# ***Worksheet***

MA 114 Spring 2013

1. Show that the curve with equation $y = \cosh 2x - 6\sinh x$ has a stationary point, and find its $x$-coordinate in logarithmic form.

   **Solution**: Find $\frac{dy}{dx}$ and where this derivative is 0.

$$\begin{aligned}
\frac{dy}{dx} &= 2\sinh(2x) - 6\cosh(x) \\
2\sinh(2x) - 6\cosh(x) &= 0 \\
\frac{e^{2x} - e^{-2x}}{2} &= 3\frac{e^{x} + e^{-x}}{2} \\
\frac{e^{2x} - e^{-2x}}{e^{x} + e^{-x}} &= 3 \\
e^{x} - e^{-x} &= 3 \\
(e^{x})^2 - 3e^{x} - 1 &= 0 \\
e^{x} &= \frac{3 \pm \sqrt{9+4}}{2} \\
&= \frac{3 + \sqrt{9+4}}{2} \\
x &= \ln\left(\frac{3 + \sqrt{9+4}}{2}\right)
\end{aligned}$$

   (Because $ln$ must take a positive argument.)

2. Find the exact solution to $3\cosh x = 4\sinh x$, giving the answer in terms of a logarithm.

   **Solution**:

$$\begin{aligned}
3\cosh(x) &= 4\sinh(x) \\
3e^{x} + 3e^{-x} &= 4e^{x} - 4e^{-x} \\
e^{x} - 7e^{-x} &= 0 \\
(e^{x})^2 - 7 &= 0 \\
e^{x} &= \sqrt{7} \\
x &= \frac{1}{2}\ln 7
\end{aligned}$$

3. Find the equation of the tangent to the curve $y = \cosh(2x)$ at $x = 1$.

   **Solution**: $\frac{dy}{dx} = 2\sinh(2x)$. Set $x = 1$ and $m = y'(1) = 2\sinh 2$, so the equation of the tangent line is

$$y - \cosh 2 = 2\sinh 2(x - 1).$$

4. Find the equation of the tangent to the curve $y = \sinh(x) + \cosh(x)$ at $x = 0$.

   **Solution**: $\dfrac{dy}{dx} = \cosh x + \sinh x$. Set $x = 0$ and $m = y'(0) = \cosh 0 + \sinh 0 = 1$, so the equation of the tangent line is
$$y - 1 = 1(x - 0),$$
   or $y = x + 1$.

5. Suppose we know that the hyperbolic cosine of a certain variable is equal to 1. What is the hyperbolic cosine of twice that variable?

   **Solution**: We are given that $\cosh a = 1$. We must find $\cosh(2a)$.

   **Method 1**: There is only one value of $a$ that will make $\cosh a = 1$ and that is $a = 0$. Then $2a = 0$ and $\cosh(2a) = 1$.

   **Method 2**: Since $\cosh^2 a - \sinh^2 a = 1$ and $\cosh a = 1$, we see that $\sinh^2 a = 0$ and thus $\sinh a = 0$. Then
$$\cosh(2a) = \cosh^2 a + \sinh^2 a = 1 + 0 = 1.$$

6. How is the hyperbolic secant of 10 related to the hyperbolic secant of $-10$?

   **Solution**: Since $\cosh x$ is an even function, it follows that $\operatorname{sech} x = 1/\cosh x$ is also even. Thus, $\operatorname{sech}(-10) = \operatorname{sech}(10)$.

7. Using the definitions of the hyperbolic functions, show that $2\cosh^2 x - \cosh 2x = 1$.

   **Solution**:
   **Method 1**: Using hyperbolic trig identities:
$$\begin{aligned} 2\cosh^2 x - \cosh 2x &= 2\cosh^2 x - (\cosh^2 x + \sinh^2 x) \\ &= \cosh^2 x - \sinh^2 x \\ &= 1 \end{aligned}$$

   **Method 2**
$$\begin{aligned} 2\cosh^2 x - \cosh 2x &= 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - \frac{e^{2x} + e^{-2x}}{2} \\ &= 2\left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) - \frac{e^{2x} + e^{-2x}}{2} \\ &= \frac{2 \cdot 2}{4} \\ &= 1 \end{aligned}$$

8. Find $\frac{d}{dt}\coth\left(e^{2tx}\right)$. **Solution**:

$$\begin{aligned}\frac{d}{dt}\coth\left(e^{2tx}\right) &= -\text{csch}^2\left(e^{2tx}\right)\left(e^{2tx}\right)(2x)\\ &= -2xe^{2tx}\text{csch}^2\left(e^{2tx}\right)\end{aligned}$$

9. $\int \cosh(3x)\,dx$

$$\int \cosh(3x)\,dx = \frac{1}{3}\sinh(3x) + C.$$

10. $\int \sinh(x+1)\,dx$

$$\int \sinh(x+1)\,dx = \cosh(x+1) + C.$$

11. $\int \sinh^3 x \cosh^6 x\,dx$

**Solution**: Just like regular trig functions, rewrite $\sinh^3 x = \sinh x \sinh^2 x = \sinh x(\cosh^2 x - 1)$, and then integrate.

$$\begin{aligned}\int \sinh^3 x \cosh^6 x\,dx &= \int \sinh x(\cosh^2 x - 1)\cosh^6 x\,dx\\ &= \frac{1}{9}\cosh^9 x - \frac{1}{7}\cosh^7 x + C\end{aligned}$$

12. $\int \sinh^2 x \cosh^2 x\,dx$

**Solution**: To really simplify matters, rewrite $\sinh^2 x = \frac{1}{2}(1+\cosh(2x))$ and $\cosh^2 x = \frac{1}{2}(1-\cosh(2x))$.

$$\begin{aligned}\int \sinh^2 x \cosh^2 x\,dx &= \int \frac{1}{2}(1+\cosh(2x))\frac{1}{2}(1-\cosh(2x))\,dx\\ &= -\int \frac{1}{4}(1-\cosh^2(2x))\,dx\\ &= -\int \frac{1}{4}\left(1-\frac{1}{2}(1+\cosh(4x))\right)\,dx\\ &= \int \frac{1}{8}\left(-1+\cosh(4x)\right)\,dx\\ &= \frac{1}{8}\left(-x+\frac{1}{4}\sinh(4x)\right)+C\\ &= -\frac{x}{8}+\frac{1}{32}\sinh(4x)+C\end{aligned}$$

13. Define the gudermannian function by $\mathrm{gd}(x) = \tan^{-1}(\sinh x)$. Show that $\dfrac{d}{dx}\,\mathrm{gd}(x) = \operatorname{sech} x$.

**Solution**: We differentiate.

$$\begin{aligned}
\frac{d}{dx}\left(\mathrm{gd}(x)\right) &= \frac{d}{dx}\left(\tan^{-1}(\sinh x)\right) \\
&= \frac{\cosh x}{1+\sinh^2 x} \\
&= \frac{\cosh x}{\cosh^2 x} \\
&= \frac{1}{\cosh x} \\
&= \operatorname{sech} x
\end{aligned}$$

14. Let $f(x) = 2\tan^{-1}(e^x) - \dfrac{\pi}{2}$. Prove that $\mathrm{gd}(x) = f(x)$ by showing that they have the same derivative and that $f(0) = \mathrm{gd}(0)$.

**Solution**: We differentiate.

$$\begin{aligned}
\frac{d}{dx}\left(2\tan^{-1}\left(e^x\right) - \frac{\pi}{2}\right) &= \frac{2e^x}{1+e^{2x}} \\
&= \frac{2e^x}{1+e^{2x}}\frac{e^{-x}}{e^{-x}} \\
&= \frac{2}{e^x + e^{-x}} \\
&= \frac{1}{\cosh x} \\
&= \operatorname{sech} x \\
&= \frac{d}{dx}\left(\mathrm{gd}(x)\right)
\end{aligned}$$

Thus, since the functions have the same derivative, they must differ only by a constant, i.e., $\mathrm{gd}(x) + C = 2\tan^{-1}(e^x) - \dfrac{\pi}{2}$. By comparing the two functions at $x = 0$ we can find the value of this constant.

$$\begin{aligned}
\mathrm{gd}(0) + C &= 2\tan^{-1}(e^0) - \frac{\pi}{2} \\
\tan^{-1}(\sinh 0) + C &= 2\tan^{-1}(1) - \frac{\pi}{2} \\
\tan^{-1}(0) + C &= 2\left(\frac{\pi}{4}\right) - \frac{\pi}{2} \\
0 + C &= \frac{\pi}{2} - \frac{\pi}{2} \\
C &= 0
\end{aligned}$$

Thus, the two functions are equal.

15. Show that

i. $\cosh(x) = \sec(\mathrm{gd}(x))$.

$$\begin{aligned}
\sec(\mathrm{gd}(x)) &= \sec\left(\tan^{-1}(\sinh x)\right) \\
&= \frac{\sqrt{1 + \sinh^2(x)}}{1} \\
&= \sqrt{\cosh^2(x)} \\
&= \cosh x
\end{aligned}$$

ii. $\tanh(x) = \sin(\mathrm{gd}(x))$.

$$\begin{aligned}
\sin(\mathrm{gd}(x)) &= \sin\left(\tan^{-1}(\sinh x)\right) \\
&= \frac{\sinh x}{\sqrt{1 + \sinh^2(x)}} \\
&= \frac{\sinh x}{\cosh(x)} \\
&= \tanh x
\end{aligned}$$

iii. $\coth(x) = \csc(\mathrm{gd}(x))$.

$$\begin{aligned}
\csc(\mathrm{gd}(x)) &= \csc\left(\tan^{-1}(\sinh x)\right) \\
&= \frac{\sqrt{1 + \sinh^2(x)}}{\sinh x} \\
&= \frac{\sqrt{\cosh^2(x)}}{\sinh x} \\
&= \frac{\cosh x}{\sinh x} \\
&= \coth x
\end{aligned}$$

iv. $\operatorname{sech}(x) = \cos(\operatorname{gd}(x))$.

$$\begin{aligned}\cos(\operatorname{gd}(x)) &= \cos\left(\tan^{-1}(\sinh x)\right)\\ &= \frac{1}{\sqrt{1+\sinh^2(x)}}\\ &= \frac{1}{\sqrt{\cosh^2(x)}}\\ &= \frac{1}{\cosh x}\\ &= \operatorname{sech} x\end{aligned}$$

v. $\operatorname{csch}(x) = \cot(\operatorname{gd}(x))$.

$$\begin{aligned}\cot(\operatorname{gd}(x)) &= \cot\left(\tan^{-1}(\sinh x)\right)\\ &= \frac{1}{\sinh(x)}\\ &= \operatorname{csch} x\end{aligned}$$

16. Find

i. $\frac{d}{dx}\big(\ln(\cosh x)\big)^5$

$$\begin{aligned}\frac{d}{dx}\big(\ln(\cosh x)\big)^5 &= 5\big(\ln(\cosh x)\big)^4 \frac{1}{\cosh x}\sinh x\\ &= 5\tanh(x)\big(\ln(\cosh x)\big)^4\end{aligned}$$

ii. $\frac{d}{dx}\sinh(\ln x)$

$$\begin{aligned}\sinh(\ln x) &= \frac{1}{2}\left(x - \frac{1}{x}\right)\\ \frac{d}{dx}\sinh(\ln x) &= \frac{1}{2}\left(1 + \frac{1}{x^2}\right)\end{aligned}$$

iii. $\frac{d}{dx}e^{\tanh x}$

$$\frac{d}{dx}e^{\tanh x} = \operatorname{sech}^2 x \cdot e^{\tanh x}$$

iv. $\frac{d}{dx}\tanh(e^x)$

$$\frac{d}{dx}\tanh(e^x) = e^x \operatorname{sech}^2\left(e^x\right)$$

v. $\dfrac{d}{dx}\sinh(\cosh^3 x)$

$$\frac{d}{dx}\sinh(\cosh^3 x) = \cosh\left(\cosh^3 x\right) 3\cosh^2 x \sinh x$$

17. Show that

$$(\cosh(x) + \sinh(x))^n = \cosh(nx) + \sinh(nx)$$

for any integer $n$.

**Solution**:

$$\begin{aligned}(\cosh(x) + \sinh(x))^n &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)^n \\ &= \left(\frac{2e^x}{2}\right)^n \\ &= e^{nx}\end{aligned}$$

Likewise:

$$\begin{aligned}\cosh(nx) + \sinh(nx) &= \left(\frac{e^{nx} + e^{-nx}}{2} + \frac{e^{nx} - e^{-nx}}{2}\right) \\ &= \left(\frac{2e^{nx}}{2}\right) \\ &= e^{nx}\end{aligned}$$

Thus, clearly $(\cosh(x) + \sinh(x))^n = \cosh(nx) + \sinh(nx)$ for any integer $n$.

18. At what point on the curve $y = \cosh x$ does the tangent have slope 1?

**Solution**: We need to find the value of $x$ so that $\dfrac{d}{dx}\cosh x = 1$.

$$\begin{aligned}\frac{d}{dx}\cosh x &= \sinh x \\ \sinh x &= 1 \\ \frac{e^x - e^x}{2} &= 1 \\ e^x - e^x &= 2 \\ (e^x)^2 - 2e^x - 1 &= 0 \\ e^x &= \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}\end{aligned}$$

Thus, $x = \ln\left(1 + \sqrt{2}\right)$.