

Fast Matrix Exponentiation

With Applications

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1 Introduction

2 Matrix exponentiation

3 Applications

Matrix Multiplication

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$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nm} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{bmatrix} * \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \dots & \dots & \dots & \dots \\ b_{k1} & b_{k2} & \dots & b_{km} \end{bmatrix}$$

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Thus, C is an $n * m$ dimensional matrix.

Which is calculated as: $c_{ij} = \sum_{r=1}^k a_{ir} * b_{rj}$

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Thus, we can obtain C in $O(n * m * k)$, given A and B .
- If $n = m = k$ (i.e. both A and B have n rows and n columns), then C has n rows and n columns, and can be computed in $O(n^3)$.

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- If you have a matrix with n rows and n columns, then multiplying it by I_n gives the same matrix.
 - i. e. $I_n * A = A * I_n = A$.Where I_n is a matrix with n rows and n columns of such form:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

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We can do so via:

```
function matpow_naive(A, p):  
    result = I_n  
    for i = 1..p:  
        result = result * A  
    return result
```

Which will run in $O(n^3 * p)$

Fast Matrix exponentiation

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Yes, we can apply the *BinPower* algorithm here:

```
function matBinPow(A, p):  
    result = I_n  
    while p > 0:  
        if p % 2 == 1:  
            result = result * A  
        A = A * A  
        p = p / 2  
    return result
```

Which will run in $O(n^3 * \log p)$

Outline

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Finding Nth Fibonacci number

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- $F_i = F_{i-1} + F_{i-2}$ for $i > 1$.

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- $F_0 = F_1 = 1$
- $F_i = F_{i-1} + F_{i-2}$ for $i > 1$.

We want to calculate $F_n \bmod M$, where $n < 10^{18}$ and $M = 10^9 + 7$.

Finding Nth Fibonacci number

Suppose we have a vector (matrix with one row and several columns) of (F_{i-2}, F_{i-1}) and we want to multiply it by some matrix, so that we get (F_{i-1}, F_i) as a result.

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Let's call this matrix M :

$$(F_{i-2}, F_{i-1}) * M = (F_{i-1}, F_i)$$

Two questions we should answer arise immediately:

- 1 What are the dimensions of M ?
- 2 What are the exact values in M ?

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The result is (F_{i-1}, F_i) , which has 1 row and 2 columns.

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By definition, if we multiply a matrix with N rows and K columns by a matrix with K rows and L columns, we get a matrix with N rows and L columns.

Therefore, matrix M has $K = 2$ rows and $L = 2$ columns.

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Moreover, we know that the result of this multiplication must be (F_{i-1}, F_i) :
 $(a * F_{i-2} + c * F_{i-1}, b * F_{i-2} + d * F_{i-1}) = (F_{i-1}, F_i)$

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Thus, we obtain:

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Now, can you see the pattern?

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You can find the reference code at <https://devnur.me/campunist>

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Side note: sequence A_i is called **recurrent**, because computing A_i requires computing A_j for some $j < i$. Also, A_i is called **linear**, because it depends linearly on A_j for some $j < i$.

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If we obtain matrix M , such that:

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we can get A_N as follows:

$$(a_0, a_1, \dots, a_{k-1}) * M^{N-k+1} = (A_{N-k+1}, A_{N-k+2}, \dots, A_N)$$

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- 2 What are the values in M ? Let's denote them as x_{ij} :

$$M = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ \dots & \dots & \dots & \dots \\ x_{k1} & x_{k2} & \dots & x_{kk} \end{bmatrix}$$

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From which we can easily obtain $x_{i,i-1} = 1$ for $i = 2, 3, \dots, k$ and $x_{ij} = 0$ for $i = 1, 2, 3, \dots, k$ and $j \leq k - 1, j \neq i - 1$.

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The last equation looks like the definition of A_i . Based on that, we get $x_{ik} = c_{k-i+1}$:

$$M = \begin{bmatrix} 0 & 0 & \dots & 0 & c_k \\ 1 & 0 & \dots & 0 & c_{k-1} \\ 0 & 1 & \dots & 0 & c_{k-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & c_1 \end{bmatrix}$$

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Now, when we have matrix M , there are no more obstacles.
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We can implement the solution just like the original *Fibonacci problem*.
If you use *Fast Maxtrix Exponentiation* you get $O(k^3 * \log N)$ solution.

Thank you!