MTH26001 Assignment 3

Nurseiit Abdimomyn – 20172001

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1. (Sec 4.2 - 3)

   Given $a \equiv b \pmod{n}$ implies that $k * n = a - b$ for some k. Because $gcd(b, n) \mid b$ and $gcd(b, n) \mid n$, it's also true that $gcd(b, n) \mid a = k*n+b$.

   Since $gcd(b, n) \mid a$, it should imply that $gcd(b, n) \mid gcd(a, n)$.

   Then, we could also prove similar for $b = a-k*n$. Because $gcd(a, n) \mid a$ and $gcd(a, n) \mid n$, it's should follow that $gcd(a, n) \mid b = a - k * n$.

   Since $gcd(a, n) \mid b$, it should imply that $gcd(a, n) \mid gcd(b, n)$.

   Thus, $gcd(a, n) \mid gcd(b, n)$ and $gcd(b, n) \mid gcd(a, n)$ implies that $gcd(a, n) = gcd(b, n)$.

2. (Sec 4.2 - 5)

   $53^{103} \equiv 1^{103} \pmod{1}3$ and $103^{53} \equiv (-1)^{53} \pmod{13}$.

   Thus, $53^{103} + 103^{53} \equiv 1^{103} + (-1)^{53} \equiv 0 \pmod{13}$.

   Also, $53^{103} \equiv (-1)^{103} \pmod{3}$ and $103^{53} \equiv 1^{53} \pmod{3}$.

   Thus, $53^{103} + 103^{53} \equiv (-1)^{103} + 1^{53} \equiv 0 \pmod{3}$.

   $111^{333} + 4^{111} \equiv (-1)^{333} + 4^{111} \pmod{7}$

   $\equiv (-1)^{333} + (4^3)^{37} \pmod{7}$

   $\equiv (-1)^{333} + 1^{37} \equiv 0 \pmod{7}$ [as $64 \equiv 1 \pmod{7}$].

3. (Sec 4.2 - 11)

   $0, 2^0, 2^1, ..., 2^9 \equiv 0, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512$

$\equiv 0, 1, 2, 4, 8, 5, 10, 9, 7, 3, 6 \pmod{11}$, thus a complete residue set modulo 11.

But, $0, 1^2, 2^2, ..., 10^2$ is not a complete residue set modulo 11 because $1^2 \equiv 10^2 \pmod{11}$.

4. (Sec 4.3 - 3)

   Let's look for the period of $9^n \pmod{100}$:

   so powers of nine modulo hundred are $1, 9, 81, 29, 61, 49, 41, 69, 21, 89, 1$, with a period of 10.

   Since $9^9 \equiv 9 \pmod{10}$ (here 10 is our period), $9^{9^9} \equiv 89 \pmod{100}$.

5. (Sec 4.3 - 10)

   All numbers n with digits adding up to 15 are $n \equiv 6 \pmod 9$. However, for all a, it's true that $a^3 \equiv 0, 1, 8 \pmod 9$ and $a^2 \equiv 1, 4, 7 \pmod 9$.

   Thus, n can not be a cube or a square.

6. (Sec 4.3 - 11)

   $495 = 5 * 9 * 11$, the resulting number should be divisible by all three 5, 9, and 11.

   Since $273x49y5$ ends with 5, it already is divisible by 5.

   In case of 9, sum of digits should be divisible by 9:

   $2 + 7 + 3 + x + 4 + 9 + y + 5 \equiv 0 \pmod 9$

   $\equiv 3 + x + y \equiv 0 \pmod 9$.

   In case of 11, alternating sum of digits should be divisible by 11:

   $2 + 3 + 4 + y - (7 + x + 9 + 5) \equiv 0 \pmod{11}$

   $x - y + 1 \equiv 0 \pmod{11}$.

   Thus, since $0 \leqslant x, y \leqslant 9$, there is only one solution:

   $x = 7, y = 8$.

   Indeed, $27374985 \equiv 0 \pmod{495}$.

7. (Sec 4.3 - 25)

Let's first prove that for any prime p, that it's either of form $6k+1$ or $6k-1$.

Considering all numbers in a form $6k+r$ we see that:

$6k+0$ is divisible by 6.

$6k+1$ has no apparent divisor.

$6k+2$ is divisible by 2.

$6k+3$ is divisible by 3.

$6k+4$ is divisible by 2.

$6k+5$ has no apparent divisor [i.e $6k-1 \pmod 6$].

Let's prove the original problem for when $p = 6k+1$ first by induction, so it becomes: $10^{2(6k+1)} - 10^{6k+1} + 1 \equiv 0 \pmod{13}$.

(a). $k = 1$. $10^{14} - 10^7 + 1 \equiv 9 - 10 + 1 \equiv 0 \pmod{13}$.

(b). Assume $10^{2(6k+1)} - 10^{6k+1} + 1 \equiv 0 \pmod{13}$,

then, $10^{2(6(k+1)+1)} - 10^{6(k+1)+1} + 1 \equiv$

$\equiv 10^{2(6k+7)} - 10^{6k+7} + 1 \pmod{13}$

$\equiv 10^{2*6} * 10^{2(6k+1)} - 10^6 * 10^{6k+1} + 1 \equiv 0 \pmod{13}$.

Now, similarly for $p = 6k-1$,

$10^{2(6k-1)} - 10^{6k-1} + 1 \equiv 0 \pmod{13}$.

(a). $k = 1$. $10^{10} - 10^5 + 1 \equiv 3 - 4 + 1 \equiv 0 \pmod{13}$.

(b). Assume $10^{2(6k-1)} - 10^{6k-1} + 1 \equiv 0 \pmod{13}$,

then, $10^{2(6(k+1)-1)} - 10^{6(k+1)-1} + 1 \equiv$

$\equiv 10^{2(6k+5)} - 10^{6k+5} + 1 \pmod{13}$

$\equiv 10^{2*6} * 10^{2(6k-1)} - 10^6 * 10^{6k-1} + 1 \equiv 0 \pmod{13}$.