CSE232: Discrete Mathematics
Assignment 2: Suggested answers

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1. Is the following compound proposition a tautology or a contradiction? Justify your answer.

$$(p \to q) \to (\neg q \to \neg p).$$

Answer. This statement is a tautology. We now prove it by a direct proof. So suppose that $p \to q$ is true. As $\neg q \to \neg p$ is the contrapositive of $p \to q$, then $\neg q \to \neg p$ is also true.

2. Is the following statement true? justify your answer.

$$\exists x \forall y (xy < 0 \to x < y)$$

where the domain for x and y is $\mathbb{R}$.

Answer. Yes, it is true. We choose $x = -1$. Then for all y, if $xy < 0$, we have $y > 0$. Therefore $x < y$.

3. Let n be an integer. Prove or disprove the following: If $3n$ is odd, then n is odd.

Answer. We make a proof by contraposition. So we assume that n is not odd, and we want to prove that $3n$ is not odd.

If n is not odd, then it is even. So there exists an integer k such that $n = 2k$. It follows that $3n = 6k = 2 \times 3k$, so $3n$ is also even.

4. Prove or disprove the following statement: If $g : A \to B$ and $f : B \to C$ are both one-to-one, the $f \circ g$ is also one-to-one.

Answer. It is true. We make a direct proof. So we want to show that, if $(f \circ g)(a) = (f \circ g)(b)$, then $a = b$.

So suppose that $(f \circ g)(a) = (f \circ g)(b)$. It means that $f(g(a)) = f(g(b))$. As f is one-to-one, it implies that $g(a) = g(b)$. As g is also one-to-one, it implies that $a = b$.

5. Prove or disprove the following:

$$\forall n \in \mathbb{Z} \ (\lfloor n/2 \rfloor + \lceil n/2 \rceil = n)$$

Answer. We make a proof by case: either n is even or it is odd.

- If n is even, then there exists $k \in \mathbb{Z}$ such that $n = 2k$. It follows that

$$\begin{aligned} \lfloor n/2 \rfloor + \lceil n/2 \rceil &= \lfloor k \rfloor + \lceil k \rceil && \text{because } k \text{ is an integer} \\ &= k + k \\ &= n. \end{aligned}$$

- If n is odd, then there exists $k \in \mathbb{Z}$ such that $n = 2k + 1$. It follows that

$$\begin{aligned} \lfloor n/2 \rfloor + \lceil n/2 \rceil &= \left\lfloor k + \frac{1}{2} \right\rfloor + \left\lceil k + \frac{1}{2} \right\rceil \\ &= k + \left\lfloor \frac{1}{2} \right\rfloor + k + \left\lceil \frac{1}{2} \right\rceil && \text{because } k \text{ is an integer} \\ &= k + 0 + k + 1 \\ &= n. \end{aligned}$$

6. How many integers in $\{1, 2, \dots, 1000\}$ are divisible by 4 or 9? Justify your answer.

Answer. Let A be the set of integers in $\{1, 2, \dots, 1000\}$ are divisible by 4, and let B be the set of those that are divisible by 9. Our goal is to find

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

So we have $|A| = \lfloor 1000/4 \rfloor = 250$ and $|B| = \lfloor 1000/9 \rfloor = 111$. The set $A \cap B$ consists of numbers divisible by 4 and 9, in other words numbers divisible by 36. Therefore, $|A \cap B| = \lfloor 1000/36 \rfloor = 27$. So the result is

$$|A \cup B| = 250 + 111 - 27 = 334.$$

7. How many integers in $\{1, 2, \dots, 1000\}$ are divisible by 4 or 6? Justify your answer.

Answer. We use the same approach as above, except that we need to observe that a number is divisible by 4 and 6 if and only if it is divisible by 12. Thus the answer is

$$\lfloor 1000/4 \rfloor + \lfloor 1000/6 \rfloor - \lfloor 1000/12 \rfloor = 333.$$

8. How many two-to-one functions are there from $\{1, 2, 3, 4, 5, 6, \}$ to $\{a, b, c\}$? Justify your answer.

Answer. The preimage of a consists of a pair $\{x, y\}$. There are 6 choices for x and 5 choices for y, but then we are counting each pair $\{x, y\} = \{y, x\}$ twice, so by the division rule, there are in total $6 \times 5/2$ choices for the preimages of a.

The same approach for b shows that there are $4 \times 3/2$ choices for its preimages, and then the preimage of c is fixed. So the answer is

$$\frac{6 \times 5 \times 4 \times 3}{2 \times 2} = 90.$$

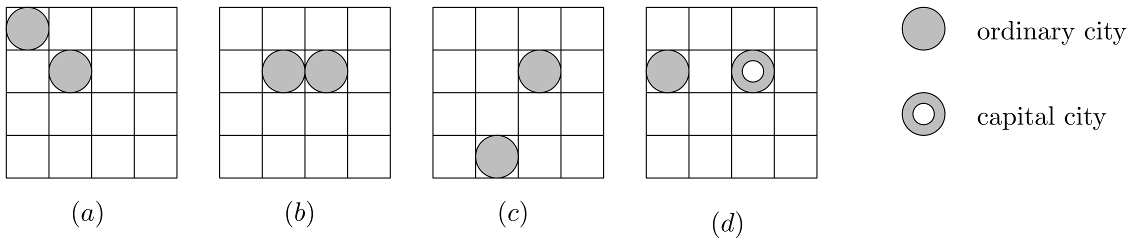

Figure 1: Examples where $n = 4$. (a) and (b) represent illegal placements because the two cities are adjacent. (c) and (d) represent legal placements.

9. In the game Civilization, you can build cities on a map consisting of square tiles. One city plays a special role—the capital city—and we assume that the other cities are identical. (See Figure 1.) The constraint is that no two cities can be in the same square, or in adjacent squares. (Two squares are adjacent if they touch along an edge, or diagonally at a corner.) Assume that you want to build cities on an $n \times n$ grid.

(a) How many different ways are there of placing your capital city and one ordinary city? Justify your answer.

(b) How many different ways are there to place two ordinary cities? Justify your answer.

Answer a. We consider 3 cases.

- Suppose that the capital city is at a corner. There are 4 choices for this placement, and then for each placement of the capital city, the other city can be anywhere except for the 4 squares at this corner. So there are $n^2 - 4$ ways of placing the ordinary city. So in total, by the product rule, this case consist of $4n^2 - 16$ possible placements.
- Suppose that the capital city is along the border, but not on a corner. There are $4n - 8$ possible placements for the capital, and for each such placement, we can place the ordinary city anywhere except for the square of the capital and the 5 neighboring squares. So this case contributes a total of $(4n - 8)(n^2 - 6)$ placement.
- The third case is when the capital city does not lies on the border. Then There are $(n - 2)^2$ ways of placing it, and for each of these ways there are $n^2 - 9$ ways of placing the ordinary city. So in total it yields $(n - 2)^2(n^2 - 9)$.

By the sum rule, the total number of placements is

$$4n^2 - 16 + (4n - 8)(n^2 - 6) + (n - 2)^2(n^2 - 9) = n^4 + 2n^3 - 9n^2 - 6n - 4.$$

Answer b. Let A be the set of legal placements from (a) and let B be the set of legal placements for this question. Any placement $p \in A$ can be mapped to a placement $f(p) \in B$ such that the capital city is replaced with an ordinary city. The mapping f is two-to-one, because we can choose the capital city to be any of the two cities. So by the division rule, we have

$$|B| = \frac{|A|}{2} = \frac{n^4 + 2n^3 - 9n^2 - 6n - 4}{2}.$$