CSE232: Discrete Mathematics
Midterm: Suggested Answers

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Question 1. Circle the correct answer: **T** (True) or **F** (False). No justification is needed.

(a) **T** **F**

$(1, 2, 3) = (3, 2, 1)$

(b) **T** **F**

$\{1, 2, 3\} = \{3, 2, 1\}$

(c) **T** **F**

$\mathbb{Z} \cup \mathbb{R} = \mathbb{R}$

(d) **T** **F**

$A - (B - C) = (A \cup C) - B$ for all sets A, B and C

(e) **T** **F**

$\neg(p \vee \neg q) \equiv \neg p \wedge q.$

(f) **T** **F**

$(p \wedge q) \vee \neg p \vee \neg q$ is a tautology

(g) **T** **F**

$(p \wedge q) \vee (\neg p \wedge \neg q)$ is a tautology

(h) **T** **F**

$\binom{5}{3} = 10$

(i) **T** **F**

$P(4, 2) = 8$

(j) **T** **F**

$C(n, k) \leqslant P(n, k)$ for all integers $1 \leqslant k \leqslant n$

Answers. (a) F (b) T (c) T (d) F (e) T (f) T (g) F (h) T (i) F (j) T

Question 2. Find sets A and B such that:

(a) $A \cup B = \{1, 2, 3, 4\}$, $A - B = \{1\}$ and $B - A = \{2, 4\}$.

(b) $A \cup B = A \cap B$

(c) $A^2 - B^2 = \{(0, 0), (0, 1), (0, 2), (1, 0), (2, 0)\}$.

Answer a. $A = \{1, 3\}$ and $B = \{2, 3, 4\}$. (It can be found easily by drawing the Venn diagram.)

Answer b. $A = \emptyset$ and $B = \emptyset$.

Answer c. $A = \{0, 1, 2\}$ and $B = \{1, 2\}$.

Question 3. You toss 5 coins. What is the probability that you get, in total, 2 heads and 3 tails?

Answer. The probability of having 2 successes in 5 Bernoulli trials with probability of success $p = 1/2$ is:

$$B(2; 5, 1/2) = \binom{5}{2} \frac{1}{2^2} \frac{1}{2^3} = \frac{5}{16}.$$

Question 4. State the converse, contrapositive, and inverse of the following statement:

"If there is a typhoon, then I will not go to the beach."

Converse: "If I don't go to the beach, then there will be a typhoon."

Contrapositive: "If I go to the beach, then there will be no typhoon."

Inverse: "If there is no typhoon, then I will go to the beach."

Question 5. Rewrite this statement so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

$$\neg\exists y(\exists x R(x, y) \vee \forall x S(x, y))$$

Answer.

$$\begin{aligned}\neg\exists y(\exists x R(x,y) \vee \forall x S(x,y)) &= \forall y \neg(\exists x R(x,y) \vee \forall x S(x,y)) \\ &= \forall y(\neg\exists x R(x,y) \wedge \neg\forall x S(x,y)) \\ &= \forall y(\forall x \neg R(x,y) \wedge \exists x \neg S(x,y))\end{aligned}$$

Question 6. How many subsets of $\{1,2,3,4,5,6,7,8\}$ contain exactly 2 even numbers and 3 odd numbers? (For instance, $\{1,2,3,6,7\}$ is one such subset.)

Answer. There are $\binom{4}{2} = 6$ ways of choosing the even numbers, and $\binom{4}{3} = 4$ ways of choosing the odd numbers. So in total there are **24** such sets.

Question 7. How many numbers must be selected from the set $\{1,2,3,4,5,6\}$ to guarantee that at least one pair of these numbers add up to 7? Justify your answer.

Answer. The answer is: **at least 4**. We now prove it.

First observe that 3 is not enough, as we could pick 1, 3 and 5 for instance. Now suppose that we pick at least 4 numbers. Consider 3 boxes $\{1,6\}$, $\{2,5\}$, and $\{3,4\}$. By the pigeonhole principle, 2 numbers we picked must fall in the same box. Then these two numbers we picked add up to 7.

Question 8. Prove that $(p \to r) \wedge (q \to r)$ and $(p \vee q) \to r$ are logically equivalent

(a) using a truth table,

(b) and using another method.

Answer a.

p	q	r	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$	$p \vee q$	$(p \vee q) \to r$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

Answer b.

$$\begin{aligned}(p \vee q) \to r &= \neg(p \vee q) \vee r \\ &= (\neg p \wedge \neg q) \vee r && \text{by de Morgan's law} \\ &= (\neg p \vee r) \wedge (\neg q \vee r) && \text{by distributive law} \\ &= (p \to r) \wedge (q \to r)\end{aligned}$$

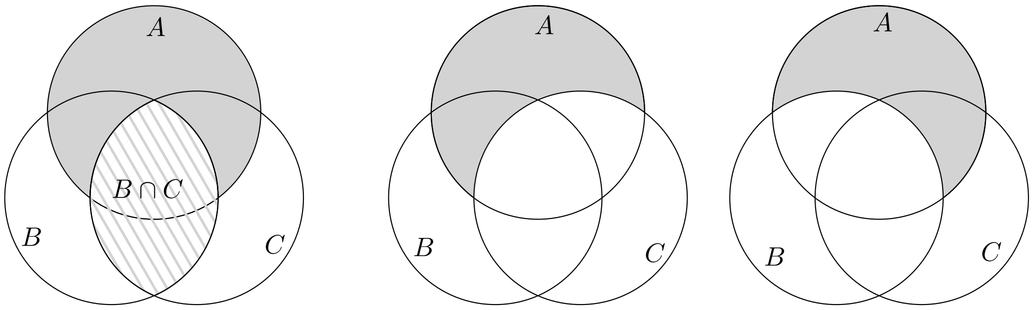

Figure 1: The shaded area on the left is $A - (B \cap C)$. It is the union of the shaded areas in the middle and on the right, which are $A - B$ and $A - C$.

Question 9. The goal of this question is to prove that

$$A - (B \cap C) = (A - B) \cup (A - C) \tag{1}$$

for all sets A, B and C.

(a) Show that (1) is true using a Venn diagram.

(b) Prove that (1) is true using another method.

Answer a. See Figure 1.

Answer b.

$$\begin{aligned} A - (B \cap C) &= A \cap \overline{B \cap C} \\ &= A \cap (\bar{B} \cup \bar{C}) && \text{by de Morgan's law} \\ &= (A \cap \bar{B}) \cup (A \cap \bar{C}) && \text{by distributive law} \\ &= (A - B) \cup (A - C) && \text{by de Morgan's law} \end{aligned}$$

10. Does there exist a set S and a function $f : S \to S$ such that f is two-to-one? Justify your answer.

Answer. Yes: take $S = \mathbb{N}$ and $f(n) = \lfloor n/2 \rfloor$ for all $n \in \mathbb{N}$.

11. In a factory there are two machines manufacturing bolts. The first machine manufactures 80% of the bolts and the second machine manufactures the remaining 20%. From the first machine 4% of the bolts are defective and from the second machine 2% of the bolts are defective. A bolt is selected at random. What is the probability that the bolt came from the first machine, given that it is defective?

Answer. Let M be the event that the bolt is from the first machine, and let D be the event that it is defective. The problem statement tells us that

$$\begin{aligned} p(M) &= 0.8 \\ p(D|M) &= 0.04 \\ p(D|\bar{M}) &= 0.02 \end{aligned}$$

and we want to find $p(M|D)$. Thus we apply Baye's theorem:

$$\begin{aligned} p(M|D) &= \frac{p(M)p(D|M)}{p(M)p(D|M) + p(\bar{M})p(D|\bar{M})} \\ &= \frac{0.8 \times 0.04}{0.8 \times 0.04 + 0.2 \times 0.02} \\ &= \frac{8 \times 4}{8 \times 4 + 2 \times 2} \\ &= \frac{32}{36} = \mathbf{\frac{8}{9}} \end{aligned}$$

12. Alice, Bob and Charlie play the following game. Each of them throws a die, and if one of them gets a higher number than the other two, then he or she wins the game. On the other hand, if more than one of them gets the highest number, then the game is declared to be a draw (i.e. nobody wins).

For instance, if Alice gets 4, Bob gets 5 and Charlie gets 2, then Bob wins. It Charlie gets 5 and Alice and Bob get 3, then Charlie wins. If Alice and Charlie get 4 and Bob gets 1, then it is a draw. If all three of them get 2 then it is also a draw.

What is the probability that Alice wins?

Answer. Our sample set is the set of triples of (not necessarily distinct) numbers in $\{1, \ldots, 6\}$, so $|S| = 6^3$. Let A, B and C be the events where Alice, Bob and Charlie wins, respectively. Let D be the event that there is a draw. As Alice, Bob and Charlie play the same role, we must have $P(A) = P(B) = P(C)$. As $p(A) + p(B) + p(C) + p(D) = 1$, it follows that $p(A) = (1 - p(D))/3 = p(\bar{D})/3$.

Let us determine $p(\bar{D})$, that is, the probability that there is no draw. There are two cases: Either the three numbers are different (event E), or two numbers are equal and the third is larger (event F), and thus we have $|\bar{D}| = |E| + |F|$.

- By definition of permutations, $|E| = P(6, 3)$.
- For event F, there are $C(6, 2)$ ways of choosing the two numbers, and then the largest could be either the die of Alice, Bob or Charlie. So $|F| = 3C(6, 2)$.

Thus we have

$$p(A) = \frac{|\bar{D}|}{3|S|} = \frac{P(6,3) + 3C(6,2)}{3 \cdot 6^3} = \frac{6 \cdot 5 \cdot 4 + 3 \cdot 3 \cdot 5}{3 \cdot 6 \cdot 6 \cdot 6} = \frac{2 \cdot 5 \cdot 4 + 3 \cdot 5}{6 \cdot 6 \cdot 6} = \mathbf{\frac{55}{216}}.$$

Question 13. Give a combinatorial proof of the following identity:

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Answer. Let $A = \{1, 2, \ldots, n\}$ and $B = \{n+1, \ldots, 2n\}$. Let $E = A \cup B$, hence $|E| = 2n$. The number of subsets of E of size n is $\binom{2n}{n}$. We are now going to count them in a different way.

In order to determine a subset S of E of size n, we first choose the number k of elements of $S \cap A$, so we have $k \in \{0, \ldots, n\}$. Then there are $\binom{n}{k}$ possible choices for $S \cap A$. We then choose the $n - k$ elements of $S \cap B$. This is equivalent to choosing which elements of S are *not* in B, and since there are $n - (n - k) = k$ such elements, there are $\binom{n}{k}$ ways of choosing them. So overall their are

$$\sum_{k=0}^{n} \binom{n}{k}^2$$

of choosing S, and thus it is equal to $\binom{2n}{n}$.