Practical 01: Probability Generating function (Binomial)

Find the corresponding probabilities by using the probability generating function when an unbiased coin is tossed.

a) One time
b) Two times
c) Three times
d) Four times
e) Five times
f) Six time

Solution:

Let us consider a simple fair coin tossing experiment and X_n is the outcome of nth toss ($n = 1,2,3,4,5,6$) which represents the total number of "H" (heads). Suppose 1 represents "H" and 0 represents "T" (tails). The corresponding probability of getting head (probability of success) is p and the probability of failure is q.

Now X be a random variable having Binomial distribution with parameters n and p is,

$$Pr(X = x) = \binom{n}{x} p^x q^{n-x}; x = 0,1, \dots, n$$

where $p = 0.5$ and $q = 1 - p = 0.5$. and n is the number of tosses.

The probability generating function (pgf) is:

$$p(s) = (q + ps)^n$$

where,

- p = probability of success (getting head)
- q = probability of failure (getting tail)
- s = auxiliary variable
- n = number of trials (tosses)

Now, the probabilities, $p_i = \frac{1}{i!}[\frac{d^i}{ds^i} p(s)]_{s=0}$

Here,

n	p_i
1	$p_i = np(q + ps)^{n-1}$
2	$p_i = \frac{n(n-1)}{2!} p^2(q + ps)^{n-2}$
3	$p_i = \frac{n(n-1)(n-2)}{3!} p^3(q + ps)^{n-3}$
4	$p_i = \frac{n(n-1)(n-2)(n-3)}{4!} p^4(q + ps)^{n-4}$
5	$p_i = \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} p^5(q + ps)^{n-5}$
6	$p_i = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6!} p^6(q + ps)^{n-6}$

a) For single fair coin tossed (n = 1)

$$p_0 = (q + p \times 0)^n = q^n = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$$

$$p_1 = np(q + p \times 0)^{n-1} = \frac{1}{2} \times \left(\frac{1}{2}\right)^{1-1} = \frac{1}{2}$$

$\boldsymbol{X}$	0	1
$\boldsymbol{p(X = x)}$	$\frac{1}{2}$	$\frac{1}{2}$

b) For two fair coins tossed (n = 2)

$$p_0 = (q + p \times 0)^n = q^n = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$p_1 = np(q + p \times 0)^{n-1} = 2 \times \frac{1}{2} \times \left(\frac{1}{2}\right)^{2-1} = \frac{1}{2}$$

$$p_2 = \frac{n(n-1)}{2!} p^2 (q + p \times 0)^{n-2} = \frac{2(2-1)}{2!} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{2-2} = \frac{1}{4}$$

$\boldsymbol{X}$	**0**	**1**	**2**
$\boldsymbol{p(X = x)}$	$\boldsymbol{\frac{1}{4}}$	$\boldsymbol{\frac{1}{2}}$	$\boldsymbol{\frac{1}{4}}$

c) For three fair coins tossed (n = 3)

$$p_0 = (q + p \times 0)^n = q^n = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$p_1 = np(q + p \times 0)^{n-1} = 3 \times \frac{1}{2} \times \left(\frac{1}{2}\right)^{3-1} = \frac{3}{8}$$

$$p_2 = \frac{n(n-1)}{2!} p^2 (q + p \times 0)^{n-2} = \frac{3(3-1)}{2!} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{3-2} = \frac{3}{8}$$

$$p_3 = \frac{n(n-1)(n-2)}{3!} p^3 (q + p \times 0)^{n-3} = \frac{3(3-1)(3-2)}{3!} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{3-3} = \frac{1}{8}$$

$\boldsymbol{X}$	0	1	2	3
$\boldsymbol{p(X = x)}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

d) For four fair coins tossed (n = 4)

$$p_0 = (q + p \times 0)^n = q^n = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

$$p_1 = np(q + p \times 0)^{n-1} = 4 \times \frac{1}{2} \times \left(\frac{1}{2}\right)^{4-1} = \frac{1}{4}$$

$$p_2 = \frac{n(n-1)}{2!} p^2 (q + p \times 0)^{n-2} = \frac{4(4-1)}{2!} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2} = \frac{3}{8}$$

$$p_3 = \frac{n(n-1)(n-2)}{3!} p^3 (q + p \times 0)^{n-3} = \frac{4(4-1)(4-2)}{3!} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{4-3} = \frac{1}{4}$$

$$p_4 = \frac{n(n-1)(n-2)(n-3)}{4!} p^4 (q + p \times 0)^{n-4} = \frac{4(4-1)(4-2)(4-3)}{4!} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{4-4} = \frac{1}{16}$$

X	0	1	2	3	4
$p(X=x)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

e) For five fair coins tossed (n = 5)

$$p_0 = (q + p \times 0)^n = q^n = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

$$p_1 = np(q + p \times 0)^{n-1} = 5 \times \frac{1}{2} \times \left(\frac{1}{2}\right)^{5-1} = \frac{5}{32}$$

$$p_2 = \frac{n(n-1)}{2!} p^2 (q + p \times 0)^{n-2} = \frac{5(5-1)}{2!} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{5-2} = \frac{5}{16}$$

$$p_3 = \frac{n(n-1)(n-2)}{3!} p^3 (q + p \times 0)^{n-3} = \frac{5(5-1)(5-2)}{3!} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{5-3} = \frac{5}{32}$$

$$p_4 = \frac{n(n-1)(n-2)(n-3)}{4!} p^4 (q + p \times 0)^{n-4} = \frac{5(5-1)(5-2)(5-3)}{4!} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{5-4} = \frac{5}{32}$$

$$p_5 = \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} p^5 (q + p \times 0)^{n-5} = \frac{5(5-1)(5-2)(5-3)(5-4)}{5!} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{5-5} = \frac{1}{32}$$

X	0	1	2	3	4	5
$p(X=x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

f) For six fair coins tossed (n = 6)

$$p_0 = (q + p \times 0)^n = q^n = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

$$p_1 = np(q + p \times 0)^{n-1} = 6 \times \frac{1}{2} \times \left(\frac{1}{2}\right)^{6-1} = \frac{3}{32}$$

$$p_2 = \frac{n(n-1)}{2!} p^2 (q + p \times 0)^{n-2} = \frac{6(6-1)}{2!} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{6-2} = \frac{15}{64}$$

$$p_3 = \frac{n(n-1)(n-2)}{3!} p^3 (q + p \times 0)^{n-3} = \frac{6(6-1)(6-2)}{3!} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{6-3} = \frac{5}{16}$$

$$p_4 = \frac{n(n-1)(n-2)(n-3)}{4!} p^4 (q + p \times 0)^{n-4} = \frac{6(6-1)(6-2)(6-3)}{4!} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{6-4} = \frac{15}{64}$$

$$p_5 = \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} p^5 (q + p \times 0)^{n-5} = \frac{6(6-1)(6-2)(6-3)(6-4)}{5!} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{6-5} = \frac{3}{32}$$

$$p_6 = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6!} p^6 (q + p \times 0)^{n-6} = \frac{6(6-1)(6-2)(6-3)(6-4)(6-5)}{6!} \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^{6-6} = \frac{1}{64}$$

X	0	1	2	3	4	5	6
$p(X=x)$	$\frac{1}{64}$	$\frac{3}{32}$	$\frac{15}{64}$	$\frac{5}{16}$	$\frac{15}{64}$	$\frac{3}{32}$	$\frac{1}{64}$

R code

```r
# Function to generate Binomial probabilities using PMF concept
generate_binomial_probs <- function(n, p = 0.5) {
  q <- 1 - p
  k <- 0:n
  probs <- choose(n, k) * p^k * q^(n - k)
  return(probs)
}
```

```
# Generate and print for n = 1 to 6
for (i in 1:6) {
  cat("\n--- n =", i, "---\n")
  probs <- generate_binomial_probs(i)
  df <- data.frame(X = 0:i, Probability = round(probs, 4))
  print(df)
}


--- n = 1 ---
  X Probability
1 0         0.5
2 1         0.5

--- n = 2 ---
  X Probability
1 0        0.25
2 1        0.50
3 2        0.25

--- n = 3 ---
  X Probability
1 0       0.125
2 1       0.375
3 2       0.375
4 3       0.125

--- n = 4 ---
  X Probability
1 0      0.0625
2 1      0.2500
3 2      0.3750
4 3      0.2500
5 4      0.0625

--- n = 5 ---
  X Probability
1 0      0.0312
2 1      0.1562
3 2      0.3125
4 3      0.3125
5 4      0.1562
6 5      0.0312

--- n = 6 ---
  X Probability
1 0      0.0156
2 1      0.0938
3 2      0.2344
4 3      0.3125
5 4      0.2344
6 5      0.0938
7 6      0.0156
```

```r
# Generate binomial probabilities using PGF derivative formula
prob_pgf <- function(n, p = 0.5) {
  q <- 1 - p
  s <- 0

  pgf <- expression((q + p*s)^n)
  probs <- numeric(n + 1)

  for (i in 0:n) {

    # Take i derivatives wrt s
    for (j in 1:i) {
      di <- D(pgf, "s")
    }
    # Evaluate at s = 0, then divide by i!
    val <- eval(di, list(s = s, p = p, q = q, n = n))
    probs[i + 1] <- val / factorial(i)
  }

  return(probs)
}

# Run for n = 1 to 6
for (n in 1:6) {
  cat("\n--- n =", n, "---\n")
  probs <- prob_pgf(n, p = 0.5)
  df <- data.frame(X = 0:n, Probability = round(probs, 4))
  print(df)
}
```

Practical 02: Gambler's Ruin Problem

The following data shows the gambler's initial capital, combined capital, probability of win and probability of loss [capital in p]

Initial Capital (z)	Combined Capital (a)	Probability of Win (p)	Probability of Loss (q)
9	10	0.5	0.5
90	100	0.5	0.5
900	1000	0.5	0.5
950	1000	0.5	0.5
8000	10000	0.5	0.5
9	10	0.45	0.55
90	100	0.45	0.55
99	100	0.45	0.55
90	100	0.4	0.6
99	100	0.4	0.6

a) Find the probability of the gambler's ultimate ruin and success.

b) Determine the expected duration of the game.

c)
   i. Interpret your conclusions from (a) when win and loss probabilities are equal and unequal.

   ii. In what condition would the gambler never be ruined, but ruther grow endlessly rich?

   iii. If $p \leq 0.5$ (each gamble is not in his favour), then with probability one the gambler will get ruined. Are you agreeing with this agreement? If so, explain

Solution:

In the classical Gambler's Ruin setup:

- Let the gambler start with z units of capital.
- Let a be tehe combined capital of gambler and opponent (goal).
- At each step: With probability p, the gambler wins 1 unit, and with probability $q = 1 - p$, the gambler loses 1 unit.

a) Probability of Ultimate Ruin and Success:

Let q_z be the **probability of ruin** starting with capital z.

Condition	**Probability of Ruin q_z**
$p = q = 0.5$	$q_z = \frac{a-z}{a}$
$p \neq q$	$q_z = \frac{\left(\frac{p}{q}\right)^a - \left(\frac{q}{p}\right)^z}{\left(\frac{q}{p}\right)^a - 1}$

Then the **probability of success**: $p_z = 1 - q_z$

b) Expected Duration of the Game:

Let D_z be the **expected number of steps (duration** until the gambler is ruined or reaches the goal.

Condition	Expected Duration D_z
$p = q = 0.5$	$D_z = z(a - z)$
$p \neq q$	$D_z = \frac{z}{q-p} - \frac{a}{q-p} \times \frac{1-\left(\frac{q}{p}\right)^z}{1-\left(\frac{q}{p}\right)^a}$

R Code:

```
prob_ruin <- function(z, a, p) {
  q <- 1 - p
  if (p == 0.5) {
    return((a - z) / a)  # Symmetric case
  } else {
    return(( (q/p)^a - (q/p)^z ) / ( (q/p)^a - 1 ))  # Asymmetric case
  }
}

expected_duration <- function(z, a, p) {
  q <- 1 - p
  if (p == 0.5) {
    return(z * (a - z))  # Symmetric case
  } else {
    term1 = z / (q - p)
    term2 = (a / (q - p)) * (1 - (q/p)^z) / (1 - (q/p)^a)
    return(term1 - term2)  # Asymmetric case
  }
}

data <- data.frame(
  z = c(9, 90, 900, 950, 8000, 9, 90, 99, 90, 99),
  a = c(10, 100, 1000, 1000, 10000, 10, 100, 100, 100, 100),
  p = c(0.5, 0.5, 0.5, 0.5, 0.5, 0.45, 0.45, 0.45, 0.4, 0.4),
  q = c(0.5, 0.5, 0.5, 0.5, 0.5, 0.55, 0.55, 0.55, 0.6, 0.6)
)
qz = round(mapply(prob_ruin, data$z, data$a, data$p), 3)
data$qz <- qz
data$pz <- 1 - qz

data$Dz <- round(mapply(expected_duration, data$z, data$a, data$p), 2)
data$Dz <- format(data$Dz, scientific = FALSE)

data

      z     a    p    q    qz    pz          Dz
1     9    10 0.50 0.50 0.100 0.900        9.00
2    90   100 0.50 0.50 0.100 0.900      900.00
3   900  1000 0.50 0.50 0.100 0.900    90000.00
4   950  1000 0.50 0.50 0.050 0.950    47500.00
5  8000 10000 0.50 0.50 0.200 0.800 16000000.00
6     9    10 0.45 0.55 0.210 0.790       11.01
7    90   100 0.45 0.55 0.866 0.134      765.57
8    99   100 0.45 0.55 0.182 0.818      171.82
9    90   100 0.40 0.60 0.983 0.017      441.33
10   99   100 0.40 0.60 0.333 0.667      161.67
```

c) i)

When win and loss probabilities are equal ($p = 0.5$), ruin probability depends on the ratio z/a. If both z and a increase proportionally, ruin probability remains the same, but expected duration increases significantly. For example, when $z = 9$ and $a = 10$, ruin probability is 0.1 and duration is 9. When $z = 90$ and $a = 100$, ruin is still 0.1 but duration becomes 900.

However, when win probability is less than loss probability ($p < 0.5$), ruin probability increases, and success becomes less likely even with high capital. Also, expected duration decreases compared to the fair case, as ruin happens faster.

When total capital a is fixed, increasing the initial capital z reduces the probability of ruin and increases the chance of success. For example, when $a=100$, if $z=90$, ruin probability is 0.1; but if $z=99$, ruin probability drops further to 0.01 in the fair case ($p = 0.5$).
Similarly, with unequal probabilities (e.g., $p=0.45$), increasing z from 90 to 99 reduces ruin probability from 0.866 to 0.182. This shows that higher starting capital gives a clear advantage, even when the game is unfair.

ii) If the game has infinite total capital (i.e., there is no absorbing upper barrier), and the probability of winning each round is greater than losing (i.e., $p > 0.5$), then the gambler has a positive probability of growing endlessly rich and may never be ruined.

iii) If $p<0.5$, i.e., each gamble is not in the gambler's favor, then with probability 1, the gambler will eventually be ruined.

Practical 03: Markov chain

Let $\{X_n, n \geq 0\}$ be a Markov chain with states $\Omega = \{1,2,3\}$ and the transition probability matrix is:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

a) Verify the states of P by drawing transition graph.

b) Verify the states of P using mathematical formula in R-code and compare with the result of (a) and also calculate the mean recurrence time.

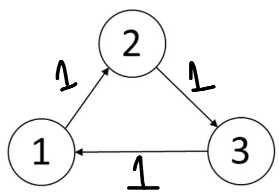
c) Is the transition probability matrix P periodic? Is so find the period.

d) Comment on your result.

Solution:

a) Here, the states are 1, 2, and 3.

<u>Transition Graph:</u>



Here, $s = \{1, 2, 3\}$ is **closed** and **irreducible**. So, states 1, 2, 3 are **persistent**.

From the graph,

$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 1$

$2 \longrightarrow 3 \longrightarrow 1 \longrightarrow 2$

$3 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3$

Here, we can see that, all 3 states 1, 2, 3 are persistent from the transition graph.

b) Now we want to verify the states of P by using mathematical formula

In order to classify the transition probability matrix and mean recurrence time, we have to use the following mathematical formula:

i) $f_{ij}^{(n)} = p_{ij}^{(n)} - \sum_{r=1}^{n-1} f_{ij}^{(r)} p_{jj}^{(n-r)}$

ii) $F_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$

iii) Mean recurrence time, $\mu_{ij} = \sum_{n=1}^{\infty} n\, f_{ij}^{(n)}$

```
library(MASS)
x1 <- matrix(c(0, 1, 0, 0, 0, 1, 1, 0, 0), nrow = 3, byrow = TRUE)

# Recurrence relation for matrix power
matrix_power <- function(mat, p) {
  if(p == 1) {
    return(mat)
  } else {
    return(mat %*% matrix_power(mat, p - 1))
  }
}

x2 = matrix_power(x1, 2)
x3 = matrix_power(x1, 3)
x4 = matrix_power(x1, 4)

f1 = matrix(0, 3, 3)
f2 = matrix(0, 3, 3)
f3 = matrix(0, 3, 3)

sumF = c()
mu = c()
for (i in 1:3) {
  f1[i, i] = x1[i, i]
  f2[i, i] = x2[i, i] - f1[i, i]*x1[i, i]
  f3[i, i] = x3[i, i] - f1[i, i]*x2[i, i] - f2[i, i]*x1[i, i]

  sumF[i] = f1[i, i] + f2[i, i] + f3[i, i]
  mu[i] = 1*f1[i, i] + 2*f2[i, i] + 3*f3[i, i]
}

data.frame(State = 1:3, Fii = sumF, Result = ifelse(sumF == 1, "Persistent", "Not pers
istent"))

  State Fii     Result
1     1   1 Persistent
2     2   1 Persistent
3     3   1 Persistent

data.frame(State = 1:3, muii = mu, Result = ifelse(mu < Inf, "Non-null persistent", "N
ull or Transient"))

  State muii              Result
1     1    3 Non-null persistent
2     2    3 Non-null persistent
3     3    3 Non-null persistent
```

Therefore all states are persistent

And previously we have found from the transition graph that states 1, 2, 3, are persistent. So it has been confirmed that all states are persistent.

And all μ are finite, so all states are non-null persistent.

c) Periodicity of Transition Probability Matrix:

```
perriodicity <- function(mat, n=2, max_n = 100) {
  null_mat <- matrix(0, nrow = nrow(mat), ncol = ncol(mat))

  if(identical(mat, matrix_power(mat, n))) {
    return(n-1)
  } else if (n < max_n) {
    return(perriodicity(mat, n + 1, max_n))
  } else {
    return(NA)  # No periodicity found within max_n steps
  }
}

period <- perriodicity(x1)
period

[1] 3
```

Comment: The Markov chain is periodic and the period is 3. This means that the chain returns to its initial state every 3 steps, indicating a cyclic behavior among the states.

d) Comment on Results:

At first from the transition graph, we can see states 1, 2, 3 are irreducible and persistent. Then by using mathematical formulas we can also find that the states are persistent and specifically they non-null persistent.

After that, we found that the transition probability matrix is periodic with a period of 3, indicating that the states cycle through every 3 steps. This cyclic behavior is consistent with the structure of the transition matrix, where each state transitions to the next in a fixed order.

Practical 04: Markov chain with absorbing states

Let, $X_n, n \geq 0$ be a Markov chain with states $\Omega = 1{,}2{,}3{,}4{,}5$ and the transition probability matrix is:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{7}{10} & 0 & \frac{3}{10} & 0 & 0 \\ 0 & \frac{7}{10} & 0 & \frac{3}{10} & 0 \\ 0 & 0 & \frac{7}{10} & 0 & \frac{3}{10} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

a) Classify the states of t.p.m by drawing transition graph.

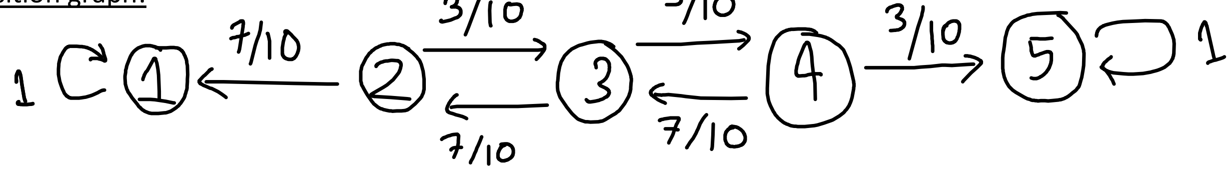
b) Verify the states of P using mathematical formula in R-code and compare with the result of (a) and also calculate the mean recurrence time.

c) Is the transition probability matrix P periodic? If so, find the period.

Solution:

a) Here, the states are 1, 2, 3, 4, and 5

<u>Transition graph:</u>



Comment:

$1 \rightarrow 1$: so, the state 1 is persistent

$2 \rightarrow 1, 1 \nrightarrow 2$: so, the state 2 is transient

$4 \rightarrow 5, 5 \nrightarrow 4$: so, the state 4 is transient

$3 \rightarrow 4 \rightarrow 5 \nrightarrow 3$: so, the state 3 is transient

$5 \rightarrow 5$: so, the state 5 is persistent

b) Now, we want to verify the states of P using mathematical formulas.

In order to classify the transition probability matrix, we have to use the following formulas:

i) $f_{ij}^{(n)} = p_{ij}^{(n)} - \sum_{r=1}^{n-1} f_{ij}^{(r)} p_{jj}^{(n-r)}$

ii) $F_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$

iii) Mean recurrence time, $\mu_{ij} = \sum_{n=1}^{\infty} n\, f_{ij}^{(n)}$

```
library(MASS)
x1 = matrix(c(1, 0, 0, 0, 0,
               7/10, 0, 3/10, 0, 0,
               0, 7/10, 0, 3/10, 0,
               0, 0, 7/10, 0, 3/10,
               0, 0, 0, 0, 1), nrow = 5, byrow = TRUE)
```

```
# fractions(x1)  # Display the matrix with fractions

# Recurrence relation for matrix power
matrix_power <- function(mat, p) {
  if(p == 1) {
    return(mat)
  } else {
    return(mat %*% matrix_power(mat, p - 1))
  }
}

n_max <- 10  # up to P^10

X <- list()
for (i in 1:n_max) {
  X[[i]] <- matrix_power(x1, i)
}

f <- vector("list", length = nrow(x1))  # f[[i]] will store all f_ii^(n)

for (i in 1:5) {
  f[[i]] <- numeric(n_max)  # empty vector for each state

  for (n in 1:n_max) {
    if (n == 1) {
      f[[i]][n] <- X[[1]][i, i]
    } else {
      sum_term <- 0
      for (r in 1:(n - 1)) {
        sum_term <- sum_term + f[[i]][r] * X[[n - r]][i, i]
      }
      f[[i]][n] <- X[[n]][i, i] - sum_term
    }
  }
}


Fii <- sapply(f, sum)
muii <- sapply(f, function(v) sum(seq_along(v) * v))

result <- data.frame(
  State = 1:5,
  Fii = round(Fii, 5),
  Persistence = ifelse(Fii == 1, "Persistent", "Transient"),
  muii = round(muii, 5),
  Type = ifelse(Fii == 1 & muii < Inf, "Non-null Persistent", "Null or Transient")
)

print(result)

  State     Fii Persistence    muii                Type
1     1 1.00000  Persistent 1.00000 Non-null Persistent
2     2 0.26571   Transient 0.67161   Null or Transient
3     3 0.42000   Transient 0.84000   Null or Transient
4     4 0.26571   Transient 0.67161   Null or Transient
5     5 1.00000  Persistent 1.00000 Non-null Persistent
```

c) Periodicity of Transition Probability Matrix:

```
perriodicity <- function(mat, n=2, max_n = 100) {

  if(identical(mat, matrix_power(mat, n))) {
    return(n-1)
  } else if (n < max_n) {
    return(perriodicity(mat, n + 1, max_n))
  } else {
    return(NA)  # No periodicity found within max_n steps
  }
}

period <- perriodicity(x1)
period
```

```
[1] NA
```

Comment: The transition probability matrix is not periodic, as the period is NA. This indicates that the states do not return to their initial state in a fixed number of steps, which is consistent with the presence of absorbing states (state 1 and state 5).

Practical 05: Class Property of Markov Chain

Let $X_n, n \geq 0$ is a Markov chain with state space, $\Omega = \{1,2,3,4,5,6,7\}$ and transition probability matrix is:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/2 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

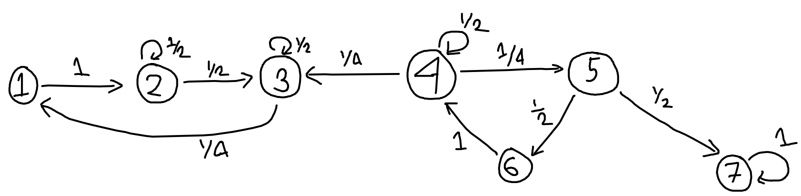
a) Classify the states of t.p.m by drawing transition graph.

b) Verify the states of P using mathematical formula in R-code and compare with the result of (a) and also calculate the mean recurrence time.

c) Is the transition probability matrix P periodic? If so, find the period.

Solution:

a) Here the states are 1, 2, 3, 4, 5, 6, and 7



Comment:

$S = \{1, 2, 3\}$ is closed and irreducible, so state 1, 2, and 3 are persistent.

$4 \rightarrow 3, 3 \nrightarrow 4$: so, state 4 is transient

$5 \rightarrow 7, 7 \nrightarrow 5$: so, state 5 is transient

$6 \rightarrow 4 \rightarrow 3$: so, state 6 is transient

$7 \rightarrow 7$: so, state 7 is persistent

b) Now, we want to verify the states of P using mathematical formulas.

In order to classify the transition probability matrix we have to use the following formulas:

i) $f_{ij}^{(n)} = P_{ij}^{(n)} - \sum_{r=1}^{n-1} f_{ij}^{(n)} P_{ij}^{(n-r)}$

ii) $F_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$

iii) Mean recurrence time, $\mu_{ij} = \sum_{n=1}^{\infty} n\, f_{ij}^{(n)}$

```r
library(MASS)
x1 <- matrix(c(0, 1, 0, 0, 0, 0, 0,
               0, 1/2, 1/2, 0, 0, 0, 0,
               1/2, 0, 1/2, 0, 0, 0, 0,
               0, 0, 1/4, 1/2, 1/4, 0, 0,
               0, 0, 0, 0, 0, 1/2, 1/2,
               0, 0, 0, 1,   0,   0,   0,
               0, 0, 0,   0,   0,   0 ,1), nrow =7 , byrow = TRUE)

# fractions(x1)


# Recurrence relation for matrix power
matrix_power <- function(mat, p) {
  if(p == 1) {
    return(mat)
  } else {
    return(mat %*% matrix_power(mat, p - 1))
  }
}

n_max <- 20  # up to P^20

X <- list()
for (i in 1:n_max) {
  X[[i]] <- matrix_power(x1, i)
}

f <- vector("list", length = nrow(x1))  # f[[i]] will store all f_ii^(n)

for (i in 1:ncol(x1)) {
  f[[i]] <- numeric(n_max)  # empty vector for each state

  for (n in 1:n_max) {
    if (n == 1) {
      f[[i]][n] <- X[[1]][i, i]
    } else {
      sum_term <- 0
      for (r in 1:(n - 1)) {
        sum_term <- sum_term + f[[i]][r] * X[[n - r]][i, i]
      }
      f[[i]][n] <- X[[n]][i, i] - sum_term
    }
  }
}


Fii <- sapply(f, sum)
muii <- sapply(f, function(v) sum(seq_along(v) * v))

result <- data.frame(
  State = 1:ncol(x1),
  Fii = round(Fii, 4),
  Persistence = ifelse(round(Fii, 4) == 1, "Persistent", "Transient"),
  muii = round(muii, 4),
  Type = ifelse(round(Fii, 4) == 1 & muii < Inf, "Non-null Persistent", "Null or Trans
ient")
)
```

```r
print(result)

  State   Fii Persistence   muii                 Type
1     1 1.000  Persistent 4.9992 Non-null Persistent
2     2 1.000  Persistent 2.5000 Non-null Persistent
3     3 1.000  Persistent 2.5000 Non-null Persistent
4     4 0.625   Transient 0.8750   Null or Transient
5     5 0.250   Transient 1.0000   Null or Transient
6     6 0.250   Transient 1.0000   Null or Transient
7     7 1.000  Persistent 1.0000 Non-null Persistent
```

Comment: In graphical method, we show that state 1, 2, 3, and 7 are persistent and state 4, 5 and 6 are Transient. So, it confirmed.

d) Periodicity of Transition Probability Matrix:

```r
perriodicity <- function(mat, n=2, max_n = 100) {


  if(identical(mat, matrix_power(mat, n))) {
    return(n-1)
  } else if (n < max_n) {
    return(perriodicity(mat, n + 1, max_n))
  } else {
    return(NA)  # No periodicity found within max_n steps
  }
}

period <- perriodicity(x1)
period

[1] NA
```

Comment: Therefore the transition probability matrix is not periodic, as the period is NA.

Practical 06: Poisson Process

Traffic accidents in Chittagong city have been found to occur us a Poisson process at the following rates:

7:00 AM to 10:00 AM - 2 accidents per hour

10:00 AM to 4:00 PM - 4 accidents per hour

What is the probability that there will be 10 or fewer accidents on a day between 7:00 AM and 4:00 PM?

Solution:

Based on the issue, 07:00 AM to 10:00 AM has a rate of 2 accidents per hour, and from 10:00 AM to 4:00 PM has a rate of 4 accidents per hour. The total time from 7:00 AM to 4:00 PM is 9 hours, with the first 3 hours at a rate of 2 accidents per hour and the remaining 6 hours at a rate of 4 accidents per hour. So,

$\lambda_1 t_1 = 2 \times 3 = 6$ and $\lambda_2 t_2 = 4 \times 6 = 24$

Following the Poisson additive property, 07:00 AM to 04:00 PM has a rate of $\lambda t = \lambda_1 \mathrm{t}_1 + \lambda_2 \mathrm{t}_2 = 6 + 24 = 30$ accidents.

Now, the probability that there will be 10 or fewer accidents on a day between 7:00 AM and 4:00 PM can be calculated using the cumulative distribution function (CDF) of the Poisson distribution:

$$\Pr\{N(t) \leq 10\} = \sum_{n=0}^{10} \frac{e^{-\lambda t}(\lambda t)^n}{n!} = \sum_{n=0}^{10} \frac{e^{-30}(30)^n}{n!}$$

$$= e^{-30}\left(1 + \frac{30}{1!} + \frac{(30)^2}{2!} + \cdots + \frac{(30)^{10}}{10!}\right)$$

$$= 0.89$$

∴ The probability that there will be 10 or fewer accidents on a day between 07:00 AM to 04:00 PM is 0.89