Homework 4

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1 Question 1

Let $G \leq S_n$ be a subgroup of the permutation group S_n acting on the index set $X = \{1, 2, \ldots, n\}$ where n is a finite natural number. An element g acts on a vector $v \in \mathbb{R}^n$ on the coordinate i by $(gv)_i = v_{g^{-1}(i)}$. Consider a G-linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Thus, $T(gv) = gT(v)$. Therefore, T must fulfill $Tgv = gTv$ for all $g \in G$ and $v \in \mathbb{R}^n$. As a result $Tg = gT$ and equivalently, $g^{-1}Tg = T$ for all $g \in G$ (in the lectures we used W). Applying g from the right on the columns of T and g^{-1} from the left on the rows of T results in the following condition:

$$T_{i,j} = T_{g^{-1}(i),g^{-1}(j)}. \tag{1}$$

g acts on T and on each orbit of T the requirement is that T is constant. Thus, the condition requires that $T_{i,j} = T_{l,m}$ if $Orb_G\left((i,j)\right) = Orb_G\left((l,m)\right)$. Therefore the number of free parameters is the number of orbits.

We define $U = X \times X$. Burnside's Lemma states that

$$|U/G| = \frac{1}{|G|}\sum_{g \in G} |U^g| = \frac{1}{|G|}\sum_{g \in G} |X^g|^2$$

where $|U/G|$ is the number of orbits and $U^g = \{u \in U \mid gu = u\}$ is the set of elements in U that g doesn't change. We have the property of $|U^g| = \chi_\sigma(g)$,

2 Question 2

2.1

The natural group here is the cross product of the permutation groups $S_n \times S_m$ acts on the Netflix rating matrices $A \in \mathbb{R}^{n \times m}$. It can be explained by the fact that the order of the m movies and the order of the n users is arbitrary and both can be changed independently.

2.2

Since we look at the G-linear transformation from matrices in $\mathbb{R}^{n \times m}$ to itself, it will be more convenient to consider the vectorization of these matrices. Thus

$\vec{(A)} = [a_{1,1}, \ldots, a_{n,1}, \ldots, a_{1,m}, \ldots, a_{n,m}]^T \in \mathbb{R}^{nm}$. Let $g = (g_l, g_r) \in S_n \times S_m$ be a group element where $g_l \in S_n$ is a left action that acts on the rows of A and $g_r \in S_m$ acts on the columns of A as follows:

$$gA = g_l A g_r^T = P_l A P_r^T$$

where $P_l \in \mathbb{R}^{n\times n}$ and $P_r \in \mathbb{R}^{m\times m}$ are the permutations of g_l and g_r, respectively. Define $v = \vec{(A)} \in \mathbb{R}^{nm}$. In index notation we have $(gA)_{i,j} = A_{g_l^{-1}(i), g_r^{-1}(j)}$. We are looking at the affine transformation that is equivariant, thus $f(v) = Wv + b$, that fulfills for all $g \in S_n \times S_m$ and $v \in \mathbb{R}^{nm}$

$$\begin{aligned} f(gv) &= W\,(gv) + b = g\,(Wv + b) \\ &\Rightarrow Wgv + b = gWv + gb \end{aligned} \tag{2}$$

Since (2) needs to be fulfilled for all $v \in \mathbb{R}^{nm}$ it leads to $Wg = gW$ and equivalently $gWg^T = W$. Moreover, it requires that $gb = b$ for all $g \in S_n \times S_m$. The group $S_n \times S_m$ is isomorphic to a subgroup $G \in S_{nm}$ and now let us look at the homomorphism between the two groups, where $S_n \times S_m$ acts as described above and S_{nm} acts on vectors in $\mathbb{R}^{nm}$. We have

$$gA = g_l A g_r^T = P_l A P_r^T.$$

By vectorizing the above we obtain

$$\text{vec}\left(P_l A P_r^T\right) = (P_r \otimes P_l)\,\vec{(A)} = (P_r \otimes P_l)\,v$$

For b, we can see from (2) that this means that b is constant, $b = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} c$. This can be seen by applying the inverse vectorization operation on b and seeing that the condition (2) requires that all values of b will be the same since $S_n \times S_m$ is two times transitive on $\mathbb{R}^{n\times m}$.

For W we have $Wg = gW \Rightarrow gWg^{-1} = W$. Thus, for the permutation P_g representing $g \in S_{nm}$ which fulfills $P = P_r \otimes P_l$, we have $W_{i,j} = W_{g^{-1}(i), g^{-1}(j)}$, therefore we look at the orbits of W induced by G.

Let us look at the action of the permutation $g^{-1} = (g_l^{-1}, g_r^{-1})$ on the vector v. We take the inverse for convenience of notation. Originally, the element with the index $A_{i,j}$ moves to $v_{i+n\cdot j}$, therefore

$$\left(g^{-1}A\right)_{i,j} = A_{g_l(i), g_r(j)} = v_{g_l(i) + n(g_r(j)-1)}.$$

Consequently, for an index $1 \le k \le nm$, define $j = k//n$ as the truncating integer division and $i = k\%n$ as the modulo operation, thus $\left(g^{-1}v\right)_k = v_{g_l(i) + n\cdot(g_r(j)-1)}$. Let k, l be indices in $[1, \ldots, nm]$ with the decompositions $k = i_1 + nj_1$ and $l = i_2 + nj_2$. Then $\left(g^{-1}Wg\right)_{k,l} = W_{g_l(i_1)+n\cdot(g_r(j_1)-1), g_l(i_2)+n(g_r(j_2)-1)}$.

2.2.1 Orbit analysis

Let us look at the possible orbits in the index space $X^2 = \{1, 2, \ldots, nm\}^2$. Indices (k', l') can be on the same orbits as (k, l) if and only if there exists a permutation $(g_l, g_r) \in S_n \times S_m$ such that

$$\begin{cases} k' &= g_l(i_1) + n\left(g_r(j_1) - 1\right) \\ l' &= g_l(i_2) + n\left(g_r(j_2) - 1\right) \end{cases}.$$

Define i_3, i_4, j_3, j_4 such that $k' = i_3 + n(j_3 - 1)$ and $l' = i_4 + n(j_4 - 1)$. It is clear that we can create equivalence groups by the rule:

$$\left(i_1 \overset{\neq}{=} i_2\right) \wedge \left(j_1 \overset{\neq}{=} j_2\right)$$

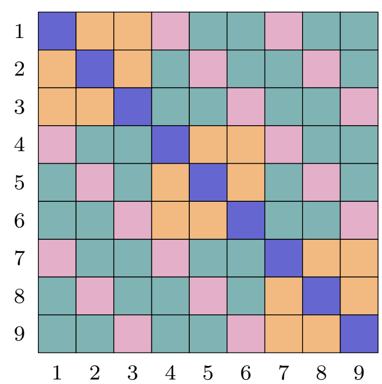
. It is clear that $i_3 = i_4$ holds if and only if $i_1 = i_2$, since by applying g_l we have $g_l(i_1) = g_l(i_2) = i_3 = i_4$, and since g_l is a bijection. The same goes for $j_1 = j_2$.

These are the orbits and the explanation for why they are the only orbits is as follows: if $i_1 = i_2$ then we can move to any other k', l' that has $i_3 = i_4$ by applying the right $g_l(i_1) = i_3$. If $i_1 \neq i_2$ we can move to any other indices $i_3 \neq i_4$ since S_n is 2-transitive. The same goes for the indices j. Therefore we will have only these 4 orbits:

$$\begin{gathered} O_{=i=j} : i_1 = i_2, j_1 = j_2 \\ O_{=i\neq j} : i_1 = i_2, j_1 \neq j_2 \\ O_{\neq i=j} : i_1 \neq i_2, j_1 = j_2 \\ O_{\neq i\neq j} : i_1 \neq i_2, j_1 \neq j_2 \end{gathered}$$

2.3

For $n = m = 3$ we will obtain $W \in \mathbb{R}^{9\times 9}$ and $b \in \mathbb{R}^9$



2.4

In the case of a linear invariant layer, for all $v \in \mathbb{R}^{nm}$ and $g \in G$ we have:

$$\begin{gathered} f(gv) = f(v) \\ \Rightarrow w^T gv + b = w^T v + b \\ \Rightarrow g^T w = w \end{gathered}$$

This is true for all biases $b \in \mathbb{R}$. Consider index $k = i + n(j-1)$. For w we have $\left(g^T w\right)_k = w_{g(k)} = w_{g_l(i)+n(g_r(j)-1)}$. In this case the orbit is the entire vector w which means that $w = \lambda \mathbf{1}$.

2.5

In the case of 3-tensors with the symmetric group $S_{N_1} \times S_{N_2} \times S_{N_3}$ applied independently to each index axis of the tensor, we have:

$$v = \vec{(}A)$$

and

$$v_m = a_{i_1,i_2,i_3}$$

where $m = i_1+N_1(i_2-1)+N_1N_2(i_3-1)$. Define $k = i_1^k+N_1(i_2^k-1)+N_1N_2(i_3^k-1)$ and $l = i_1^l + N_1(i_2^l - 1) + N_1N_2(i_3^l - 1)$. We will have the same derivation as in the previous clause 2.2, thus

$$W_{k,l} = W_{g(k),g(l)}$$

The indices (k, l) can transform to

$$\begin{aligned} k' &= g_1(i_1^k) + N_1(g_2(i_2^k) - 1) + N_1N_2(g_3(i_3^k) - 1) \\ l' &= g_1(i_1^l) + N_1(g_2(i_2^l) - 1) + N_1N_2(g_3(i_3^l) - 1) \end{aligned}$$

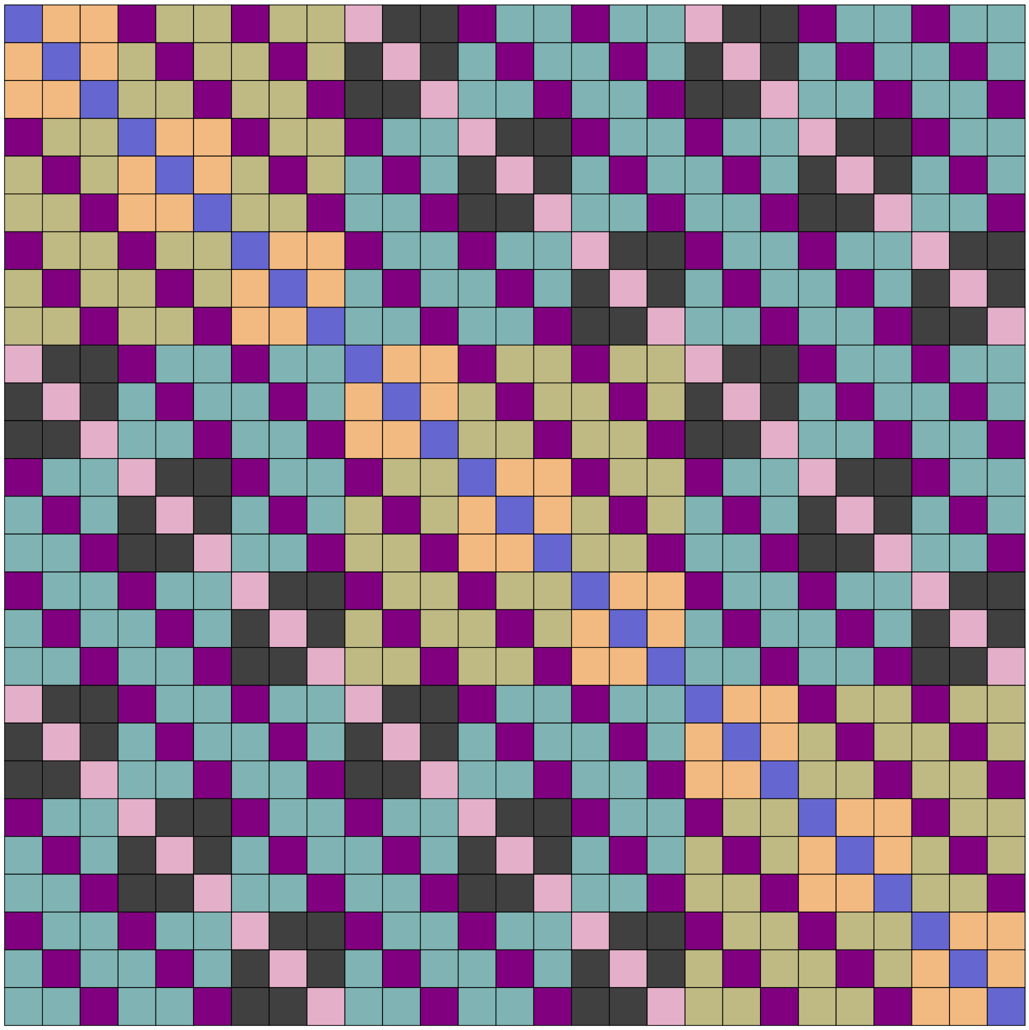
where the equivalence relations follow the same pattern as before, thus the orbits are

$$\wedge_{n=1}^{3} i_n^k \underset{=}{\neq} i_n^l.$$

In each coordinate in the original matrix A, the permutation is 2-transitive, therefore we can move to any other configuration. As a result, the orbits are:

$$\begin{aligned} O_{=i_1=i_2=i_3} &: i_1^k = i_1^l, i_2^k = i_2^l, i_3^k = i_3^l \\ O_{=i_1=i_2\neq i_3} &: i_1^k = i_1^l, i_2^k = i_2^l, i_3^k \neq i_3^l \\ O_{=i_1\neq i_2=i_3} &: i_1^k = i_1^l, i_2^k \neq i_2^l, i_3^k = i_3^l \\ &\vdots \\ O_{=i_1=i_2=i_3} &: i_1^k \neq i_1^l, i_2^k \neq i_2^l, i_3^k \neq i_3^l \end{aligned}$$

For $N_1 = N_2 = N_3 = 3$ we will have $W \in \mathbb{R}^{27\times27}$ and $b \in \mathbb{R}^{27} = \lambda\mathbf{1}$. W can be visualized as follows:



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