Modeling Asset-Liability Management

Computational Optimization Final Project, Spring 2014

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1 Introduction

The primary responsibilities of any financial institute can be divided into two broad categories: 1) asset-liability management (ALM) and 2) risk management. Financial institutes utilize their assets to make profit by getting involved in various financial activities. These activities, however, are always shadowed by associated risks. While financial health of an institute is reflected in the balance sheets via assets and liabilities, risk reflects the vulnerability of the company against market uncertainties (Cornuejols & Tütüncü 2007). To make profit from the uncertain market financial institutes usually employ various optimization techniques to understand and manage risks associated with assets and liabilities.

Financial optimization models finds the best strategies to reduce investment risks and increase investment return. The conventional approach is to optimize a performance measure (e.g. expected investment return, sharp ratio) while constraining the risk measures (e.g. portfolio variance, value-at-risk, conditional value-at-risk). The mean-variance optimization models of risk management, however, are mostly single-period static models and do not incorporate the dynamic nature of asset management and multiple liabilities with different maturities faced by financial institutions. In the paradigm of ALM, therefore, a multi-period model is required to emphasize the need to meet liabilities in each period. Since asset returns and liabilities have random components, an optimal management of these metrics require stochastic programming (SP). Note that risk management can also be modeled using SP. The optimization problems in ALM try to maximize the expected wealth of a company at the end of given time period after taking care of all liabilities in the most effective manner (Cornuejols & Tütüncü 2007).

The organization of the write-up is as follows: section 2 provides a summary of ALM modeling techniques. It also introduces the most widely used ALM models in finance. Section 3 provides general algebraic framework of an ALM model and its implementation in AMPL for both deterministic and probabilistic cases. Section 4 highlights the primary model developed in this project. It also briefly discuss the outcome and the limitations of the model. A summary is provided in section 5.

2 ALM Modeling Techniques

Financial market is inherently uncertain. It thus provides opportunities for innovative investment strategies and various risk management methods. The market uncertainties limit the capabilities of deterministic programming (DP) models to predict the market effectively. The SP has become an ideal tool in modeling dynamic financial decision-making processes under uncertainty. Due to multi-dimensional uncertainty and multi-periods forecasting, many financial problems are relatively complex and need optimization models. In many cases, SP provide efficient and powerful alternative approaches to replace the existing modeling and analysis methods used in finance (Wu 1997).

The ALM models are the most widely discussed and applied models in financial application of SP. Finding optimal solutions to ALM problems are crucial to pension funds, insurance companies and banks where business involves large amount of liquidity. The financial institutions apply ALM to guarantee their liabilities while pursuing profit.

The management of assets and liabilities is the practice of managing risks that arise due to mismatches between assets and liabilities with a goal to maximize wealth. Many forms of the objective function are used in various ALM models. These include 1) expected terminal wealth functions, 2) expected utility of terminal wealth, 3) mean-variance of returns, and 4) expected shortfall. Risk is included in ALM models in the form of various penalties, e.g. under-funding, under-performance etc., if the goals of the models are not met.

2.1 Major ALM Models

The major empirical ALM applications utilizing SP include 'The Russell-Yasuda Kasai Model' (Cariño et al. 1994) and 'The CALM stochastic programming model for dynamic asset-liability management' (Consigli & Dempster 1998). To develop a better understanding of ALM modeling techniques I primarily focus on the methodologies used in the Russel-Yasuda Kasai Model (RYK) model and in a similar model developed by Yang et al. (2011).

In the RYK model the management of assets and liabilities are modeled as a four-stage problem and the objective function maximizes expected terminal wealth and minimizes accumulated expected shortfalls. The model includes hard constraints which were not allowed to be violated and soft constraints which were allowed to be violated, but the violations were penalized in the objective function. Compared with other ALM models, the RYK model also adds end effect constraints which will reasonably allocate assets at the terminal period. The uncertainty of the model is represented by a four-time periods scenario tree with random variables for prices and income returns for all the assets as well as policy crediting rates. Some scenario reduction techniques are employed to curb the rapid growth of the scenario tree. When a complete scenario tree is obtained, the total number of scenarios then is reduced by aggregating the branches at each node such that the reduction of the number of scenarios will still maintain the same mean and variance of the complete scenario tree. The number of branches per node of the simplified scenario tree is smaller in the later stages than in the earlier stages. This aggregation substantially reduces the size of the scenario tree (Wu 1997).

In the RYK model the asset classes contain loans and investments. The liabilities are savings-oriented policies issued by the company. Each new policy sold represents an inflow of funds. Interest is periodically credited until maturity, typically for three to five years, at which time the principal amount plus the credited interest is refunded to the policy-holders. The random elements include price return and interest income for each asset class and policy crediting rates.

A full deterministic version of the RYK model is provided in appendix A. I should state here that instead of reproducing the exact RYK model for the project I instead build a model with a slightly different formulation that is widely used in literature (Mitra and Schwaiger 2011; Yang et al. 2011; Wallace and Ziemba 2005).

3 Algebraic Formulation of ALM

3.1 ALM Modeling : Deterministic Case

A deterministic ALM model can be stated as follows:
An investor faces the problem of creating a portfolio allocating a set of assets belonging to a universe I. Each assets class is characterized by price x. The goal of the investor is to maximize the portfolio wealth at the end of a predefined time horizon T. He needs to take into account future obligations (liabilities) L, and the fact that each trade has an associated transaction cost expressed by the fraction g. In each time period of the time horizon, and for each asset considered, the investor needs to decide:

- The amount of assets to buy
- The amount of assets to hold
- The amount of assets to sell

To control risk, a measure known as the 'downside risk', is introduced into the model. This risk is measured as the deviation of our portfolio wealth from a predefined target level. Downside risk, as opposed to the mean variance framework, penalizes only under-performance. According to this modeling approach the investor is not interested in avoiding investment opportunities with very high levels of return but is only concerned in investments that will drive the portfolio to smaller returns.

Algebraic Formulation:
Indices:
$i = 1 \ldots I$, Number of assets
$t = 1 \ldots T$, Number of time stages

Decision Variables:
B_{ti} = Quantity of assets i to buy in time stage t
H_{ti} = Quantity of assets i to hold in time stage t
S_{ti} = Quantity of assets i to sell in time stage t

Problem Parameters:
x_{ti} = Price of asset i in stage t
g = Transaction cost, percent of trade value
L_t = Liability valuation at end of stage t
$H0_i$ = Initial composition of the portfolio
F_t = Funding in stage t

A_t = Target wealth at the end of stage t
r_t = Maximum deviation from target accepted by the investor in stage t

Objective Function:

$$\text{maximize wealth} = \sum_{i=1}^{N} x_{Ti} \cdot H_{Ti} \tag{1}$$

Constraints:
1) Asset Holding :
$H_{ti} = H0_i + B_{ti} - S_{ti}$ for $i = 1...I, t = 1$
$H_{ti} = H_{t-1,i} + B_{ti} - S_{ti}$ for $i = 1...I, t = 2...T$

2) Fund Balance:
$(1-g)\sum_{i=1}^{I} x_{ti} \cdot S_{ti} + F_t = L_t + (1-g)\sum_{i=1}^{I} x_{ti} \cdot B_{ti}$ for $t = 1...T$

3) Downside Risk :
$A_t - \sum_{i=1}^{I} X_{ti} \cdot H_{ti} \leq r_t \cdot A_t$ for $t = 2...T$

4) Non-negativity :
$H_{ti}, B_{ti}, S_{ti} \geq 0$

3.2 ALM Modeling : Introducing Uncertainty

The deterministic model presented above provide optimal decisions assuming the model parameters are known with certainty. However, it has been known how a wrong estimation of the model parameters, e.g., future asset prices, can lead to a non-optimal decision, which translates into a loss of profits. Therefore, the uncertainty in the asset prices has to be considered and introduced in the model in order to obtain a robust solution.

A first refinement of the framework illustrated above is to introduce a more sophisticated method for the estimation of the asset prices. Empirical evidence demonstrates that the normal approximation employed earlier does not reflect the reality, as the returns of the financial assets exhibit fat tails (rare events do occur in the markets), and the volatility (described as the standard deviation) is not itself constant. The sub-models of randomness, introduced in SP, do not impose such restrictions. This results in more accurate representation of the behavior of assets.

The decision models in SP is a combination of a deterministic model and model of randomness. The modeling of randomness utilizes the set of available past data with the aim of building sub-models for each individual stochastic parameter. These sub-models are then employed to generate a set of scenarios that encapsulate the investor's perception about the future. The deterministic model representing the needs and constraints of an institution or an investor is modeled initially. This is represented in a linear form and includes a risk measure usually in the form of a penalty function. The linear programming representation coupled with the model of randomness implied by the scenarios provides the instrument for dealing with uncertainty. However, once scenarios have been calculated, they have to be incorporated into the optimization model (Valente et al. 2005).

Algebraic Formulation:
Following Valente et al. we consider a 2-stage model with a scenario tree with 8 possible outcomes. We assume that each scenario (s) has the same probability, $\mathrm{p_s} = 1/8$. This optimization model is, therefore, a refinement of the deterministic ALM model. The introduction of the scenarios requires the addition to the model of a new index scenarios, ranging from s = 1 to s = 8.

Indices:
$\mathrm{t = 1...T}$, Number of time stages
$\mathrm{i = 1...I}$, Number of assets
$\mathrm{s = 1...S}$, Number of scenario

Decision Variables:
$\mathrm{H_{tis}}$ = Quantity of assets i to hold in time stage t in scenario s
$\mathrm{S_{tis}}$ = Quantity of assets i to sell in time stage t in scenario s
$\mathrm{B_{tis}}$ = Quantity of assets i to buy in time stage t in scenario s

Problem Parameters:
$\mathrm{p_s}$ = Probability of scenario s
$\mathrm{x_{tis}}$ = Price of asset i in stage t in scenario s
g = Transaction cost, percent of trade value
$\mathrm{L_t}$ = Liability valuation at end of stage t
$\mathrm{H0_i}$ = Initial composition of the portfolio
$\mathrm{F_t}$ = Funding in stage t
$\mathrm{A_t}$ = Target wealth at the end of stage t
$\mathrm{r_t}$ = Maximum deviation from target accepted by the investor in stage t

Objective Function:

$$\text{maximize wealth} = \sum_{\mathrm{s=1}}^{\mathrm{S}} \mathrm{p_s} \left(\sum_{\mathrm{i=1}}^{\mathrm{I}} (\mathrm{x_{Tis}} \cdot \mathrm{H_{Tis}}) \right) \tag{2}$$

Constraints:
1) Asset Holding :
$\mathrm{H_{tis} = H0_i + B_{tis} - S_{tis}}$ for $\mathrm{t = 1, i = 1...I, s = 1...S}$
$\mathrm{H_{tis} = H_{t-1,is} + B_{tis} - S_{tis}}$ for $\mathrm{t = 2...T, i = 1...I, s = 1...S}$

2) Fund Balance:
$\mathrm{(1-g)\sum_{i=1}^{I} x_{tis} \cdot S_{tis} + F_t = L_t + (1-g)\sum_{i=1}^{I} x_{tis} \cdot B_{tis}}$ for $\mathrm{t = 1...T,\ s = 1...S}$

3) Downside Risk :
$A_t - \sum_{i=1}^{I} X_{tis} \cdot H_{tis} \leq r_t \cdot A_t$ for $t = 2...T,\ s = 1...T$

4) Non-negativity :
$H_{tis}, B_{tis}, S_{tis}, g, H0_i, L_t, F_t, r_t, A_t \geq 0$

3.3 AMPL Implementation of ALM Models

The implementation ALM model requires various inputs. The following is a list general purpose input data. The list may also include additional input parameters depending on the requirements of the model:
1. identification of assets to invest;
2. estimates of the returns on these assets;
3. estimates of capital gains (losses) as a function of minimum time period to hold the assets;
4. identification of liabilities to sell;
5. estimates of the costs of these liabilities;
6. the rate at which deposits are withdrawn;
7. an estimated weighted cost of funds to determine the discount rate;
8. pertinent legal constraints;
9. parameters used in the development of liquidity constraints;
10. policy constraints used by the bank;
11. estimates of the marginal distributions of the stochastic resources;
12. unit penalties incurred for shortage or surplus in the stochastic constraints.

We provide AMPL implementation of the deterministic ALM model for 4 time stages ($t = 4$) and for 10 assets ($i = 10$). Various model inputs such as asset prices, funding, and liabilities are provided in the model and data file (almLP01.mod and almLP01.dat). The model output can be obtained by running the script file, almLP01.run.

We provide AMPL implementation of the probabilistic ALM model for 2 time stages ($t = 2$) and for 4 assets ($i = 4$). The necessary model inputs are provided in almLP02.mod and almLP02.dat files.

4 The ALM Model Studied in the Project

Constructing and implementing effective measure of risk management in the context of ALM is a challenging task. As mentioned earlier there are various ways to measure risk such as variance and expected shortfall. However, in recent years stochastic dominance has gained substantial interest from the research community an alternative measure (Levy 2006). The most attractive features of this measure are: 1) it is consistent with utility functions and 2) it considers the entire probability distribution.

Stochastic dominance constraints link variables which are associated with different nodes at the same stage in the event tree. It, therefore, involves in comparison of (nonlinear) probability distribution functions which makes the straightforward application difficult. An application of the first-order stochastic dominance in SP leads to a non-convex mixed integer programming formulation. In contrast, the second-order stochastic dominance can be incorporated in a form of linearized constraints which makes it easily applicable to financial optimization problems. Ruszczynski and collaborators analyzed various aspects of the use of stochastic dominance (Yang et al. 2011 and reference therein).

In recent study Yang et al. applied stochastic dominance as a risk measure in modeling ALM. The authors also developed a chance constraint from relaxed interval second-order stochastic dominance and show that it is an intermediate dominance constraint between first-order and second-order in the problem with discrete probability distribution. The authors demonstrated that by combining second-order stochastic dominance and relaxed interval second-order stochastic dominance, the model help generating portfolio strategies with better management of risk and better control of under-funding.

In this project I only consider the use of second-order stochastic dominance to control the risk of overall performance of a portfolio compared to a benchmark portfolio. I will explore application of chance constraints in the future.

Algebraic Formulation of Yang et al. Model:
To implement the model of Yang et al. we incorporate two major changes compared to the formulation shown in section 3. First, for this model we follow the prescription of Dempster et al. (2005) to generate nodes and scenario partitions. This method, complementary to the method presented in section 3, keeps tracks of the number of nodes in a given stage. I assume a 1-8-4-2 tree structure with 4 distinct time stages along the time horizon. I label the first stage containing the root node as stage 1. I generate a total of 64 scenarios at the time final time stage 4. All scenarios of the leaf nodes are equally likely, i.e., $\mathrm{p_s} = 1/64$. Second, we include the linearized second-order stochastic dominance as a constraint.

Indices:
$\mathrm{t} = 1...\mathrm{T}$, Number of time stages
$\mathrm{i} = 1...\mathrm{I}$, Number of assets
$\mathrm{s} = 1...\mathrm{S}$, Number of scenarios
$\mathrm{k} = 1...10$, Number of benchmarks

Decision Variables:
$\mathrm{H_{t,is}}$ = Quantity of assets i to hold in time stage t in scenario s
$\mathrm{S_{t,is}}$ = Quantity of assets i to sell in time stage t in scenario s
$\mathrm{B_{t,is}}$ = Quantity of assets i to buy in time stage t in scenario s

Problem Parameters:
p_s = Probability of scenario s
$x_{t,is}$ = Price of asset i in stage t in scenario s
g = Transaction cost, percent of trade value
L_t = Liability valuation at end of stage t
$H0_i$ = Initial composition of the portfolio
F_t = Funding in stage t
A_t = Target wealth at the end of stage t
r_t = Maximum deviation from target accepted by the investor in stage t
M_k = Market index used as the benchmark
π_k = Probability of market index used as the benchmark

Objective Function:

$$\text{maximize wealth} \ = \ \sum_{s=1}^{S} p_s \left(\sum_{i=1}^{I} (x_{T,is} \cdot H_{T,is}) \right) \tag{3}$$

Constraints:
1) Asset Holding :
$H_{t,is} = H0_i + B_{t,is} - S_{t,is}$ for $t = 1, i = 1...I, s = 1...S$
$H_{t,is} = H_{t-1,is} + B_{t,is} - S_{t,is}$ for $t = 2...T, i = 1...I, s = 1...S$

2) Fund Balance:
$(1 - g) \sum_{i=1}^{I} x_{t,is} \cdot S_{t,is} + F_t = L_t + (1 - g) \sum_{i=1}^{I} x_{t,is} \cdot B_{t,is}$ for $t = 1...T,\ s = 1...S$

3) Downside Risk :
$A_t - \sum_{i=1}^{I} x_{t,is} \cdot H_{t,is} \leq r_t \cdot A_t$ for $t = 2...T,\ s = 1...T$

4a) Second-order Stochastic Dominance: Overall Performance w.r.t. a Benchmark
$\sum_{i=1}^{I} x_{t,is} \cdot H_{t,is} + s_{n,k} \geq M_k$ for $t = 2...T,\ s = 1...S,\ k = 1...10$
where M_k is one dimensional array of the market index, π_k is the probability of the individual index, and $s_{n,k}$ is the shortfall. For simplicity we have consider only 10 realization of the market ($k = 1...10$).

4b) Second-order Stochastic Dominance: Area Under the Curve of Cum. Distribution
$\sum_{s=1}^{S} p_s \cdot s_{n,k} \leq \sum_{l=1}^{10} \pi_k (M_k - M_l)_+$ for $t = 2...T,\ k = 1...10$
where $(M_k - M_l)_+ = \max(M_k - M_l, 0)$

5) Non-negativity :
$H_{t,is}, B_{t,is}, S_{t,is}, g, H0_i, L_t, F_t, r_t, A_t, s_{n,k} \geq 0$

4.1 The Output of the Model

In this project I develop a basic ALM model which contains a portfolio of six assets. These include two domestic stocks, one fixed income (government security, two cash securities (money market and cash), and one real estate. The model extends to a time horizon of one year in the future which is further divided into four quarters. Note that I keep all asset prices and the market indices as realistic as possible to the real data.

Initial Funding: \$150,000

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time stage = 1 quarter
time Horizon = 1 year

Initial Assets Prices (\$):
MoneyMkt 70
bond 40
cash 100
fstock 58
mstock 49
realEstate 65

Final Wealth : \$217,625

It is evident that the model increases the initial wealth, which is a combination of an initial fund of \$150,000 plus additional assets values, to the final wealth of \$217,625. The model starts with six assets without any holding. However, using the data and the generated scenarios it moves to the final stage. The model suggests to hold the stocks named 'mstock' and the bonds by the amount of 2865 and 48, i.e.

$$\mathrm{H}[4, (\mathrm{stock1}, \mathrm{stock2}, \mathrm{bond}, \mathrm{moneyMarket}, \mathrm{cash}, \mathrm{realEstate}] = [4, (0, 2865, 48, 0, 0, 0)]$$

where 4 represents the final stage.

4.2 Limitations of the Model

In this section I highlights several features that I did not include in the model. However, I intended to improve the model by including these features in the future.

1) The model does not consider trading cost at the parent node.
2) The number of securities in the portfolio remains static in each stage. I do not attempt to change the portfolio composition by dropping an asset with poor performance or add a new asset at any stage after the initial stage.
3) No legal, policy, or regulatory constraint is added to the model.

4) The model does not contain the effects of taxes.
5) Unlike RYK model, which uses multiple accounts (general and savings) to divide the assets, this model only consider one account that comprises all assets.
6) No correlations among assets prices are considered at any given stage.
7) Most importantly, the model only considers the randomness in the asset returns. No randomness is assigned to the liabilities.

5 Conclusion

Successful optimization of various problems associated with the ALM is of vital importance to pension funds, insurance companies and banks where business involves large amount of liquidity. The financial institutions apply ALM to guarantee their liabilities while pursuing profit. I use two successful ALM models currently existing in the financial market as the benchmark models and develop a unique model to test my understanding of various techniques associated with ALM modeling. I use a particular risk measure, known as the stochastic dominance, to control the overall performance of the model w.r..t the market. Using synthetic data, which resembles the market data as closely as possible, I show that the model successfully generates profit and thus increase the wealth.

Acknowledgment
For this project I resort to various paper,s texts and online resources. One particular web site that I find very useful for this project is maintained by OptiRisk System[1]. The site includes a collection of white papers on various techniques of ALM modeling by various researchers of the company.
I am indebted to the class instructor Dr. M. Lejeune for his encouragement and thoughtful suggestion to choose ALM as the class project. I enjoyed the challenge offered by the tricks and techniques associated with the scenario generation in SP.

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Appendix A: The Russel-Yasuda Kasai Model

In this appendix we reproduce the algebraic formulation of deterministic (single scenario) RYK model from Cariño et al. (1994). The objective of the model is to allocate fund value among available assets to maximize expected wealth at the end of the planning horizon T less expected penalized shortfalls accumulated through the planning horizon. The model presentation does not include additional types of shortfall, indirect investments, regulatory restrictions, multiple accounts, loan assets, the effects of taxes, and end effects.

Random Variables:
$\mathrm{RP_{it}}$ = price return of asset i in period t
$\mathrm{RI_{it}}$ = income return of asset i in period t
$\mathrm{F_t}$ = deposit inflow from period t-1 to period t
$\mathrm{P_t}$ = principal payout from period t-1 to period t
$\mathrm{I_t}$ = income pay out from period t-1 to period t
$\mathrm{g_t}$ = rate of interest paid on policies from period t-1 to period t
$\mathrm{L_t}$ = liability valuation at end of period t

Decision Variables:
$\mathrm{x_{it}}$ = market value held in asset i in period t
$\mathrm{w_t}$ = interest income short-fall at $\mathrm{t \geq 1}$
$\mathrm{v_t}$ = interest income surplus $\mathrm{t \geq 1}$

Objective Function:
maximize $\mathrm{E}\left[\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{x_{iT}} - \sum_{\mathrm{t=1}}^{\mathrm{T}} \mathrm{c_t(w_t)}\right]$,
where $\mathrm{c_t}$ is a piecewise linear convex cost function.

Constraints:
1) Asset accumulation :
$\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{x_{it}} - \sum_{\mathrm{i=1}}^{\mathrm{N}}(1 + \mathrm{RP_{it}} + \mathrm{RI_{it}}) \cdot \mathrm{x_{i,t-1}} = \mathrm{F_t} - \mathrm{P_t} - \mathrm{I_t}$ for $\mathrm{t} = 1, 2, \ldots\mathrm{T}$.

2) Liability accumulation :
$\mathrm{L_t} = (1 + \mathrm{g})\mathrm{L_{t-1}} + \mathrm{F_t} - \mathrm{P_t} - \mathrm{I_t}$ for $\mathrm{t} \geq 1$.

3) Interest income short-fall :
$\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{RI_{it}} \cdot \mathrm{x_{i,t-1}} + \mathrm{w_t} - \mathrm{v_t} = \mathrm{g_t} \cdot \mathrm{L_{t-1}}$ for $\mathrm{t} = 1, 2, \ldots\mathrm{T}$.

4) Non-negativity :
$\mathrm{x_{it}} \geq 0, \mathrm{w_t} \geq 0$, and $\mathrm{v_t} \geq 0$