HW2 Solution

1. Let $(V, \|\cdot\|)$ be a normed vector space.

   (a) Prove that, for all $\boldsymbol{x}, \boldsymbol{y} \in V$,
   $$|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x} - \boldsymbol{y}\|.$$

   (b) Let $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}}$ be a convergent sequence in V with limit $\boldsymbol{x} \in V$. Prove that
   $$\lim_{k\to\infty} \|\boldsymbol{x}_k\| = \|\boldsymbol{x}\|.$$
   (*Hint: Use part (a).*)

   (c) Let $\{\boldsymbol{x}^{(k)}\}_{k\in\mathbb{N}}$ be a sequence in V and $\boldsymbol{x}, \boldsymbol{y} \in V$. Prove that, if
   $$\boldsymbol{x}^{(k)} \to \boldsymbol{x}, \quad \text{and} \quad \boldsymbol{x}^{(k)} \to \boldsymbol{y},$$
   then $\boldsymbol{x} = \boldsymbol{y}$. (In other words, the limit of the same sequence in a normed vector space is unique.)

Solution: (a) Since $(V, \|\cdot\|)$ is a normed vector space,
$$\|u+v\| \leq \|u\| + \|v\| \quad \text{for any } u, v \in V.$$
Let $u = x-y$, $v = y$. Then
$$\|x-y+y\| \leq \|x-y\| + \|y\|$$
That is $\|x\| - \|y\| \leq \|x-y\|$. ①

Similarly, let $u = y-x$, $v = x$. We have
$$\|y-x+x\| \leq \|y-x\| + \|x\|.$$
That is $\|y\| - \|x\| \leq \|x-y\|$ ②

Combining ①, ②, we have
$$|\|x\| - \|y\|| \leq \|x-y\|.$$

(b) Since $\lim_{k\to\infty} x_k = x$, for any $\varepsilon > 0$, there exists K such that for all $k > K$,
$$\|x_k - x\| \leq \varepsilon.$$
Since $|\|x_k\| - \|x\|| \leq \|x_k - x\| \leq \varepsilon$, we have
$$\lim_{k\to\infty} \|x_k\| = \|x\|.$$

(c). Since $\lim_{k\to\infty} x^{(k)} = x$, $\lim_{k\to\infty} x^{(k)} = y$, for any $\varepsilon > 0$, there exist K such that for any $k > K$, we have

$$\|x^{(k)} - x\| \leq \varepsilon \quad \text{and} \quad \|x^{(k)} - y\| \leq \varepsilon.$$

Note that
$$\begin{aligned} \|x - y\| &= \|x - x^{(k)} + x^{(k)} - y\| \\ &\leq \|x - x^{(k)}\| + \|x^{(k)} - y\| \\ &\leq 2\varepsilon. \end{aligned}$$

Since ε can be arbitrary small, we have $\|x - y\| = 0$.

That is, $x = y$.

2. Let V be a vector space, and $\langle \cdot, \cdot \rangle$ be an inner product on V. Use the definition of inner product to prove the following.

(a) Prove that $\langle \mathbf{0}, \boldsymbol{x} \rangle = \langle \boldsymbol{x}, \mathbf{0} \rangle = 0$ for any $\boldsymbol{x} \in V$. Here $\mathbf{0}$ is the zero vector in V.

(b) Prove that the second condition

$$\langle \alpha \boldsymbol{x}_1 + \beta \boldsymbol{x}_2, \boldsymbol{y} \rangle = \alpha \langle \boldsymbol{x}_1, \boldsymbol{y} \rangle + \beta \langle \boldsymbol{x}_2, \boldsymbol{y} \rangle, \qquad \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y} \in V\ , \alpha, \beta \in \mathbb{R}$$

is equivalent to

$$\langle \boldsymbol{x}_1 + \boldsymbol{x}_2, \boldsymbol{y} \rangle = \langle \boldsymbol{x}_1, \boldsymbol{y} \rangle + \langle \boldsymbol{x}_2, \boldsymbol{y} \rangle \quad \text{and} \quad \langle \alpha \boldsymbol{x}, \boldsymbol{y} \rangle = \alpha \langle \boldsymbol{x}, \boldsymbol{y} \rangle, \qquad \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}, \boldsymbol{y} \in V,\ \alpha \in \mathbb{R}.$$

Solution: (a) From the definition, we have $\langle 0, x \rangle = \langle x, 0 \rangle$.

Furthermore,
$$\begin{aligned} \langle 0, x \rangle &= \langle 0 + 0, x \rangle \\ &= \langle 0, x \rangle + \langle 0, x \rangle \end{aligned}$$

which implies $\langle 0, x \rangle = 0$.

Hence, $\langle 0, x \rangle = \langle x, 0 \rangle = 0$.

(b) If $\langle \alpha x_1 + \beta x_2, y \rangle = \alpha \langle x_1, y \rangle + \beta \langle x_2, y \rangle$, $\forall x_1, x_2, y \in V$ $\alpha, \beta \in \mathbb{R}$ holds, we have

$\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$ by choosing $\alpha = \beta = 1$.

Furthermore, choose $x_1 = x$, $x_2 = 0$, $\beta = 0$, we have $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

On the other hand, if, for any $\alpha \in \mathbb{R}$, $x_1, x_2, y \in V$, $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$, $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ holds.

we have $\langle \alpha x_1 + \beta x_2, y\rangle = \langle \alpha x_1, y\rangle + \langle \beta x_2, y\rangle$

$= \alpha\langle x_1, y\rangle + \beta\langle x_2, y\rangle$

3. $\mathbb{R}^{m\times n}$ is a vector space over $\mathbb{R}$. Show that $\langle \boldsymbol{A}, \boldsymbol{B}\rangle = \text{trace}(\boldsymbol{A}^T\boldsymbol{B})$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times n}$ is an inner product on $\mathbb{R}^{m\times n}$. Here trace$(\cdot)$ is the trace of a matrix, i.e., the sum of all diagonal entries.

Solution: ① $\text{trace}(A^TB) = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{ji}B_{ji}$, $\forall A, B \in \mathbb{R}^{m\times n}$

$\langle A, A\rangle = \text{trace}(A^TA) = \sum_{i=1}^{n}\sum_{j=1}^{m} (A_{ji})^2 \geqslant 0$

Furthermore if $\langle A, A\rangle = 0$, then $\sum_{i=1}^{n}\sum_{j=1}^{n} (A_{ji})^2 = 0$

$A_{ji} = 0$ for all $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$

thus $A = 0$

② For any $\alpha, \beta \in \mathbb{R}$, $A, B, C \in \mathbb{R}^{m\times n}$

$$\begin{aligned}
\langle \alpha A + \beta B, C\rangle &= \text{trace}(\alpha A + \beta B)^T C \\
&= \text{trace}(\alpha A^TC + \beta B^TC) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} (\alpha A_{ji}C_{ji} + \beta B_{ji}C_{ji}) \\
&= \alpha \sum_{i=1}^{n}\sum_{j=1}^{m} A_{ji}C_{ji} + \beta \sum_{i=1}^{n}\sum_{j=1}^{m} B_{ji}C_{ji} \\
&= \alpha\langle A, C\rangle + \beta\langle B, C\rangle
\end{aligned}$$

③
$$\begin{aligned}
\langle A, B\rangle &= \text{trace}(A^TB) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} A_{ji}B_{ji} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} B_{ji}A_{ji} \\
&= \text{trace}(B^TA) \\
&= \langle B, A\rangle
\end{aligned}$$

Thus $\langle A, B\rangle = \text{trace}(A^TB)$ for $A, B \in \mathbb{R}^{m\times n}$ is an inner product on $\mathbb{R}^{m\times n}$.

4. Consider the polynomial kernel $K(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}^T\boldsymbol{y} + 1)^2$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$. Find an explicit feature map $\phi : \mathbb{R}^2 \to \mathbb{R}^6$ satisfying $\langle \phi(\boldsymbol{x}), \phi(\boldsymbol{y})\rangle = K(\boldsymbol{x}, \boldsymbol{y})$, where the inner product the standard inner product in $\mathbb{R}^6$.

Solution: Since $x, y \in \mathbb{R}^2$, we denote $x = (x_1, x_2)$, $y = (y_1, y_2)$.

Then $k(x,y) = (x^T y + 1)^2 = (x_1 y_1 + x_2 y_2 + 1)^2$

$$= x_1^2 y_1^2 + 2 x_1 x_2 y_1 y_2 + 2 x_1 y_1 + 2 x_2 y_2 + x_2^2 y_2^2 + 1$$

Let $\phi(x) = (x_1^2, x_2^2, 1, \sqrt{2} x_1, \sqrt{2} x_2, \sqrt{2} x_1 x_2)$.

Then $\langle \phi(x), \phi(y) \rangle = x_1^2 \cdot y_1^2 + x_2^2 y_2^2 + 1 \cdot 1 + \sqrt{2} x_1 \cdot \sqrt{2} y_1 + \sqrt{2} x_2 \cdot \sqrt{2} y_2$

$$+ \sqrt{2} x_1 x_2 \cdot \sqrt{2} y_1 y_2$$

$$= x_1^2 y_1^2 + x_2^2 y_2^2 + 1 + 2 x_1 y_1 + 2 x_2 y_2 + 2 x_1 x_2 y_1 y_2$$

$$= k(x, y)$$