1) $\xi \sim R[\theta, 2\theta]$

a) ОММ:

$$\xi \sim p(x) = \frac{1}{2\theta-\theta}\{[\theta, 2\theta]\} = \frac{1}{\theta}\{[\theta, 2\theta]\}$$

$$\alpha_1 = M[\xi] = \int_{\theta}^{2\theta} \frac{1}{\theta} x\,dx = \frac{1}{\theta}\frac{x^2}{2}\Big|_{\theta}^{2\theta} = \frac{1}{2\theta}(4\theta^2-\theta^2) = \frac{3\theta^2}{2\theta} = \frac{3}{2}\theta$$

$$\tilde{\alpha}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \alpha_1 = \frac{3\tilde{\theta}}{2} = \bar{x} \Rightarrow \tilde{\theta}_1 = \frac{2}{3}\bar{x}$$

ОМП:

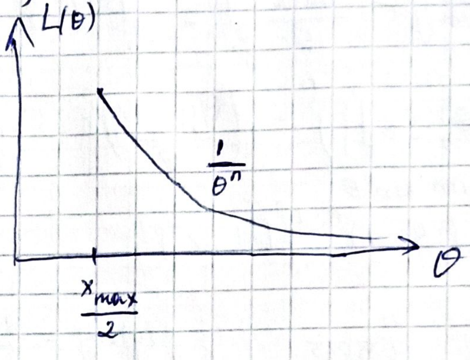
$$L(\vec{x}_n, \theta) = \frac{1}{\theta^n}\{[\theta, 2\theta]\}$$

$\theta \le x \le 2\theta$

$\Downarrow$

$2\theta \ge \max(x_i) = x_{max}$

$\Rightarrow \tilde{\theta}_2 = \frac{x_{max}}{2}$



в) $\boxed{\tilde{\theta}_1:}$ $M[\tilde{\theta}_1] = M[\frac{2}{3}\bar{x}] = \frac{2}{3}M[\xi] = \frac{2}{3}\cdot\frac{3\theta}{2} = \theta \Rightarrow$ несмещ.

$$D[\tilde{\theta}_1] = \frac{4}{9}D[\frac{1}{n}\sum_{i=1}^{n} x_i] = \frac{4}{9}\frac{1}{n}D[\xi] = \frac{4}{9n}(M[\xi^2]-M^2[\xi]) =$$

$$= \frac{4}{9n}\left(\frac{1}{\theta}\frac{x^3}{3}\Big|_{\theta}^{2\theta} - \theta^2\right) = \frac{4}{9n}\left(\frac{1}{3\theta}(8\theta^3-\theta^3)-\theta^2\right) = \frac{4}{9n}\left(\frac{7}{3}\theta^2-\theta^2\right) = \frac{16\theta^2}{27n} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ состоят.

$\boxed{\tilde{\theta}_2:}$

$$M[\tilde{\theta}_2] = M[\frac{x_{max}}{2}] = \frac{1}{2}\int_{-\infty}^{+\infty} x\, q(x)\,dx$$

$$\Psi(x) = (F(x))^n = \left(\int_{\theta}^{x}\frac{1}{\theta}dy\right)^n = \left(\frac{x-\theta}{\theta}\right)^n$$

$F' = p$

$$q(y) = \Psi'(x) = n\left(\frac{x-\theta}{\theta}\right)^{n-1}\cdot\frac{1}{\theta}\{[\theta, 2\theta]\}$$

$$M[\tilde{\theta}_2] = \frac{1}{2}\int_{\theta}^{2\theta} x\,\frac{n}{\theta}\left(\frac{x-\theta}{\theta}\right)^{n-1}dx = \frac{2\theta n+\theta}{2n+2} \Rightarrow \text{смещ.}$$

$$\Rightarrow \tilde{\theta}_2' = \frac{(2n+1)\theta}{2n+2} \Rightarrow \tilde{\theta}_2' = \frac{2n+2}{2n+1}\cdot\frac{x_{max}}{2} \text{ - несмещ.}$$

$$D[\tilde{\theta}_2'] = M[\tilde{\theta}_2'^2] - M^2[\tilde{\theta}_2']$$

$$M[\tilde{\theta}_2'^2] = \left(\frac{n+1}{2n+1}\right)^2\int_{\theta}^{2\theta} x^2\,\frac{n}{\theta}\left(\frac{x-\theta}{\theta}\right)^{n-1}dx = \frac{(n+1)^2}{(2n+1)^2}\cdot\frac{2\theta^2(2n(n+2)+1)}{(n+1)(n+2)} =$$

$$= \frac{(n+1)\,2\theta^2\,2n\cancel{(n+2)}}{(2n+1)^2\cancel{(n+2)}} + \frac{(n+1)\,2\theta^2\,2n}{(n+2)(2n+1)^2}$$

$$D[\tilde{\theta}_2'] = M[\tilde{\theta}_2'^2] - \theta^2 = \frac{(n+1)\,2\theta^2(2n(n+2)+1)}{(2n+1)^2(n+2)} - \theta^2 =$$

$$= \frac{((n+1)2(2n(n+2)+1) - (2n+1)^2(n+2))\theta^2}{(2n+1)^2(n+2)} = \frac{n\theta^2}{\underbrace{(2n+1)^2(n+2)}_{=4n^3+12n^2+9n+2}} \xrightarrow[n\to\infty]{} 0 \Rightarrow \text{состоят. по дост. усл}$$

c) $D[\tilde{\theta}_1] = \frac{16\theta^2}{27n} \sim \frac{16}{27n}, \; n\to\infty$

$D[\tilde{\theta}_2'] = \frac{n\theta^2}{4n^3+12n^2+9n+2} \sim \frac{1}{4n^2}, \; n\to\infty$

$\Rightarrow \tilde{\theta}_2'$ - эффективнее

d) $\exists\, f(\theta, \vec{x}_n) = \frac{x_{max}}{\theta} = \eta$

$$\eta \sim F(y) = P\left(y > \frac{x_{max}}{\theta}\right) = P(\theta y > x_{max}) = F_{max}(\theta y) =$$

$$= \left(R(\theta y)\{[\theta, 2\theta]\}\right)^n = R^n(\theta y)\{[\theta, 2\theta]\} = \left\{\begin{matrix} \rho = \frac{1}{2\theta-\theta} = \frac{1}{b-a} \\ R = \int\limits_a^x \frac{1}{b-a} = \frac{x-a}{b-a} \end{matrix}\right\} = \left(\frac{\theta y - \theta}{\theta}\right)^n = (y-1)^n$$

$q(y) = n(y-1)^{n-1}\{[1,2]\}$ — не зависит от θ $\quad \{\theta \le \theta y \le 2\theta \Rightarrow 1 \le y \le 2\}$

$$t_1 = \eta_{\frac{1-\beta}{2}}^{\beta=0{,}95} = \eta_{0{,}025}: \int\limits_1^{t_1} q(y)\,dy = \frac{1-\beta}{2} \Rightarrow (t_1 - 1)^n = \frac{1-\beta}{2} \Rightarrow t_1 = \sqrt[n]{\frac{1-\beta}{2}} + 1$$

$$t_2 = \eta_{0{,}975}: \int\limits_1^{t_2} q(y)\,dy = \frac{1+\beta}{2} \Rightarrow t_2 = \sqrt[n]{\frac{1+\beta}{2}} + 1$$

$$\sqrt[n]{\frac{1-\beta}{2}} + 1 < \frac{x_{max}}{\theta} < \sqrt[n]{\frac{1+\beta}{2}} + 1 \Rightarrow \frac{x_{max}}{1+\sqrt[n]{\frac{1+\beta}{2}}} < \theta < \frac{x_{max}}{1+\sqrt[n]{\frac{1-\beta}{2}}}$$

$$\Rightarrow \frac{x_{max}}{1+\sqrt[n]{0{,}975}} < \theta < \frac{x_{max}}{1+\sqrt[n]{0{,}025}}$$

e) ОМП нельзя, т.к $R\{[\theta, 2\theta]\}$ - не регулярная модель (не выполняется перестанов. $\int$ и $\frac{\partial}{\partial\theta}$)

по ОММ: $\tilde{\theta} = \frac{2}{3}\bar{x} \Rightarrow \tilde{\theta} = g(\tilde{\alpha}) = \frac{2}{3}\tilde{\alpha}_1$

$g(\alpha) = \frac{2}{3}\alpha_1$

$$\sigma(\alpha) = \sqrt{\nabla^T g(\alpha)\, K\, \nabla g(\alpha)} = \sqrt{\frac{2}{3}(\alpha_2 - \alpha_1^2)\frac{2}{3}}$$

$$\sqrt{n}\,\frac{g(\tilde{\alpha}) - g(\alpha)}{\sigma(\alpha)} \sim N(0,1) \Rightarrow \sqrt{n}\,\frac{\frac{2}{3}\tilde{\alpha}_1 - \frac{2}{3}\alpha_1}{\sqrt{\frac{2}{3}(\alpha_2 - \alpha_1^2)\frac{2}{3}}} = \sqrt{n}\,\frac{\tilde{\alpha}_1 - \alpha_1}{\sqrt{\alpha_2 - \alpha_1^2}} =$$

$$= \sqrt{n}\,\frac{\tilde{\alpha}_1 + \frac{3}{2}\theta}{\sqrt{\alpha_2 - \alpha_1^2}}\,\frac{\sqrt{\alpha_2 - \alpha_1^2}}{\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}} \overset{\text{по лемме Слуцкого}}{=} \sqrt{n}\,\frac{\tilde{\alpha}_1 + \frac{3}{2}\theta}{\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}} \sim N(0,1)$$

$$-1{,}96 < \sqrt{n}\,\frac{\tilde{\alpha}_1 - \frac{3}{2}\theta}{\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}} < 1{,}96$$

$$-\frac{2}{3}\left(\frac{-1{,}96\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}}{\sqrt{n}} - \tilde{\alpha}_1\right) > \theta > -\frac{2}{3}\left(\frac{1{,}96\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}}{\sqrt{n}} - \tilde{\alpha}_1\right)$$

$$\frac{2}{3}\left(-1{,}96\sqrt{\frac{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}{n}} + \tilde{\alpha}_1\right) < \theta < \frac{2}{3}\left(1{,}96\sqrt{\frac{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}{n}} + \tilde{\alpha}_1\right)$$

5) $$\rho(x) = \begin{cases} \frac{\theta-1}{x^\theta}, & x \geq 1 \\ 0, & x < 1 \end{cases}, \theta > 1$$

$$\rho(x) = \frac{\theta-1}{x^\theta}\{x \geq 1\}, \theta > 1$$

$$F(x) = \int_1^x \frac{\theta-1}{y^\theta} dy = 1 - x^{1-\theta} \Rightarrow x = (1 - F(x))^{\frac{1}{1-\theta}}$$

а) ММП: $L(\vec{x}_n, \theta) = \prod_{i=1}^n \left(\frac{\theta-1}{x_i^\theta}\right)\{x \in [1, +\infty)\}$

$$\ln L(\theta) = \sum_{i=1}^n \ln\left(\frac{\theta-1}{x_i^\theta}\right) = n \cdot \ln(\theta-1) - \sum_{i=1}^n \ln x_i^\theta = n\ln(\theta-1) - \theta \sum_{i=1}^n \ln x_i$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta-1} - \sum_{i=1}^n \ln x_i \overset{\text{т.к. ищем max}}{=} 0 \Rightarrow \tilde{\theta} = \frac{n}{\sum_{i=1}^n \ln x_i} + 1$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n}{(\theta-1)^2} < 0 \Rightarrow max$$

в) $f(\vec{x}_n, \theta) = ?$

med: $\int_{-\infty}^{x_p} \rho(x)dx = \frac{1}{2} \Rightarrow \int_1^{x_p} \frac{\theta-1}{x^\theta} dx = -\frac{1}{x^{\theta-1}}\Big|_1^{x_p} = -x_p^{1-\theta} + 1 = \frac{1}{2} \Rightarrow x_p^{1-\theta} = \frac{1}{2}$

$$med(\theta) = 2^{\frac{1}{\theta-1}}$$

ОМП: проверим регулярность:

1) $\rho(x, \theta)$ - непр. дифф. по θ на Θ

2) $\int_1^\infty \frac{\partial}{\partial \theta} \frac{\theta-1}{x^\theta} dx = \int_1^\infty \frac{x^\theta - (\theta-1)\theta x^{\theta-1}}{x^{2\theta}} dx = -\theta+1+\theta-1 = 0 \Rightarrow$ перестановочно

3) $I(\theta) - ?$

$$\ln \rho(x, \theta) = \ln(\theta-1) - \theta \ln x$$

$$\frac{\partial \ln \rho(x,\theta)}{\partial \theta} = \frac{1}{\theta-1} - \ln x$$

$$I(\theta) = \int_1^{+\infty} \left(\frac{1}{\theta-1} - \ln x\right)^2 \frac{\theta-1}{x^\theta} dx = \int_1^{+\infty}\left(\frac{1\,\cancel{(x-1)}}{(\theta-1)^2 x^\theta} - \frac{2\ln x\,\cancel{(\theta-1)}}{\cancel{(\theta-1)}\,x^\theta} + \frac{(\theta-1)\ln^2 x}{x^\theta}\right)dx =$$

$$= \int_1^\infty\left(\frac{1}{(\theta-1)x^\theta} - \frac{2\ln x}{x^\theta} + \frac{2\ln x\,(\theta-1)}{x^\theta}\right)dx = \frac{1}{(\theta-1)^2}$$ - непр. на Θ

и $I(\theta) > 0$ на Θ

$\Rightarrow$ модель регулярна.

$$\sqrt{n}\ \frac{f(\tilde\theta)-f(\theta)}{\sigma(\theta)} \rightsquigarrow N(0,1) \quad , \quad \sigma = \sqrt{\nabla^T f(\theta)\, I^{-1}(\theta) \nabla f(\theta)}$$

$$\text{J}\quad f(\theta) = med\,\theta \Rightarrow \nabla f(\theta) = 2^{\frac{1}{\theta-1}} \cdot \ln 2 \left(-\frac{1}{(\theta-1)^2}\right)$$

$$\sigma = \sqrt{\left(2^{\frac{1}{\theta-1}} \ln 2 \left(-\frac{1}{(\theta-1)^2}\right)\right)^2 (\theta-1)^2} = \frac{2^{\frac{1}{\theta-1}} \ln 2}{\theta-1}$$

$$\sqrt{n}\ \frac{2^{\frac{1}{\tilde\theta-1}} - 2^{\frac{1}{\theta-1}}}{\frac{2^{\frac{1}{\tilde\theta-1}} \ln 2}{\tilde\theta-1}} \rightsquigarrow N(0,1)$$

$$2^{\frac{1}{\tilde\theta-1}} - \frac{2^{\frac{1}{\tilde\theta-1}} \ln 2}{(\tilde\theta-1)\sqrt{n}}\, t_2 < med < 2^{\frac{1}{\tilde\theta-1}} - \frac{2^{\frac{1}{\tilde\theta-1}} \ln 2}{(\tilde\theta-1)\sqrt{n}}\, t_1$$

c) $f(\theta) = \theta$

$$f(\tilde\theta) = \tilde\theta = \frac{n}{\sum_{i=1}^{n} \ln x_i} + 1$$

$$\sigma(\theta) = \sqrt{I^{-1}(\theta)} = (\theta-1) \Rightarrow \sqrt{n}\ \frac{\tilde\theta-\theta}{\tilde\theta-1} \rightsquigarrow N(0,1)$$

$$t_1 < \sqrt{n}\ \frac{\tilde\theta-\theta}{\tilde\theta-1} < t_2 \Rightarrow \tilde\theta - \frac{t_2(\tilde\theta-1)}{\sqrt{n}} < \theta < \tilde\theta - \frac{t_1(\tilde\theta-1)}{\sqrt{n}}$$