MSBD 5011

Homework - '2'

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Find conjugate priors: -

1. Normal distribution with a known mean 'μ', but unknown variance

Likelihood: $f(y_1, \ldots, y_n | \mu, \sigma^2) = f(y_n | \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y_i-\mu)^2}{2\sigma^2}\right\}$

$$f(y_n | \mu, \sigma^2) = (2\pi)^{-n/2} (\sigma^2)^{-n/2} e^{-1/\sigma^2 \sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2}}$$ {Alternative way of writing the likelihood function}

Inverse Gamma: $f(y; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot y^{-(\alpha+1)} e^{-\beta/y}$ {Inverse gamma in terms of 'y'}

$$g(\sigma^2) = \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-(\alpha+1)} e^{-\beta/\sigma^2}$$ be the prior

-> prior * likelihood $\propto$ posterior

-> Ignore constants : [prior : $\frac{\beta^\alpha}{\Gamma(\alpha)}$; likelihood : $(2\pi)^{-n/2}$] for convinience.

-> prior * likelihood $\propto (\sigma^2)^{-(\alpha+1)} \cdot e^{-\beta/\sigma^2} * (\sigma^2)^{-n/2} e^{-1/\sigma^2 \sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2}}$

$$\equiv (\sigma^2)^{-(\alpha+n/2+1)} e^{-1/\sigma^2 \left(\beta + \sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2}\right)}$$

$\propto$ posterior

-> It's in the form of inverse gamma with parameters $\left(\alpha+n/2, \beta+\sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2}\right)$

$$\rightarrow g(\sigma^2 | y_n, \mu) = \frac{\left[\beta + \sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2}\right]^{\alpha+n/2}}{\Gamma(\alpha+n/2)} \cdot (\sigma^2)^{-(\alpha+n/2+1)} \cdot e^{-1/\sigma^2 \left(\beta + \sum_{i=1}^{n} \frac{(y_i-\mu)^2}{2}\right)}$$

② Poisson Model

Likelihood : $f(y|\lambda) = \dfrac{\lambda^y e^{-\lambda}}{y!} \propto \lambda^y e^{-\lambda}$ ——— ①

-> Conjugate prior is the gamma distribution.

$$g(\lambda, \gamma, \nu) = \frac{\nu^\gamma \lambda^{\gamma-1} e^{-\nu\lambda}}{\Gamma(\gamma)} \propto \lambda^{\gamma-1} \cdot e^{-\nu\lambda} \quad \text{——— ②}$$

-> Prior * Likelihood :-

$$= \frac{\nu^\gamma}{\Gamma(\gamma)} \cdot \lambda^{\gamma-1} \cdot e^{-\nu\lambda} \times \frac{1}{y!} \lambda^y e^{-\lambda}$$

$$\propto \lambda^{\gamma-1} \cdot e^{-\nu\lambda} \lambda^y e^{-\lambda}$$

grouping similar items together:-

$$= \lambda^{\gamma+y-1} \cdot e^{-(\nu+1)\lambda} \quad \text{——— ③}$$

From ③ & ②, It is similar,

$$\text{Gamma}\left(\lambda; (\gamma+y), (\nu+1)\right)$$

$$= \frac{(\nu+1)^{\gamma+y} \lambda^{(\gamma+y-1)} \cdot e^{-(\nu+1)\lambda}}{\Gamma(\gamma+y)}$$

(3) Exponential model :-

Likelihood : $L(\theta; y) = \prod_{i=1}^{n} \theta e^{-\theta y_i} = \theta^n \cdot e^{-\sum_{i=1}^{n} y_i \cdot \theta}$

→ Similarly, for the exponential model, gamma distribution is the conjugate prior.

Prior:-

$$g(\theta; \alpha, \beta) = \frac{\beta^\alpha \theta^{\alpha-1} \cdot e^{-\beta\theta}}{\Gamma(\alpha)} \propto \theta^{\alpha-1} \cdot e^{-\beta\theta} \quad \text{——— (1)}$$

→ Prior × Likelihood :-

$$\Rightarrow \frac{\beta^\alpha \cdot \theta^{\alpha-1} \cdot e^{-\beta\theta}}{\Gamma(\alpha)} \times \theta^n \cdot e^{-\theta \cdot \sum_{i=1}^{n} y_i}$$

⇒ By eliminating constants to get the posterior proportionality

$$\propto \theta^{\alpha-1} \cdot e^{-\beta\theta} \times \theta^n \cdot e^{-\theta \sum_{i=1}^{n} y_i}$$

→ Grouping similar items together:-

$$= \theta^{n+\alpha-1} \cdot e^{-\theta\left(\sum_{i=1}^{n} y_i + \beta\right)} \quad \text{——— (2)}$$

From (1) & (2), they are Proportional

$$\text{Gamma}\left(\theta; (n+\alpha), \left(\sum_{i=1}^{n} y_i + \beta\right)\right)$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} y_i\right)^{n+\alpha} \cdot \theta^{(n+\alpha-1)} \cdot e^{-\left(\sum_{i=1}^{n} y_i + \beta\right)\theta}}{\Gamma(\alpha+n)}$$