MSBD5011

Homework - 2(a)

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① Find sufficient stat if $x_1, x_2 - - - x_n \overset{iid}{\sim} f_\theta(x)$

→ $f_\theta(x) = \frac{1}{b} e^{-\frac{(x-a)}{b}} \quad I\{x \geq a\}$, $\theta = (a, b)$

→ The pdf of the given function is:-

$$f_\theta(x_1, -- x_n) = \frac{1}{b^n} e^{-\sum_{i=1}^{n} (x_i - a)/b} \quad I\{x \geq a\}$$

$$= \frac{e^{na/b}}{b^n} e^{-\sum_{i=1}^{n} x_i/b} \quad I\{x \geq a\}$$

→ Assuming $h(x) = 1$, $g_\theta(x) = \frac{e^{na/b}}{b^n} e^{-\sum_{i=1}^{n} x_i/b}$

$$\therefore g_{(a,b)}(T(x)) = \frac{e^{na/b}}{b^n} e^{-\sum_{i=1}^{n} x_i/b} \quad x_{(1)} \geq a.$$

$\therefore$ $T(x) = \left(\sum_{i=1}^{n} x_i , x_1\right)$ are the sufficient Statistics

$\Rightarrow T(x) = (T_1, T_2)$

(2) Find sufficient stat if $X_1, \dots, X_n \overset{iid}{\sim} f_\theta(x)$,

$$f_\theta(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \cdot e^{-x/\beta}, \quad x \geq 0 \quad ; \quad \theta = (\alpha, \beta).$$

-> From the definition, a probability density function is $f_\theta(x)$, and T is sufficient for 'θ' if and only if non-negative functions 'g' and 'h' can be found that,

$$f_\theta(x) = h(x)\, g_\theta(T(x))$$

$$\therefore f(x) = f(x_1, \dots, x_n) = \prod_{i=1}^{n} \left(\frac{1}{\Gamma(\alpha)\beta^\alpha} x_i^{\alpha-1} \cdot e^{-\frac{x_i}{\beta}} \right) = \left(\frac{1}{\Gamma(\alpha)\beta^\alpha} \right)^n \left(\prod_{i=1}^{n} x_i \right)^{\alpha-1} \cdot e^{-1/\beta \sum_{i=1}^{n} x_i}$$

-> Assuming $h(x) = 1$, then the whole expression is $g_\theta(T(x))$, $\theta = (\alpha, \beta)$

$$\therefore g_{\alpha,\beta}(T(x)) = \left(\frac{1}{\Gamma(\alpha)\beta^\alpha} \right)^n \left(\prod_{i=1}^{n} x_i \right)^{\alpha-1} \cdot e^{-1/\beta \sum_{i=1}^{n} x_i}$$

-> From the expression, $g_{\alpha,\beta}(T(x))$ depends on the drawn simple only through $\prod_{i=1}^{n} x_i$ and $\sum_{i=1}^{n} x_i$, then these two are the sufficient statistics. i.e;

$$T(x) = \left(\prod_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i \right)$$

③ "**Exponential tilting**" :-

Denoting $X \sim p(x)$ (poisson distribution function)

$A(\theta) = \log E e^{\theta x}$ (cumulative generative function)

$\therefore$ $E \cdot e^{\theta x} = M(\theta)$ (Moment generating function)

-> Using the conditions above :-

-> we can generate (or) define a family of densities by:-

$$P(x ; \theta) = e^{(\theta x - A(\theta))} p(x)$$

① Proof:-

As $\{\theta : M(\theta) < \infty\}$, these densities ~~are~~ can intregates to 1 as the exponential functions are non-negative.

-> Family is indexed by θ.

$$P_\theta(x) = c \cdot e^{\theta x} p(x)$$

$$\therefore \int e^{\theta(x) + c(x)} \cdot dx = e^{A(\theta)} \quad [\text{from } A(\theta)]$$

$$\Rightarrow \int e^{\theta x - A(\theta) + c(x)} \cdot dx = 1$$

$$\therefore P_\theta(x) = e^{\theta x - A(\theta) + c(x)}$$

$$= e^{(\theta x - A(\theta))} \cdot px$$

$$= e^{-A(\theta)} \cdot e^{\theta x} \cdot px$$

$$\therefore \underline{c = e^{-A\theta}}$$

(2) we have written the expression of $P_\theta(x)$ as:-

from $c = e^{-A\theta}$

$$P_\theta(x) = c \cdot e^{\theta x} \cdot p(x)$$

$$= e^{-A\theta} \cdot e^{\theta x} \cdot p(x)$$

$$= e^{\theta x - A(\theta)} \cdot p(x)$$ (proven in the 'proof' section)

(3) $\dot{A}(\theta) = E_\theta X \equiv \mu$

Proof:-

$$\frac{d}{d\theta} A(\theta) = E_\theta X$$

$$\rightarrow \frac{d}{d\theta} A(\theta) = \frac{d}{d\theta} \log E e^{\theta X} = \frac{1}{e^{A\theta}} \frac{d}{d\theta} E \cdot e^{\theta X}$$

$$= \frac{1}{e^{A\theta}} \cdot E \cdot \frac{d}{d\theta} e^{\theta X}$$

$$= \frac{1}{e^{A\theta}} \cdot E \cdot X e^{\theta X}$$

$$\therefore e^{-A\theta} \cdot \int x \cdot e^{\theta x} p(x) \cdot dx = \int x \cdot e^{\theta x - A\theta} \cdot p(x) \cdot dx$$

$$= \underline{\underline{E_\theta X}}$$

(4) $\underline{\underline{\mathrm{Var}_\theta(X) = A''(\theta) = \left(\frac{d}{d\theta} E_\theta X\right)}}$