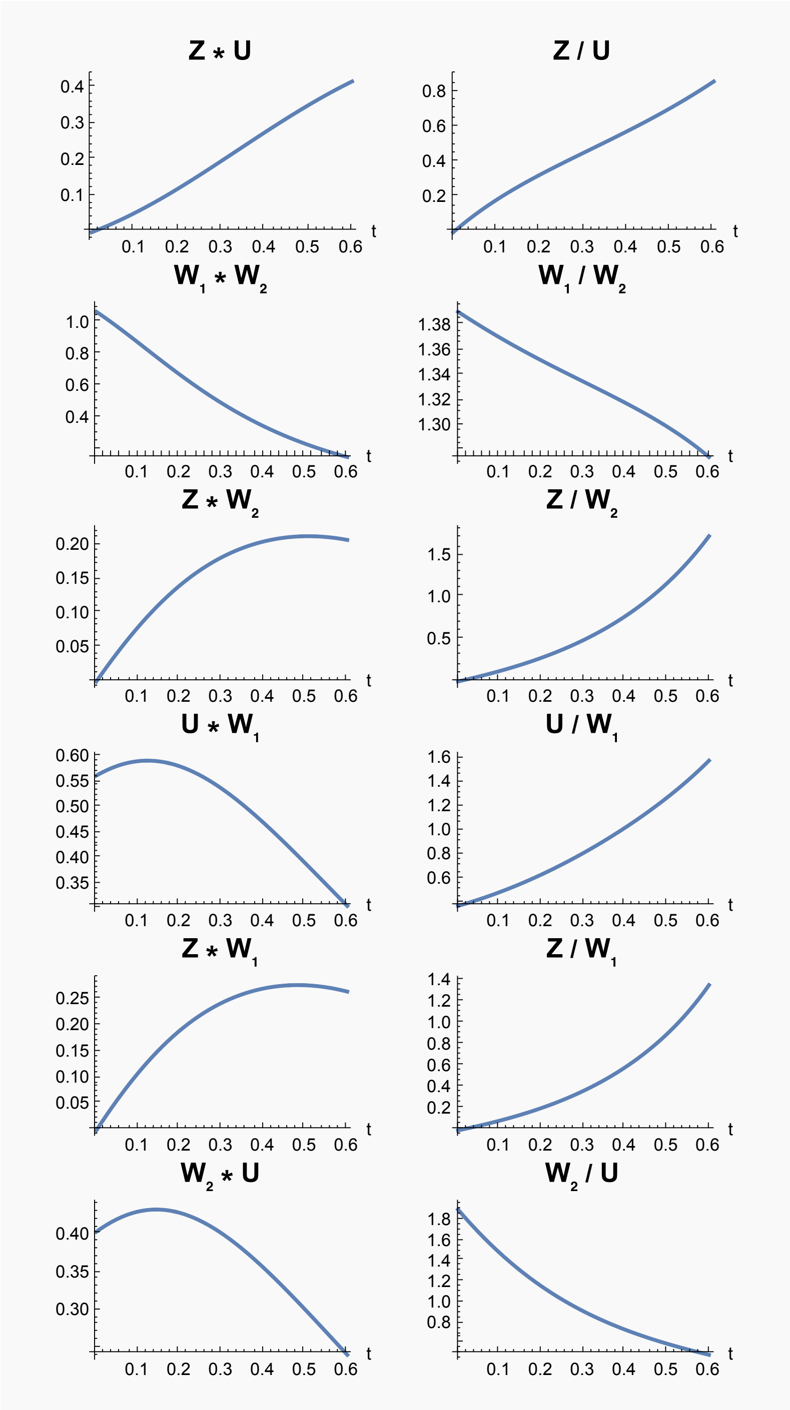
**============= NOT LINEAR COMPOUND =============**

Above we consider the first pair of equations ($P_1(t, W_1) = 0$ and $P_2(t, W_2) = 0$). Solving them as quadratic equations in $W_1$ and $W_2$, respectively we parametrize the solutions by the first two  and fourth expressions in Solutions 1 and 2 in a neighborhood of any point ($W_1$, t, $W_2$) such that the expression in the square root and denominators are not zero. Here, we choose any continuous branch of the square root in this neighborhood, and the signs in ± need Not agree (this means we consider all 4 pairs we describe above). We conclude that NO component of the solution set of the first pair of Bricard's equations satisfies $W_1/W_2$=const NOR $W_1W_2$=const.

Analogously, NO component of the solution set of the other pair of equations ($P_3(U, W_2) = 0$ and $P_4(U, W_1) = 0$) satisfies $W_1/W_2$=const NOR $W_1W_2$=const. As a result, NO component of the solution set of all four equations satisfies $W_1/W_2$=const NOR $W_1W_2$=const. So, our example does not belong to the linear compound class, even after switching the boundary strips.

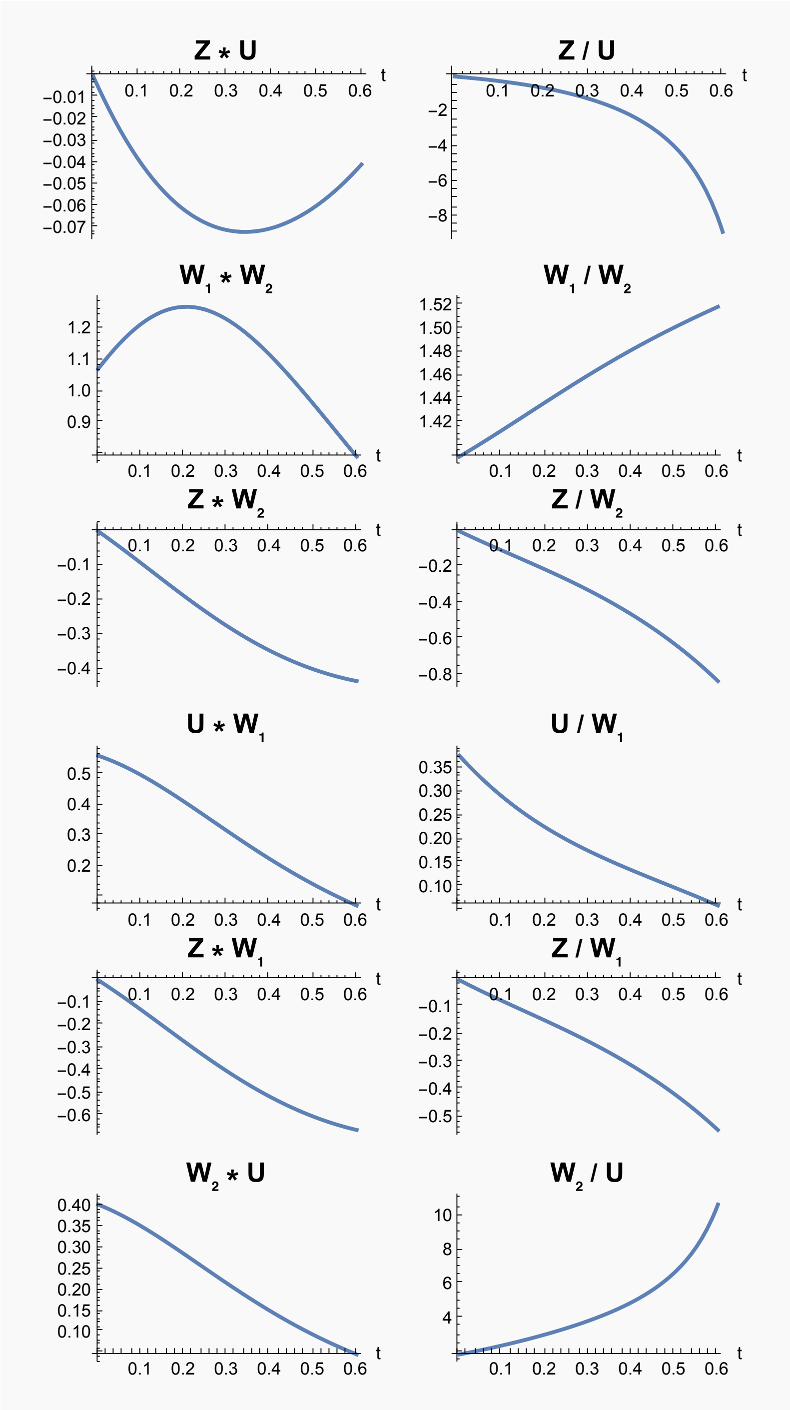
Below, we also present the plots of functions ZU, Z/U, $W_1W_2$, $W_1/W_2$, $ZW_2$, $Z/W_2$, $UW_1$, $U/W_1$, $ZW_1$, $Z/W_1$, $W_2U$, $W_2/U$.

**Solution 1:**


Z * U
Z / U
W1 * W2
W1 / W2
Z * W2
Z / W2
U * W1
U / W1
Z * W1
Z / W1
W2 * U
W2 / U
t

Out[324]=

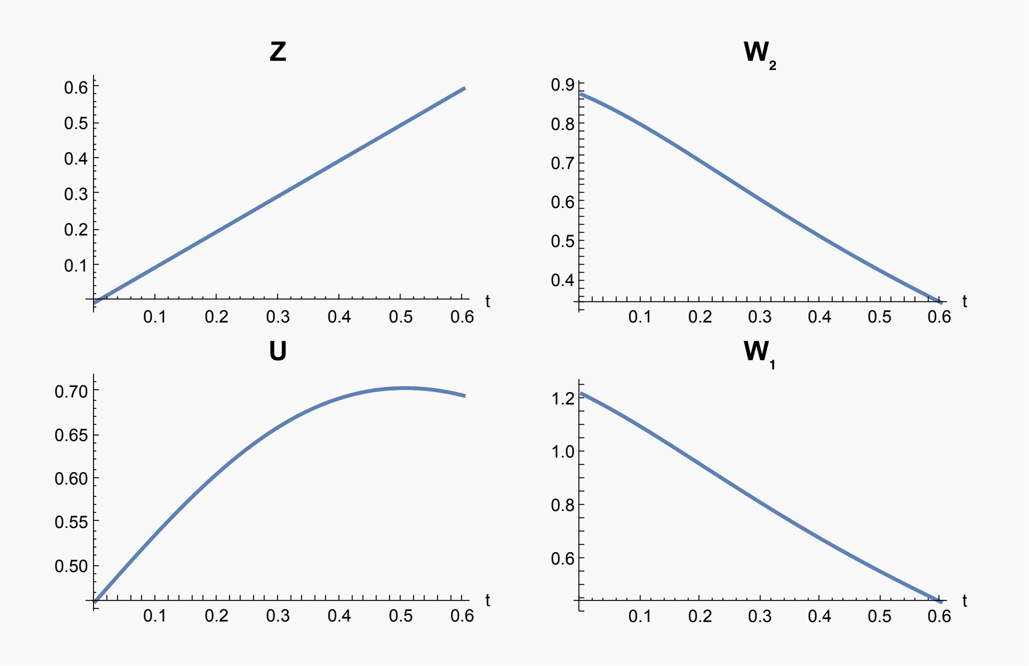
**Solution 2:**



Out[327]=

================= NOT TRIVIAL (FLEXION 1) =================

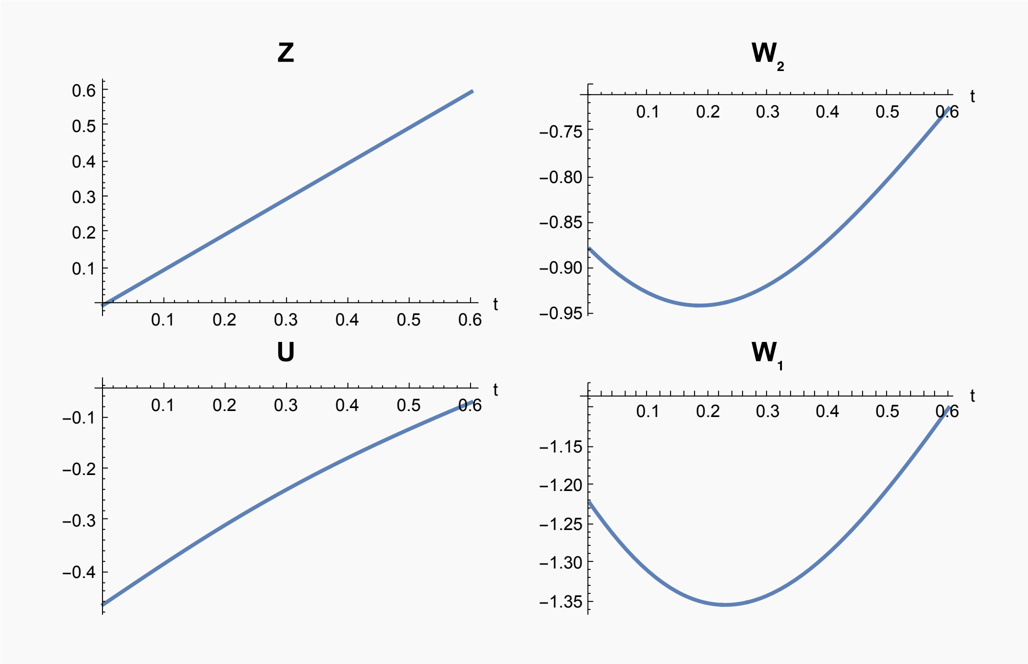
This configuration does not belong to the trivial
class – even after switching the boundary strips –
since none of the functions Z, W2, U, or W1 is constant.



Out[330]=

**================= NOT TRIVIAL (FLEXION 2) =================**

This configuration does not belong to the trivial
class — even after switching the boundary strips —
since none of the functions Z, W2, U, or W1 is constant.




Out[337]=

**================= NOT CONIC & NOT CHIMERA & NOT LINEAR CONJUGATE & NOT ISOGONAL=================**

Condition (N.0) is satisfied for all i=1,...,4
⇒ NOT equimodular-conic, NOT chimera, NOT isogonal
and NOT linear conjugate. Applying any boundary-strip
switch still preserves (N.0), so no conic, no chimera,
no isogonal and no linear conjugate form emerges.

Out[338]=

CONDITION (N.0) AFTER SWITCHING BOUNDARY STRIPS
**Right:** Condition (N.0) is still satisfied.
**Left:** Condition (N.0) is still satisfied.
**Lower:** Condition (N.0) is still satisfied.
**Upper:** Condition (N.0) is still satisfied.
**Right + Left:** Condition (N.0) is still satisfied.
**Right + Lower:** Condition (N.0) is still satisfied.
**Right + Upper:** Condition (N.0) is still satisfied.
**Left + Lower:** Condition (N.0) is still satisfied.
**Left + Upper:** Condition (N.0) is still satisfied.
**Lower + Upper:** Condition (N.0) is still satisfied.
**Right + Left + Lower:** Condition (N.0) is still satisfied.
**Right + Left + Upper:** Condition (N.0) is still satisfied.
**Right + Lower + Upper:** Condition (N.0) is still satisfied.
**Left + Lower + Upper:** Condition (N.0) is still satisfied.
**Right + Left + Lower + Upper:** Condition (N.0) is still satisfied.

Out[339]=

================== **ORTHOGONALITY CHECK** ==================

$\cos(\alpha_i)\cdot\cos(\gamma_i) \neq \cos(\beta_i)\cdot\cos(\delta_i)$ for
at least one $i = 1,\ldots, 4 \Rightarrow$ NOT orthodiagonal.
Switching boundary strips does not correct this.

**Initial anglesDeg (no switches):**

-> Condition met: At least one difference is non-zero.

Out[343]=

NON-ORTHOGONALITY CHECK AFTER SWITCHING BOUNDARY STRIPS
**Right:** Condition met (at least one difference is non-zero).
**Left:** Condition met (at least one difference is non-zero).
**Lower:** Condition met (at least one difference is non-zero).
**Upper:** Condition met (at least one difference is non-zero).
**Right + Left:** Condition met (at least one difference is non-zero).
**Right + Lower:** Condition met (at least one difference is non-zero).
**Right + Upper:** Condition met (at least one difference is non-zero).
**Left + Lower:** Condition met (at least one difference is non-zero).
**Left + Upper:** Condition met (at least one difference is non-zero).
**Lower + Upper:** Condition met (at least one difference is non-zero).
**Right + Left + Lower:** Condition met (at least one difference is non-zero).
**Right + Left + Upper:** Condition met (at least one difference is non-zero).
**Right + Lower + Upper:** Condition met (at least one difference is non-zero).
**Left + Lower + Upper:** Condition met (at least one difference is non-zero).
**Right + Left + Lower + Upper:**
Condition met (at least one difference is non-zero).

Out[344]=

================== **CONJUGATE-MODULAR CHECK** ==================

$M1 = M2 = M3 = M4 = M$ and $M \neq 2 \Rightarrow$ NOT
conjugate-modular. Boundary-strip switches preserve this.

**Initial anglesDeg (no switches):**

-> Condition met: All $M_i$ are equal and $M \neq 2$.

Out[348]=

CONJUGATE-MODULAR CHECK AFTER SWITCHING BOUNDARY STRIPS

**Right:** Condition met (All $M_i$ are equal and $M \neq 2$).
**Left:** Condition met (All $M_i$ are equal and $M \neq 2$).
**Lower:** Condition met (All $M_i$ are equal and $M \neq 2$).
**Upper:** Condition met (All $M_i$ are equal and $M \neq 2$).
**Right + Left:** Condition met (All $M_i$ are equal and $M \neq 2$).
**Right + Lower:** Condition met (All $M_i$ are equal and $M \neq 2$).
**Right + Upper:** Condition met (All $M_i$ are equal and $M \neq 2$).
**Left + Lower:** Condition met (All $M_i$ are equal and $M \neq 2$).
**Left + Upper:** Condition met (All $M_i$ are equal and $M \neq 2$).
**Lower + Upper:** Condition met (All $M_i$ are equal and $M \neq 2$).
**Right + Left + Lower:** Condition met (All $M_i$ are equal and $M \neq 2$).
**Right + Left + Upper:** Condition met (All $M_i$ are equal and $M \neq 2$).
**Right + Lower + Upper:** Condition met (All $M_i$ are equal and $M \neq 2$).
**Left + Lower + Upper:** Condition met (All $M_i$ are equal and $M \neq 2$).
**Right + Left + Lower + Upper:** Condition met (All $M_i$ are equal and $M \neq 2$).