Мат статистика

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N1 $\xi \sim R(0,\theta)$

$\tilde{\theta}_1 = 2\bar{x} = 2\frac{1}{n}\sum_{i=1}^{n} x_i$ $\qquad$ $\tilde{\theta}_3 = x_{max}$

$\tilde{\theta}_2 = x_{min}$ $\qquad$ $\tilde{\theta}_4 = x_1 + \frac{1}{n-1}\sum_{k=2}^{n} x_k$

1) $\tilde{\theta}_1$: несмещенность: $M[\tilde{\theta}_1] = \theta$

$M[2\frac{1}{n}\sum_{i=1}^{n} x_i] = \frac{2}{n} M[\sum x_i] = \frac{2}{n}\sum Mx_i = 2M[\xi] =$

$= \left\{ p(x) = \frac{1}{\theta}\{(0,\theta)\} \quad M[\xi] = \int_0^\theta x\frac{1}{\theta}dx = \frac{\theta}{2} \right\} = 2\cdot\frac{\theta}{2} = \theta \Rightarrow$ несмещённая

состоятельность:

$D[\tilde{\theta}_1] = D[\frac{2}{n}\sum x_i] = \frac{4}{n^2} D[\sum x_i] = \{\text{т.к. незав.}\} = \frac{4}{n^2}\sum Dx_i =$

$= \frac{4}{n} D\xi = \frac{4}{3n} \left\{ D\xi = M\xi^2 - M^2\xi = \int_0^\theta x^2 \frac{1}{\theta}dx - \frac{\theta^2}{2^2} = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12} \right\} = \frac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0$

$\Rightarrow$ по теор. (дост. усл. сост.) $\Rightarrow$ состоятельна.

2) $\tilde{\theta}_2$: несмещённость:

$\tilde{\theta}_2 = x_{min} = x_{(1)}$; $M[\tilde{\theta}_2] = \int_{-\infty}^{+\infty} y\, q(y)\, dy$

$x_i \sim R(0,\theta)$, $x_i \sim F(x)$, $x_{(1)} \sim \underbrace{1-(1-F(y))^n}_{F(y)}$

$q(y) = n(1-(1-F(y)))^{n-1}F'(y) = n(1-\frac{y}{\theta})^{n-1}\frac{1}{\theta}\{(0,\theta)\}$

$M[\tilde{\theta}_2] = \int_{-\infty}^{+\infty} y\, q(y)\, dy = \int_0^\theta y\, n(1-\frac{y}{\theta})^{n-1}\frac{1}{\theta}dy = \{t = 1-\frac{y}{\theta}\} =$

$= -\int_1^0 \theta(1-t)nt^{n-1}\frac{1}{\cancel{\theta}}\cancel{\theta}\, dt = n\theta(\int_0^1 t^{n-1}dt - \int_0^1 t^n dt) =$

$= n\theta\frac{1}{n} - n\theta\frac{1}{n+1} = \theta\frac{1}{n+1}$ $\Rightarrow$ смещённая

но $\exists\ \tilde{\theta}_2' = (n+1)\theta_2 = (n+1)x_{min} \Rightarrow M[\hat{\theta}_2'] = \theta$ - несмещён.

состоятельность

$D[\tilde{\theta}_2] = M[\tilde{\theta}_2^2] - M^2[\tilde{\theta}_2]$

$M[\tilde{\theta}_2^2] = \int_{-\infty}^{\infty} y^2 q(y)\, dy = \int_0^\theta y^2 n(1-\frac{y}{\theta})^{n-1}\frac{1}{\theta}dy = \{t = 1-\frac{y}{\theta}\} =$

$= -\int_1^0 \theta^2(1-t)^2 n t^{n-1}\frac{1}{\theta}\theta\, dt = n\theta^2(\int_0^1 t^{n-1}dt - 2\int_0^1 t^n dt +$

$+ \int_0^1 t^{n+1}dt) = n\theta^2(\frac{1}{n} - 2\frac{1}{n+1} + \frac{1}{n+2}) = n\theta^2\left(\frac{(n+1)(n+2) - 2n(n+2) + n(n+1)}{n(n+1)(n+2)}\right) =$

$$= \frac{2\theta^2}{(n+1)(n+2)}$$

$$D[\tilde\theta_2] = M[\tilde\theta_2^2] - M^2[\tilde\theta_2] = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+2)^2} = \frac{2\theta^2(n+1)-\theta^2(n+2)}{(n+1)^2(n+2)} =$$

$= \frac{\theta^2 n}{(n+1)^2(n+2)} \not\xrightarrow[n\to\infty]{} 0 \Rightarrow$ смещённая $\Rightarrow$ дост. усл. не работает

Если взять $\tilde\theta_2'$ вместо $\tilde\theta_2$

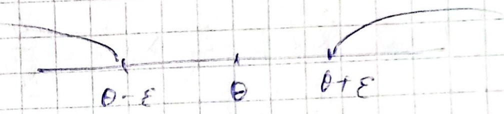
$D[\tilde\theta_2'] = D[(n+1)\tilde\theta_2] = (n+1)^2 \frac{\theta^2 n}{(n+1)^2(n+2)} \not\xrightarrow[n\to\infty]{} 0 \Rightarrow$ дост. усл. не работает.

По ~~опр~~ опр. смотрим:

$\tilde\theta_2 \xrightarrow{P} \theta \quad \forall\, \theta > 0$

$\forall\, \varepsilon > 0 \quad P(|\tilde\theta_2 - \theta| \ge \varepsilon) \xrightarrow[n\to\infty]{} 0$

$\tilde\theta_2 \ge \theta + \varepsilon$ или $\tilde\theta_2 \le \theta - \varepsilon$



$x_i \sim R(0,\theta) \qquad P(x_{min} \ge \theta+\varepsilon) = 0$

$P(x_{min} \le \theta-\varepsilon) = P(x_{min} < \theta-\varepsilon) = \Phi(\theta-\varepsilon) = \{\Phi(y) = 1-(1-F(y))^n\} =$

$= 1-(1-F(\theta-\varepsilon))^n = 1-\left(1-\frac{\theta-\varepsilon}{\theta}\right)^n = 1-\left(\frac{\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 1$

$\{$ т.к. $|\Phi| \le 1$, то $0 < \theta-\varepsilon < \theta$, $0 < \varepsilon < \theta \}$

$\Rightarrow \exists\, \varepsilon : P(|\tilde\theta_2 - \theta| \ge \varepsilon) \not\xrightarrow[n\to\infty]{} 0 \Rightarrow$ несостоятельная

Проверим $\tilde\theta_2'$

$\forall\, \varepsilon > \quad P(|\tilde\theta_2' - \theta| \ge \varepsilon) \xrightarrow[n\to\infty]{} 0$?

$P(|(n+1)x_{min} - \theta| \ge \varepsilon) = P(x_{min}(n+1) \ge \theta + \varepsilon) =$

$= P\left(x_{min} \ge \frac{\theta+\varepsilon}{n+1}\right) = 1 - P\left(x_{min} < \frac{\theta+\varepsilon}{n+1}\right) = 1 - \Phi\left(\frac{\theta+\varepsilon}{n+1}\right) =$

$= \left(1-F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n \underset{n>n_0}{=} \left(1-\frac{\theta+\varepsilon}{\theta(n+1)}\right)^n \xrightarrow[n\to\infty]{} e^{-\frac{\theta+\varepsilon}{\theta}} > 0$ – несостоят.

3) $\tilde\theta_3 = x_{max}$:

несмещённость :

$M[\tilde\theta_3] = \int_{-\infty}^{\infty} y\, q(y)\, dy = \int_0^\theta y^n \frac{n}{\theta^n}\, dy = \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1} = \frac{\theta n}{n+1}$ – смещён.

$\Psi(y) = (F(y))^n$

$q(y) = \Psi'(y) = n(F(y))^{n-1} F'(y) = n\left(\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta} \ \{(0,\theta)\}$

$\tilde\theta_3' = \frac{n+1}{n} x_{max}$ – несмещ. $M[\tilde\theta_3'] = \theta$

$n+1 \approx n$

состоятельность.

$D[\tilde{\theta}_3] = M[\tilde{\theta}_3^2] - M^2[\tilde{\theta}_3]$

$M[\tilde{\theta}_3^2] = \int_0^\theta y^2 q(y)\,dy = \int_0^\theta \frac{n}{\theta^n} y^{n+1}\,dy = \frac{n}{\theta^n} \cdot \frac{\theta^{n+2}}{n+2}$

$D[\tilde{\theta}_3] = \theta^2 \frac{n}{n+2} - \frac{\theta^2 n^2}{(n+1)^2} = \theta^2 \left(\frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2} \right) = \frac{\theta^2 n}{(n+2)(n+1)^2} \xrightarrow[n\to\infty]{} 0$

но дост усл не выполнено => смотрим по опр:

$\tilde{\theta}_3 \xrightarrow{P} \theta \quad \forall \theta > 0 \iff \forall \varepsilon > 0 \quad P(|\tilde{\theta}_3 - \theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$

$|\tilde{\theta}_3 - \theta| \geq \varepsilon \Rightarrow \left[\begin{array}{l} \tilde{\theta}_3 \geq \theta + \varepsilon \\ \tilde{\theta}_3 \leq \theta - \varepsilon \end{array} \right. \Rightarrow \left[\begin{array}{l} x_{max} \geq \theta + \varepsilon \\ x_{max} \leq \theta - \varepsilon \end{array} \right. \Rightarrow \left[\begin{array}{l} P(x_{max} \geq \theta + \varepsilon) = 0 \\ P(x_{max} \leq \theta - \varepsilon) = \end{array} \right.$

$= P(\tilde{\theta}_3 < \theta - \varepsilon) = (F(\theta - \varepsilon))^n = \left(\frac{\theta - \varepsilon}{\theta} \right)^n \xrightarrow[n\to\infty]{} 0$ => состоятельн.

Проверим $\tilde{\theta}_3'$

$D[\tilde{\theta}_3'] = D\left[\frac{n+1}{n} \tilde{\theta}_3 \right] = \frac{(n+1)^2}{n^2} \frac{\theta^2 n}{(n+2)(n+1)^2} = \frac{\theta^2}{(n+2)n} \xrightarrow[n\to\infty]{} 0$

=> состоятельная.

4) $\tilde{\theta}_4 = x_1 + \frac{1}{n-1} \sum_{i=2}^{n} x_i$

несмещенность:

$M[\tilde{\theta}_4] = M[x_1 + \frac{1}{n-1} \sum x_i] = M[x_1] + \frac{1}{n-1} \sum M[x_i] =$

$= M\xi + M\xi = 2M\xi = \theta$ => несмещённая

$\tilde{\theta}$ состоятельность

$D[\tilde{\theta}_4] = D[x_1] + \frac{1}{(n-1)^2} \sum_{i=2}^{n} D x_i = \frac{\theta^2}{12} + \frac{1}{n-1} \frac{\theta^2}{12} \xrightarrow[n\to\infty]{} \frac{\theta^2}{12}$

По опр:

$\tilde{\theta}_4 = \underbrace{x_1}_{\xi_n} + \underbrace{\frac{1}{n-1} \sum_{i=2}^{n} x_i}_{\eta_n} \xrightarrow{P} \xi + \frac{\theta}{2}$ => несостоятельная

$\left\{ \xi_n \xrightarrow{P} x_1, \quad \eta_n \xrightarrow{P} \underset{= M\xi}{M[x_i]} = \frac{\theta}{2} \right\}$

Сравнение оценок

$D[\tilde{\theta}_1] = \frac{\theta^2}{3n}$ $\quad n+2 > 3 \Rightarrow n > 1 \Rightarrow \tilde{\theta}_3'$ - эффективнее чем $\tilde{\theta}_1$

$D[\tilde{\theta}_3'] = \frac{\theta^2}{(n+2)n}$ $\quad \frac{1}{3n} > \frac{1}{n+2}$

$\boxed{N2}$

c) $\bar{x} : = \frac{1}{n} \sum_{i=1}^{n} x_i$

ЦПТ: $\{\xi_k\}$ - посл. одинаково распред. случ. вел. с кон. 2-ми моментами

$$\Rightarrow \frac{\sum_{k=1}^{n} \xi_k - M[\sum \xi_k]}{\sqrt{D[\sum \xi_k]}} \xrightarrow{F} \eta \sim N(0,1)$$

$$M[\xi_k] = \int_0^\infty x e^{-x} dx = x(-e^{-x})\Big|_0^\infty + \int_0^\infty e^{-x} dx = -\left(\frac{x+1}{e^x}\right)\Big|_0^\infty = 1$$

$$M[\xi_k^2] = \int_0^\infty x^2 e^{-x} dx = (-e^{-x}x^2)\Big|_0^\infty + 2\int_0^\infty e^{-x} x\, dx = (-2e^{-x}x)\Big|_0^\infty +$$
$$+ 2\int_0^\infty e^{-x} dx = 2\int_0^\infty e^{-x} dx = (-2e^{-x})\Big|_0^\infty = 2$$

$$D[\xi_k] = M[\xi_k^2] - M^2[\xi_k] = 2 - 1 = 1 \quad \Rightarrow \frac{\sum_{k=1}^{n} \xi_k - \cancel{M}\, 1\cdot n}{\sqrt{1\cdot n}} =$$

$$= n\frac{\frac{1}{n}\sum \xi_k - 1}{\sqrt{n}} = \sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n} \xi_k - 1\right) \sim N(0,1)$$

$$\Rightarrow \sqrt{n}\,\bar{x} - \sqrt{n} \xrightarrow{F} \eta \sim N(0,1)$$

$$\bar{x} = \frac{\eta}{\sqrt{n}} + 1 \quad \Rightarrow \bar{x} \sim N(\underset{b}{1}, \underset{a^2}{\tfrac{1}{n}})$$

$$p(\bar{x}) = \frac{1}{\sqrt{2\pi}\,a} e^{-\frac{(\bar{x}-b)^2}{2a^2}} = \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{n(\bar{x}-1)^2}{2}}$$

e) $\gamma = \frac{\mu_3}{\mu_2^{3/2}}$

$\mu_2 = D[\xi_k] = 1$

$$\mu_3 = M[(\xi_k - M[\xi_k])^3] = M[(\xi_k - 1)^3] = \int_0^\infty (x-1)^3 e^{-x} dx =$$
$$= \int_0^\infty (e^{-x}x^3 - 3e^{-x}x^2 + 3e^{-x}x - e^{-x})dx = (-e^{-x}x^3)\Big|_0^\infty + 3\int_0^\infty e^{-x} x\,dx - \int_0^\infty e^{-x} dx =$$
$$= 3\int_0^\infty e^{-x} x\,dx - \int_0^\infty e^{-x} dx = (-3e^{-x}x)\Big|_0^\infty + 2\int_0^\infty e^{-x} dx = 2\int_0^\infty e^{-x} dx =$$
$$= (-2e^{-x})\Big|_0^\infty = 2$$

$$\gamma = \frac{2}{1^{3/2}} = 2.$$

f) ~~[crossed out]~~

$\int_0^t p(x)dx = e^{-t}+1$

$P(med) = p(x_{(13)}) = C_{24}^{12}\,(F(t))^{12}\cdot(1-F(t))^{12}\cdot 25\cdot e^{-x} =$

$= C_{24}^{12}\cdot 25\,(1-1-e^{-x})^{12}\,(e^{-x}+1)e^{-x}$

$\boxed{N3}$

$$p(x) = \begin{cases} \frac{e^{-x/\theta}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad \theta > 0 \qquad n = 3.$$

$\tilde{\theta}_1 = \bar{x}, \quad \tilde{\theta}_3 = x_{(2)}$

a) ① $M[\hat{\theta}_1] = \theta$ - ?

$M[\hat{\theta}_1] = M[\bar{x}] = M[\frac{1}{n}\sum_{i=1}^{n} x_i] = \frac{1}{n}\cdot n\cdot M[\xi] = M[\xi] =$

$= \int_{-\infty}^{+\infty} x\, p(x)dx = \int_0^{+\infty} x\,\frac{e^{-x/\theta}}{\theta}\,dx = \theta\int_0^{\infty}\frac{x}{\theta}e^{-x/\theta}\,d\frac{x}{\theta} = \{t = \frac{x}{\theta}\} =$

$= \theta\int_0^{\infty} t e^{-t}\,dt = \theta(-te^{-t}|_0^{+\infty} - \int_0^{\infty} -e^{-t}dt) = -\theta(0 + e^{-t}|_0^{\infty}) = \theta$

$\Rightarrow$ несмещенная

② $M[\tilde{\theta}_3] = M[x_{(2)}] = \int_0^{\infty} t\, n\, p(t)\, C_2^1\, F(t)\,(1-F(t))dt =$

$= 6\int_0^{\infty}(t\frac{1}{\theta})e^{-t/\theta}\,(1-e^{-t/\theta})\,(-1+1+e^{-t/\theta})dt = \{F(t) = \int_{-\infty}^{t} p(x)dx = -e^{-x/\theta}|_0^t =$

$= 1-e^{-t/\theta}\} = 6\int_0^{\infty}\frac{t}{\theta}e^{-\frac{2t}{\theta}}(1-e^{-t/\theta})dt = 6\theta\,(\frac{1}{4}\int_0^{\infty}\frac{2t}{\theta}e^{-\frac{2t}{\theta}}\,d(\frac{2t}{\theta}) -$

$- \frac{1}{9}\int_0^{\infty}\frac{3t}{\theta}e^{-\frac{3t}{\theta}}\,d(\frac{3t}{\theta})) = \left\{\begin{array}{l}\frac{2t}{\theta} = u \\ \frac{3t}{\theta} = v\end{array}\right\} = 6\theta\,(\frac{1}{4}\int_0^{\infty} u e^{-u}du -$

$- \frac{1}{9}\int_0^{\infty} v e^{-v}dv = 6\theta\cdot\frac{5}{36}\,(-ue^{-u}|_0^{\infty} - e^{-u}|_0^{\infty}) = \frac{-5\theta}{6}\,(0-1) = \frac{5\theta}{6}$

$\Rightarrow$ смещенная.

$\Rightarrow \tilde{\theta}_3' = \frac{6}{5}\, x_{(2)}$

б) $D[\tilde{\theta}_1] = M[\tilde{\theta}_1^2] - M^2[\tilde{\theta}_1] = M[\bar{x}^2] - M^2[\bar{x}] =$

$= M[(\frac{1}{n}\sum x_i)^2] - M^2[\xi] = \frac{1}{9}M[x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 +$

$+ 2x_3x_1] - \theta^2 = M[\xi^2] - \theta^2 = \int_{-\infty}^{\infty} x^2 p(x)dx = 2\theta^2 - \theta^2 = \theta^2$

$D[\tilde{\theta}_3'] = M[\tilde{\theta}_3'^{\,2}] - M^2[\tilde{\theta}_3']$

$\mathcal{M}[\tilde\theta_2'^2] = (\frac{6}{5})^2 6 \int\limits_0^\infty t^2 \frac{1}{\theta} e^{-t/\theta}(1-e^{-t/\theta})e^{-t/\theta} dt =$

$= (\frac{6}{5})^2 6\theta^2 (\frac{1}{16} \int\limits_0^\infty (\frac{2t}{\theta})^2 e^{-2t/\theta} d\frac{2t}{\theta} - \frac{1}{81} \int\limits_0^\infty (\frac{3t}{\theta})^2 e^{-3t/\theta} d\frac{3t}{\theta}) =$

$= (\frac{6}{5})^2 6\, \theta^2 \frac{81-16}{81\cdot 16} \int\limits_0^\infty u^2 e^{-u} du = (\frac{6}{5})^2 \frac{65}{216} \theta^2 (-u^2 e^{-u}|_0^\infty +$

$+ \int\limits_0^\infty 2u e^{-u} du) = (\frac{6}{5})^2 \frac{65}{108} \theta^2 \cdot \cancel{8} = \frac{6\cdot 65}{5\cdot 108} \theta^2 = (\frac{6}{5}) \frac{13}{18} \theta^2 =$

$= \frac{38}{25}\theta^2$

$\mathcal{M}^2[\tilde\theta_3'] = (\frac{6}{5})^2 \mathcal{M}^2[x_{(2)}] = (\frac{6}{5})^2 \cdot (\frac{5}{6})^2 \theta^2 = \theta^2$

$D[\tilde\theta_3'] = \frac{38}{25}\theta^2 - \theta^2 = \frac{13}{25}\theta^2$

$1 > \frac{13}{25} \Rightarrow \tilde\theta_3'$ - эфективнее чем $\tilde\theta_1$

c)