Portfolio Optimization Techniques

Problem Formulation

Invest the entirety of capital K in an asset universe comprising N securities. No short-sell is allowed. Denote the proportion of the capital invested in asset i by W_i. The allocation policy is based on the variants of the mean-variance (Markowitz) portfolio optimization model. The problem is to minimize the variance of the portfolio's return given that an expected return will be at least or equal to a threshold value R.

a) Formulate portfolio optimization problem as a quadratic optimization problem.
b) Provide an equivalent second-order cone programming formulation of part a).
c) Provide an AMPL code for the formulation of parts a) and b). Add a constraint that stipulates that one can invest in at most 12 securities and set the minimal level of expected return to 5

Optimization Techniques:

1) Quadratic Porgramming (QP): Let us denote the division of the capital K among N securities by $\mathrm{x_1, x_2 ... x_N}$ where,

$\mathrm{w_1 = \frac{x_1}{K}}$, proportion invested in security 1,
$\mathrm{w_2 = \frac{x_2}{K}}$, proportion invested in security 2,
... ...
$\mathrm{w_N = \frac{x_N}{K}}$, proportion invested in security N.

In portfolio optimization problem w's are the decision variables because an investor needs to decide how much of the total capital needs to be invested in a specific security. The objective function of the mean-variance optimization (MVO) problem of a portfolio where the variance is minimized can be written as,

$$\text{minimize} \quad \mathbf{w^T Q w}. \tag{1}$$

In the expression above $\mathbf{w} = \{\mathrm{w_1, w_2 ... w_N}\}$ is the N-dimensional vector showing fractional investment and $\mathbf{Q}$ is a $\mathrm{N \times N}$ covariance matrix which incorporates the interactions, i.e., correlations, of securities. The covariance matrix can be written as,

$$\begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \dots & \sigma_{1,\mathrm{N}} \\ \sigma_{2,1} & \sigma_{2,2} & \dots & \sigma_{2,\mathrm{N}} \\ \dots & \dots & \dots & \dots \\ \sigma_{\mathrm{N},1} & \sigma_{\mathrm{N},2} & \dots & \sigma_{\mathrm{N,N}} \end{pmatrix},$$

where $\sigma_{1,1}$ is the portfolio risk associated with the random variations of the security 1 and $\sigma_{1,\mathrm{N}}$ is the risk associated with the variations of securities 1 and N when both of them are correlated. For uncorrelated securities $\mathbf{Q}$ contains only diagonal terms. We note here that portfolio MVO problem is a quadratic program (QP) since the objective function is quadratic in decision variable. However, the optimal value of the objective function is a scalar quantity.

The constraints of this quadratic optimization problem are as follows:
1) Budget constraint: This constraint asserts that the securities invested in the market must add up to the total capital K. In other words the w's invested must add up to 1.

$\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{w_i} = 1$

2) Expected portfolio return: This constraint states that the expected return of the entire portfolio return at least equal to a threshold value. In other words,

$\sum_{\mathrm{i=1}}^{\mathrm{N}} \mu_{\mathrm{i}} \cdot \mathrm{w_i} \geq \mathrm{R}$, where μ_i is the expected return of security i and R is the desired threshold.

3) Non-negativity and no short-selling: This constraint imposes the restriction that there is no short-selling of the securities by imposing that the weights must be non-negative real numbers. In other words, $\mathrm{w_i} \geq 0$.

2) Second-order Cone Programming (SOCP): In SOCP paradigm the objective function is a linear function of the decision variables. The set of constraints of the model, however, contain linear as well quadratic functions. The objective function of the MVO problem is a scalar quantity. To convert it into a SOCP model, therefore, we need to incorporate a new decision variable. Let us denote it by t. This variable will substitute the quadratic term of the objective function. However, it will be tied to the quadratic term as a constraint. The equivalent SOCP model of the MVO problem with the same optimal solutions and optimal value can be written as follows:

$$\text{minimize} \quad \mathrm{t} \tag{2}$$

The set of constraints of the SOCP model is:
1) Convex non-linear cone membership:

$$\mathbf{w^T Q w} \leq \mathrm{t} \quad \Leftrightarrow \quad \|[\begin{matrix} 2\mathrm{Q}^{1/2}\mathrm{w} \\ 1 - \mathrm{t} \end{matrix}]\| \leq 1 + \mathrm{t}$$

2) Budget constraint: $\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{w_i} = 1$

3) Expected portfolio return: $\sum_{\mathrm{i=1}}^{\mathrm{N}} \mu_{\mathrm{i}} \cdot \mathrm{w_i} \geq \mathrm{R}$

4) Non-negativity and no short-selling: $\mathrm{w_i} \geq 0$.

4) Non-negativity of t: $\mathrm{t} \geq 0$.

Equivalence of QP and SOCP: The QP with linear constraints and SOCP with both linear and non-linear constraints are equivalent programming models because both of these solve optimization problems defined in a convex space. In QP the objective function is convex as long as the covariance matrix $\mathbf{Q}$ is positive definite. In SOCP, however, the objective function is linear and hence convex. Both of these models contain linear constraints which are always convex. The addition of the quadratic constraint in the SOCP model simply confines the feasibility space into a second-order cone which is a convex region in N-dimensional space.