DEPARTMENT OF MATHEMATICS

MATH 223–CALCULUS II

Exercise 1

1. **Verifying the Rolle's Theorem:** Find an interval on which Rolle's Theorem applies to $f(x) = x^3 - 7x^2 + 10x$. Then find all the points on that interval at which $f'(c) = 0$ for some $c \in \mathbb{R}$.

2. Sketch the graph of a function that illustrates why the continuity condition of the Mean Value Theorem is needed. Sketch the graph of a function that illustrates why the differentiability condition of the Mean Value Theorem is needed.

3. The lapse rate is the rate at which the temperature Tdecreases in the atmosphere with respect to increasing altitude z It is typically reported in units of $°C$/km and is defined by

$$\gamma = -\frac{dT}{dz}.$$

   When the lapse rate rises above $7°C$/km in a certain layer of the atmosphere, it indicates favorable conditions for thunderstorm and tornado formation, provided other atmospheric conditions are also present. Suppose the temperature at $z = 2.9$km is $T = 7.6\,°C$ and the temperature at $z = 5.26$km is $T = -14.3\,°C$. Assume also that the temperature function is continuous and differentiable at all altitudes of interest. What can a meteorologist conclude from these data?

4. Show that the real-valued function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 3x^2 + 3x + 1$ increases for all $x \in \mathbb{R}$.

5. Use the Mean Value Theorem (MVT) to establish the following inequalities.

   (a) $|\sin x - \sin y| \leq |x - y| \quad \forall x \in \mathbb{R}.$

   (b) $\frac{x}{1+x} < \ln(1+x) < x$ for $-1 < x < 0$ and for $x > 0$.

6. Find the interval of increase and decrease of the following functions:

i.	$f(x) = x^3 + 4x + 1$	ii.	$f(x) = x^3(5-x)^2$
iii.	$f(x) = x + \sin x$	vi.	$f(x) = \left(x^2 - 4\right)^2$

7. Prove that

$$\frac{\tan x}{x} > \frac{x}{\sin x}$$

   Whenever $0 < x < \frac{\pi}{2}$.

8. Find the values of k for which the function

$$f(x) = -4x^2 + (4k-1)x - k^2 + 4$$

   is negative for all values of x.

9. Show that, for real values of x,

$$f(x) = \frac{3\sin x}{2 + \cos x}$$

   cannot have a value greater than $\sqrt{3}$ or a value less than $-\sqrt{3}$.

10. If $\varepsilon \in \mathbb{R}$ and the function

$$f(x) = \varepsilon x - \frac{x^3}{1+x^2}$$

   is increasing $\forall x \in \mathbb{R}$, show also that $\varepsilon \geq \frac{9}{8}$.

11. Suppose $f'(x) = c$ for all x, where c is a constant. Show that, for some constant d, $f(x) = cx + d$.

12. Without evaluating derivatives, which of the following functions have the same derivative: $f(x) = \ln x, \quad g(x) = \ln 2x, \quad h(x) = \ln x^2, \quad p(x) = \ln 10x^2$.

13. Suppose $3 \leq f'(x) \leq 5$ for all x, show that $18 \leq f(8) - f(2) \leq 30$.

14. Prove that
$$\frac{2}{\pi} < \frac{\sin x}{x} < 1$$
whenever $0 < x < \frac{\pi}{2}$.

15. A function f is one-to-one. Does this mean it has an inverse? If yes show why. If No, what can you do to make it possible for the function to have an inverse (Justify your answer with a proof).

16. Determine intervals on which $f(x) = x^2 - 1$ has an inverse function.

17. Do all linear functions have inverses? If yes give a proof of that. If No, give a proof for the ones that have inverses and the ones that do not have inverses.

18. The following functions are defined on the entire real line. Determine which ones have inverses and which ones do not. In the case where the inverses exist, find them. If they do not exist, provide restrictions to their domain, range, or co-domain so that they would have inverses and find them.

    (a) $f(x) = x^2 - 2x$
    (b) $g(x) = \frac{1}{x}$
    (c) $f(x) = 10 - 3x$
    (d) $g(x) = \mid x \mid$
    (e) $h(x) = 1 + \cos x$.

19. For each of the functions below, find $(f^{-1})'(a)$ where possible.

    (a) $f(x) = x + 5$; $a = 5$
    (b) $f(x) = 2x^2 + 3$; $a = 11$
    (c) $f(x) = x^3$; $a = 27$
    (d) $f(x) = 16 - x^2, \quad 0 \leq x \leq 3$, $a = 15$
    (e) $x^3 + 3\sin x + 2\cos x$; $a = 2$
    (f) $2x^3 + 3x^2 + 7x + 4$; $a = 4$.

20. Verify the relationships $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ in the following functions

    (a) $f(x) = 4x + 7$
    (b) $f(x) = 4x^3$
    (c) $f(x) = \sqrt{x + 2}$
    (d) $f(x) = x^2 + 4$ for $x \geq 0$.