

0.1 Finding Infinite Limits: Order of Magnitude Analysis

In order to work with infinite limits it is helpful to introduce the set of *affinely extended real numbers*, which contains all of the real numbers as well as $\pm\infty$. Using set notation, we write

$$\mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\}.$$

One way in which we encounter limits where

$$\lim_{x \rightarrow x_0} f(x) = \infty$$

is when $x_0 \in \mathbb{R}$, and f contains a division by 0 at the point x_0 . In such a situation $f(x_0)$ is often undefined, but because the denominator of $f(x)$ approaches 0 as $x \rightarrow x_0$, the output of $f(x)$ grows without bound as $x \rightarrow x_0$.

Example 1. Find $\lim_{x \rightarrow -4} \frac{1}{(x+4)^4}$.

Solution In analyzing this function, the first thing to notice is that we have a fourth power in the denominator, $(x+4)^4$. Since $(x+4)$ is raised to an even power, the denominator is always non-negative (it is 0 when $x = -4$ and positive otherwise). Since the numerator is always positive as well, it means that this function is always non-negative. As $x \nearrow -4$ and $x \searrow -4$ the denominator approaches 0, so correspondingly the values of the function grow without bound (so the limit does not exist). We write

$$\lim_{x \rightarrow -4} \frac{1}{(x+4)^4} = \infty$$

to signify the function values grow without bound as $x \rightarrow -4$ from both sides.

Example 2. Find $\lim_{x \rightarrow 3} \frac{3-x}{(x-3)^4}$.

Solution The first thing for us to do is simplify this fraction as much as possible. If we factor -1 from the numerator, we find that

$$\lim_{x \rightarrow 3} \frac{3-x}{(x-3)^4} = \lim_{x \rightarrow 3} \frac{-(x-3)}{(x-3)^4} = \lim_{x \rightarrow 3} \frac{-1}{(x-3)^3}.$$

Note that above we were able to cancel the factor of $(x-3)$ because in looking at the limit as $x \rightarrow 3$, we know that $x \neq 3$, so the factor will never be 0 (we, of course, cannot divide by 0). In this case, unlike the previous one, we have an odd power in the denominator; that is, $(x-3)$ is raised to the third power. Because of this, we will have

$$(x-3)^3 < 0 \text{ when } x < 3 \quad \text{and} \quad (x-3)^3 > 0 \text{ when } x > 3.$$

Since the sign of the denominator is different depending on what side $x \rightarrow 3$ from, we should look at the one-sided limits rather than trying to calculate a two-sided limit directly (because they will have different signs, so unless they are both 0, the two-sided limit will not exist). As $x \nearrow 3$ the denominator is negative, so the entire fraction is positive (because there is a -1 in the numerator). As $x \searrow 3$ the denominator is always positive, so the entire function is negative. In both cases the denominator approaches 0, so we find that

$$\lim_{x \nearrow 3} \frac{3-x}{(x-3)^4} = \lim_{x \nearrow 3} \frac{-1}{(x-3)^3} = \infty \quad \text{and} \quad \lim_{x \searrow 3} \frac{3-x}{(x-3)^4} = \lim_{x \searrow 3} \frac{-1}{(x-3)^3} = -\infty.$$

Since our function has no consistent behavior as $x \rightarrow 3$, all we can say is that the limit does not exist.

Example 3. Find $\lim_{x \rightarrow 3} \frac{4x^2 - 9x - 9}{x - 3}$.

Solution As $x \rightarrow 3$, the denominator approaches 0, but so does the numerator. For any polynomial to have a zero at 3 means it has a factor of $(x - 3)$. In fact,

$$4x^2 - 9x - 9 = (x - 3)(4x + 3).$$

Now we can cancel a factor of $(x - 3)$ from numerator and denominator, to find that

$$\lim_{x \rightarrow 3} \frac{4x^2 - 9x - 9}{x - 3} = \lim_{x \rightarrow 3} (4x + 3) = 15.$$

Suppose that we have two functions, f and g , with

$$\lim_{x \rightarrow x_0} |f(x)| = \lim_{x \rightarrow x_0} |g(x)| = \infty,$$

where $x_0 \in \mathbb{R}^*$. What happens when we look at

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)}?$$

In this situation we cannot apply the quotient rule for limits, because

$$\lim_{x \rightarrow x_0} f(x) \quad \text{and} \quad \lim_{x \rightarrow x_0} g(x)$$

do not actually exist (recall that writing a limit is ∞ means that the limit doesn't exist precisely because the function values grow without bound). Nevertheless, just as when we were faced with a limit of a quotient where both the limits of the numerator and denominator were 0, we suspect that it is possible for the limit of this quotient to be finite. For instance, if we let

$$f(x) = g(x) = x,$$

then we have two functions both approaching ∞ as $x \rightarrow \infty$, yet the limit of their ratio is 1. Before delving into this problem in more generality, we will note that there is a simple trick for evaluating this type of limit when we are dealing with rational functions (a ratio of polynomials). If we divide by the highest power of x in the denominator we can evaluate the limit.

Example 4. Find $\lim_{x \rightarrow \infty} \frac{2x^2 - 1}{5x^2 - x}$.

Solution As before, we want to multiply by a convenient choice of 1 in order to rewrite the expression into something we can manipulate. Since the highest power of x in the denominator is 2 (ie, x^2) we multiply by

$$\frac{1/x^2}{1/x^2},$$

in order to remove all terms in the denominator approaching $\pm\infty$, reducing the limit into something we can analyze (because considering something of the form ∞/∞ is meaningless). Doing so we find

$$\lim_{x \rightarrow \infty} \frac{2x^2 - 1}{5x^2 - x} = \lim_{x \rightarrow \infty} \frac{2x^2 - 1}{5x^2 - x} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \rightarrow \infty} \frac{2 - 1/x^2}{5 - 1/x} = \frac{2 - 0}{5 - 0} = \frac{2}{5}.$$

Example 5. Find $\lim_{x \rightarrow \infty} \frac{2x - 1}{5x^2 - x}$.

Solution Using the same trick as before we find

$$\lim_{x \rightarrow \infty} \frac{2x - 1}{5x^2 - x} = \lim_{x \rightarrow \infty} \frac{2x - 1}{5x^2 - x} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \rightarrow \infty} \frac{2/x - 1/x^2}{5 - 1/x} = \frac{0 - 0}{5 - 0} = 0.$$

Example 6. Find $\lim_{x \rightarrow -\infty} \frac{2x^4 - x^3}{5x^3 - x}$.

Solution Yet again we perform the same trick, finding that

$$\lim_{x \rightarrow -\infty} \frac{2x^4 - x^3}{5x^3 - x} = \lim_{x \rightarrow -\infty} \frac{2x^4 - x^3}{5x^3 - x} \cdot \frac{1/x^3}{1/x^3} = \lim_{x \rightarrow -\infty} \frac{2x - 1}{5 - 1/x^2} = -\infty.$$

In the above examples, it was the relative order of magnitude of the polynomials (the highest power of x) in the numerator and denominator that determined the limit. This is true in general. If the numerator has a higher order of magnitude than the denominator, then the rational function will approach $\pm\infty$. If the numerator has the same order of magnitude as the denominator, then the limit will be the ratio of the coefficients of the highest order terms in the numerator and denominator. Finally, if the numerator has a lower order of magnitude than the denominator, the limit will approach 0.

The key to evaluating limits of $f(x)/g(x)$ in the more general case is extending this idea of order of magnitude to functions other than polynomials. We will call this order of magnitude analysis of growth and decay. The basic idea is that when we look at functions that approach ∞ , we can distinguish functions based on the *rate* at which they approach ∞ . For a simple example, think of the functions $f(x) = x$ and $g(x) = x^2$. As x gets larger, both functions grow without bound, but $g(x)$ grows much more rapidly than $f(x)$. This distinction is not difficult to make because they functions are so familiar. The goal is to extend this classification to many more functions, in order to make evaluating limits much easier. We formalize the idea of rate of growth with the following definition.

Definition 0.1.1 (Relative Rates of Growth). Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$, $x_0 \in \mathbb{R}^*$, and

$$\lim_{x \rightarrow x_0} |f(x)| = \lim_{x \rightarrow x_0} |g(x)| = \infty.$$

We say f approaches ∞ *on a higher order of magnitude* than g if

$$\lim_{x \rightarrow x_0} \left| \frac{f(x)}{g(x)} \right| = \infty.$$

We say f approaches ∞ *on a lower order of magnitude* than g if

$$\lim_{x \rightarrow x_0} \left| \frac{f(x)}{g(x)} \right| = 0.$$

We say that f and g approach ∞ *on the same order of magnitude* if

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = a, \quad a \in \mathbb{R}, a \neq 0.$$

Now our goal is to try to put familiar functions into each of these categories. In accordance with the above examples, we already have distinctions for polynomial functions. In general, l'Hôpital's rule is extremely useful for making these classifications, but we do not yet have the tool of differentiation at our disposal. At the moment let us restrict ourselves to three classes of functions: exponentials, power functions, and logarithms.

Theorem 0.1.1 (Relative Rates of Growth). *As $x \rightarrow \infty$, the magnitude of exponential, power, and logarithmic functions approach ∞ . Moreover, the order of magnitude with which they approach ∞ is given in the following order, with exponential functions approaching the fastest, and logarithmic functions approaching the slowest.*

1. *Exponentials of the form a^x , where $a > 0, a \neq 1$. If $a > b$, then a^x approaches ∞ faster than b^x as $x \rightarrow \infty$.*
2. *Power functions, x^n , $n \in \mathbb{N}$. If $n > m$, then x^n approaches ∞ faster than x^m .*
3. *Logarithmic Functions, $\log_a(x)$, $a > 0, a \neq 1$. All logarithmic functions approach ∞ at the same rate, because $\log_a(x)/\log_b(x) = \ln(b)/\ln(a)$.*

If any of the above functions is multiplied by a nonzero constant, it does not change the relative order of magnitude at which the magnitude of the function approaches ∞ .

Using this information, we can immediately evaluate limits such as

$$\lim_{x \rightarrow \infty} \frac{e^x}{\ln(x)} = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\ln(x)}{e^x} = 0.$$

There is one additional fact we need in order to really utilize this tool. The key thing is that when we look at sums and differences of these functions, it is only the term that grows that fastest that matters - this is called the *dominant term*.

Definition 0.1.2 (Dominant Term). Suppose f can be written as a linear combination of functions $\{f_1, f_2, \dots, f_n\}$, and

$$\lim_{x \rightarrow x_0} |f(x)| = \infty,$$

where $x_0 \in \mathbb{R}^*$. The *dominant term* of f is the function f_i which approaches ∞ on the highest order of magnitude, denoted by \hat{f} .

Recall that a linear combination of functions is simply a sum of constant multiples of the functions. For instance, a polynomial is simply a linear combination of power functions.

Theorem 0.1.2 (Dominant Terms). *Consider $f(x)$ and $g(x)$, with*

$$\lim_{x \rightarrow x_0} |f(x)| = \lim_{x \rightarrow x_0} |g(x)| = \infty.$$

If \hat{f} and \hat{g} are the dominant terms of f and g respectively, then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{\hat{f}(x)}{\hat{g}(x)}.$$

Now for any given limit we simply need to look at the dominant terms. For instance,

$$\lim_{x \rightarrow \infty} \frac{e^x - 12x^2 + x}{x^4 + 1} = \lim_{x \rightarrow \infty} \frac{e^x}{x^4} = \infty.$$

We can also apply this same idea to limits in which both the numerator and denominator approach 0.

Definition 0.1.3 (Relative Rates of Decay). Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$, $x_0 \in \mathbb{R}^*$, and

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0.$$

We say f approaches 0 *on a higher order of magnitude* than g if

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0.$$

We say f approaches 0 *on a lower order of magnitude* than g if

$$\lim_{x \rightarrow x_0} \left| \frac{f(x)}{g(x)} \right| = \infty.$$

We say that f and g approach 0 *on the same order of magnitude* if

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = a, \quad a \in \mathbb{R}, a \neq 0.$$

The most simple case is to look at the reciprocals of exponential and power functions. The relative rates of decay of such functions follow immediately from the relative rates at which they grow. For instance, consider the functions e^{-x} and x^{-2} . As $x \rightarrow \infty$, e^x grows faster than x^2 , so it follows that its reciprocal decays faster, because in viewing e^{-x} as $1/e^x$ and x^{-2} as $1/x^2$, we see that the magnitude of the denominator of the first fraction is growing the fastest. Thus, the magnitude of the overall fraction is decaying the fastest. Another way to evaluate limits involving these reciprocal functions is simply to rewrite them in terms of the growing functions.

Example 7. Evaluate $\lim_{x \rightarrow \infty} \frac{e^{-x} + 1}{x^{-2}}$.

Solution In order to evaluate this limit we simply rewrite the constituent functions in terms of their reciprocals.

$$\frac{e^{-x} + 1}{x^{-2}} = \frac{e^{-x}}{x^{-2}} + \frac{1}{x^{-2}} = \frac{x^2}{e^x} + x^2$$

Now we simply evaluate the limit, using our knowledge of the relative rates of growth of the given functions, and see that

$$\lim_{x \rightarrow \infty} \left(\frac{x^2}{e^x} + x^2 \right) = \infty.$$

In some more complicated cases we can also evaluate infinite limits by using substitution.

Example 8. Find $\lim_{x \rightarrow \infty} \sin\left(\frac{1}{x}\right)$

Solution Here we will use the fact that as $x \rightarrow \infty$ we have $1/x \rightarrow 0$, and introduce the variable $t = 1/x$. Thus,

$$\lim_{x \rightarrow \infty} \sin\left(\frac{1}{x}\right) = \lim_{t \rightarrow 0^+} \sin(t) = 0$$

The reason we need to write this as a one-sided limit is because $x \rightarrow \infty$ from only one side.