Maths Challenge No. 6

14 November, 2022

Question 1. Consider the following function of N variables

$$S(P_1, \ldots, P_N) = -\sum_{i=1}^{N} P_i \ln P_i \qquad \text{subject to} \qquad \sum_{i=1}^{N} P_i = 1 \tag{1}$$

Find the maximum value of S, and the values taken by P_i at which this maximum occurs.

Let $a_1, \ldots, a_N$ be constants, and again consider (1), but now with an additional constraint

$$\sum_{i=1}^{N} a_i P_i = 1.$$

Show that the maximum value of S now occurs at

$$P_i = \frac{e^{-\beta a_i}}{\sum_{i=1}^{N} e^{-\beta a_i}},$$

where β is some constant.

Question 2. Let $\mathbb{N}_+$ be the set of positive integers. Consider the function $\Delta : \mathbb{N}_+ \to \mathbb{N}$ such that $\Delta(n) =$ number of divisors of n. For example, $\Delta(1) = 1$, $\Delta(2) = 2$, $\Delta(6) = 4$, $\Delta(10) = 4$, and so on.

Is Δ surjective (onto)?

Let $\xi : \mathbb{N}_+ \to \mathbb{N}$ be a function such that $\xi(k) = \min\{n \in \mathbb{N} | \Delta(n) \geq k\}$. For example, $\xi(2) = 2$, $\xi(3) = 4$, $\xi(4) = 6$, and so on.

Compute $\xi(5)$, $\xi(50)$, and $\xi(500)$.