



UNIVERSITY OF GHANA
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BSc/BA, FIRST SEMESTER EXAMINATIONS: 2016/2017

DEPARTMENT OF MATHEMATICS

MATH 223: CALCULUS II (4 credits)

INSTRUCTION:

ANSWER ANY FIVE(5) OUT OF THE FOLLOWING SEVEN(7) QUESTIONS

TIME ALLOWED:

TWO HOURS AND THIRTY MINUTES $\left(2\frac{1}{2} \text{ hours}\right)$

1. State the Fundamental Theorem of Calculus (Part I and Part II). Hence or otherwise:

(a) If u is a differentiable function of x over I , obtain a formula for

$$\frac{d}{dx} \int_a^u f(t) dt.$$

Hence find $\frac{d^2}{dx^2} \left[\int_0^x \left(\int_1^{\sin t} \sqrt{1+u^4} du \right) dt \right].$

(b) Evaluate

$$\lim_{x \rightarrow 3} \left(\frac{x}{x-3} \int_3^x \frac{\sin t}{t} dt \right).$$

(Hint: You may use $\int_3^x \frac{\sin t}{t} dt = F(x) - F(3)$ for some antiderivative F of $\frac{\sin t}{t}$)

(c) By Using L'hospital's rule,

i. Evaluate the limit $\lim_{z \rightarrow 0} \frac{\sin(2z) + 7z^2 - 2z}{z^2(z+1)^2}.$

2. Let $a, b, c \in \mathbb{R}^+$ and distinct.

(a) Let $x > 0$, By using the definition of the natural logarithmic function,

i. Prove that $x(1+x) > (1+x) \ln(1+x) > x.$

ii. Hence show that $\ln(1+x) > x - x^2 + x^3 - x^4 + \dots$

(b) Let $a, b, c \in \mathbb{R}^+$ and distinct. Show that there are two different real values of x which satisfy the equation

$$a \cosh x + b \sinh x = 0$$

only if $b^2 < a^2 < b^2 + c^2.$

(c) Let $x \in \mathbb{R} \setminus \{0\}$. Find the values of x that satisfy the equation

$$\sinh 2x + \sinh x (\cosh^2 x - 1) = 0.$$

3. (a) Using $n = 6$ subintervals, approximate the value of

$$\int_0^4 \cos(1 + \sqrt{x}) dx,$$

using the Upper, Lower, and Midpoint Riemann sums. Compare the results and describe.

- (b) By interpreting as an integral, evaluate

$$\lim_{n \rightarrow \infty} \frac{1}{n^{3/2}} \left(1 + \sqrt{2} + \dots + \sqrt{n-1} \right).$$

4. (a) Show that

$$\int t^3 \sin(2t^4) dt = -\frac{1}{8} \cos(2t^4) + C$$

hence show that

$$\int_0^{\pi^{1/4}} t^7 \sin(2t^4) dt = -\frac{\pi}{8}.$$

- (b) Show that if $I_n = \int \sec^n x dx$

$$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}.$$

Hence or otherwise, evaluate

$$\int_0^{\pi/4} \sec^4 x dx.$$

5. (a) Determine the exact value of the definite integral

$$\int_{-\infty}^{\infty} \frac{6w^3}{(w^4 + 1)^2} dw.$$

- (b) Find the surface area produced when the curve

$$x = 3 \cos(\pi t) \quad y = 5t + 2, \quad 0 \leq t \leq \frac{1}{2}$$

is rotated about the y -axis.

(You may use without proof $\int \sec^3 x dx = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| + C$.)

- (c) A particle travels along a path defined by the curve

$$x = 3t + 1 \quad y = 4 - t^2, \quad -2 \leq t \leq 0.$$

Determine the total length of the curve.

6. (a) Given that

$$y = x + \ln \left(\frac{(x-3)^5}{(y-1)^2} \right),$$

show that

$$(xy - 3y + x - 3) \frac{dy}{dx} = xy + 2y - x - 2.$$

- (b) Suppose an object of mass m kg falling near the surface of the earth is retarded by air resistance proportional to its velocity $v = v(t)$ at any given time t . Then Newton's second law of motion states that

$$m \frac{dv}{dt} = mg - kv,$$

where g is the acceleration due to gravity near the surface of the earth and k is a constant. If the object starts from rest,

- find the velocity for any $t > 0$;
 - what is the limiting value of $v(t)$?
- (c) Let $g(x)$ be a differentiable function over an interval I and $g(x) > 0$ on I . Show that

$$\frac{d}{dx} 2\sqrt{g(x)} = \frac{g'(x)}{\sqrt{g(x)}}.$$

Hence or otherwise, show that the integral of the function

$$\frac{2x+1}{\sqrt{x^2+2x-1}} dx$$

is

$$2\sqrt{x^2+2x-1} - \ln |\cot \theta + \csc \theta| + K,$$

for some angle θ .

7. (a) Consider the substitution $t = \tanh \frac{x}{2}$.

- Find expressions for dx , $\sinh x$ and $\cosh x$ in terms of t .
- Setup the integral

$$\int \frac{\sinh x + \cosh x + 1}{\sinh x + \cosh x + 2} dx.$$

as the integral of a rational function of t using the substitution above. (There is no need to evaluate the integral.)

- (b) Solve the initial value problem

$$2u \frac{du}{dx} - \frac{6x}{x^2+1} u^2 = 2x^3 u,$$

for $u(0) = 2$.

- (c) Find the volume of the solid formed when the plane figure bounded by $r = 2a \cos \theta$ and the radius vectors at $\theta = 0$ and $\theta = \frac{\pi}{2}$ rotates about the initial line. Leave your answer in terms of π and a .