MTH26001 Assignment 1

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1. (Sec 2.2 - 8) Observe that any term of the form 111...111 can be written as $100m + 11 = 4 * 25m + 11$ and $4 * 25m + 11 \equiv 3 \pmod 4$ for some m. (a)

   As all terms of the form 111...111 are odd, so must be the square roots of them (i.e n^2 is odd, then so is n). Transform $n = 2 * k + 1$ and $n^2 = 4 * k^2 + 4k + 1$ which by modulo 4 is not 3 which we got from (a). To be more precise, $n^2 = 4 * k^2 + 4k + 1 \equiv 1 \neq 3 \pmod 4$.

2. (Sec 2.2 - 9) For a number to be both a square and a cube it must of the form n^6. We could prove that n^6 can only be either of $7k$ or $7k + 1$ by Fermat's little theorem picking $p = 7$. But let's keep it short and observe that the congruent clasess of $n^6 \pmod 7$ is a set of $\{0, 1\}$.

3. (Sec 2.3 - 4d) Let's first establish a base case for induction:

   $21 \mid 4^{n+1} + 5^{2n-1} = 4^{1+1} + 5^{2-1} = 21$ for $n = 1$.

   Assume it holds true for some n. Then it should follow for $n + 1$ that $21 \mid 4^{(n+1)+1} + 5^{2*(n+1)-1} = 4^{n+2} + 5^{2n+1} = 4 * 4^{n+1} + 5^2 * 5^{2n-1} = 4 * (4^{n+1} + 5^{2n-1}) + 21 * 5^{2n-1}$.

4. (Sec 2.3 - 8b) Prove that $k * (k + 1) * (k + 2) * (k + 3) = n^2 - 1$ for some k, n;

   Let's rearrange it like so: $k * (k + 3) * (k + 1) * (k + 2) = (k^2 + 3k) * (k^2 + 3k + 2) = ((k^2 + 3k + 1) - 1) * ((k^2 + 3k + 1) + 1) = (k^2 + 3k + 1)^2 - 1$ so n is $(k^2 + 3k + 1)$.

5. (Sec 2.3 - 15) We know that $gcd(a, b) = gcd(a, b - a)$.

So, $gcd(2a-3b, 4a-5b) = gcd(2a-3b, 4a-5b-(2a-3b)) = gcd(2a-3b, 4a-5b-(2a-3b)-(2a-3b)) = gcd(2a-3b, 4a-5b-2*(2a-3b)) = gcd(2a-3b, b)$.

Thus, $gcd(2a-3b, 4a-5b) \mid b$. Let's assign $b = -1$, then it follows that $d = 1$ and $gcd(2a-3b, 4a-5b) = gcd(2a+3, 4a+5) = 1$.

6. (Sec 2.4 - 4c) Let $d = gcd(a+b, a^2+b^2)$. By definition of gcd, we know that both $d \mid (a+b)$ and $d \mid (a^2+b^2)$. Then, by rearranging a^2+b^2 as $(a+b)*(a-b)+2b^2$ and $(a+b)*(b-a)+2a^2$ we learn that $d \mid 2b^2$ and $d \mid 2a^2$.

   Consequently, $d \mid 2a^2 \wedge d \mid 2b^2 \implies d \mid gcd(2a^2, 2b^2)$. $gcd(2a^2, 2b^2) = 2 * gcd(a^2, b^2) = 2 * (gcd(a, b))^2 = 2 * 1^2 = 2$.

   Thus, d is a divisor of 2 (i.e 1 and 2).

   ps. proof of $gcd(a^2, b^2) = (gcd(a, b))^2$ is omitted for clarity.

7. (Sec 2.4 - 6) If $gcd(a, b) = 1$, prove that $gcd(a+b, ab) = 1$.

   Let $d = gcd(a+b, ab)$. Now, since $d \mid a+b$ and $d \mid ab$, it follows that $d \mid a*(a+b) - ab = a^2$ – similar for $d \mid b^2$.

   Similar to the proof in (Sec 2.4 - 4c), $d \mid a^2 \wedge d \mid b^2 \implies d \mid gcd(a^2, b^2) = (gcd(a, b))^2 = 1^2 = 1$. So $d = gcd(a+b, ab) = 1$.

8. (Sec 2.5 - 3b) $54x + 21y = 906$. $gcd(54, 21) = 3 \mid 906$, so there's a solution. Simplify to $18x + 7y = 302$. By brute force find that $x = 2, y = 38$ is one solution.

   Substituting 7 for x for 16 in y we find other solutions. So $x = 2, y = 38; x = 16, y = 2; x = 9, y = 20;$

9. (Sec 2.5 - 5c) $6x + 9y = 126$ and $9x + 6y = 114$, then subtracting one from the other we get $3y - 3x = 12$, dividing each side by 3 is $y - x = 4$, $y = x + 4$.

   Substituting for y in the initial equation we get $9x + 6*(x+4) = 114$. Solving for $x = 6$. Then $y = x + 4$, so $y = 6 + 4 = 10$. Thus, $x = 6, y = 10$.