Q. 1

a) i) $$\lim_{x\to 0} (\sinh x)^x = \lim_{x\to 0} \exp\left[x\ln(x\sinh x)\right]$$

$$= \lim_{x\to 0} \exp\left[\frac{x}{\sinh x} \times \left[(\sinh x)\ln(\sinh x)\right]\right]$$

$$= \exp(1 \times 0) = e^0 = 1$$

since $\displaystyle\lim_{x\to 0} \frac{x}{\sinh x} = 1$

and $\displaystyle\lim_{x\to 0} (\sinh x)\ln(\sinh x) = 0$

ii) $$\lim_{x\to 0} \frac{\tanh(5x)}{\sinh(2x)} = \lim_{x\to 0} \frac{\dfrac{\tanh(5x)}{5x}}{\dfrac{\sinh(2x)}{2x}} \times \frac{5x}{2x}$$

$$= \frac{5}{2} \times \frac{\displaystyle\lim_{x\to 0}\left(\frac{\tanh 5x}{5x}\right)}{\displaystyle\lim_{x\to 0}\left(\frac{\sinh 2x}{2x}\right)}$$

$$= \frac{5}{2} \times \frac{1}{1} = \frac{5}{2}$$

N.B: One can also use the L'hopital's rule provided a justification is given.

b) Let $f(t) = (1+t)\ln(1+t)$ $t > 0$

Then f is continuous on $[0,\infty)$ and differentiable functions $(1+t)$ and $\ln(1+t)$. Hence, in particular, for any $x > 0$, f is continuous on $[0,x]$ and differentiable on $(0,x)$. Thus by the MVT, there exists $c \in (0,x)$ such that

$$(c) = \frac{f(x) - f(0)}{x - 0} = \frac{(1+x)\ln(1+x)}{x}$$

t $f'(c) = \ln(1+c) + (1+c) \times \frac{1}{(1+c)}$

$= \ln(1+c) + 1 > 1$ since $c > 0$

nce,

$$1 < f'(c) = \frac{(1+x)\ln(1+x)}{x}$$

ich is equivalent to

$$\frac{x}{1+x} < \ln(1+x)$$

call that $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$ for $|x| < 1$

ence, for $0 < x < 1$

$$\ln(1+x) > \frac{x}{1+x} = x(1 - x + x^2 - x^3 + \ldots)$$

e,

$$\ln(1+x) > x - x^2 + x^3 - x^4 + \ldots$$

Let $f(x) = 3\cosh x - 5\sinh x$, $x \geq 0$

$$f(\ln 2) = 3 \times \frac{e^{\ln 2} + e^{-\ln 2}}{2} - 5 \times \frac{e^{\ln 2} - e^{-\ln 2}}{2}$$

$$= \frac{3}{2}\left(2 + \frac{1}{2}\right) - \frac{5}{2}\left(2 - \frac{1}{2}\right)$$

$$= \frac{15}{4} - \frac{15}{4}$$

$$= 0$$

Thus, $\ln 2$ satisfies the equation $f(x) = 0$

ii) By i), $f^{-1}(0) = \ln 2$

Hence, $(f^{-1})'(0) = \dfrac{1}{f'(f^{-1}(0))} = \dfrac{1}{f'(\ln 2)}$

but $f'(x) = 3\sinh x - 5\cosh x$

and so, $f'(\ln 2) = \frac{3}{2}(e^{\ln 2} - e^{-\ln 2}) - \frac{5}{2}(e^{\ln 2} + e^{-\ln 2})$

$= \frac{3}{2}(2 - \frac{1}{2}) - \frac{5}{2}(2 + \frac{1}{2})$

$= \frac{9}{4} - \frac{25}{4}$

$= -\frac{16}{4}$

$= -4$

Hence, $(f^{-1})'(0) = -\frac{1}{4}$

Q.2

a) By definition,

$$\ln(e^x) = \int_1^{e^x} \frac{1}{t}\,dt$$

For $1 < t < e^x$, $e^{-x} < \frac{1}{t} < 1$

Hence, $$\int_1^{e^x} e^{-x}\,dt < \int_1^{e^x} \frac{1}{t}\,dt < \int_1^{e^x} 1\,dt$$

ie, $$e^{-x}\int_1^{e^x} dt < \ln(e^x) < \int_1^{e^x} dt$$

ie, $$e^{-x}(e^x - 1) < \ln(e^x) < e^x - 1$$

OR

$$1 - e^{-x} < x < e^x - 1$$

Taking $x = \ln 2$, we obtain

$$1 - e^{-\ln 2} < \ln 2 < e^{\ln 2} - 1$$

OR

$$1 - \frac{1}{2} < \ln 2 < 2 - 1$$

ie, $$\frac{1}{2} < \ln 2 < 1$$

b) Let $|x| < 1$ and put $y = \tanh^{-1} x$. Then

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

which is equivalent to

$$(x-1)e^{y} = -(x+1)e^{-y}$$

<u>OR</u> $$e^{2y} = \frac{x+1}{1-x}$$

ie, as $|x| < 1$,

$$e^{2y} = \exp\left[\ln\left(\frac{x+1}{1-x}\right)\right]$$

and so,

$$2y = \ln\left(\frac{x+1}{1-x}\right)$$

<u>OR</u> $$y = \frac{1}{2}\ln\left(\frac{x+1}{1-x}\right)$$

<u>ie</u> $$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{x+1}{1-x}\right)$$

$$\tanh^{-1}x = 1 \iff \frac{1}{2}\ln\left(\frac{x+1}{1-x}\right) = 1$$

$$\iff \ln\left(\frac{x+1}{1-x}\right) = 2$$

$$\iff \frac{x+1}{1-x} = e^{2}$$

$$\iff (e^{2}+1)x = e^{2} - 1$$

$$x = \frac{e^{2}-1}{e^{2}+1}$$