DEPARTMENT OF MATHEMATICS
(2014/2015)SEMESTER 1

MATH 223–CALCULUS II

Exercise 4

1. Suppose that n is a fixed positive integer. Show that $\lim_{x\to\infty}\left(\sqrt{x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n} - x\right) = \frac{a_1}{n}$.

2. Determine the constants a and b so that $\frac{1+a\cos 2x+b\cos 4x}{x^4}$ has a finite limit as $x \to 0$. Find the value of the limit.

3. If $y = \tanh^{-1}(1/x)$, express y as a natural logarithmic function of x. Sketch the graph of $\tanh^{-1}(1/x)$ showing its general characteristics.

4. Express each of the following functions as a natural logarithmic function and sketch the graph
a) $\operatorname{cosech}^{-1}x$ b) $\operatorname{sech}^{-1}x$ c) $\coth^{-1} x$

5. Let $f(x) = \sinh x - (x-1)\cosh x$.

   (a) Find the local maximum and minimum values of f.

   (b) Find the point of inflection and sketch the graph of f.

6. Prove that if $p = \frac{1}{2}\ln(2+\sqrt{5})$ and $q = \ln(1+\sqrt{2})$ then
$\tanh x < \sinh x < \operatorname{sech} x < \cosh x < \operatorname{cosech} x < \coth x$ if $0 < x < p$, and
$\tanh x < \operatorname{sech} x < \sinh x < \operatorname{cosech} x < \cosh x < \coth x$ if $p < x < q$.

7. If $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and a is any real constant, show that the equation $\sin x = \tanh a$ has just one solution, and prove that $\tan x = \sinh a$ and $\sec x = \cosh a$ for this value of x.

8. By expressing the hyperbolic functions in terms of the exponential functions, prove the following identities

   (a) $\cosh(x+y) = \cosh x\cosh y + \sinh x \sinh y$

   (b) $\sinh x + \sinh y = 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$

   (c) $\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$

9. If $y = \ln\tan x$, prove the following
(a) $\sin ny = \frac{1}{2}(\tan^n x - \cot^n x)$ (b) $2\cosh ny \operatorname{cosec} 2x = \cosh(n+1)y + \cosh(n-1)y$

10. Solve the following equations
(i) $10\cosh x = 2\sinh x = 11$ (ii) $3\tanh x = 4(1-\operatorname{sech} x)$
(iii) $4\tanh x = \coth x$

11. Determine the real values of x which satisfy the equation
$$\exp\left(\sin^{-1} x\right) = 1 + \exp\left(\cosh^{-1} x\right).$$

12. By making it a quadratic in e^x, show that the equation
$$a\cosh x + b\sinh x = 1$$
has no solution if $a^2 - b^2 > 1$, and that, *if* $a^2 - b^2 < 1$, it has two solutions, one solution or no solution, depending on whether $a+b$ and $a-b$ are both positive, of opposite signs, or both negative.

13. Prove that
(a) $\alpha = \ln\tan\beta \iff \tanh\alpha = -\cos 2\beta$ (b) $\alpha = \ln\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right) \iff \tanh\alpha = \sin\beta$.

14. Prove that in the range $0 < \theta < \frac{\pi}{2}$, the equation

$$\cosh^{-1}(\sec\theta) + \ln(\sin 2\theta) = 0$$

has just one solution, viz. $\theta = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$.

15. Find the real values of x and y which satisfy the equations

$$\sinh x + \sinh y = \frac{25}{12}, \qquad \cosh x - \cosh y = \frac{5}{12}.$$

16. If $f(x) = \cosh^{-1} x - \sinh^{-1} x$ for $x > 1$, prove that $f(x)$ increases with x and that for large values of x the value of $f'(x)$ is very nearly x^{-3}. Find the range of values of $f(x)$ a sx ranges from 1 to ∞.

17. Differentiate the following expressions with respect to x.

   (a) $x\sqrt{1+x^2} + \sinh^{-1} x$
   (b) $\ln\sinh\left(x + \cosh^2 x\right)$
   (c) $x\sqrt{x^2 - a^2} - a^2\cosh^{-1}\left(\frac{x}{a}\right)$
   (d) $\ln\sinh\left(x = \cosh^2 x\right)$

18. Express $\mathrm{cosech}^{-1} x$ in logarithmic form and hence solve the equation

$$\mathrm{cosech}^{-1} x + \ln x = 3.$$

19. Verify the inequalities

$$\mathrm{sech} x < \mathrm{cosech} x < \coth x, \qquad x > 0.$$