Fast Matrix Exponentiation

With Applications

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Outline

1. Introduction
2. Matrix exponentiation
3. Applications

Matrix Multiplication

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Thus, C is an $n * m$ dimensional matrix.

Which is calculated as: $c_{ij} = \sum_{r=1}^{k} a_{ir} * b_{rj}$

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  Thus, we can obtain C in $O(n * m * k)$, given A and B.
- If $n = m = k$ (i.e. both A and B have n rows and n columns), then C has n rows and n columns, and can be computed in $O(n^3)$.

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  Where I_n is a matrix with n rows and n columns of such form:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

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We can do so via:

```
function matpow_naive(A, p):
  result = I_n
  for i = 1..p:
    result = result * A
  return result
```

Which will run in $O(n^3 * p)$

Fast Matrix exponentiation

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Yes, we can apply the *BinPower* algorithm here:

```
function matBinPow(A, p):
  result = I_n
  while p > 0:
    if p % 2 == 1:
      result = result * A
    A = A * A
    p = p / 2
  return result
```

Which will run in $O(n^3 * \log p)$

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Finding Nth Fibonacci number

Fibonacci numbers, F_n are defined as:

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We want to calculate $F_n \ mod \ M$, where $n < 10^{18}$ and $M = 10^9 + 7$.

Finding Nth Fibonacci number

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Let's call this matrix M:
$(F_{i-2}, F_{i-1}) * M = (F_{i-1}, F_i)$

Two questions we should answer arise immediately:

1. What are the dimensions of M?
2. What are the exact values in M?

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  By definition, if we multiply a matrix with N rows and K columns by a matrix with K rows and L columns, we get a matrix with N rows and L columns.

  Therefore, matrix M has $K = 2$ rows and $L = 2$ columns.

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Moreover, we know that the result of this multiplication must be (F_{i-1}, F_i):
$(a * F_{i-2} + c * F_{i-1}, b * F_{i-2} + d * F_{i-1}) = (F_{i-1}, F_i)$

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Now we can write the system of equations:

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The easiest solution is to set:
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Thus, we obtain:

$$(F_{i-2}, F_{i-1}) * \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = (F_{i-1}, F_i)$$

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Now, can you see the pattern?

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In general, we get:
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Computing M^k takes $O(2^3 * log(k))$ time.
Thus, we can now find N-th Fibonacci number in $O(log(N))$ time.

General Solution for Recurrence Relations

Thank you!