



UNIVERSITY OF GHANA

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BSc/BA, FIRST SEMESTER EXAMINATIONS: 2017/2018

DEPARTMENT OF MATHEMATICS

MATH 223: CALCULUS II (3 credits)

INSTRUCTION:

ANSWER ANY 4 OUT OF THE FOLLOWING 6 QUESTIONS

TIME ALLOWED:

TWO HOURS AND THIRTY MINUTES $\left(2\frac{1}{2} \text{ hours}\right)$

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1. (a) Use the Mean Value Theorem to establish the following inequality

$$e^a(x-a) < e^x - e^a < e^x(x-a),$$

if $a < x$.

[15 Marks]

- (b) By considering the derivative of the function $f : [-1, 1] \rightarrow \mathbb{R}$ defined by

$$f(x) = \frac{2x}{x^2 + 1},$$

show that f^{-1} exists and find $(f^{-1})'(\frac{4}{5})$.

[15 Marks]

- (c) Evaluate the following limit

$$\lim_{x \rightarrow \infty} \left(\frac{x+1}{x+2}\right)^x.$$

[20 Marks]

2. (a) Express $5 \sinh x + \cosh x$ in the form $Ae^x + Be^{-x}$, where A and B are integers. [10 Marks]

- (b) Solve the equation $5 \sinh x + \cosh x + 5 = 0$, giving your answer in the form $\ln a$, where $a \in \mathbb{R}$. [15 Marks]

- (c) Differentiate the following functions with respect to x

(i) $\sinh x \tanh x$ (ii) $\int_{\sqrt{x}}^x \frac{\sin t}{t} dt.$

[25 Marks]

3. (a) Use the Riemann sum to calculate the area under the curve $y = -x^2 + 3x + 2$ between $x = -1$ and $x = 2$. [20 Marks]

(b) Let $R = \lim_{n \rightarrow \infty} \left[\sum_{j=0}^{n-1} \left(3 + \frac{4j}{n} \right)^2 \cdot \frac{4}{n} \right]$, write R as a definite integral and hence evaluate R .

[15 Marks]

- (c) If $x^y = e^{x-y}$, prove that

$$\frac{dy}{dx} = \frac{\ln x}{(1 + \ln x)^2}.$$

[15 Marks]

4. (a) Evaluate the following integrals

(i) $\int \frac{x+1}{\sqrt{x^2-x+1}} dx$ (ii) $\int \frac{dx}{5+3\cos x}$ (Hint: use the substitution $t = \tan(x/2)$)

[25 Marks]

- (b) Evaluate the following definite integrals

(i) $\int_0^2 \frac{dx}{(4+x^2)^2}$ (Hint: Use the substitution $x = 2 \tan \theta$)

(ii) $\int_1^2 x^4 (\ln x)^2 dx$ (Hint: Use integration by parts).

[25 Marks]

5. (a) By writing $3 \cos x + 4 \sin x = \lambda \frac{d}{dx}(4 \cos x + 5 \sin x) + \mu(4 \cos x + 5 \sin x)$, where λ and μ are constants, find the values of λ and μ and hence evaluate the integral

$$\int \frac{3 \cos x + 4 \sin x}{4 \cos x + 5 \sin x} dx.$$

[15 marks]

- (b) Let $I_n = \int_0^1 x^n \sqrt{1-x^2} dx$, $n \in \mathbb{N}$. Show that $(n+2)I_n = (n-1)I_{n-2}$, $n \geq 2$. [20 Marks]

- (c) If $g = \sin(\sin x)$, prove that

$$\frac{d^2 y}{dx^2} + \tan x \frac{dy}{dx} + y \cos^2 x = 0.$$

[15 Marks]

6. (a) Evaluate the following improper integrals

(i) $\int_0^3 \frac{dx}{(x-1)^{2/3}}$ (ii) $\int_0^\infty (1+2x)e^{-x} dx$ (iii) $\int_{-\infty}^\infty \frac{6x^3}{(x^4+1)^2} dx.$

[35 Marks]

- (b) Evaluate the following integral using partial fractions

$$\int \frac{3x^2 + 2x}{(x+2)(x^2+4)} dx.$$

[15 Marks]