CSE232: Discrete Mathematics
Assignment 1: Suggested answers

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1. Use set builder notation to give a description of each of these two sets:

(a) $\{0, 1, 4, 9, 16, 25\}$

(b) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0)\}$

Answer a. $\{x^2 \mid x \in \mathbb{N} \text{ and } x \leqslant 5\}$

Answer b. $\{(x, y) \in \mathbb{N}^2 \mid x + y \leqslant 2\}$

2. For each of the statements below, determine whether it is true or false, and justify your answer.

(a) $\{1\} = \{1, 1\}$

(b) $\{1, 2\} = \{2, 1\}$

(c) $\{1, 2\} = (1, 2)$.

Answer a. This is true because in a set, the multiplicity of the elements does not matter.

Answer b. This is true because in a a set, the order of the elements does not matter.

Answer c. This is false because $\{1, 2\}$ is a set, and $(1, 2)$ is a 2-tuple. Said differently, $\{1, 2\}$ is unordered and $(1, 2)$ is ordered.

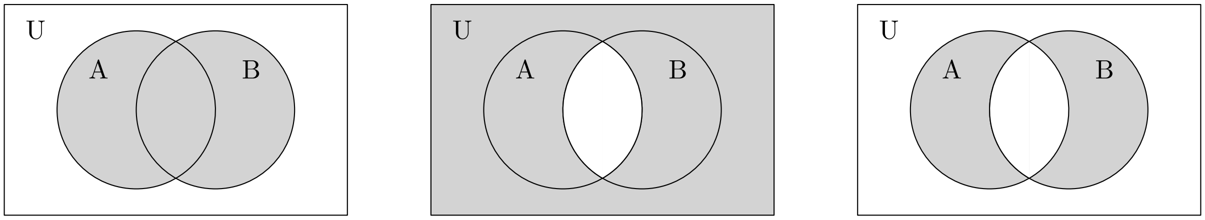
3. Is it true that for all sets A and B, we have $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$? Justify your answer.

Answer. This statement is wrong. A counterexample is $A = \{a\}$ and $B = \{b\}$. Then $\{a, b\} \in \mathcal{P}(A \cup B)$, but $\{a, b\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

4. Prove that $(A - B) \cup (B - A) = (A \cup B) \cap (\bar{A} \cup \bar{B})$

(a) using Venn diagrams,

(b) and using set identities.



$A \cup B$ $\qquad$ $\bar{A} \cup \bar{B}$ $\qquad$ $(A - B) \cup (B - A)$

Figure 1: Proof that $(A - B) \cup (B - A) = (A \cup B) \cap (\bar{A} \cup \bar{B})$ using a Venn diagram.

Answer a. The diagrams are on Figure 1. We can see that the intersections of the shaded areas on the left and middle diagrams yield the shaded area on the right, that is, $(A - B) \cup (B - A) = (A \cup B) \cap (\bar{A} \cup \bar{B})$.

Answer b.

$$\begin{aligned}
(A \cup B) \cap (\bar{A} \cup \bar{B}) &= ((A \cup B) \cap \bar{A}) \cup ((A \cup B) \cap \bar{B}) && \text{by distributive law} \\
&= (\bar{A} \cap (A \cup B)) \cup (\bar{B} \cap (A \cup B)) && \text{by commutative law} \\
&= ((\bar{A} \cap A) \cup (\bar{A} \cap B))\cup)(\bar{B} \cap A) \cup (\bar{B} \cap B)) && \text{by distributive law} \\
&= (\bar{A} \cap A) \cup (\bar{A} \cap B) \cup (\bar{B} \cap A) \cup (\bar{B} \cap B) && \text{by associative law} \\
&= \emptyset \cup (B - A) \cup (A - B) \cup \emptyset && \text{by complement law} \\
&= (A - B) \cup (B - A)
\end{aligned}$$

5. What can you say about the sets A and B when $A \times B = B \times A$? Justify your answer.

Answer. Then one of the following is true: $A = \emptyset$, $B = \emptyset$ or $A = B$. Indeed, suppose that $A \times B = B \times A$, $A \neq \emptyset$ and $B \neq \emptyset$. Then for any $a \in A$ and $b \in B$, since $(a, b) \in A \times B$ and $A \times B = B \times A$, we have $(b, a) \in B \times A$. Therefore $a \in B$ and $b \in A$. It means that A and B have the same elements, in other words $A = B$.

6. Find a compound proposition with propositional variables p, q and r that is true if and only if exactly two of these variables are true (and the other is false).

Answer a. $(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r)$

7. Prove that $(p \wedge \neg q) \vee (\neg p \wedge q) \equiv (p \vee q) \wedge (\neg p \vee \neg q)$

(a) using a truth table,

(b) and using logical equivalences.

Answer a. The 7th and 10th column in this table are identical, so the two propositions are equivalent.

p	q	$\neg q$	$p \wedge \neg q$	$\neg p$	$\neg p \wedge q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$	$p \vee q$	$\neg p \vee \neg q$	$(p \vee q) \wedge (\neg p \vee \neg q)$
T	T	F	F	F	F	F	T	F	F
T	F	T	T	F	F	T	T	T	T
F	T	F	F	T	T	T	T	T	T
F	F	T	F	T	F	F	F	T	F

Answer b.

$$\begin{aligned}
(p \wedge \neg q) \vee (\neg p \wedge q) &\equiv ((p \wedge \neg q) \vee \neg p) \wedge ((p \wedge \neg q) \vee q) && \text{by distributive law} \\
&\equiv (\neg p \vee (p \wedge \neg q)) \wedge (q \vee (p \wedge \neg q)) && \text{by commutative law} \\
&\equiv ((\neg p \vee p) \wedge (\neg p \vee \neg q)) \wedge ((q \vee p) \wedge (q \vee \neg q)) && \text{by distributive law} \\
&\equiv (\mathrm{T} \wedge (\neg p \vee \neg q)) \wedge ((q \vee p) \wedge \mathrm{T})) && \\
&\equiv (\neg p \vee \neg q) \wedge (q \vee p) && \text{by identity law} \\
&\equiv (p \vee q) \wedge (\neg p \vee \neg q) && \text{by commutative law}
\end{aligned}$$

8. Consider the subway network of a city. For any two stations x, y, $P(x, y)$ denotes "x and y are on the same line". In particular, $P(x, x)$ is true for any station x. Express the following proposition using quantifiers, logical connectives, and the predicate P:

- "It is possible go from any station to any other station without changing train more than once".

Answer. $\forall x \forall z \exists y (P(x, y) \wedge P(y, z))$

9. Write a statement that is logically equivalent to the following, and such that the negation $\neg$ appears immediately before P or Q (not any other symbol):

$$\neg \forall x \forall y (P(x, y) \vee Q(x, y)).$$

Justify your answer.

Answer.

$$\begin{aligned}
\neg \forall x \forall y (P(x, y) \vee Q(x, y)) &\equiv \exists x \exists y \neg (P(x, y) \vee Q(x, y)) && \text{by de Morgan's law for quantifiers} \\
&\equiv \exists x \exists y (\neg P(x, y) \wedge \neg Q(x, y)) && \text{by de Morgan's law}
\end{aligned}$$