CITM
Final Exam, Autumn 2017
MATVJII

SOLUTIONS

Surnames and name: ______________________________

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# Exercise 1

10 Point (a) Consider the affine transformation given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Is it a rotation plus a translation? Find the matrix that corresponds to the inverse affine transformation.

**Solution:**
If an affine transformation corresponds to a rotation $\mathbf{R}$ ($\det(\mathbf{R}) = 1$, $\mathbf{R^T R} = \mathbf{R R^T} = \mathbf{I}$) plus a translation $\mathbf{t}$,

$$\mathbf{A} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} \implies \mathbf{A^{-1}} = \begin{pmatrix} \mathbf{R^T} & -\mathbf{R^T} \cdot \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

10 Point (b) From the rotation matrix

$$\begin{pmatrix} \sqrt{6}/4 & \frac{\sqrt{6}+2\sqrt{2}}{8} & \frac{-\sqrt{6}+2\sqrt{2}}{6} \\ -\frac{\sqrt{6}}{4} & \frac{-\sqrt{6}+2\sqrt{2}}{8} & \frac{\sqrt{2}+2\sqrt{6}}{8} \\ \frac{1}{2} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \end{pmatrix}$$

find the Euler angles $\phi$, $\theta$ and $\psi$.

**Solution:**
$\theta = \arcsin -r_{31} = \arcsin -\frac{1}{2} = -30^\circ$
$\phi = \arctan 2(\frac{r_{32}}{\cos\theta}, \frac{r_{33}}{\cos\theta}) = -120^\circ$
$\psi = \arctan 2(\frac{r_{21}}{\cos\theta}, \frac{r_{11}}{\cos\theta}) = -45^\circ$

20 Point (c) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix}$$

1. Demonstrate that $\mathbf{A}$ is a rotation matrix.
2. What happens to the vector $\mathbf{p} = [-1, -1, 0]^T$ under the rotation?
3. Determine rotation angle and rotation axis.

**Solution:**

1. $\det \mathbf{A} = 1$ and $\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A} = \mathbf{I}$
2. $\mathbf{A}\mathbf{p} = \mathbf{p}$, hence it is in the rotation axis.
3. Inverting the Rodrigues' formula, $\phi = 90°$ and $\mathbf{u} = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0]^T$

10 Point (d) Determine the quaternion that corresponds to the rotation matrix $\mathbf{R}([1,1,0]^T, 90°)$.

**Solution:**

$$\mathring{q} = \begin{pmatrix} \cos\frac{\phi}{2} \\ \mathbf{u}\sin\frac{\phi}{2} \end{pmatrix} \text{ for } ||\mathbf{u}|| = 1 \text{ , so } \mathring{q} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

20 Point (e) Given the quaternion $\mathring{q} = \frac{\sqrt{2}}{2} + (\frac{1}{2}, \frac{1}{2}, 0)$

1. What happens to the vector $\mathbf{p} = [1,1,0]^T$ under the associated rotation?
2. Which is the associated rotation angle $\phi$?
3. Which is the associated rotation matrix $\mathbf{R}$?

**Solution:**

1. Since $\mathring{q}$ is obtained from (d), it reamins the same.
2. $\phi = 90°$
3. Similar to $\mathbf{A}$ in (c).

10 Point (f) $\mathring{q}_1$ is the quaternion that represents the rotation of 180° about the $x$-axis and $\mathring{q}_2$ is the quaternion that represents the rotation of 180° about the $z$-axis. What rotation is represented by composite quaternion $\mathring{q} = \mathring{q}_1\mathring{q}_2$? Answer by specifying its rotation angle and axis.

**Solution:**

$$\mathring{q}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \mathring{q}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \text{ so } \mathring{q} = \mathring{q}_1\mathring{q}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

# Exercise 2

Given the two vectors $\boldsymbol{v} = [-3, 1, 4]^T$ and $\boldsymbol{n} = [\sqrt{2}/2, \sqrt{2}/2, 0]^T$, separate $\boldsymbol{v}$ into two components: (Hint: $\boldsymbol{n}$ is a unit vector)

10 Point (a) The component that is perpendicular to $\boldsymbol{n}$

10 Point (b) The component that is parallel to $\boldsymbol{n}$

**Solution:**

The right order to solve this problem is to calculate firstly the parallel component to $\boldsymbol{n}$, which is the projection of the vector $\boldsymbol{v}$ on the vector $\boldsymbol{n}$. Let's denote it as $\boldsymbol{v}_{||n}$. Next, we calculate the perpendicular one, $\boldsymbol{v}_{\perp n}$, from $\boldsymbol{v} = \boldsymbol{v}_{||n} + \boldsymbol{v}_{\perp n}$. Hence, applying the formula for the projection of the vector $\boldsymbol{v}$ over the direction of $\boldsymbol{n}$ and considering that $\boldsymbol{n}$ is a unit vector, that is $\|\boldsymbol{n}\|^2 = 1$,

$$\boldsymbol{v}_{||n} = (\boldsymbol{v}^\intercal \boldsymbol{n})\,\frac{\boldsymbol{n}}{\|\boldsymbol{n}\|^2} = (\boldsymbol{v}^\intercal \boldsymbol{n})\,\boldsymbol{n} = \Big((-3\,,1\,,4)(\sqrt{2}/2\,,\sqrt{2}/2\,,0)^\intercal\Big)\,(\sqrt{2}/2\,,\sqrt{2}/2\,,0)^\intercal \Rightarrow$$

$$\boldsymbol{v}_{||n} = -\sqrt{2}\left(\frac{\sqrt{2}}{2}\,,\frac{\sqrt{2}}{2\,,0}\right)^\intercal = (-1\,,-1\,,0)^\intercal$$

Now

$$\boldsymbol{v}_{\perp n} = \boldsymbol{v} - \boldsymbol{v}_{||n} = (-3\,,1\,,4)^\intercal - (-1\,,-1\,,o)^\intercal = (-2\,,2\,,4)^\intercal$$

# Some help

- Euler principal axis and angle to rotation matrix

$$\mathbf{R}_{\boldsymbol{u},\phi} = \mathbf{I}\cos(\phi) + (1-\cos(\phi))\,(\boldsymbol{u}\boldsymbol{u}^\intercal) + \sin(\phi)\,[\boldsymbol{u}]_\times$$

- Rotation matrix to Euler principal axis and angle

$$\phi = \arccos\left(\frac{\mathrm{trace}\,(\mathbf{R}) - 1}{2}\right); \quad [\boldsymbol{u}]_\times = \frac{\mathbf{R} - \mathbf{R}^\intercal}{2\sin(\phi)}$$

- Projection of the vector $\boldsymbol{p}$ over the direction of $\boldsymbol{u}$

$$\boldsymbol{p}' = (\boldsymbol{p}^\intercal \boldsymbol{u})\,\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|^2}$$

- Euler angles to rotation matrix

$$\mathbf{R}(\psi,\theta,\phi) = \begin{pmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\psi c_\phi s_\theta + s_\psi s_\phi \\ c_\theta s_\psi & s_\psi s_\theta s_\phi + c_\phi c_\psi & c_\phi s_\psi s_\theta - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{pmatrix}$$

- Quaternion multiplication

$$\mathring{r}\mathring{s} = \begin{pmatrix} r_0 s_0 - \boldsymbol{r}^\intercal \boldsymbol{s} \\ r_0 \boldsymbol{s} + s_0 \boldsymbol{r} + \boldsymbol{r} \times \boldsymbol{s} \end{pmatrix}$$

| Question | Points | Score |
|---|---|---|
| 1 | 80 | |
| 2 | 20 | |
| Total: | | |