DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GHANA, LEGON

MATH 223: CALCULUS II

LECTURE NOTES

Cartious Enyonyoge Aziedu
(CEKA)

August 10, 2013

Course Outline

1 The Mean Value Theorem (MVT) and the Applications

- State, without proof, the MVT as a generalisation of Rolle's theorem.

1.1 Corollaries (Consequencies of the MVT)

- Functions with zero derivatives
- Functions with equal derivatives
- Increasing and decreasing functions

1.2 Applications of the MVT

- Use the MVT to establish the validity or otherwise of an inequality

Consult: [1], pages 190-200; [2], pages 163-176; [3], pages 178-189.

2 Inverse Functions

- Reflective property of inverse functions
- The existence of an inverse function
- The derivative of an inverse functions

Consult: [1], pages 344-360; [2], pages 239-337; [3], pages 59-62,113-115.

3 Logarithmic and Exponential Functions

- The natural logarithmic function defined as an integral:

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 1.$$

- The graphs of the natural logarithmic and exponential functions.
- Proofs of the basic properties of the natural logarithmic function using the defintion.
- Proofs of the basic properties of the exponential function using the logarithmic properties.
- The definition of the number e.

- Behaviour of the exponential and logarithmic functions, $\exp(x)$ at $\pm\infty$, and $\ln x$ at 0^+, $+\infty$.
- General exponential and logarithmic functions.
- Representation of the natural exponential function as a limit:

$$\ln y = \lim_{x \to 0} \frac{y^x - 1}{x}, \; y > 0.$$

- Representation of the natural exponential function as a limit:

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n, \quad e^x = \lim_{n \to 0} (1 + x)^{\frac{1}{n}}.$$

- Logarithmic inequalities and limits.
- Scales of infinity: Comparison of order of magnitude of two functions.
- Logarithmic differentiation.

Consult: [1], pages 360-396; [2], pages 311-320; [3], pages 360-381; [5], Chapter 2.

4 Indeterminate Forms and l'Hôpital's Rule

Consult: [1], pages 419-428; [2], pages 523-532; [3], pages 542-548.

5 Hyperbolic Functions

- Definition of odd even and odd functions.
- Definition of hyperbolic functions in terms of the exponential function.
- Sketch of graphs of hyperbolic functions using their definitions.
- Hyperbolic identities and Osborn's Rule.
- Inverses of hyperbolic functions and their domains.
- Inverse hyperbolic functions as natural logarithmic functions.
- Derivatives of hyperbolic functions and their inverses.

Consult: [1], pages 413-418; [2], pages 392-400; [5], pages Chapter 20.

6 Integration

6.1 Fundamental Theorems of Calculus (FTCs)

- Intuitive idea of the FTCs
- MVT for integrals
- Part I (with proof) and Part II (without proof) of the FTC.

6.2 Riemann Sums and the Definite Integral: Integration as a Sum

- The existence of the integral
- The definite integral and its properties
- Evaluation of basic integrals (with polynomials of at most degree three being the integrand) using the Riemann sum.
- Interpreting infinite limits as definite integrals

Consult: [1], pages 258-295; [2], pages 252-287; [3], pages 242-273.

6.3 Techniques of Integration

- ### Method of Substitution
  - **Trigonometric Integrals**
    - Integrals of sines and cosines
    - Integrals of tangents and secants
    - Integrals of cotangents and cosecants
  - **Hyperbolic Integrals**
    - Some standard hyperbolic integrals (as a result of antiderivatives of the hyperbolic functions)
    - Integrals of hyperbolic sines and cosines
    - Integrals of hyperbolic tangents and secants
    - Integrals of hyperbolic cotangents and cosecants
  - **Trigonometric and Hyperbolic Substitutions**
    - Substitions for integrands containing $\sqrt{a^2 - b^2x^2}$, $\sqrt{a^2 + b^2x^2}$ and $\sqrt{b^2x^2 - a^2}$.
  - **Integration of Rational Functions using Partial Fractions**
    - Integration of proper and improper rational functions
    - Special cases of rational function integrands where partial fractions can be avoided; e.g.

$$\int \frac{x^3+1}{(x-2)^4}dx, \text{ let } y = x-2; \quad \int \frac{1}{x^4(1-x)}dx, \text{ expand } (1-x1)^{-1}; \quad \int \frac{1}{x(x^\alpha+1)}dx, \text{ e}$$

  - **Integrals of the Form**

$$\int \frac{px+q}{ax^2+bx+c}dx, \quad \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx, \quad \int (px+q)\sqrt{ax^2+bx+c}dx,$$

    where $a, b, c, p, q \in \mathbb{R}$ and ax^2+bx+c cannot be factored into linear expressions.
  - **Integrals of the Form**

$$\int \frac{\sin^2 x + \cos x}{\cos^2 x + \sin x}dx, \quad \int \frac{a\sin x + b\cos x + c}{a_1 \sin x + b_1 \cos x + c_1}dx, \quad \int \frac{a\sinh x + b\cosh x + c}{a_1 \sinh x + b_1 \cosh x + c_1}dx$$

    where the substitutions $t = \tan\frac{x}{2}$ and $t = \tanh\frac{x}{2}$ are useful.

- **Integrals of the Form**

$$\int \frac{a\sin x + b\cos x}{a_1 \sin x + b_1 \cos x} dx$$

where the substitution $a\sin x+b\cos x = \lambda \frac{d}{dx}\left(a_1 \sin x + b_1 \cos x\right)+\mu\left(a_1 \sin x + b_1 \cos x\right)$ for some scalars λ and μ is useful.

Consult: [1], Chapter 7; [3], Chapter 9, [5], Chapters 1, 13, 20.

- **Integration by Parts**
- Method of Bubstitution and Integration by Parts for Definite Integrals.
- Proofs and Use of the the following Properties in Simplifying and Evaluating Definite Integrals

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx; \quad \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx.$$

Consult: [1], pages 437-422; [3], pages 463-465; [5], pages 282,283.

7 The Reduction Reduction

7.1 Reduction Formulae for Indefinite Integrals

- Reduction formulae for integrals of the form

$$\int x^n e^x dx, \quad n \in \mathbb{N}.$$

- Reduction formulae for integrals of the form

$$\int \frac{dx}{(x^2+a^2)^n}, \quad n \in \mathbb{N}.$$

7.2 Reduction Formulae for Trigonometric and Hyperbolic Integrals

- Reduction formulae for integrals of the forms

$$\int \cos^n x dx, \quad \int \sinh^n x dx, \quad n \in \mathbb{N}, n \geq 2, \text{ etc.}$$

- Reduction formulae for integrals of the forms

$$\int \sin^m x \cos^n x dx, \quad \int \sin^m x \cos nx dx, \quad m, n \in \mathbb{N}, \text{ etc.}$$

7.1 Reduction Formulae for Definite Integrals

- Reduction formulae for integrals of thee forms

$$\int_0^{\frac{\pi}{2}} x \sin^n x dx, \quad \int_0^{\frac{\pi}{2}} x^n \sqrt{1-x} dx, \text{ etc.}$$

Consult: [1], pages 440; [3], pages 462-469, 494; [5], pages 284-289.

8 Improper Integrals

Integrals of the form

$$\int_a^b f(x)\, dx$$

where:

- f is not defined at a or b or in between a and b;
- the interval of integration is infinite.

Consult: [1], pages 487-495; [2], pages 533-541; [3], pages 562-568.

9 Applications of Integration

- Arc length and area of surface of revolution

Consult: [1], pages 514-524; [2], pages 435-444; [3], pages 321-329.

10 Ordinary Differential Equations

- Application of first and first order differential equations: Mixtures, Logistic equations, Growth and Decay
- Formation of ordinary differential equations
- Second order linear differential equations with constant coefficients
- Algebra of linear operators
- Complementary functions and particular integrals; complete solutions
- simultaneous ordinary differental equations.

Consult: [1], Chapter 15; [3], Chapter 18; [4], Chapters 8-10; [5], Chapter 17.

Consult: [1], pages 344-360; [2], pages 239-337; [3], pages 59-62,113-115.

Bibliography

[1] James Stewart, *Calculus, Third Edition,* COLE Publishing Company, 1995.

[2] R. E. Larson, B. H. Edwards, R. P. Hostetler, *Calculus of a single Variable, Sixth Edition,* Houghton Mifflin Company, 1998.

[3] C. H. Edwards, Jr., D. E. Penney, *Calculus and Analytic Geometry, Second Edition,* Prentice-Hall, Inc., Englewood Cliffs, New Jersey 07632.

[4] G. S. Sharma, K. L. Ahuja, I. J. S. Sarna, *Advanced Mathematics for Engineers and Scientists, Second Edition,* Jain Bhola Nath Nagar Shahdra, Dehli - 110032.

[5] J. K. Backhouse, S. P. T. Houldsworth and B. E. D. Cooper, *Pure Mathematics, A Second Course, SI Edition,* Harrow-on-the-Hill, November 1962.

Chapter 1

The Mean Value Theorem

Recall the the following two important theorems from Calculus I (MATH 122):

Theorem 1.1. Maximum-Minimum (Extreme Value) Theorem

If $f(x)$ is continuous on the closed, finite interval $[a,b]$, then $\exists p,q \in [a,b]$ such that $\forall x \in [a,b]$,

$$f(p) \leq f(x) \leq f(q) \tag{1.1}$$

and

$$(b-a)\, f(p) \leq \int_a^b f(x)\, dx \leq (b-a)\, f(q)\,. \tag{1.2}$$

That is, f has the absolute minimum value $m = f(p)$ and absolute maximum value $M = f(q)$.

Theorem 1.2. Critical Points

If f is defined on the open interval (a,b) and f attains a maximum (or minimum) at the point $c \in (a,b)$, and if $f'(c)$ exists, then $f'(c) = 0$.

The value of x for which $f'(x) = 0$ are called the ***critical points*** *of f.*

Theorem 1.3. Rolle's Theorem

Let f be a function such that

1. *f is continuous on the closed interval $[a,b]$;*
2. *f is differentiable on the open interval (a,b);*
3. *and $f(a) = f(b) = 0$.*

Then there is a number c in (a,b) such that $f'(c) = 0$.

Proof. If $f(x) = f(a)\, \forall x \in [a,b]$, then f is a constant function and so $f'(x) = 0\; \forall x \in (a,b)$.

Now, suppose $\exists x \in (a,b)$ such that $f(x) \neq f(a)$. Suppose further that $f(x) > f(a)$. And since f is continuous on $[a,b]$, then by the maximum-minimum theorem, f must have a maximum value at some point $c \in [a,b]$ and so $f(c) \geq f(x) > f(a) = f(b)$. Similarly, if $f(x) < f(a)$, then f must have a minimum value at some point $c \in [a,b]$. Therefore, since $f(c) \leq f(x) < f(a) = f(b)$. Thus, c can neither be a nor b, i.e. $c \in (a,b)$ and so differentiable at c. By Theorem 1.2, c must be a critical point of f, i.e. $f'(c) = 0$. □

Theorem 1.4. The Mean Value Theorem

Let f be a function such that

1. *f is continuous on $[a,b]$;*
2. *f is differentiable on (a,b).*

Then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

or

$$f(b) - f(a) = (b-a)\, f'(c).$$

Proof.

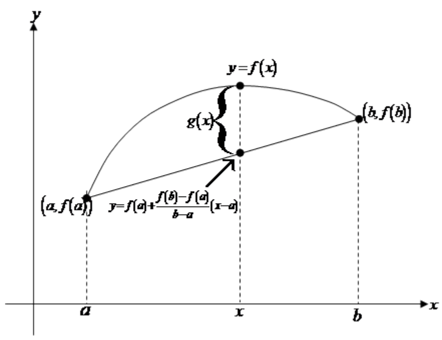


Figure 1.1: MVT

Let $y = f(x)$ be a continuous curve on $[a,b]$ and differentiable on (a,b). Then for any $x \in [a,b]$, the equation of the chord joining the points $(a, f(a))$ and $(b, f(b))$ is given by $y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$. Let us denote by $g(x)$ is the vertical distance between the curve $y = f(x)$ and this chord. Then

$$g(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right).$$

Suppose f satisfies the conditions of the MVT, then g is also continuous on $[a,b]$ and differentiable on (a,b). And since $g(a) = g(b) = 0$, we have by Rolle's theorem that $\exists c \in (a,b)$ such that $g'(c) = 0$. Hence, since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b-a},$$

we have that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

□

Corollary 1.5. The Generalised MVT

Suppose f and g are two continuous functions on $[a,b]$ and differentiable on (a,b). If $\forall x \in (a,b)$ $g'(x) \neq 0$, then $\exists c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Let

$$h(x) = (f(b) - f(a))(g(x) - g(a)) - (g(b) - g(a))(f(x) - f(a)).$$

Then $h(a) = h(b) = 0$ and by the MVT, there must exist some $c \in (a,b)$ such that $h'(c) = 0$; that is,

$$(f(b) - f(a))\, g'(c) - (g(b) - g(a))\, f'(c) = 0$$

or

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

□

Corollary 1.6. *If $f'(x) = 0\, \forall x \in (a,b)$, then f is constant on (a,b).*

Proof. Let x_1, x_2 be two points in (a,b) with $x_1 < x_2$. Then since f is differentiable on (a,b), so must it be on (x_1, x_2) and continuous on $[x_1, x_2]$. So by the MVT, $\exists c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1)\, f'(c). \tag{1.3}$$

Since $f'(x) = 0 \;\; \forall x \in (a,b)$, we have $f'(c) = 0$. Hence, (1.3) becomes

$$\begin{aligned} f(x_2) - f(x_1) &= 0 \\ \text{or} \quad f(x_1) &= f(x_2). \end{aligned}$$

Thus, f has the same value at any two different points on (a,b) and so is constant. □

Corollary 1.7. *If $f'(x) = g'(x)$ $\forall x \in (a,b)$, the $f - g$ is constant on (a,b); that is $f(x) = g(x) + k$, where k is a constant.*

Proof. Let $h(x) = f(x) - g(x)$. Then

$$h'(x) = f'(x) - g'(x) = 0 \;\; \forall x \in (a,b).$$

Hence, by Corollary 1.6, h is constant on (a,b); that is,

$$\begin{aligned} f(x) - g(x) &= k, \;\; \text{a constant} \\ \text{or} \quad f(x) &= g(x) + k. \end{aligned}$$

□

Some Other Useful Variations of the MVT

1. $f(b) = f(a) + (b-a) f'(c)$ for some $c \in (a, b)$.
2. $f(x) = f(a) + (x-a) f'(c)$ for some $c \in (a, x)$.
3. $f(b) = f(a) + (b-a) f'(a + \theta(b-a))$ for some θ such that $0 < \theta < 1$ and $c = a + \theta(b-a) \in (a, b)$.
4. $f(a+h) = f(a) + hf'(a + \theta h)$ for some θ such that $0 < \theta < 1$ and $h = b - a$.

Applications of the MVT

Definition 1.8. Increasing and Decreasing (Monotone) Functions

Suppose the function f is defined on an interval I and that $x_1, x_2 \in I$. Then f is said to be

1. *strictly increasing* on I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
2. *strictly decreasing* on I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.
3. *increasing* on I if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$.
4. *decreasing* on I if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$.

Note: Some literature use *increasing* (respectively *decreasing*) to rather mean *strictly increasing* (respectively *strictly decreasing*) and *nondecreasing* (respectively *nonincreasing*) to rather mean *increasing* (respectively *decreasing*).

Theorem 1.9. Monotone Functions

Let f be a continuous function on a closed interval $[a, b]$ and differentiable on (a, b). Then if

1. $f'(x) > 0\ \forall x \in (a, b)$ *then f is strictly increasing* on $[a, b]$.
2. $f'(x) < 0\ \forall x \in (a, b)$ *then f is strictly decreasing* on $[a, b]$.
3. $f'(x) \geq 0\ \forall x \in (a, b)$ *then f is increasing* on $[a, b]$.
4. $f'(x) \leq 0\ \forall x \in (a, b)$ *then f is decreasing* on $[a, b]$.

Proof. We prove (1) and (2); the others can be proved similarly. Let $x_1, x_2 \in [a, b]$ with $a \leq x_1 \leq x_2 \leq b$. Then by the MVT, $\exists c \in (x_1, x_2)$ such that (1.3) holds; that is

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(c).$$

1. Suppose $f'(c) > 0$. Since $x_1 < x_2$, then from (1.3), we have $f(x_1) < f(x_2)$. Hence,

$$x_1 < x_2 \implies f(x_1) < f(x_2).$$

So f is strictly increasing on $[a,b]$.

2. If $f'(c) < 0$, then since $x_1 < x_2$, we have $f(x_1) > f(x_2)$, by (1.3). Therefore, we have

$$x_1 < x_2 \implies f(x_1) > f(x_2).$$

So f is strictly decreasing on $[a,b]$. $\square$

Example 1.10. Show that $\sin x < x \ \forall x > 0$.

Proof. We prove the case for $x > 2\pi$ and then for $0 < x \leq 2\pi$. Suppose $x > 2\pi$, then since $\sin x \leq 1 < 2\pi < x$, we have that $\sin x < x \ \forall x \in (2\pi, \infty)$.

Also, if $0 < x \leq 2\pi$, then by the MVT, $\exists c \in (0, 2\pi)$ such that for $f(x) = \sin x$,

$$\frac{f(x) - f(0)}{x - 0} = f'(c).$$

That is,

$$\begin{aligned} \frac{\sin x - \sin 0}{x - 0} &= \frac{d}{dx}(\sin x)\,|_{x=c} \\ \implies \frac{\sin x}{x} &= \cos c \leq 1 \\ &< 1 \\ \text{i.e. } \sin x &< x \quad x \in (0, 2\pi]. \end{aligned}$$

Hence, $\sin x < x \ \forall x \in (0, \infty)$.

<u>Alternative Proof</u>. Let $f(x) = \sin x - x$. Then

$$\begin{aligned} f'(x) &= \cos x - 1 \\ &< 0 \ \forall x \in (0, \infty). \end{aligned}$$

That is f is strictly decreasing on $(0, \infty)$. Therefore,

$$\begin{aligned} 0 &< x \Rightarrow f(0) > f(x) \\ &\Rightarrow 0 > \sin x - x \\ &\text{i.e. } \sin x < x \ \text{ for } x \in (0, \infty). \end{aligned}$$

$\square$

Example 1.11. Show that $\sqrt{1+x} < 1 + \frac{x}{2}$ for $x > 0$ and for $-1 \leq x < 0$.

Proof. Let $f(x) = \sqrt{1+x} - \frac{x}{2} - 1$. Then $f'(x) = \frac{1}{2}\left((1+x)^{-\frac{1}{2}} - 1\right)$.
For $x > 0$, $\sqrt{(1+x)} > 1$ and so $(1+x)^{-\frac{1}{2}} < 1$. In effect, we have that $f'(x) = \frac{1}{2}\left((1+x)^{-\frac{1}{2}} - 1\right) < 0$ and so f strictly decreases on $(0, \infty)$. Hence,

$$\begin{aligned} x > 0 &\Rightarrow f(x) < f(0) \\ &\Rightarrow \sqrt{1+x} - \frac{x}{2} - 1 < \sqrt{1+0} - 1 = 0 \\ &\Rightarrow \sqrt{1+x} < 1 + \frac{x}{2} \ \forall x \in (0, \infty). \end{aligned}$$

Also, if $-1 \leq x < 0$, the $0 \leq \sqrt{(1+x)} < 1$ and so $(1+x)^{-\frac{1}{2}} > 1$ so that $f'(x) = \frac{1}{2}\left((1+x)^{-\frac{1}{2}} - 1\right) > 0$, and so f strictly increases on $[-1, 0)$. Hence,

$$\begin{aligned} -1 \leq x < 0 \quad &\Rightarrow \quad f(x) < f(0) = 0 \\ &\Rightarrow \quad \sqrt{1+x} < 1 + \frac{x}{2} \ \forall x \in [-1, 0). \end{aligned}$$

□

Example 1.12. Let $r > 1$. If $x > 0$ or $-1 \leq x < 0$, show that $(1+x)^r > 1 + rx$. Also, show that if $0 < r < 1$ and $x > 0$ or $-1 \leq x < 0$, show that $(1+x)^r < 1 + rx$.

Proof. Let $f(x) = (1+x)^r - rx - 1$, where $r > 1$. Then $f'(x) = r(1+x)^{r-1} - r = r\left((1+x)^{r-1} - 1\right)$.

If $-1 < x < 0$, then $(1+x)^{r-1} < 1 \ \forall r > 1$, so we have $f'(x) < 0$. Thus f is strictly decreasing for $-1 < x < 0$ and so

$$\begin{aligned} 0 < x < -1 \quad &\Rightarrow \quad f(x) > f(0) \\ &\Rightarrow \quad (1+x)^r - rx - 1 > (1+0)^r - 1 = 0 \\ &\Rightarrow \quad (1+x)^r > 1 + rx \text{ for } 0 < x < -1. \end{aligned}$$

Also, if $x > 0$, then $(1+x)^{r-1} > 1 \ \forall r > 1$ and so $f'(x) > 0$. Hence, f is strictly increasing for $x > 0$, thus

$$\begin{aligned} x > 0 \quad &\Rightarrow \quad f(x) > f(0) \\ &\Rightarrow \quad (1+x)^r - rx - 1 > (1+0)^r - 1 = 0 \\ &\Rightarrow \quad (1+x)^r > 1 + rx \text{ for } x>0. \end{aligned}$$

Hence, for $r > 1$, if $x > 0$ or $-1 \leq x < 0$, then we have $(1+x)^r > 1 + rx$.

Similarly, for $0 < r < 1$, if $-1 < x < 0$, then $(1+x)^{r-1} > 1$, so we have $f'(x) > 0$. Thus f is strictly increasing for $-1 < x < 0$ and so

$$\begin{aligned} 0 < x < -1 \quad &\Rightarrow \quad f(x) < f(0) = 0 \\ &\Rightarrow \quad (1+x)^r < 1 + rx \text{ for } 0 < x < -1. \end{aligned}$$

And, if $x > 0$, we have $(1+x)^{r-1} < 1$ for $0 < r < 1$. So $f'(x) < 0$, that is, f is strictly decreasing. So we have

$$\begin{aligned} x > 0 \quad &\Rightarrow \quad f(x) < f(0) = 0 \\ &\Rightarrow \quad (1+x)^r < 1 + rx \text{ for } x > 0. \end{aligned}$$

Hence, for $0 < r < 1$, if $x > 0$ or $-1 \leq x < 0$, then we have $(1+x)^r < 1 + rx$. □

How to Determine the Interval of Increase and Decrease of a Function

Recall that for a function f to be increasing, $f'(x) > 0$ and for it to be increasing, $f'(x) < 0$. Hence, we solve for x for which $f'(x) > 0$ to get the interval of increase and $f'(x) < 0$ to get the interval of decrease of f.

Example 1.13. Find the interval of increase and decrease of the following functions:

i. $f(x) = x^2 + 2x + 2$ iii.. $f(x) = \dfrac{1}{x^2+1}$

ii . $f(x) = x - 2\sin x$ vi. $f(x) = x^3 - 4x + 1$.

Proof.
i. $f(x) = x^2 + 2x + 2 \Longrightarrow f'(x) = 2x + 2 = 2(x+1)$.

Therefore,

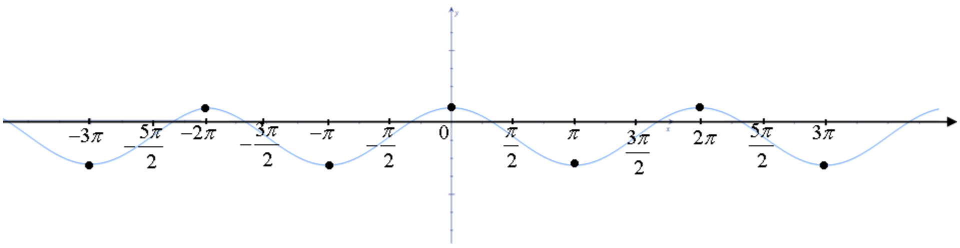
$$\begin{aligned} f'(x) > 0 &\Rightarrow 2(x+1) > 0 \\ &\Rightarrow x > -1, \end{aligned}$$

and

$$\begin{aligned} f'(x) < 0 &\Rightarrow 2(x+1) < 0 \\ &\Rightarrow x < -1. \end{aligned}$$

Hence, f increases on $(-1, \infty)$ and decreases on $(-\infty, -1)$.

ii. $f(x) = x - 2\sin x \Longrightarrow f'(x) = 1 - 2\cos x = 0$ at $x = 2n\pi \pm \frac{\pi}{3}$, $n \in \mathbb{Z}$.



Therefore,

$$\begin{aligned} f'(x) > 0 &\Rightarrow 1 - 2\cos x > 0 \\ &\Rightarrow \cos x - \frac{1}{2} < 0 \\ &\Leftrightarrow x \in \left(\frac{\pi}{3} + 2n\pi, -\frac{\pi}{3} + (2n+1)\pi\right), \end{aligned}$$

and

$$\begin{aligned} f'(x) < 0 &\Rightarrow 1 - 2\cos x < 0 \\ &\Rightarrow \cos x - \frac{1}{2} > 0 \\ &\Leftrightarrow x \in \left(-\frac{\pi}{3} + 2n\pi, \pi + 2n\pi\right). \end{aligned}$$

Hence, f is increasing on $\left(\frac{\pi}{3}+2n\pi, -\frac{\pi}{3}+(2n+1)\pi\right)$ and decreasing on $\left(-\frac{\pi}{3}+2n\pi, \pi+2n\pi\right)$.

iii. $\quad f(x)=\dfrac{1}{x^2+1} \Longrightarrow f'(x)=\dfrac{-2x}{(x^2+1)^2}$.

Now, for f to be increasing, we have

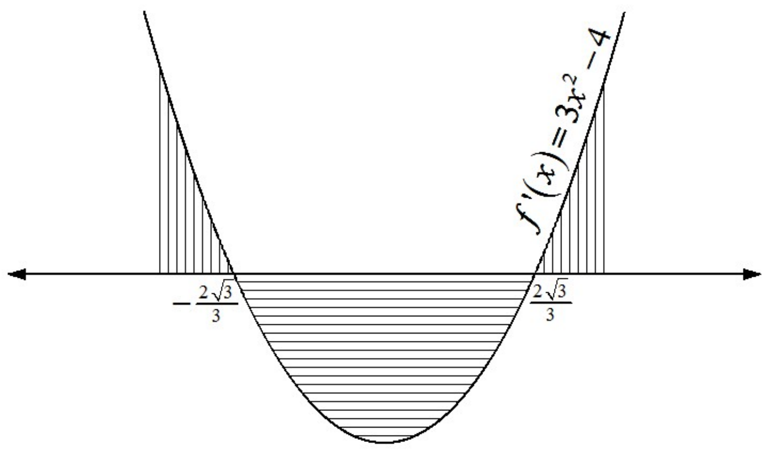
$$\begin{aligned} f'(x)>0 &\Rightarrow \frac{-2x}{(x^2+1)^2}>0 \\ &\Leftrightarrow \frac{2x}{(x^2+1)^2}<0 \\ &\Leftrightarrow 2x<0 \quad \because \left(x^2+1\right)^2>0\ \forall x \\ &\Leftrightarrow x<0, \end{aligned}$$

and for f to be decreasing, we have

$$\begin{aligned} f'(x)<0 &\Rightarrow \frac{-2x}{(x^2+1)^2}<0 \\ &\Leftrightarrow 2x>0 \quad \because \left(x^2+1\right)^2>0\ \forall x \\ &\Leftrightarrow x>0, \end{aligned}$$

Hence, f increases on $(-\infty,0)$ and decreases on $(0,\infty)$.

iv. $\quad f(x)=x^3-4x+1 \Longrightarrow f'(x)=3x^2-4=0 \Leftrightarrow x=\pm\frac{2\sqrt{3}}{3}$.



From the diagram, we have that,

$$f'(x)>0 \quad \Leftrightarrow \quad x>\frac{2\sqrt{3}}{3} \text{ and } x<-\frac{2\sqrt{3}}{3},$$

(shaded vertically), and

$$f'(x) < 0 \Leftrightarrow -\frac{2\sqrt{3}}{3} < x < \frac{2\sqrt{3}}{3}$$

(shaded horizontally). Hence, f increases on $\left(-\infty, \frac{2\sqrt{3}}{3}\right) \cup \left(\frac{2\sqrt{3}}{3}, \infty\right)$ and decreases on $\left(\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$. □

Example 1.14. Prove that $\frac{1}{2}\sin x \tan x - \ln \sec x$ is positive and increasing for $0 < x < \frac{\pi}{2}$.

Proof. Let $f(x) = \frac{1}{2}\sin x \tan x - \ln \sec x$. Then

$$\begin{aligned} f'(x) &= \tfrac{1}{2}\left(\cos x \tan x + \sin x \sec^2 x\right) - \frac{\sec x \tan x}{\sec x} \\ &= \tfrac{1}{2}\left(\cos x \frac{\sin x}{\cos x} + \sin x \sec^2 x\right) - \frac{\sin x}{\cos x} \\ &= \tfrac{1}{2}\sin x \left(\sec^2 x + 1\right) - \sin x \sec x \\ &= \tfrac{1}{2}\sin x \left(\sec^2 x - 2\sec x + 1\right) \\ &= \tfrac{1}{2}\sin x \left(\sec x - 1\right)^2. \end{aligned}$$

Now, for $0 < x < \frac{\pi}{2}$, $\sin x > 0$; and since $(\sec x - 1)^2 > 0\ \forall x$, we conclude that $f'(x) > 0\ \forall x \in \left(0, \frac{\pi}{2}\right)$. Thus, f increases for $0 < x < \frac{\pi}{2}$.

Hence,

$$\begin{aligned} x > 0 &\Rightarrow f(x) > f(0) = 0 \\ &\Rightarrow \tfrac{1}{2}\sin x \tan x - \ln \sec x > \tfrac{1}{2}\sin(0)\tan(0) - \ln(\sec(0)) \\ &\Rightarrow \tfrac{1}{2}\sin x \tan x - \ln \sec x > \tfrac{1}{2} \times 0 \times 0 - \ln(1) \\ &\Rightarrow \tfrac{1}{2}\sin x \tan x - \ln \sec x > 0. \end{aligned}$$

Hence, $\frac{1}{2}\sin x \tan x - \ln \sec x$ is positive and increasing on $\left(0, \frac{\pi}{2}\right)$. □

Example 1.15. By applying the MVT to the function $f(x) = \cos x + \dfrac{x^2}{2}$, show that $\cos x > 1 - \dfrac{x^2}{2}$ for $x > 0$.

Proof. The function $f(x) = \cos x + \dfrac{x^2}{2}$ for $x > 0$. is continuous on $[0, \infty)$and differentiable on $(0, \infty)$. Hence, by the MVT, for $x > 0$, $\exists c \in (0, \infty)$ such that

$$f(x) - f(0) = (x - 0) f'(c) = x f'(c).$$

Now, $f'(x) = -\sin x + x > 0\ \forall x > 0$ (see Example 1.10) and so $x f'(c) > 0$. Hence,

$$\begin{aligned} f(x) &> f(0). \\ \Rightarrow \cos x + \frac{x^2}{2} &> \cos(0) + 0 = 1 \\ \therefore\ \cos x &> 1 - \frac{x^2}{2}. \end{aligned}$$

□

Example 1.16. Show that $\tan x > x$ for $0 < x < \frac{\pi}{2}$.

Proof. Let $f(x) = \tan x - x$ for $0 < x < \frac{\pi}{2}$. Then

$$\begin{aligned} f'(x) &= \sec^2 x - 1 \\ &= \tan^2 x > 0 \ \forall x \in \left(0, \tfrac{\pi}{2}\right). \end{aligned}$$

Therefore, f is strictly increasing on $\left[0, \frac{\pi}{2}\right)$. No that we exclude $\frac{\pi}{2}$ from the domain because f is clearly not defined there. Hence,

$$\begin{aligned} x > 0 &\Rightarrow f(x) > f(0) \\ &\Rightarrow \tan x - x > 0, \ \because f(0) = \tan(0) - 0 = 0 \\ &\Rightarrow \tan x > x \text{ for } 0 < x < \tfrac{\pi}{2}. \end{aligned}$$

□

Problem 1

1. Show that the real-valued function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 3x^2 + 3x + 1$ increases for all $x \in \mathbb{R}$.

2. Use the MVT to establish the following inequalities.

   (a) $|\sin x - \sin y| \leq |x - y| \quad \forall x \in \mathbb{R}$.

   (b) $\dfrac{x}{1+x} < \ln(1+x) < x$ for $-1 < x < 0$ and for $x > 0$.

3. Find the interval of increase and decrease of the following functions:

i.	$f(x) = x^3 + 4x + 1$	ii..	$f(x) = x^3 (5 - x)^2$
iii.	$f(x) = x + \sin x$	vi.	$f(x) = (x^2 - 4)^2$

4. Prove that $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ whenever $0 < x < \dfrac{\pi}{2}$.

5. Find the values of k for which the function

   $$f(x) = -4x^2 + (4k - 1)x - k^2 + 4$$

   is negative for all values of x (Question 8a of 2012/13 MATH 121: Algebra and Trigonometry).

6. Show that, for real values of x, $f(x) = \dfrac{3 \sin x}{2 + \cos x}$ cannot have a value greater than $\sqrt{3}$ or a value less than $-\sqrt{3}$.

7. If $\varepsilon \in \mathbb{R}$ and the function $f(x) = \varepsilon x - \dfrac{x^3}{1+x^2}$ is increasing $\forall x \in \mathbb{R}$. Show also that $\varepsilon \geq \frac{9}{8}$.

8. Suppose $f'(x) = c$ for all x, where c is a constant. Show that, for some constant d, $f(x) = cx + d$.

9. Suppose $3 \leq f'(x) \leq 5$ for all x, show that $18 \leq f(8) - f(2) \leq 30$.
10. Prove that $\dfrac{2}{\pi} < \dfrac{\sin x}{x} < 1$ whenever $0 < x < \frac{\pi}{2}$.
11. m
12. m
13. m
14. m
15. m

Chapter 2

Inverse Functions

Definition 2.1. One-to-one (Injective) Functions
A function f is said to be one-to-one if $f(x_1) \neq f(x_2)$ whenever $x_1, x_2 \in \text{dom}f$ and $x_1 \neq x_2$, or equivalently,

$$f(x_1) = f(x_2) \Longrightarrow x_1 = x_2 \ \forall x_1, x_2 \in \text{dom}f.$$

Note that a function is injective on an interval if it is monotone on that interval.

Definition 2.2. Inverse of a Function
A function $g : D_g \to R_g$ is said to be the inverse of the function $f : D_f \to R_f$ if and only if

$$D_g = R_f \ \text{ and } \ D_f = R_g,$$

and

$$f(g(x)) = x \quad \forall x \in D_g \text{ and } \ g(f(x)) = x \ \ \forall x \in D_f,$$

where D_f is the domain of f and R_f is the range of f, etc. We write $g = f^{-1}$ and $f = g^{-1}$. Thus, the definition of inverse says that,

$$f^{-1}(x) = y \Leftrightarrow f(y) = x.$$

Note: The -1 in f^{-1} is not an exponent, and so $f^{-1}(x) \neq \frac{1}{f(x)}$, the reciprocal of $f(x)$. To mean a reciprocal, we would rather write $[f(x)]^{-1}$.

Example 2.3. Show that the functions

$$h(x) = \frac{x^3}{4} \text{ and } k(x) = (4x)^{\frac{1}{3}}$$

are inverses of one another.

Proof. We need to show that $h(k(x)) = x = k(h(x))$. Now,

$$\begin{aligned} h(k(x)) &= h\left((4x)^{\frac{1}{3}}\right) \\ &= \frac{\left((4x)^{\frac{1}{3}}\right)^3}{4} \\ &= \frac{(4x)^{\frac{1}{3}\times 3}}{4} \\ &= \frac{4x}{4} = x. \end{aligned}$$

Similarly,

$$\begin{aligned} k\left(h\left(x\right)\right) &= k\left(\frac{x^3}{4}\right) \\ &= \left(4\frac{x^3}{4}\right)^{\frac{1}{3}} = \left(x^3\right)^{\frac{1}{3}} = x^{3\times\frac{1}{3}} \\ &= x. \end{aligned}$$

Hence, k and h are inverses of one another. □

Definition 2.4. The Identity Function

Let S be a non-empty set. Then the function $I_S : S \to S$, defined by

$$I_S\left(x\right) = x \quad \forall x \in S,$$

is called the identity function on S.

That is, the identity function is any function that maps each element in a set onto itself.

Hence, since

$$f^{-1}\left(f\left(x\right)\right) = \left(f^{-1} \circ f\right)\left(x\right) = x \quad \forall x \in D_f$$

and

$$f\left(f^{-1}\left(x\right)\right) = \left(f \circ f^{-1}\right)\left(x\right) = x \quad \forall x \in D_{f^{-1}},$$

we conclude that the composed functions $f^{-1} \circ f$ and $f \circ f^{-1}$ are identity maps on D_f and $D_{f^{-1}}$ respectively.

Theorem 2.5. Existence of an Inverse of a Function

Suppose f is a strictly monotone function onto function on an interval $[a, b]$. Then f^{-1} exists and is strictly monotone on $[a, b]$.

Proof. Suppose f is strictly increasing (resp. decreasing) on $[a, b]$ and that $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. Then $f\left(x_1\right) < f\left(x_2\right)$ (resp. $f\left(x_1\right) > f\left(x_2\right)$). And since $x_1 \neq x_2$, we also have that $f\left(x_1\right) \neq f\left(x_2\right)$. Thus, f is one-to-one and onto, and so f^{-1} exists.

To prove the monotonicity, let y_1, y_2 be any two elements in the range of f with $y_1 < y_2$. Then since f is onto, $\exists x_1, x_2 \in [a, b]$ such that $y_1 = f\left(x_1\right)$ and $y_2 = f\left(x_2\right)$ with $f\left(x_1\right) < f\left(x_2\right)$ (resp. $f\left(x_1\right) > f\left(x_2\right)$). By the monotonicity of f, we have

$$x_1 < x_2 \Longrightarrow f\left(x_1\right) < f\left(x_2\right) \qquad (\text{resp. } f\left(x_1\right) > f\left(x_2\right)). \tag{2.1}$$

Since f^{-1} exists, we have that $x_1 = f^{-1}\left(y_1\right)$ and $x_2 = f^{-1}\left(y_2\right)$, and so, (2.1) becomes

$$f^{-1}\left(y_1\right) < f^{-1}\left(y_2\right) \Longrightarrow y_1 < y_2$$

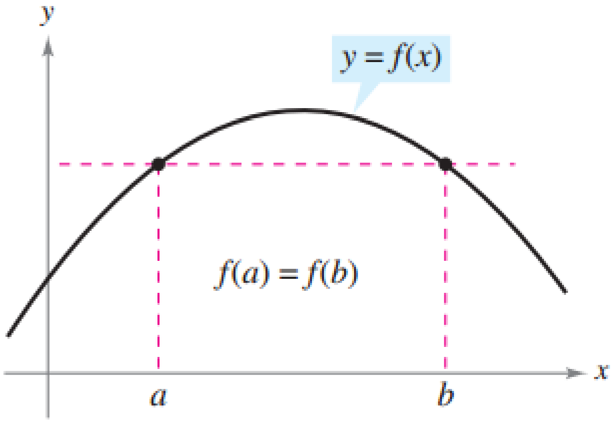
or

$$y_1 > y_2 \Longrightarrow f^{-1}\left(y_1\right) > f^{-1}\left(y_2\right) \qquad (\text{resp. } f^{-1}\left(y_1\right) < f^{-1}\left(y_2\right))$$

Hence, thus f^{-1} is strictly monotone. □

The Horizontal Line Test

Also, a function $y = f(x)$ is said to be one-to-one if and only if its graph intersets each horizontal line *at most once*.



If a horizontal line intersects the graph of f twice, then f is not one-to-one.

2.1 Finding the Inverse of a Function

- Determine whether the inverse exists, that is, if the function is injective.
- Let $y = f(x)$ and solve for x in terms of y.
- Interchange x and y, and let $y = f^{-1}(x)$.

Fact. *All monotone ((strictly) increasing and decreasing) functions are one-to-one and the inverse of a function exists if and only if it is one-to-one, therefore, all monotone functions have inverses.*

Example 2.6. Show that the function $f(x) = 2x - 1$ has an inverse and find the inverse.

Solution. Since $f'(x) = 2 > 0 \ \forall x \in \mathbb{R}$, f is increasing on $\mathbb{R}$ and so is one-to-one, and hence has an inverse on $\mathbb{R}$. Now, let $y = f(x)$. Then $x = \frac{1}{2}(y+1)$. Interchanging x and y, we have $y = \frac{1}{2}(x+1)$. Thus, $f^{-1}(x) = \frac{1}{2}(x+1)$.

Example 2.7. m

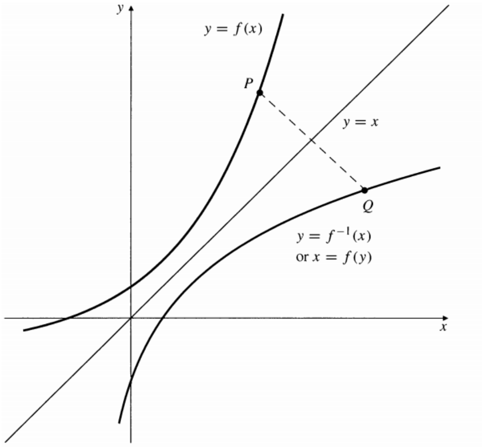
Solution.

Example 2.8. m

Solution.

2.2 Reflective Property of Inverse Functions

If the functions f and g are inverses of each other (i.e. $g = f^{-1}$) then the graph of $y = f(x)$ is a reflection of the graph of $y = g(x)$ in the line $y = x$, that is, if (a, b) is a point on the graph of $y = f(x)$, then (b, a) is a point on the graph of $y = f^{-1}(x)$.



2.3 Derivative of Inverse Functions

Theorem 2.9. Differentiating an Inverse Function

Let f be a differentiable function at every point of an interval I and that $f'(x) \neq 0$ at any point of I. Then f^{-1} is differentiable on $f(I)$ and for any $a \in I$,

$$\left.\frac{df^{-1}}{dx}\right|_{x=a} = \frac{1}{\left.\frac{df}{dy}\right|_{y=f^{-1}(a)}}$$

or

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}.$$

In short,

$$\left(f^{-1}\right)' = \frac{1}{f'\left(f^{-1}\right)}.$$

Proof. If $y = f^{-1}(x)$, then $f(y) = x$; and if $b = f^{-1}(a)$, then $f(b) = a$. Hence, by definition,

$$\left.\frac{df^{-1}}{dx}\right|_{x=a} = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a}.$$

Now, since f is differentiable on I, it is also continuous, so f^{-1} is continuous. Therefore, as $x \to a$ so does $f^{-1}(x) = y \to f^{-1}(a) = b$. Hence,

$$\begin{aligned}
\left.\frac{df^{-1}}{dx}\right|_{x=a} &= \lim_{y\to b} \frac{y-b}{f(y)-f(b)} \\
&= \frac{1}{\lim_{y\to b} \frac{f(y)-f(b)}{y-b}} \\
&= \frac{1}{\left.\frac{df}{dy}\right|_{y=b}} \\
&= \frac{1}{\left.\frac{df}{dy}\right|_{y=f^{-1}(a)}}.
\end{aligned}$$

Alternative Proof. Let f^{-1} be the inverse of f on I. Then $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ $\forall x \in I$. So, by the chain rule, we have

$$\begin{aligned}
f'\left(f^{-1}(x)\right) \cdot \frac{d}{dx}\left(f^{-1}(x)\right) &= 1 \\
\therefore \frac{d}{dx}\left(f^{-1}(x)\right) &= \frac{1}{f'(f^{-1}(x))}.
\end{aligned}$$

□

Example 2.10. Find $(f^{-1})'(1)$ if $f(x) = x^3 - x^2 + 1, \;\; x > \frac{2}{3}$.

Solution. First of all, we have to establish the existence or not of f. Now,

$$\begin{aligned}
f(x) = x^3 - x^2 + 1 \Rightarrow f(x) &= 3x^2 - 2x \\
&= 3x\left(x - \frac{2}{3}\right) > 0 \quad \forall x > \frac{2}{3}.
\end{aligned}$$

That is, f is increasing on $\left(\frac{2}{3}, \infty\right)$ and hence is $1-1$. Therefore, f^{-1} exists.

Now, by definition,

$$\left.\left(f^{-1}(x)\right)'\right|_{x=a} = \frac{1}{f'(f^{-1}(a))}$$

and so,

$$\left.\left(f^{-1}(x)\right)'\right|_{x=1} = \frac{1}{f'(f^{-1}(1))}.$$

Let $f^{-1}(1) = x \Rightarrow f(x) = 1$. To get the value of $f^{-1}(1)$, we solve for x in the equation $f(x) = 1$, that is,

$$\begin{aligned}
x^3 - x^2 + 1 &= 1 \\
\Rightarrow x^2(x-1) &= 0 \\
\Rightarrow x = 0 \quad &or \quad x = 1.
\end{aligned}$$

But $0 \notin \left(\frac{2}{3}, \infty\right)$, so we choose $x = 1$ so that $f^{-1}(1) = 1$. Therefore,

$$\begin{aligned}\left(f^{-1}\right)'(1) &= \frac{1}{f'\left(f^{-1}(1)\right)} \\ &= \frac{1}{f'(x)|_{x=1}} \\ &= \frac{1}{3x^2 - 2x|_{x=1}} \\ &= \frac{1}{3-2} = 1.\end{aligned}$$

Example 2.11. Show that $f(x) = x\sqrt{3+x^2}$ is invertible when $x > 0$ and hence find $\left(f^{-1}\right)'(2)$.

Solution.

$$\begin{aligned}f'(x) &= \sqrt{3+x^2} + x^2\left(3+x^2\right)^{-\frac{1}{2}} = \sqrt{3+x^2} + \frac{x^2}{\sqrt{3+x^2}} \\ &= \sqrt{3+x^2}\left(1 + \frac{x^2}{3+x^2}\right) \\ &= \frac{\sqrt{3+x^2}}{3+x^2}\left(2x^2+3\right).\end{aligned}$$

Since $x^2 > 0$ for all $x \in \mathbb{R}$, we have $\sqrt{3+x^2} > 0$ and $2x^2 + 3 > 0$ for all $x \in \mathbb{R}$, and so $f'(x) > 0 \ \ \forall x > 0$. Hence, f strictly increases for all real numbers and so is $1-1$, and is thus invertible. Hence,

$$\left(f^{-1}\right)'(2) = \frac{1}{f'\left(f^{-1}(2)\right)}.$$

Now, let $f^{-1}(2) = x$ so that $f(x) = 2$. That is,

$$x\sqrt{3+x^2} = 2.$$

Squaring both sides, we have

$$\begin{aligned}x^2\left(x^2+3\right) &= 4 \\ \Rightarrow x^4 + 3x^2 - 4 &= 0 \\ \Rightarrow (x-1)\left(x^3 + x^2 + 4x + 4\right) &= 0 \\ \Rightarrow (x-1)(x+1)\left(x^2+4\right) &= 0,\end{aligned}$$

so that $x = 1$ for $x > 0$, i.e. $f^{-1}(2) = 1$. Therefore,

$$\begin{aligned}\left(f^{-1}\right)'(2) &= \frac{1}{f'\left(f^{-1}(2)\right)} \\ &= \frac{1}{f'(x)|_{x=1}} \\ &= \frac{1}{\sqrt{3+x^2} + \frac{x^2}{\sqrt{3+x^2}}\Big|_{x=1}} \\ &= \frac{1}{\sqrt{3+1} + \frac{1}{\sqrt{3+1}}} \\ &= \frac{1}{2+\frac{1}{4}} = \frac{4}{9}.\end{aligned}$$

Example 2.12. Show that $f(x) = \int_2^x \sqrt{1+t^2}dt$ is one-to-one for $x > 0$ and find $\left(f^{-1}\right)'(2)$.

Solution. If $f(x) = \int_2^x \sqrt{1+t^2}dt$, then the FTC I (as we shall later see) states that

$$\begin{aligned} f'(x) &= \sqrt{1+x^2} \\ &> 0 \ \forall x > 0. \end{aligned}$$

Therefore, f is monotone on $(0,\infty)$ line and so one-to-one, thus, f has an inverse. Hence

$$\left(f^{-1}\right)'(2) = \frac{1}{f'\left(f^{-1}(2)\right)}.$$

Note that if $f^{-1}(0) = x$, then $f(x) = 0$, or

$$\int_2^x \sqrt{1+t^2}dt = 0.$$

Differentiating (just doing the integration by writing the RHS also as an integral) both sides,

$$\int_2^x \sqrt{1+t^2}dt = \int_2^2 dt$$

we have

$$\begin{aligned} \sqrt{1+x^2} &= 2 \\ \Rightarrow 1+x^2 &= 4 \\ \Rightarrow x &= \pm\sqrt{3}. \end{aligned}$$

Therefore, for $x > 0$, we have $x = \sqrt{3}$, i.e. $f^{-1}(2) = \sqrt{3}$.

$$\begin{aligned} \therefore \ \left(f^{-1}\right)'(2) &= \frac{1}{f'\left(\sqrt{3}\right)} \\ \left(f^{-1}\right)'(2) &= \frac{1}{f'(x)\big|_{x=\sqrt{3}}} \\ &= \frac{1}{\sqrt{1+x^2}\Big|_{x=\sqrt{3}}} \\ &= \frac{1}{\sqrt{1+3}} = \frac{1}{2}. \end{aligned}$$

Problem 2

1. Prove that the function $f(x) = \frac{4x^3}{x^2+1}$ has an inverse and find $\left(f^{-1}\right)'(2)$.
2. Show that the function $f(x) = 2x^3 + 3x^2 - 36x$ has no inverse on $(-\infty,\infty)$.

3. Show that if $f(x) = x^n$, where n is odd, then f^{-1} exists.

4. If f and g have respective inverses f^{-1} and g^{-1}, show that the composite function $f \circ g$ has inverse $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

5. For what values of the constants a, b, and c is the function

$$p(x) = \frac{x-a}{bx-c}$$

is self-inverse.

6. If

$$f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}},$$

find $(f^{-1})'(0)$.

7. Let f be twice differentiable and injective on an open interval I. Show that if $g = f^{-1}$, then

$$g''(x) = \frac{f''(g(x))}{[f'(g(x))]^3}.$$

8. Let

$$f(x) = \frac{ax+b}{cx+d}.$$

Show that f is invertible if and only if $bc - ad \neq 0$. Hence, given that $bc - ad \neq 0$, find f^{-1} and determine the values of a, b, c, and d such that $f = f^{-1}$.

9. m

10. m

Chapter 3

Logarithemic and Exponential Functions

3.1 The Integral Definition of the Natural Logarithmic Function

Recall the definite integral as a function of x:

$$f(x) = \int_a^x f(t)\,dt;$$

where a is a constant, and the general power rule:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.$$

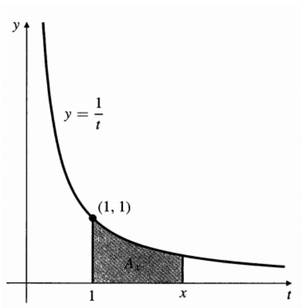
The case $n = -1$ is neither a rational nor trigonometric function; it is a new function.

Definition 3.1. The **natural logarithmic function** is defined by

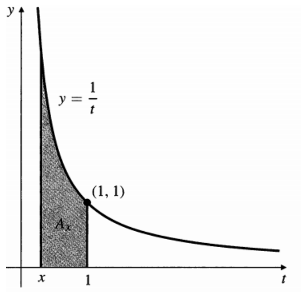
$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0.$$

Some Characteristics of $\ln x$:

- The domain is the set of all positive real numbers, $\mathbb{R}^+$.
- $\ln x$ is the area bounded by the curve $y = \frac{1}{t}$, the t-axis and the lines $t = 1$ and $t = x > 1$.



- $\ln x < 0$ for $0 < x < 1$.



For, if $0 < x < 1$, then

$$\begin{aligned} \ln x &= \int_1^x \frac{1}{t} dt \\ &= -\int_x^1 \frac{1}{t} dt \\ &< 0. \end{aligned}$$

- If $x = 1$, then $\ln x = 0$, that is,

$$\begin{aligned} \ln(1) &= \int_1^1 \frac{1}{t} dt \\ &= 0. \end{aligned}$$

- If $x < 0$, then $\ln x = \infty$.

Theorem 3.2. Properties of the Log Function.

Let $x > 0$ and $y > 0$. Then

1. $\frac{d}{dx}(\ln x) = \frac{1}{x}$,
2. $\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$, *provided $f'(x)$ exists.*
3. $\ln(xy) = \ln x + \ln y$,
4. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$.
5. *If $x_i > 0$ for $i \in (1, \ldots, n)$, then*

$$\ln\left(\prod_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \ln x_i$$

and hence

$$\ln x^r = r \ln x.$$

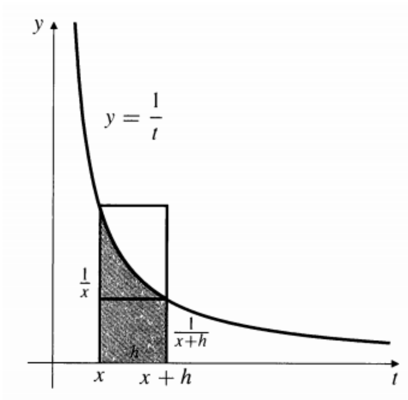
6. $$\lim_{x\to\infty} \ln x = +\infty.$$

7. $$\lim_{x\to 0^+} \ln x = -\infty.$$

Proof.
i. Let $x > 0$ and $h > 0$. Then $\ln(x+h) - \ln x$ gives the area of the shaded region between the curve $y = \frac{1}{t}$, and the lines $t = x$ and $t = x+h$ in the graph below; that is

$$\begin{aligned} \text{Shaded Araea} = \int_x^{x+h} \frac{1}{t} dt &= \int_x^1 \frac{1}{t} dt + \int_1^{x+h} \frac{1}{t} dt \\ &= \int_1^{x+h} \frac{1}{t} dt - \int_1^x \frac{1}{t} dt \\ &= \ln(x+h) - \ln x. \end{aligned}$$

.



Comparing this area with that of the two rectangles, we see that

$$h\left(\frac{1}{x+h}\right) < \ln(x+h) - \ln x < h\left(\frac{1}{x}\right),$$

or

$$\frac{1}{x+h} < \frac{\ln(x+h) - \ln x}{h} < \frac{1}{x}.$$

Therefore, as $h \to 0$ from the right, we obtain by the Squeeze Theorem that

$$\begin{aligned} \frac{d}{dx}(\ln x) &= \lim_{h \to 0^+} \frac{\ln(x+h) - \ln x}{h} \\ &= \frac{1}{x}. \end{aligned}$$

Similarly, for $0 < x+h < x$, we have

$$\frac{1}{x} < \frac{\ln(x+h) - \ln x}{h} < \frac{1}{x+h}$$

and

$$\frac{d}{dx}(\ln x) = \lim_{h \to 0^-} \frac{\ln(x+h) - \ln x}{h} = \frac{1}{x}.$$

Alternative Proof.
Since by definition

$$\ln x = \int_1^x \frac{1}{t} dt,$$

then by the FTC I, we have

$$\begin{aligned}\frac{d}{dx}(\ln x) &= \frac{d}{dx}\int_1^x \frac{1}{t}dt \\ &= \frac{1}{x}.\end{aligned}$$

ii. Suppose f is a differentiable function. Then by the chain rule, we have

$$\begin{aligned}\frac{d}{dx}(\ln f(x)) &= \frac{d}{df(x)}(\ln f(x)) \cdot \frac{d}{dx}f(x) \\ &= \frac{1}{f(x)} \cdot f'(x) \\ &= \frac{f'(x)}{f(x)}.\end{aligned}$$

iii. Let $x > 0$ and $y > 0$. Then by definition

$$\begin{aligned}\ln(xy) &= \int_1^{xy} \frac{1}{t}dt \\ &= \int_1^x \frac{1}{t}dt + \int_x^{xy} \frac{1}{t}dt, \quad \because x > 0, y > 0 \Rightarrow 0 < x < xy \\ &= \ln x + \int_x^{xy} \frac{1}{t}dt. \end{aligned} \tag{3.1}$$

Now, when we substitute $t = xu$, we have that when $t = x$, $u = 1$; when $t = xy$, $u = y$; and $dt = xdu$. Hence, (3.1) becomes

$$\begin{aligned}\ln(xy) &= \ln x + \int_1^y \frac{1}{xu} \cdot xdu \\ &= \ln x + \int_1^y \frac{1}{u}du \\ &= \ln x + \ln y.\end{aligned}$$

Alternative Proof.
If $y > 0$ is a constant, then by the chain rule, we have

$$\frac{d}{dx}(\ln(xy) - \ln x) = \frac{y}{xy} - \frac{1}{x} = 0 \quad \forall x > 0.$$

Thus, $\ln(xy) - \ln x = k$, a constant for $x > 0$. If we set $x = 1$, then $\ln y = k$.

$$\therefore \quad \ln(xy) = \ln x + \ln y \quad \forall x > 0, y > 0.$$

iv. For $x > 0$ and $y > 0$, we have

$$\begin{aligned}\ln\left(\frac{x}{y}\right) &= \int_1^{\frac{x}{y}} \frac{1}{t}dt \\ &= \int_1^x \frac{1}{t}dt + \int_x^{\frac{x}{y}} \frac{1}{t}dt \\ &= \ln x + \int_x^{\frac{x}{y}} \frac{1}{t}dt.\end{aligned}$$

Let $t = \frac{x}{u}$. Then $t = x \Rightarrow u = 1$, $t = \frac{x}{y} \Rightarrow u = y$, and $dt = -\frac{x}{u^2}du$, so that

$$\begin{aligned}
\ln\left(\frac{x}{y}\right) &= \ln x + \int_1^y \frac{u}{x} \cdot \left(-\frac{x}{u^2}du\right) \\
&= \ln x - \int_1^y \frac{1}{u}du \\
&= \ln x - \ln y.
\end{aligned}$$

Notice that, when $x = 1$, then

$$\begin{aligned}
\ln\left(\frac{1}{y}\right) &= \ln 1 - \ln y \\
&= -\ln y.
\end{aligned}$$

Alternative Proof 1.
Let $z = \frac{x}{y}$. Then $x = yz$ and so, by (iii) we have

$$\begin{aligned}
\ln(yz) &= \ln y + \ln z \\
\Rightarrow \ln z &= \ln(yz) - \ln y \\
\text{i.e.} \quad \ln\left(\frac{x}{y}\right) &= \ln x - \ln y.
\end{aligned}$$

Alternative Proof 2.
If $y > 0$ is a constant, then by the chain rule, we have

$$\begin{aligned}
\ln\left(\frac{x}{y}\right) &= \ln\left(x \cdot \frac{1}{y}\right) \\
&= \ln(x) + \ln\left(\frac{1}{y}\right) \\
&= \ln(x) - \ln y.
\end{aligned}$$

v. We prove that $\ln\left(\prod_{i=1}^n x_i\right) = \sum_{i=1}^n \ln x_i$ by induction. First, when $n = 2$, then we have by (iii) that

$$\ln(x_1 x_2) = \ln x_1 + \ln x_2.$$

Suppose the result holds for some $n = k > 2$, then

$$\ln\left(\prod_{i=1}^k x_i\right) = \sum_{i=1}^k \ln x_i.$$

Now, consider the case when $n = k + 1$. Then

$$\begin{aligned}
\ln\left(\prod_{i=1}^{k+1} x_i\right) &= \ln\left(\prod_{i=1}^k x_i \cdot x_{k+1}\right) \\
&= \ln\left(\prod_{i=1}^k x_i\right) + \ln x_{k+1}, \quad (\text{by } (iii)) \\
&= \sum_{i=1}^k \ln x_i + \ln x_{k+1} \\
&= \sum_{i=1}^{k+1} \ln x_i.
\end{aligned}$$

That is, $n = k$ true $\implies$ $n = k+1$ true. Hence, by the principle of mathematical induction, we

$$\ln\left(\prod_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \ln x_i \quad \forall n \geq 2.$$

Now, if $x_i = x \ \ \forall i \in (1, \ldots, r)$, then

$$\begin{aligned}
\ln\left(\prod_{i=1}^{r} x_i\right) &= \ln\left(\underbrace{x \cdot x \cdots x}_{r\ times}\right) \\
&= \ln\left(x^r\right) \\
&= \sum_{i=1}^{r} \ln x \\
&= \underbrace{\ln x + \cdots + \ln x}_{r\ times} \\
&= r \ln x.
\end{aligned}$$

Alternative Proof. Suppose $x > 0$. Then for $r \in \mathbb{R}$, we have

$$\ln\left(x^r\right) = \int_1^{x^r} \frac{1}{t} dt.$$

Let $t = u^r \Rightarrow dt = ru^{r-1}du$. Then $t = x^r \Rightarrow u = x$ and $t = 1 \Rightarrow u = 1$.

$$\begin{aligned}
\therefore \quad \ln\left(x^r\right) &= \int_1^{x^r} \frac{1}{t} dt \\
&= \int_1^{x} \frac{1}{u^r} \cdot ru^{r-1} du \\
&= r \int_1^{x} \frac{1}{u} du \\
&= r \ln x.
\end{aligned}$$

To prove (vi) and (vii), notice that

$$\frac{d}{dx} \ln x = \frac{1}{x} > 0 \ \ \forall x > 0$$

implies that $\ln x$ is strictly increases on $(0, \infty)$.

vi. Suppose $x > 2^n$ for $x > 1$, $n \in \mathbb{N}$. Then because $\ln x$ is an increasing function, we have

$$\begin{aligned}
x > 2^n \quad &\implies \quad \ln x > \ln 2^n \\
&\text{i.e.} \quad \ln x > n \ln 2.
\end{aligned}$$

As $n \to \infty$, $x \to \infty$ and $n \ln 2 \to \infty$, and so $\ln x \to \infty$.

Hence,

$$\lim_{x \to \infty} \ln x = \infty.$$

vii. Also, suppose $x < 2^{-n} = \frac{1}{2^n}$ for $0 < x < 1$, $n \in \mathbb{N}$. Then, by the increasing nature of $\ln x$, we have

$$\begin{aligned} x < 2^{-n} \quad &\Longrightarrow \quad \ln x < \ln 2^{-n} \\ &\Longrightarrow \quad \ln x < -n \ln 2 \\ &\text{i.e.} \quad -\ln x > n \ln 2. \end{aligned}$$

As $n \to \infty$, $x \to \frac{1}{2^n} = 0^+$ and $n \ln 2 \to \infty$, $-n \ln 2 \to -\infty$ and so $\ln x \to -\infty$.

Therefore,

$$\lim_{x \to 0^+} \ln x = -\infty.$$

□

Graph of $f(x) = \ln x$

By definition,

$$f(x) = \ln x = \int_1^x \frac{1}{t} dt, \quad x > 0.$$

- Intercepts:

  To find the x-intercept, put $f(x) = 0$ and solve for x, thus, $\ln x = 0 \Leftrightarrow x = 1$. Therefore, $(1, 0)$ is the x-intercept.

  There is no y-intercept since $\nexists x \in \mathbb{R}^+$ such that $\ln(0) = x$. In other words, $\ln x$ is not defined at $x = 0$, and so, $f(x) \to \infty$ as $x \to 0$.

- Turning Points:

  $$f'(x) = \frac{1}{x} > 0 \quad \forall x > 0.$$

  Therefore, $f(x)$ strictly increases for all $x > 0$. However, since $\nexists x \in \mathbb{R}^+$ such that $f'(x) = \frac{1}{x} = 0$, $f(x)$ has no turning point on $(0, \infty)$.

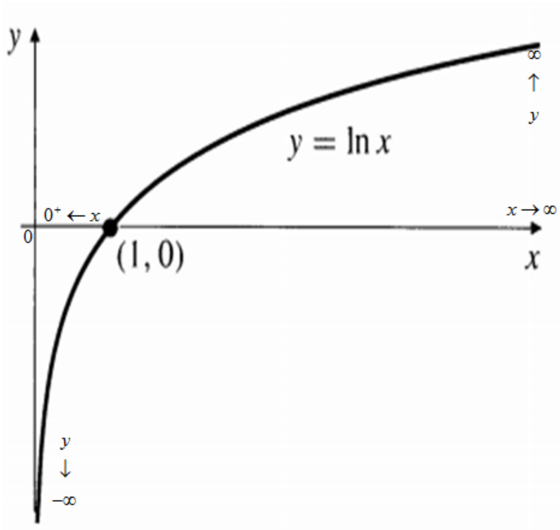
  Also,

  $$\begin{aligned} f''(x) &= -\frac{1}{x^2} \\ &< 0 \quad \forall x > 0. \end{aligned}$$

  Thus, the graph of $f(x)$ is concave downward for $x > 0$.

- Asymptotes:

  As $x \to \infty$, $f(x) \to \infty$ and as $x \to 0^+$, $f(x) \to -\infty$. Therefore, the line $x = 0$ is a vertical asymptote, but there is no horizontal asymptote.



3.2 The (Natural) Exponential Function.

Note that the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f(x) = \ln x$$

increases strictly and so is bijective. It, thus, has an inverse, f^{-1}, called the (natural) exponential function.

Definition 3.3. The (natural) exponential function, denoted $\exp(x)$, and is defined as

$$\exp : \mathbb{R} \to \mathbb{R}^+,$$

by

$$\exp(x) = y \quad \Leftrightarrow \quad \ln y = x,$$

that is,

$$\exp : \mathbb{R} \to \mathbb{R}^+ \mid \exp(\ln x) = x \ \ \forall x \in \mathbb{R}.$$

Some Characteristics of $\exp x$

- The domain is the set of all real numbers, $\mathbb{R} \equiv (-\infty, \infty)$.
- The range is the set of all real numbers, $\mathbb{R}^+ \equiv (0, \infty)$, thus the exp is a positive function.
- $\exp x$ is a bijective function, since it is the inverse of the bijective function $\ln x$, i.e. $\exp = \ln^{-1}$.

Theorem 3.4. Properties of the Exp Function.

Let $x, y \in \mathbb{R}$. Then

1. $\exp(x) > 0 \quad \forall x \in \mathbb{R}$
2. $\exp(x)$ is strictly increasing for all $x \in \mathbb{R}$.
3. $\exp(x+y) = \exp(x) \cdot \exp(y)$,
4. $\exp(0) = 1$,
5. $\exp(-x) = \frac{1}{\exp(x)}$,
6. $\exp(x-y) = \frac{\exp(x)}{\exp(y)}$.
7. $$\lim_{x\to\infty} \exp(x) = \infty$$
   and
   $$\lim_{x\to-\infty} \exp(x) = 0$$

Proof.

(i) Suppose $\ln y = x$. Then by definition, $y = \exp(x)$. But since $x = \ln y$, we have that $y > 0$, by the

definition of $\ln y$. That is, $y = \exp(x) > 0$ for all $x \in \mathbb{R}$.

(ii) Let $f(x) = \exp(x)$. Then

$$x = \ln f(x).$$

Differentiating both sides implicitly with respect to x, we get

$$\begin{aligned} 1 &= \frac{f'(x)}{f(x)} \\ \Rightarrow f'(x) &= f(x) \qquad (3.2) \\ &= \exp(x) > 0 \ \forall x \in \mathbb{R}. \end{aligned}$$

$\therefore f(x) = \exp(x)$ is strictly increasing on $\mathbb{R}$.

(iii) By (iii) of Theorem 3.2, we have

$$\begin{aligned} \ln(\exp(x)\cdot\exp(y)) &= \ln(\exp(x)) + \ln(\exp(y)) \\ &= x + y, \qquad \text{by Definition 4.3.} \end{aligned}$$

Taking exp of both sides and using Definition 3.3 again, we have

$$\begin{aligned} \exp(\ln(\exp(x)\cdot\exp(y))) &= \exp(x+y) \\ \text{i.e.} \quad \exp(x)\cdot\exp(y) &= \exp(x+y). \end{aligned}$$

(iv) Let $\exp(0) = x$. Then $\ln x = 0$. But

$$\ln(1) = \int_1^1 \frac{1}{t}dt = 0.$$

Therefore, $x = 1$, and so $\exp(0) = 1$.

(v) We can write

$$\begin{aligned}\exp(0) &= \exp(x+(-x))\\ &= \exp(x)\cdot\exp(-x)\\ \Rightarrow \exp(-x) &= \frac{\exp(0)}{\exp(x)}\\ &= \frac{1}{\exp(x)}.\end{aligned}$$

(vi) By (iv) of Theorem 3.2, we have

$$\begin{aligned}\ln\left(\frac{\exp(x)}{\exp(y)}\right) &= \ln(\exp(x)) - \ln(\exp(y))\\ &= x-y, \qquad \text{by Definition 4.3.}\end{aligned}$$

Taking exp of both sides and using Definition 3.3 again, we have

$$\begin{aligned}\exp\left(\ln\left(\frac{\exp(x)}{\exp(y)}\right)\right) &= \exp(x-y)\\ \text{i.e. } \frac{\exp(x)}{\exp(y)} &= \exp(x-y).\end{aligned}$$

Alternative Proof. Notice that we can write $\exp(x-y)$ as $\exp(x+(-y))$, hence, by (iii), we have

$$\begin{aligned}\exp(x-y) &= \exp(x)\cdot\exp(-y)\\ &= \exp(x)\cdot\frac{1}{\exp(y)}, \text{ by (v)}\\ &= \frac{\exp(x)}{\exp(y)}.\end{aligned}$$

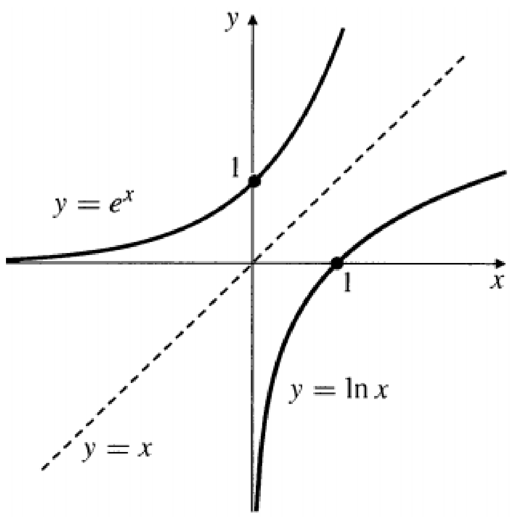
(vii) Since $\exp(x)$ is an increasing function, we have that $\exp(x)\to\infty$ as $x\to\infty$ and $\exp(x)\to 0$ as $x\to-\infty$. Therefore,

$$\lim_{x\to\infty}\exp(x) = \infty$$

and

$$\lim_{x\to-\infty}\exp(x) = 0.$$

□



3.3 The Number e

Definition 3.5. The number e is the number whose natural logarithm is 1. That is,

$$e \in \mathbb{R} : \ln e = \int_1^e \frac{1}{t} dt = 1,$$

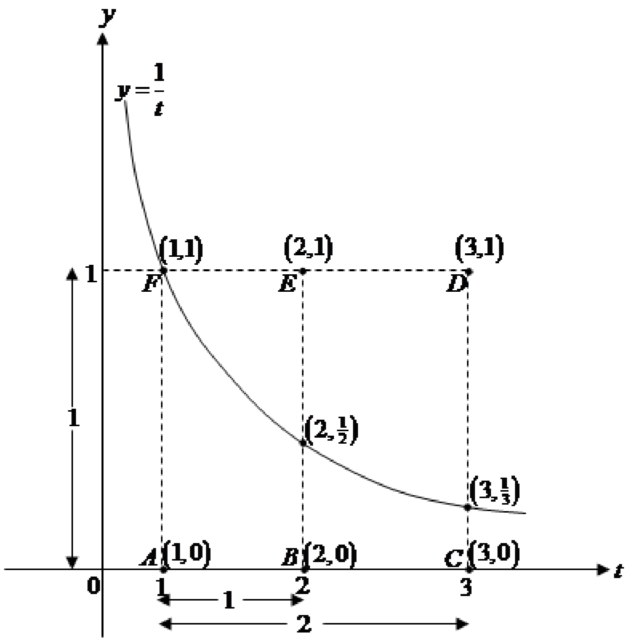
or

$$e \in \mathbb{R} : e = \exp(1) = \exp(\ln e).$$

Theorem 3.6. *The number e lies strictly between 2 and 3, i.e.*

$$2 < e < 3.$$

Proof. Consider the diagram below:



From the diagram, area of rectangle $ABEF$ is

$$A_S = 1 \times 1 = 1 = \ln e \text{ sq. units}$$

and that of rectangle $ACDF$ is

$$A_B = 1 \times 2 = 2 \text{ sq. units.}$$

Now, using the diagram and the definition of the ln function, we have

$$\begin{aligned} \ln 2 &= \int_1^2 \frac{1}{t} dt \\ &< A_s = 1 = \ln e. \end{aligned}$$

That is,

$$\begin{aligned} \ln 2 &< \ln e \\ \Rightarrow 2 &< e. \end{aligned} \tag{3.3}$$

Also,

$$\begin{aligned} \ln 3 &= \int_1^3 \frac{1}{t} dt \\ &= \int_1^2 \frac{1}{t} dt + \int_2^3 \frac{1}{t} dt, \quad \because 1 < 2 < 3. \end{aligned}$$

We evaluate these integrals by making the following substitutions:
For $\int_1^2 \frac{1}{t} dt$, let $t = 2 - 2\sin\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$, then $dt = -2\cos\theta d\theta$; and when $t = 1$, $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$, when $t = 2$, $\sin\theta = 0 \Rightarrow \theta = 0$.
Therefore,

$$\int_1^2 \frac{1}{t} dt = \int_{\frac{\pi}{6}}^0 \frac{1}{2 - 2\sin\theta} \left(-2\cos\theta d\theta\right) = \int_0^{\frac{\pi}{6}} \frac{\cos\theta}{1 - \sin\theta} d\theta.$$

For, $\int_2^3 \frac{1}{t} dt$, let $t = 2 + 2\sin\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$, then $dt = 2\cos\theta d\theta$; and when $t = 2$, $\sin\theta = 0 \Rightarrow \theta = 0$, when $t = 3$, $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$.
Hence,

$$\int_2^3 \frac{1}{t} dt = \int_0^{\frac{\pi}{6}} \frac{1}{2 + 2\sin\theta} \left(2\cos\theta d\theta\right) = \int_0^{\frac{\pi}{6}} \frac{\cos\theta}{1 + \sin\theta} d\theta.$$

Therefore,

$$\begin{aligned} \ln 3 = \int_1^3 \frac{1}{t} dt &= \int_0^{\frac{\pi}{6}} \frac{\cos\theta}{1 - \sin\theta} d\theta + \int_0^{\frac{\pi}{6}} \frac{\cos\theta}{1 + \sin\theta} d\theta \\ &= \int_0^{\frac{\pi}{6}} \frac{\cos\theta\left(1 + \sin\theta\right) + \cos\theta\left(1 - \sin\theta\right)}{1 - \sin^2\theta} d\theta \\ &= 2\int_0^{\frac{\pi}{6}} \sec\theta d\theta \\ &= 2\left[\ln\left|\sec\theta + \tan\theta\right|\right]_0^{\frac{\pi}{6}} \\ &= 2\ln\left(\frac{\sec\frac{\pi}{6} + \tan\frac{\pi}{6}}{\sec 0 + \tan 0}\right) \\ &= 2\ln\left(\frac{\frac{2\sqrt{3}}{3} + \frac{\sqrt{3}}{3}}{1 + 0}\right) \\ &= 2\ln\left(\sqrt{3}\right) \\ &> 1 = \ln e. \end{aligned}$$

That is,

$$\begin{aligned} \ln e &< \ln 3 \\ \text{i.e. } e &< 3. \end{aligned} \tag{3.4}$$

Hence, combining (3.3) and (3.4), we get the required result:

$$2 < e < 3.$$

$\square$

Representation of e as a Limit.

The binomial expansion of $(1+a)^n$ for any positive integer n may be written as

$$(1+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k.$$

Specifically, note that

$$\begin{aligned}
\left(1+\frac{1}{2}\right)^2 &= \sum_{k=0}^{2} \binom{2}{k} \left(\frac{1}{2}\right)^k = 1 + 2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 = 2 + \frac{1}{4} \\
\left(1+\frac{1}{3}\right)^3 &= \sum_{k=0}^{3} \binom{3}{k} \left(\frac{1}{3}\right)^k = 1 + 3\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 = 2 + \frac{10}{27} \\
\left(1+\frac{1}{4}\right)^4 &= \sum_{k=0}^{4} \binom{4}{k} \left(\frac{1}{4}\right)^k = 1 + 4\left(\frac{1}{4}\right) + 6\left(\frac{1}{4}\right)^2 + 4\left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 = 2 + \frac{113}{256} \\
&\vdots \\
\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)^n = 2 + \varepsilon,
\end{aligned}$$

where $0 < \varepsilon < 1$.

Observation. Since $\frac{1}{4} < \frac{10}{27} < \frac{113}{256}$, etc., we have

$$2 < \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} < 3, \ n \in \mathbb{N}_{\geq 2}.$$

Theorem 3.7. *The number e is given as*

$$\begin{aligned}
e &= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \\
&= \lim_{n\to 0} (1+n)^{\frac{1}{n}}, \ \ n \in \mathbb{N} \\
&\approx 2.71828182845904523536028747135266249775724 7\cdots
\end{aligned}$$

Definition 3.8. The exponential function $\exp : \mathbb{R} \to \mathbb{R}^+$ is written simply as

$$f(x) = e^x.$$

3.4 The General Exponential Function.

Let $f(x) = a^x$, $a > 0$. Then $\ln f(x) = x \ln a$, and so $f(x) = \exp(x \ln a) = e^{x \ln a}$.

Deductions:

- Since, $\frac{d}{dx}(e^x) = e^x$, (see equation (3.2)), we have by the chain rule that

$$\begin{aligned} \frac{d}{dx}(a^x) &= \frac{d}{dx}\left(e^{x \ln a}\right) \\ &= e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) \\ &= e^{x \ln a} \ln a. \end{aligned} \tag{3.5}$$

- Hence,

$$\int a^x dt = \frac{a^x}{\ln a} + C.$$

Definition 3.9. If $a > 1$, then the logarithm of x to the base a is defined by

$$\log_a x = \frac{\ln x}{\ln a}.$$

Theorem 3.10. *For any positive number x,*

$$\log_e x = \ln x.$$

Proof. By definition,

$$\begin{aligned} \log_e x &= \frac{\ln x}{\ln e} \\ &= \ln x, \quad \because \ln e = 1. \end{aligned}$$

□

Theorem 3.11. *For $a > 0$,*

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} = \frac{\log_a e}{x}.$$

Proof.

$$\begin{aligned} \frac{d}{dx}\log_a x &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\ &= \frac{1}{x \ln a}. \end{aligned}$$

Note that since,

$$\log_a x = \frac{\log_e x}{\log_e a}$$

we have

$$\log_e a = \ln a = \frac{\log_x a}{\log_x e} = \frac{1}{\log_x e}.$$

$$\begin{aligned} \therefore \frac{d}{dx}(\log_a x) &= \frac{1}{x \ln a} \\ &= \frac{\log_x e}{x}. \end{aligned}$$

□

Theorem 3.12. Representation of the Natural Logarithm as a Limit.
Let $a > 0$. Then

$$\ln a = \lim_{x \to 0} \frac{a^x - 1}{x}.$$

Proof. Suppose $a > 0$. Then from equation (3.5), we have

$$\frac{d}{dx}\left(a^x\right) = e^{x \ln a} \ln a.$$

Also, for any $h > 0$, we have, by definition of the derivative that

$$\frac{d}{dx}\left(a^x\right) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}. \tag{3.6}$$

Now, as $x \to 0$, $a^x \to 1$, $a^{x+h} \to a^h$ and $e^{x \ln a} \ln a \to \ln a$. Hence, from the above two equations, we have

$$\begin{aligned} \ln a &= \lim_{h \to 0} \frac{a^h - 1}{h}, \\ \text{or } \ln a &= \lim_{x \to 0} \frac{a^x - 1}{x}. \end{aligned}$$

□

Theorem 3.13. Representation of e^x as a Limit.
Let $x \in \mathbb{R}$. Then

$$e^x = \lim_{n \to 0} (1 + nx)^{\frac{1}{n}} = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n.$$

and so

$$e = \lim_{n \to 0} (1 + n)^{\frac{1}{n}} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

Proof. Let $x \in \mathbb{R}$ and the function $\ln(1 + xt)$. Then

$$\frac{d}{dt}\left(\ln(1 + xt)\right) = \frac{x}{1 + xt}.$$

Also, by differentiating from first principles, we have

$$\begin{aligned} \frac{d}{dt}\left(\ln(1 + xt)\right) &= \lim_{h \to 0} \frac{\ln(1 + x(t + h)) - \ln(1 + xt)}{h} \\ &= \lim_{h \to 0} \left\{ \frac{1}{h} \ln\left(\frac{1 + x(t + h)}{(1 + xt)}\right) \right\}. \end{aligned}$$

As $t \to 0$, $\frac{1+x(t+h)}{(1+xt)} \to 1 + xh$ and $\frac{x}{1+xt} \to x$. Hence,

$$\begin{aligned} \frac{d}{dt}\left(\ln(1 + xt)\right) &= \lim_{h \to 0} \left\{ \frac{1}{h} \ln(1 + xh) \right\} \\ &= \lim_{h \to 0} \left\{ \ln(1 + xh)^{\frac{1}{h}} \right\} \\ & \ln\left\{ \lim_{h \to 0} (1 + xh)^{\frac{1}{h}} \right\}. \end{aligned}$$

Therefore, combining the two derivatives, we have

$$\begin{aligned} x &= \ln\left\{\lim_{h\to 0}(1+xh)^{\frac{1}{h}}\right\} \\ \Rightarrow e^x &= \lim_{h\to 0}(1+xh)^{\frac{1}{h}}, \end{aligned}$$

proving the first part. To prove the second part is just a matter of changing variables; let $n = \frac{1}{h}$. Then as $h \to 0$, $n \to \infty$, and so, we have

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

Now, when $x = 1$, we get

$$e = \lim_{n\to 0}(1+n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

An *alternative* way to prove the last part (in its own right) is to consider the function $f(x) = \ln x$ which has the derivative $f'(x) = \frac{1}{x}$, so that $f'(1) = 1$.

Now, $f(x) = \ln x$, we have that

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{\ln(x+h) - \ln x}{h} \\ \therefore \quad f'(1) &= \lim_{h\to 0}\frac{\ln(1+h) - \ln(1)}{h} \\ &= \lim_{n\to 0}\frac{\ln(1+n) - \ln(1)}{n} \\ &= \lim_{n\to 0}\left(\frac{1}{n}\ln(1+n)\right) \\ &= \lim_{n\to 0}\left(\ln(1+n)^{\frac{1}{n}}\right) \\ &= \ln\left(\lim_{n\to 0}(1+n)^{\frac{1}{n}}\right). \end{aligned}$$

But $f'(1) = 1$.

$$\begin{aligned} \therefore \quad \ln\left(\lim_{n\to 0}(1+n)^{\frac{1}{n}}\right) &= 1 \\ \Rightarrow \quad \lim_{n\to 0}(1+x)^{\frac{1}{n}} &= e^1 = e. \end{aligned}$$

□

3.5 Logarithmic Inequalities

Example 3.14. Prove that

$$\frac{x}{1+x} < \ln(1+x) < x, \quad \forall x > 0 \tag{3.7}$$

and $\forall x \in (0,1)$,

$$x < -\ln(1-x) < \frac{x}{1-x}. \tag{3.8}$$

Proof. By the definition of the ln function, we have

$$\ln(1+x) = \int_1^{1+x} \frac{1}{t} dt, \quad x > 0.$$

Now, since $x > 0$, we have that $x + 1 > t > 1$ and so $1 < \frac{1}{t} < \frac{1}{1+x}$. Therefore, integrating all sides with respect to t, we have

$$\int_1^{1+x} 1 dt < \int_1^{1+x} \frac{1}{t} dt < \int_1^{1+x} \frac{1}{1+x} dt,$$

or

$$\begin{aligned} [x]_1^{1+x} < \ln(1+x) < \frac{1}{1+x}\left([x]_1^{1+x}\right) \\ \Rightarrow 1 + x - 1 < \ln(1+x) < \frac{1}{1+x}(1 + x - 1) \\ \Rightarrow x < \ln(1+x) < \frac{x}{1+x}. \end{aligned}$$

Also,

$$-\ln(1-x) = -\int_1^{1-x} \frac{1}{t} dt, \quad 0 < x < 1.$$

Let us make the substitution $t = 1 - u$. Then $dt = -du$; and whenever $t = 1$, $u = 0$ and whenever $t = 1 - x$, $u = x$. Hence,

$$\begin{aligned} -\ln(1-x) &= \int_0^x \frac{1}{1-u} du \\ &= \int_0^x \sum_{k=0}^{\infty} u^k du \equiv \int_0^x \left(1 + u + u^2 + u^3 + \cdots\right) du \\ &= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \equiv x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \cdots \\ &> x, \quad \forall x \in (0,1), \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} -\ln(1-x) &= \sum_{k=1}^{\infty} \frac{x^k}{k} \equiv x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \cdots \\ &= x \sum_{k=1}^{\infty} \frac{x^{k-1}}{k} \equiv x\left(1 + \tfrac{1}{2}x + \tfrac{1}{3}x^2 + \tfrac{1}{4}x^3 + \cdots\right) \\ &< x \sum_{k=0}^{\infty} x^k \equiv x\left(1 + x + x^2 + x^3 + \cdots\right), \quad \forall 0 < x < 1 \\ &= x(1-x)^{-1} \\ &= \frac{x}{1-x}. \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we obtain the result. □

Example 3.15. Show that for all values of $x > 0$, we have

$$x - \tfrac{1}{2}x^2 < \ln(1+x),$$

and for all $x > 1$, we have

$$\frac{x-1}{x} < \ln x < x - 1.$$

Proof. Let us make the substitution $t = 1 + u$ in the integral

$$\ln(1+x) = \int_1^{1+x} \frac{1}{t} dt.$$

Then $dt = du$, and $t = 1 \Rightarrow u = 0$, whiles $t = 1 + x \Rightarrow u = x$, whence

$$\begin{aligned}
\ln(1+x) &= \int_0^x \frac{1}{1+u} du \\
&= \int_0^x \sum_{k=0}^{\infty} (-u)^k \, du \\
&= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \\
&> x - \tfrac{1}{2}x^2, \quad \forall x > 0;
\end{aligned}$$

Similarly, for

$$\ln x = \int_1^x \frac{1}{t} dt,$$

we make the substitution $t = 1 - u$. Then $dt = -du$, $t = 1 \Rightarrow u = 0$ and $t = x \Rightarrow u = 1 - x$.

$$\begin{aligned}
\therefore \quad \ln x &= \int_0^{1-x} \frac{1}{u-1} du = -\int_0^{1-x} \frac{1}{1-u} du \\
&= -\int_0^x \sum_{k=0}^{\infty} u^k du \\
&= -\left[\sum_{k=1}^{\infty} \frac{u^k}{k}\right]_0^{1-x} \\
&= -\left(1 - x + \tfrac{1}{2}(1-x)^2 + \tfrac{1}{3}(1-x)^3 + \cdots\right) \\
&= x - 1 - \left(\tfrac{1}{2}(1-x)^2 + \tfrac{1}{3}(1-x)^3 + \cdots\right) \\
&= x - 1 + \left(-\tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 + \tfrac{1}{5}(x-1)^5 - \cdots\right) \\
&= x - 1 + \left(\tfrac{1}{3}(x-1)^3 - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{5}(x-1)^5 - \tfrac{1}{4}(x-1)^4 + \cdots\right) \\
&= x - 1 + \left((x-1)^2\left(\tfrac{1}{3}(x-1) - \tfrac{1}{2}\right) + (x-1)^4\left(\tfrac{1}{5}(x-1) - \tfrac{1}{4}\right) + \cdots\right) \\
&< x - 1, \quad \forall x > 1.
\end{aligned}$$

Also,

$$\begin{aligned}
\ln x &= x - 1 - \tfrac{1}{2}(1-x)^2 - \tfrac{1}{3}(1-x)^3 - \cdots \\
&= \frac{x-1}{x}\left(x - \tfrac{1}{2}\frac{x(1-x)^2}{x-1} - \tfrac{1}{3}\frac{x(1-x)^3}{x-1} - \cdots\right) \\
&> \frac{x-1}{x}, \quad \forall x > 1.
\end{aligned}$$

□

Some Important Limits Involving Logarithms and Exponentials.

Theorem 3.16. *Let $x > 0$. Then*

1.
$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1 \tag{3.11}$$
or
$$\lim_{x\to 1}\frac{\ln x}{x-1}=1. \tag{3.12}$$

2.
$$\lim_{x\to\infty}\frac{\ln x}{x}=0. \tag{3.13}$$

3.
$$\lim_{x\to\infty}\frac{\ln x}{x^k}=0,\quad \forall k>0. \tag{3.14}$$

4.
$$\lim_{x\to\infty}\frac{x^k}{e^x}=0,\quad \forall k>0. \tag{3.15}$$

Proof.

i. From (3.7), i.e.,
$$\frac{x}{1+x}<\ln(1+x)<x,\quad \forall x>0,$$
we have
$$\frac{1}{1+x}<\frac{\ln(1+x)}{x}<1,\quad \forall x>0.$$
$$\therefore \lim_{x\to 0}\frac{1}{1+x}<\lim_{x\to 0}\frac{\ln(1+x)}{x}<\lim_{x\to 0}1$$
$$\text{i.e.}\quad 1<\lim_{x\to 0}\frac{\ln(1+x)}{x}<1,$$
and so, by the Sandwich (Squeeze) theorem, we have that
$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1.$$
Also, when we make the substitution $x=y-1$, where $y\to 1$ as $x\to 0$, we have
$$\lim_{y\to 1}\frac{\ln y}{y-1}=1,$$
or
$$\lim_{x\to 1}\frac{\ln x}{x-1}=1.$$

ii. Notice that $\ln x<x$ for all $x>0$, since
$$\begin{aligned}\ln x &= \int_1^x \frac{1}{t}dt=\int_0^{x-1}\frac{1}{1+u}du\\ &= \int_0^{x-1}\left(1-u+u^2-u^3+\cdots\right)du\\ &= x-1-\tfrac{1}{2}(x-1)^2+\tfrac{1}{3}(x-1)^3-\tfrac{1}{4}(x-1)^4+\cdots\\ &< x,\quad \forall x>0.\end{aligned}$$

$$\begin{aligned} \therefore \quad \ln\sqrt{x} &< \sqrt{x} \\ \Rightarrow \tfrac{1}{2}\ln x &< \sqrt{x}. \end{aligned}$$

But $\ln x > 0$ for all $x > 1$, therefore,

$$0 < \ln x < 2\sqrt{x}$$

and so,

$$0 < \frac{\ln x}{x} < \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}}.$$

Therefore,

$$0 < \lim_{x\to\infty}\frac{\ln x}{x} < \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0,$$

$$\Rightarrow \lim_{x\to\infty}\frac{\ln x}{x} = 0,$$

by the Sandwich theorem.

iii. Let $y = x^k$. Then $\ln y = k\ln x$ or $\ln x = \frac{1}{k}\ln y$. Now, as $x \to \infty$, $y \to \infty$ and so

$$\begin{aligned} \lim_{x\to\infty}\frac{\ln x}{x^k} &= \lim_{y\to\infty}\frac{\frac{1}{k}\ln y}{y} \\ &= \frac{1}{k}\lim_{x\to\infty}\frac{\ln y}{y} \\ &= \frac{1}{k}\cdot 0, \qquad \text{by (4.13)} \\ &= 0. \end{aligned}$$

Alternative Proof. Let $m \in \mathbb{Q}^+$. Then $\forall t > 0$, we have $t^{-1} < t^{m-1}$, $\forall m > 0$. Thus

$$\ln x = \int_1^x \frac{1}{t}dt < \int_1^x t^{m-1}dt$$

$$\begin{aligned} \Rightarrow \ln x &< \left.\frac{t^m}{m}\right|_1^x = \frac{x^m - 1}{m} \\ &< \frac{x^m}{m} \\ \Rightarrow 0 < \frac{\ln x}{x^k} &< \frac{x^{m-k}}{m}, \quad \forall k \gg m, \end{aligned}$$

so that as $x \to \infty$, $x^{m-k} \to \infty$ since $m \ll k$, and the result follows from the Squeeze theorem.

iv. Let $y = e^x \Leftrightarrow \ln y = x$. And $x \to \infty$, $y \to \infty$, therefore,

$$\begin{aligned} \lim_{x\to\infty}\frac{x^k}{e^x} &= \lim_{y\to\infty}\frac{(\ln y)^k}{y} \\ &= \lim_{y\to\infty}\left(\frac{\ln y}{y^{\frac{1}{k}}}\right)^k \\ &= 0^k, \qquad \text{by (4.14)} \\ &= 0. \end{aligned}$$

□

Example 3.17. Show that

$$\lim_{x\to1}(\ln(ex))^{\frac{1}{\ln x}}=e.$$

Proof. Note that

$$\begin{aligned}\lim_{x\to1}(\ln(ex))^{\frac{1}{\ln x}} &= \lim_{x\to1}(\ln e+\ln x)^{\frac{1}{\ln x}}\\ &= \lim_{x\to1}(1+\ln x)^{\frac{1}{\ln x}}, \quad \because \ln e=1.\end{aligned}$$

Now, let $t=\ln x$. Then as $x\to1$, $t=\ln x\to0$. Hence,

$$\begin{aligned}\lim_{x\to1}(\ln(ex))^{\frac{1}{\ln x}} &= \lim_{x\to1}(1+\ln x)^{\frac{1}{\ln x}}\\ &= \lim_{t\to0}(1+t)^{\frac{1}{t}}\\ &= e.\end{aligned}$$

□

3.6 Scales of Infinity

This is the process of comparing the order or size of two functions for large values of the independent variable.

Definition 3.18. Order of Magnitude

A function $f(x)$ is said to be of the *k-th order of magnitude* if

$$\frac{f(x)}{x^k}\to L\neq0$$

as $x\to\infty$.

Definition 3.19. Let $f(x)$ and $g(x)$ be any two functions such that both $f(x)\to\infty$ and $g(x)\to\infty$ as $x\to\infty$. Suppose also that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=0.$$

Then we say that $f(x)\to\infty$ *slower* than $g(x)$, or $g(x)\to\infty$ *faster* than $f(x)$. Alternatively, when

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty,$$

then $f(x)\to\infty$ *faster* than $g(x)$, or $g(x)\to\infty$ *slower* than $f(x)$. That is,

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\begin{cases}0, & \text{if } f(x)<g(x)\\ \infty, & \text{if } f(x)>g(x).\end{cases} \tag{3.16}$$

Note that to arrange any n functions in order of magnitude, we have to do ${}^nC_2$ pairwise comparisons.

Example 3.20. Since we have shown that

$$\lim_{x\to\infty} \frac{\ln x}{x^k} = 0, \quad \forall k > 0,$$

it means that $\ln x \to \infty$ slower than any positive power of x. Similarly, $e^x \to \infty$ grows faster than any positive power of x as $x \to \infty$ since

$$\lim_{x\to\infty} \frac{x^k}{e^x} = 0, \quad \forall k > 0.$$

Example 3.21. Arrange the following functions

$$x^5, x^2, x^3$$

in order of increasing magnitude for large values of x.

Solution. Compare the functions pairwise:

- For $x^5$and x^2, we have
$$\lim_{x\to\infty} \frac{x^5}{x^2} = \lim_{x\to\infty} x^3 = \infty.$$
$$\Rightarrow x^5 > x^2$$
for large values of x.
- Also x^2 and x^3, we have
$$\lim_{x\to\infty} \frac{x^2}{x^3} = \lim_{x\to\infty} \frac{1}{x} = 0.$$
$$\Rightarrow x^3 > x^2$$
for large values of x;
- and for $x^5$and x^3, we have
$$\lim_{x\to\infty} \frac{x^5}{x^3} = \lim_{x\to\infty} x^2 = \infty.$$
$$\Rightarrow x^5 > x^3$$
for large values of x. Hence the order of increasing magnitude of these functions is
$$x^2, x^3, x^5.$$

Example 3.22. Show that, for large values of x,

$$\sqrt{x} > \exp\left(\sqrt{\ln x}\right) > \ln^3 x.$$

Solution. Here,

$$\lim_{x\to\infty} \frac{\ln^3 x}{\sqrt{x}} = \lim_{x\to\infty} \left(\frac{\ln x}{x^{\frac{1}{6}}}\right)^3 = 0.$$
$$\therefore \ \sqrt{x} > \ln^3 x \quad \text{as } \ x \to \infty.$$

Also,

$$\begin{aligned}
\lim_{x\to\infty}\frac{\exp\left(\sqrt{\ln x}\right)}{\sqrt{x}} &= \lim_{x\to\infty}\frac{e^{\sqrt{\ln x}}}{e^{\ln\sqrt{x}}}, \qquad \because a\equiv e^{\ln a}\\
&= \lim_{x\to\infty} e^{\sqrt{\ln x}-\ln\sqrt{x}} = \lim_{x\to\infty} e^{\sqrt{\ln x}-\frac{1}{2}\ln x}\\
&= \lim_{x\to\infty} e^{-\frac{1}{2}\left(\ln x-2\sqrt{\ln x}\right)} = \lim_{x\to\infty} e^{-\frac{1}{2}\left(\left(\sqrt{\ln x}\right)^2-2\sqrt{\ln x}\right)}\\
&= \lim_{x\to\infty} e^{-\frac{1}{2}\left(\left(\sqrt{\ln x}-1\right)^2-1\right)} = e^{\frac{1}{2}}\lim_{x\to\infty} e^{-\left(\left(\sqrt{\ln x}-1\right)^2\right)}\\
&= \lim_{x\to\infty}\frac{e^{\frac{1}{2}}}{e^{\left(\sqrt{\ln x}-1\right)^2}}.
\end{aligned}$$

Let $z=\left(\sqrt{\ln x}-1\right)^2$, then $x\to\infty\Rightarrow z\to\infty$.

$$\begin{aligned}
\therefore\ \lim_{x\to\infty}\frac{\exp\left(\sqrt{\ln x}\right)}{\sqrt{x}} &= \lim_{x\to\infty}\frac{e^{\frac{1}{2}}}{e^{\left(\sqrt{\ln x}-1\right)^2}}\\
&= e^{\frac{1}{2}}\lim_{z\to\infty}\frac{z^0}{e^z}\\
&= e^{\frac{1}{2}}\cdot 0 = 0, \qquad \text{by} \quad (4.15).
\end{aligned}$$

Hence,

$$\sqrt{x} > \exp\left(\sqrt{\ln x}\right) \quad \text{as } x\to\infty.$$

And finally $\ln^3 x$ and $\exp\left(\sqrt{\ln x}\right)$, let $z=\sqrt{\ln x}$ and so $z^2=\ln x$, and $x\to\infty\Rightarrow z\to\infty$.

$$\therefore\ \lim_{x\to\infty}\frac{\ln^3 x}{\exp\left(\sqrt{\ln x}\right)} = \lim_{z\to\infty}\frac{z^6}{e^z}=0, \quad \text{by } (4.15).$$

$$\therefore\ \exp\left(\sqrt{\ln x}\right) > \ln^3 x \quad \text{as } x\to\infty.$$

Hence,

$$\sqrt{x} > \exp\left(\sqrt{\ln x}\right) > \ln^3 x \quad \text{as } x\to\infty.$$

3.7 Logarithmic Differentiation.

Suppose we want to differentiate functions expressed as products and quotients of many factors, such as

$$(f(x))^{g(x)}, \quad \frac{\prod_{i=1}^{n} f_i(x)}{\prod_{j=1}^{m} g_j(x)},$$

etc. Taking logarithms of such expressions reduces the products and quotients to sums and differences, and the calculations become simpler than using the Product and Quotient rules. Thus, if

$$y=(f(x))^{g(x)},$$

then

$$\ln y = g(x) \ln f(x),$$

and so differentiating both sides with respect to x and using the product rule gives us

$$\begin{aligned}
\frac{y'}{y} &= g'(x) \ln x + g(x) \cdot \frac{f'(x)}{f(x)} \\
\Rightarrow y' &= y\left(g'(x) \ln x + \frac{g(x) f'(x)}{f(x)}\right) \\
&= (f(x))^{g(x)}\left(g'(x) \ln x + \frac{g(x) f'(x)}{f(x)}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
y &= \frac{\prod_{i=1}^{n} f_i(x)}{\prod_{j=1}^{m} g_j(x)} \\
\Rightarrow \ln y &= \sum_{i=1}^{n} \ln(f_i(x)) - \sum_{j=1}^{m} \ln(g_j(x)) \\
\Rightarrow \frac{y'}{y} &= \frac{\sum_{i=1}^{n} f_i'(x)}{\sum_{i=1}^{n} f_i(x)} - \frac{\sum_{j=1}^{m} g_j'(x)}{\sum_{j=1}^{m} g_j(x)} \\
\therefore y' &= y\left(\frac{\sum_{i=1}^{n} f_i'(x)}{\sum_{i=1}^{n} f_i(x)} - \frac{\sum_{j=1}^{m} g_j'(x)}{\sum_{j=1}^{m} g_j(x)}\right) \\
&= \frac{\prod_{i=1}^{n} f_i(x)}{\prod_{j=1}^{m} g_j(x)}\left(\frac{\sum_{i=1}^{n} f_i'(x)}{\sum_{i=1}^{n} f_i(x)} - \frac{\sum_{j=1}^{m} g_j'(x)}{\sum_{j=1}^{m} g_j(x)}\right).
\end{aligned}$$

This useful technique is called *logarithmic differentiation.*

Example 3.23. Find $f'(x)$ if

$$f(x) = x^x, \quad x > 0$$

and

$$f(x) = \frac{(x+1)(x+2)(x+3)}{x+4}.$$

Solution. For $f(x) = x^x$, taking the logarithm of both sides, we have

$$\ln f(x) = x \ln x.$$

Hence, differentiating both sides with respect to x, we get

$$\begin{aligned}
\frac{f'(x)}{f(x)} &= \ln x + x\left(\frac{1}{x}\right) \\
\Rightarrow f'(x) &= f(x)(\ln x + 1) \\
&= x^x(\ln x + 1) \\
&= x^x(\ln x + \ln e) \\
&= x^x \ln ex.
\end{aligned}$$

Also, for $f(x) = \frac{(x+1)(x+2)(x+3)}{x+4}$, we have

$$\begin{aligned}
\ln|f(x)| &= \ln|x+1| + \ln|x+2| + \ln|x+3| - \ln|x+4| \\
\Rightarrow f'(x) &= f(x)\left(\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}\right) \\
&= \frac{(x+1)(x+2)(x+3)}{x+4}\left(\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}\right) \\
&\equiv \frac{1}{x+4}\left((x+2)(x+3) + (x+1)(x+3) + (x+1)(x+2) - \frac{(x+1)(x+2)(x+3)}{(x+4)}\right) \\
&= \frac{1}{x+4}\left((x+3)(2x+3) + \frac{(x+1)(x+2)}{(x+4)}\right).
\end{aligned}$$

Example 3.24. Prove that, for $0 < \theta < \pi$,

$$\frac{d}{d\theta}\left((\sin\theta)^{\ln\theta}\right) = (\sin\theta)^{\ln\theta}\left(\frac{\ln\sin\theta}{\theta} + \ln\theta \cdot \frac{\cos\theta}{\sin\theta}\right).$$

Solution. Let $f(\theta) = (\sin\theta)^{\ln\theta}$. Then $\ln f(\theta) = \ln\theta \cdot \ln\sin\theta$.

$$\begin{aligned}
\frac{f'(\theta)}{f(\theta)} &= \frac{\ln\sin\theta}{\theta} + \ln\theta \cdot \frac{\cos\theta}{\sin\theta} \\
\Rightarrow f'(\theta) &= f(\theta)\left(\frac{\ln\sin\theta}{\theta} + \ln\theta \cdot \frac{\cos\theta}{\sin\theta}\right) \\
&= (\sin\theta)^{\ln\theta}\left(\frac{\ln\sin\theta}{\theta} + \ln\theta \cdot \frac{\cos\theta}{\sin\theta}\right).
\end{aligned}$$

Problem 3

1. By using the definition of the logarithmic function, prove the following inequalities for the given intervals:

   (a) $4(x-1) - \ln x < 2x\ln x < x^2 - 1, \quad x > 1,$

   (b) $0 < \frac{1}{x} - \ln\left(\frac{x+1}{x}\right) < \frac{1}{2x^2}, \quad x > 0,$

   (c) $\frac{2}{2x+1} < \ln\left(\frac{x+1}{x}\right) < \frac{2x+1}{2x(x+1)}, \quad x > 0.$

2. Prove the following:
$$\frac{d}{dx}(\ln x)^k = \frac{k}{x(\ln x)^{1-k}}$$
   and
$$\frac{d}{dx}(\ln\ln x)^k = \frac{k}{x\ln x(\ln\ln x)^{1-k}}.$$

3. Show that the curve
$$y = x^m(\ln x)^n,$$
   where x is positive, m and n are integers larger than 1, has at least two inflexions and my have more. Sketch the curve when n is odd.

4. Show that if $x > 0$ then
$$e^x > 1 + x.$$
Hence, deduce that
$$\frac{4}{3} < \int_0^1 e^{x^2}\,dx < e.$$

5. How does the function
$$f(x) = \frac{x^a \left(\ln^b x\right)\left(\ln^c \ln x\right)}{x^i \left(\ln^j x\right)\left(\ln^k \ln x\right)}$$
behave as $x \to \infty$?

6. Arrange the functions
$$\frac{x}{\sqrt{\ln x}}, \frac{x\sqrt{\ln x}}{\ln \ln x}, \frac{x \ln \ln x}{\sqrt{\ln x}}, \frac{x \ln \ln \ln x}{\sqrt{\ln \ln x}}$$
according to their order of magnitude.

7. Find the nth derivative of the function
$$f(x) = xe^{ax},$$
hence or otherwise, show that the nth derivative of $(ax^2 + bx + c)\,e^x$ is a function of the same form but with different constants.

8. Let $f_{m,n}(x) = me^x \cos x + ne^x$. Show that $f'_{m,n}(x) = f_{m+n,n-m}(x)$. Find also $f''_{m,n}(x)$ and $f'''_{m,n}(x)$.

9. Suppose $a > 0$. Show that
$$\lim_{x\to\infty} a^{\frac{1}{x}} = 1$$
by examining $\ln\left(a^{\frac{1}{x}}\right)$. Hence, show that
$$\lim_{x\to\infty} x^{\frac{1}{x}} = 1.$$

10. Suppose that f and g are differentiable functions of x. Show, by logarithmic differentiation, that
$$\frac{d}{dx}\left(f^g\right) = g\left(f^{g-1}\right)\frac{df}{dx} + \left(f^g \ln g\right)\frac{dg}{dx}.$$
Interpret the two terms on the right in the special cases where f is constant, and where g is constant.

11. Find $f'(x)$ if $(f(x))^x = x^{f(x)}$. At what points does the tangent to the graph of this curve not exist?

12. Which is greater, $(\sqrt{n})^{\sqrt{n+1}}$ or $\left(\sqrt{n+1}\right)^{\sqrt{n}}$, when $n > 8$?

13. Show that when x is positive, then
$$\ln\left(1 + \frac{1}{x}\right) > \frac{1}{1+x}.$$

14. Evaluate the following limits:

(a) $\lim_{x\to 0^+} x\ln x$

(b) $\lim_{x\to \infty} x^{-3}e^x$

(c) $\lim_{x\to 0^+} \frac{\ln x}{x}$

(d) $\lim_{x\to \infty} x\ln x$

15. Show that the function $f(x) = xe^{-x} + 1$ has at most one root in $\left[-1, -\frac{1}{2}\right]$.

16. Solve the simultaneous equations $\ln x + \ln y^2 = 4$ and $\ln^2 x - 3\ln xy = -5$.

17. Find the solution set of the equation $2^{\frac{1}{(2x-1)}} = 10^{2x-1}$.

18. Determine the equation of the inverse of the function $y = \ln\left(2 - \frac{1}{e^x}\right)$.

19. Consider the function $f(x) = (x - m)\,e^{m-x}$. Show that the maximum value of f does not depend on the parameter m.

20. Solve the system of equations

$$\begin{aligned} e^x + e^{-y^2} &= 1, \\ e^{2x} + \sqrt{e^{-y^2}} &= 1. \end{aligned}$$

21. Sketch the graph of the function

$$y = \ln\left(\frac{e^x + 3e^{-x}}{e^x + 1}\right)$$

showing clearly all the important characteristics.

22. m

23. m

24. m

25. m

Chapter 4

Indeterminate Forms and l'Hôpital's Rule

Definition 4.1. Indeterminate Limits

The limit of an algebraic or transcendental expression is said to be indeterminate when it is one of these seven forms involving 0, 1, and ∞:

$$\frac{0}{0}, 0.\infty, \frac{\infty}{\infty}, \infty - \infty, 0^0, \infty^0, 1^\infty.$$

That is, merely knowing the limiting behaviour of the individual parts of the expression is not sufficient to determine the overall limit.

It is important to note, however, that an indeterminate form does not mean that the limit is non-existent or cannot be determined, but rather that the properties of its limits are not valid. One way of evaluating the limits of such functions is by use of l'Hôpital's Rule.

Theorem 4.2. l'Hôpital's Rule for the Indeterminate Forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.
Suppose $f(x)$ and $g(x)$ are differentiable on the interval (a,b) except possibly at $x_0 \in (a,b)$. If $\lim_{x\to x_0} f(x) = 0 = \lim_{x\to x_0} g(x)$ or if $\lim_{x\to x_0} f(x) = \infty = \lim_{x\to x_0} g(x)$, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = \frac{f'(x_0)}{g'(x_0)}. \tag{4.1}$$

Proof. Let $\lim_{x\to x_0} f(x) = 0 = \lim_{x\to x_0} g(x)$, that is, $f(x_0) = 0 = g(x_0)$. Suppose also that $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists. Then by definition, we have

$$\begin{aligned}
\frac{f'(x_0)}{g'(x_0)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} &= \frac{\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}}{\lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0}} \\
&= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} \\
&= \lim_{x\to x_0} \frac{f(x)}{g(x)}, \quad \because f(x_0) = 0 = g(x_0).
\end{aligned}$$

Similarly, if $\lim_{x\to x_0} f(x) = \infty = \lim_{x\to x_0} g(x)$, that is, $f(x_0) = \infty = g(x_0)$, then by Cauchy's Generalised MVT, we have that for $a < x_0 < x < x_1 < b$

$$\begin{aligned}
\frac{f'(c)}{g'(c)} &= \frac{f(x) - f(x_1)}{g(x) - g(x_1)}, \quad x < c < x_1 \\
&= \frac{f(x)}{g(x)} \cdot \left(\frac{1 - \frac{f(x_1)}{f(x)}}{1 - \frac{g(x_1)}{g(x)}} \right) \\
\Rightarrow \frac{f(x)}{g(x)} &= \frac{f'(c)}{g'(c)} \cdot \left(\frac{1 - \frac{g(x_1)}{g(x)}}{1 - \frac{f(x_1)}{f(x)}} \right). \\
\therefore \lim_{x\to x_0} \frac{f(x)}{g(x)} &= \lim_{x\to x_0} \left(\frac{f'(c)}{g'(c)} \cdot \left(\frac{1 - \frac{g(x_1)}{g(x)}}{1 - \frac{f(x_1)}{f(x)}} \right) \right) \\
&= \lim_{x\to x_0} \frac{f'(c)}{g'(c)} \cdot \lim_{x\to x_0} \left(\frac{1 - \frac{g(x_1)}{g(x)}}{1 - \frac{f(x_1)}{f(x)}} \right) \\
&= \lim_{x\to x_0} \frac{f'(x)}{g'(x)}, \quad \because f(x) = \infty = g(x) \Rightarrow \frac{f(x_1)}{f(x)} = 0 = \frac{g(x_1)}{g(x)}.
\end{aligned}$$

Alternative Proof. Suppose both $f(x)$ and $g(x)$ are differentiable on (a, b) and that $f(x_0) = 0 = f(x_0)$, where $x_0 \in (a, b)$. Then by Cauchy's generalised MVT,

$$\begin{aligned}
\frac{f(x)}{g(x)} &= \frac{f(x) - f(x_0)}{g(x) - g(x_0)}, \quad \because f(x_0) = 0 = g(x_0) \\
&= \frac{f'(c)}{g'(c)}, \qquad \text{for some } c \in (x_0, x).
\end{aligned}$$

Now, as $x \to x_0$, $c \to x_0$.

$$\begin{aligned}
\lim_{x\to x_0} \frac{f(x)}{g(x)} &= \lim_{x\to x_0} \frac{f'(c)}{g'(c)} \\
&= \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.
\end{aligned}$$

□

Example 4.3. To find,

$$\lim_{x\to 0} \frac{\ln(\sin x)}{\ln x},$$

notice that the limit is of the form $\frac{\infty}{\infty}$ since both $\ln(\sin x) \to \infty$ and $\ln x \to \infty$ as $x \to 0$, hence, l'Hôpital's Rule applies.

$$\begin{aligned}
\therefore \quad \lim_{x\to 0} \frac{\ln(\sin x)}{\ln x} &= \lim_{x\to 0} \frac{\frac{\cos x}{\sin x}}{\frac{1}{x}} \\
&= \lim_{x\to 0} \frac{x \cos x}{\sin x}, \qquad \left(= \frac{0}{0} \quad \text{so we apply the H. rule again} \right) \\
&= \lim_{x\to 0} \frac{\cos x - x \sin x}{\cos x} \\
&= 1.
\end{aligned}$$

Example 4.4. What is the value of

$$\lim_{x\to1}\frac{x^2-1}{x^2-x}\ ?$$

Solution. Here, notice that as $x\to 1$, the expression becomes the indeterminate form $\frac{0}{0}$, and so the condition for l'Hôpital's Rule is satisfied. Hence,

$$\begin{aligned}\lim_{x\to1}\frac{x^2-1}{x^2-x} &= \lim_{x\to1}\frac{2x}{2x-1}\\ &= \frac{2}{2-1}=2.\end{aligned}$$

Alternatively, notice that

$$\frac{x^2-1}{x^2-x}=\frac{(x+1)(x-1)}{x(x-1)}=\frac{(x+1)}{x}.$$

$$\begin{aligned}\therefore\quad \lim_{x\to1}\frac{x^2-1}{x^2-x} &= \lim_{x\to1}\frac{(x+1)}{x}\\ &= \frac{1+1}{1}=2.\end{aligned}$$

Remark. The other indeterminate forms 0^0, 1^∞, $0\cdot\infty$, $\infty-\infty$, and ∞^0 can be reduced to the forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ and Theorem 4.2 is applied.

Indeterminate Products, $0\cdot\infty$

Suppose $\lim_{x\to x_0} f(x)=0$ and $\lim_{x\to x_0} g(x)=\pm\infty$. Then write

$$\lim_{x\to x_0}\left(f(x)\cdot g(x)\right)=\lim_{x\to x_0}\frac{f(x)}{\left(\frac{1}{g(x)}\right)}$$

or

$$\lim_{x\to x_0}\left(f(x)\cdot g(x)\right)=\lim_{x\to x_0}\frac{g(x)}{\left(\frac{1}{f(x)}\right)}$$

and apply the l'Hôpital's Rule.

Example 4.5. Evaluate

$$\lim_{x\to0}x\ln x.$$

Solution. The given limit is of the form $0\cdot\infty$. Hence, we rewrite it as

$$\lim_{x\to0}\frac{\ln x}{\left(\frac{1}{x}\right)}$$

which now takes the form $\frac{\infty}{\infty}$. Therefore, applying l'Hôpital's Rule, we have

$$\begin{aligned}\lim_{x\to0}x\ln x &= \lim_{x\to0}\frac{\ln x}{\left(\frac{1}{x}\right)}\\ &= \lim_{x\to0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}\\ &= -\lim_{x\to0}x\\ &= 0.\end{aligned}$$

Indeterminate Powers, 0^0, 1^∞, ∞^0

Suppose $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = \infty$, or $\lim_{x\to x_0} f(x) = \infty$ and $\lim_{x\to x_0} g(x) = 0$, or $\lim_{x\to x_0} f(x) = 1$ and $\lim_{x\to x_0} g(x) = \infty$ and we want to find $\lim_{x\to x_0} (f(x))^{g(x)}$.

- First of all, let another variable, say y be equal to $(f(x))^{g(x)}$, i.e. $y = (f(x))^{g(x)}$.
- Take the logarithm of both sides to get $\ln y = g(x)\ln f(x) = \frac{\ln f(x)}{\left(\frac{1}{g(x)}\right)}$.
- Apply now the l'Hôpital's Rule, to obtain the limit of $\ln y$, that is, $\lim_{x\to x_0}(\ln y) = L$, say.
- Since the natural logarithm is a continuous function we have that $\lim_{x\to x_0}(\ln y) = \ln(\lim_{x\to x_0} y) = L$.
- Finally, we have
$$\lim_{x\to x_0} y = \lim_{x\to x_0} (f(x))^{g(x)} = e^L,$$
where
$$L = \lim_{x\to x_0} g(x)\ln f(x) = \lim_{x\to x_0}\left(\frac{\ln f(x)}{\left(\frac{1}{g(x)}\right)}\right).$$

Example 4.6. Find
$$\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}.$$

Solution. Since $\lim_{x\to 0}\cos x = 1$ and $\lim_{x\to 0}\frac{1}{x^2} = \infty$, the limit is of the form 1^∞. To apply l'Hôpital's Rule, let $f(x) = (\cos x)^{\frac{1}{x^2}}$ so that $\ln f(x) = \frac{1}{x^2}\ln\cos x$.

$$\begin{aligned}
\therefore \lim_{x\to 0}\ln f(x) &= \lim_{x\to 0}\frac{\ln\cos}{x^2}\\
&= \lim_{x\to 0}\frac{-\sin x}{2x\cos x} \qquad \left(= \frac{0}{0} \quad \text{so we apply the H. rule again}\right)\\
&= \lim_{x\to 0}\frac{-\cos x}{-2x\sin x + 2\cos x}\\
& \frac{-1}{0+2} = -\frac{1}{2}.
\end{aligned}$$

But

$$\begin{aligned}
\lim_{x\to 0}\ln f(x) &= \ln\lim_{x\to 0} f(x) = -\frac{1}{2}\\
\therefore \lim_{x\to 0}(\cos x)^{\frac{1}{x^2}} &= e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}.
\end{aligned}$$

Indeterminate Differences, $\infty - \infty$

Suppose $\lim_{x\to x_0} f(x) = \infty = \lim_{x\to x_0} g(x)$ and we seek to find $\lim_{x\to x_0}(f(x) - g(x))$. The trick is to rewrite the difference as a quotient, by using a common denominator, factorising out a common factor, rationalising the denominator, etc, so that the indeterminate form becomes one of either $\frac{0}{0}$ or $\frac{\infty}{\infty}$. That is, by writing

$$\begin{aligned}\lim_{x\to x_0}\left(f(x)-g(x)\right) &= \lim_{x\to x_0} f(x)\left(1-\frac{g(x)}{f(x)}\right)\\ &= \lim_{x\to x_0}\left(\frac{(f(x))^2-f(x)\,g(x)}{f(x)}\right)\end{aligned}$$

and apply the l'Hôpital's Rule.

Example 4.7. Evaluate

$$\lim_{x\to\frac{\pi}{2}^-}\left(\sec x-\tan x\right).$$

Solution. Notice that $\lim_{x\to\frac{\pi}{2}^-}(\sec x-\tan x)=\infty-\infty$. However,

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}^-}\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right) &= \lim_{x\to\frac{\pi}{2}^-}\left(\frac{1-\sin x}{\cos x}\right) \qquad \left(=\frac{0}{0}\ \text{ so we apply the H. rule}\right)\\ &= \lim_{x\to\frac{\pi}{2}^-}\frac{\cos x}{\sin x}\\ &= \frac{0}{1}=0.\end{aligned}$$

Problem 4

1. Show that, if

$$f(x)=\left(\ln(3x+2)\right)^{\sin^{-1}(2x+5)},$$

   then

$$f'(0)=\left(\frac{\pi}{4\ln 2}+\frac{2\ln\ln 2}{\sqrt{3}}\right)(\ln 2)^{\frac{\pi}{6}}.$$

2. Find the following limits:

   (a)

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}},$$

   (b)

$$\lim_{x\to 0}\left(x+e^x+e^{2x}\right)^{\frac{1}{x}},$$

   (c)

$$\lim_{x\to 0}\left(\frac{e^{2x}-2e^x+1}{\cos 3x-2\cos 2x+\cos x}\right),$$

   (d)

$$\lim_{x\to 0}\left(1+\sin 4x\right)^{\cot x}$$

3. Show that

$$\lim_{x\to 0}x^{\sin x}=1$$

4. Use l'Hôpilal's Rule to show that

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{n!}{e^x} = 0$$

5. Prove that

$$\left(\frac{d}{dx}\right)^m \left(x^n e^{-\sqrt{x}}\right) \to 0$$

when $x \to \infty$ for all integral m and n.

6. Prove that

$$e^x - 1 - x,\ e^x - 1 + x,\ \text{and}\ 1 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - (1+x)\,e^x$$

are positive and increase steadily for positive x.

7. For what values of a and b is the following equation true?

$$\lim_{x\to 0} \left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = 0.$$

8. For what values of a and b is the following equation true?

$$\lim_{x\to 0} \left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = 0.$$

9. Prove that each of the following limits evaluate to 1:

$$\lim_{x\to\infty} x^{\frac{1}{x}}, \quad \lim_{x\to 0} \sin x^{\sin x}, \quad \lim_{x\to 0} \tan^x x$$

10. Find

$$\lim_{x\to 0} \frac{\sin x + \cos x - e^x}{\ln(1+x^2)}.$$

11. m

12. m

13. m

Chapter 5

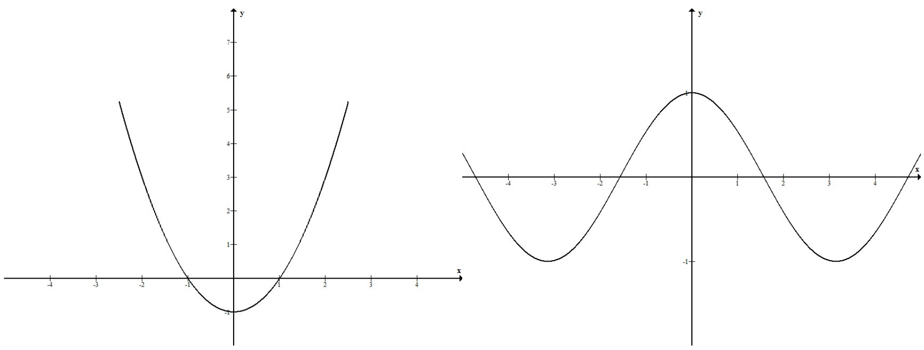
Hyperbolic Functions

5.1 Even and Odd Functions

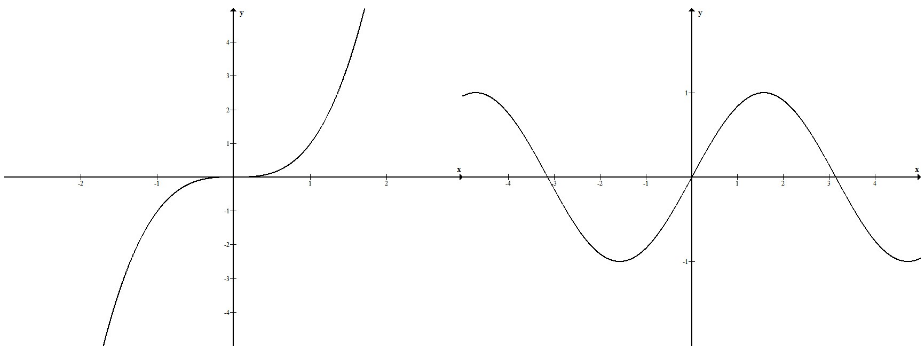
Definition 5.1. Even and Odd Functions.

A function f is said to be *even* if $f(-x) = f(x)$ and *odd* if $f(-x) = -f(x)$ for all x in its domain. Typical examples of even and odd functions are $f(x) = x^{2k}$ and $f(x) = x^{2k+1}$ respectively for some $k \in \mathbb{Z}$.

The graphs of both even and odd functions exhibit symmetries; whilst even functions are symmetric about the coordinate axes, odd functions are symmetric about the origin.



Examples of even functions showing symmetry about the vertical axis



Examples of odd functions showing symmetry about the origin

Theorem 5.2. *Let $f(x)$ be any given function. Then the functions $\frac{1}{2}(f(x) + f(-x))$ and $\frac{1}{2}(f(x) - f(-x))$ are even and odd functions respectively. That is, every function can be written as the sum of an even and an odd function.*

Proof. Let $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ and $f_o(x) = \frac{1}{2}(f(x) - f(-x))$. Then

$$\begin{aligned} f_e(-x) &= \tfrac{1}{2}(f(-x) + f(x)) \\ &= \tfrac{1}{2}(f(x) + f(-x)) \\ &= f_e(x), \end{aligned}$$

and

$$\begin{aligned} f_o(-x) &= \tfrac{1}{2}(f(-x) - f(x)) \\ &= -\tfrac{1}{2}(f(x) - f(-x)) \\ &= -f_o(x). \end{aligned}$$

Now,

$$\begin{aligned} f_e(x) + f_o(x) &= \tfrac{1}{2}(f(x) + f(-x)) + \tfrac{1}{2}(f(x) - f(-x)) \\ &= \tfrac{1}{2}(f(x) + f(-x) + f(x) - f(-x)) \\ &= f(x), \end{aligned}$$

completing the proof. □

Basic properties of Odd and Odd Functions

- The sum (or difference) of two even functions is even, and any constant multiple of an even function is even.
- The sum (or difference) of two odd functions is odd, and any constant multiple of an odd function is odd.
- The product of two even functions is an even function but the product of two odd functions is an even function. However, the product of an even function and an odd function is an odd function
- The quotient of two even or two odd functions is an even function, but the quotient of an even function and an odd function is an odd function.
- The derivative of an even function is odd and the derivative of an odd function is even.
- For any finite $a \in \mathbb{R}$, we have

  $$\int_{-a}^{a} f_o(x) = 0,$$

  and

  $$\int_{-a}^{a} f_e(x) = 2\int_{0}^{a} f_e(x).$$

  and the functions have no vertical asymptotes between $-a$ and a. These also hold true when a is infinite, provided the integrals converge.

Let us consider now the function $f(x) = e^x$. Then clearly, $e^x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})$ and $e^x = \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x})$. More importantly, $\frac{1}{2}(e^x + e^{-x})$ is even and $\frac{1}{2}(e^x - e^{-x})$ is odd.

Problem 5.1

1. m
2. m
3. m
4. m
5. m
6. m
7. m
8. m
9. m
10. m

5.2 Definitions, Derivatives and Identities of the Hyperbolic Functions

Definition 5.3. The Hyperbolic Cosine (cosh) and the Hyperbolic Sine (sinh) Functions.

The even function $\cosh : \mathbb{R} \to [1, \infty)$ defined by

$$\begin{aligned} \cosh x &= \tfrac{1}{2}\left(e^x + e^{-x}\right) \\ &= \tfrac{1}{2}\left(\frac{e^{2x} + 1}{e^x}\right) \\ &= \tfrac{1}{2}\left(\frac{1 + e^{-2x}}{e^{-x}}\right), \quad x \in \mathbb{R} \end{aligned}$$

and the odd function $\sinh : \mathbb{R} \to (-\infty, \infty)$ defined by

$$\begin{aligned} \sinh x : &= \tfrac{1}{2}\left(e^x - e^{-x}\right) \\ &= \tfrac{1}{2}\left(\frac{e^{2x} - 1}{e^x}\right) \\ &= \tfrac{1}{2}\left(\frac{1 - e^{-2x}}{e^{-x}}\right), \quad x \in \mathbb{R} \end{aligned}$$

are called the *hyperbolic cosine* and the *hyperbolic sine* respectively.

Definition 5.4. Other Hyperbolic Functions.

The Hyperbolic Tangent (tanh) Function

$\tanh : \mathbb{R} \to (-1, 1)$

$$\begin{aligned}\tanh x &= \frac{\sinh x}{\cosh x} \\ &= \frac{\frac{1}{2}\left(e^x - e^{-x}\right)}{\frac{1}{2}\left(e^x + e^{-x}\right)} = \frac{e^x\left(e^x - e^{-x}\right)}{e^x\left(e^x + e^{-x}\right)} \\ &= \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}, \quad x \in \mathbb{R}.\end{aligned}$$

The Hyperbolic Cotangent (coth) Function

$\coth : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus [-1, 1]$

$$\begin{aligned}\coth x &= \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} \\ &= \frac{\frac{1}{2}\left(e^x + e^{-x}\right)}{\frac{1}{2}\left(e^x - e^{-x}\right)} = \frac{e^x\left(e^x + e^{-x}\right)}{e^x\left(e^x - e^{-x}\right)} \\ &= \frac{e^{2x} + 1}{e^{2x} - 1} = \frac{1 + e^{-2x}}{1 - e^{-2x}}, \quad x \in \mathbb{R} \setminus \{0\}.\end{aligned}$$

The Hyperbolic Secant (sech) Function

$\text{sech} : \mathbb{R} \to (0, 1]$

$$\begin{aligned}\text{sech} x &= \frac{1}{\cosh x} \\ &= \frac{1}{\frac{1}{2}\left(e^x + e^{-x}\right)} = \frac{2}{e^x + e^{-x}} \\ &= \frac{2e^x}{e^{2x} + 1}, \quad x \in \mathbb{R}.\end{aligned}$$

The Hyperbolic Cosecant (csch) Function

$\text{cosech} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$

$$\begin{aligned}\text{csch} x &= \frac{1}{\sinh x} \\ &= \frac{1}{\frac{1}{2}\left(e^x - e^{-x}\right)} = \frac{2}{e^x - e^{-x}} \\ &= \frac{2e^x}{e^{2x} - 1} = \frac{2e^{-x}}{1 - e^{-2x}}, \quad x \in \mathbb{R} \setminus \{0\}.\end{aligned}$$

Theorem 5.5. Derivative of the Hyperbolic Functions

1. $$\frac{d}{dx}(\cosh x) = \sinh x$$

2.
$$\frac{d}{dx}(\sinh x) = \cosh x$$

3.
$$\frac{d}{dx}(\tanh x) = \text{sech}^2 x$$

4.
$$\frac{d}{dx}(\coth x) = -\text{csch}^2 x$$

5.
$$\frac{d}{dx}(\text{sech} x) = -\text{sech} x \tanh x$$

6.
$$\frac{d}{dx}(\text{csch} x) = -\text{csch} x \coth x.$$

Proof. Using the definitions of the hyperbolic functions, we have □

1.
$$\begin{aligned} \frac{d}{dx}(\cosh x) &= \frac{d}{dx}\left(\tfrac{1}{2}\left(e^x + e^{-x}\right)\right) \\ &= \tfrac{1}{2}\left(e^x - e^{-x}\right) \\ &= \sinh x \end{aligned}$$

2.
$$\begin{aligned} \frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\tfrac{1}{2}\left(e^x - e^{-x}\right)\right) \\ &= \tfrac{1}{2}\left(e^x + e^{-x}\right) \\ &= \cosh x \end{aligned}$$

3.
$$\begin{aligned} \frac{d}{dx}(\tanh x) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\ &= \frac{1}{\cosh^2 x} \\ &= \text{sech}^2 x. \end{aligned}$$

4.
$$\begin{aligned} \frac{d}{dx}(\coth x) &= \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) \\ &= \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} \\ &= -\frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x} \\ &= -\frac{1}{\sinh^2 x} \\ &= -\text{csch}^2 x. \end{aligned}$$

5.

$$\begin{aligned}
\frac{d}{dx}(\mathrm{sech}x) &= \frac{d}{dx}\left(\frac{1}{\cosh x}\right)\\
&= -\frac{\sinh x}{\cosh^2 x}\\
&= -\frac{1}{\cosh x}\cdot\frac{\sinh x}{\cosh x}\\
&= -\mathrm{sech}x\tanh x.
\end{aligned}$$

6.

$$\begin{aligned}
\frac{d}{dx}(\mathrm{csch}x) &= \frac{d}{dx}\left(\frac{1}{\sinh x}\right)\\
&= -\frac{\cosh x}{\sinh^2 x}\\
&= -\frac{1}{\sinh x}\cdot\frac{\cosh x}{\sinh x}\\
&= -\mathrm{csch}x\coth x.
\end{aligned}$$

Theorem 5.6. Hyperbolic Identities

1. $\cosh^2 x - \sinh^2 x = 1, \quad x \in \mathbb{R}$
2. $1 - \tanh^2 x = \mathrm{sech}^2 x, \quad x \in \mathbb{R}$
3. $1 + \mathrm{csch}^2 x = \coth^2 x, \quad x \in \mathbb{R}\setminus\{0\}$.

Proof. □

1.

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\tfrac{1}{2}\left(e^x + e^{-x}\right)\right)^2 - \left(\tfrac{1}{2}\left(e^x - e^{-x}\right)\right)^2\\
&= \tfrac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) - \tfrac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)\\
&= \tfrac{1}{4}(2+2)\\
&= 1.
\end{aligned}$$

2. Dividing (i) by $\cosh^2 x$, we have

$$\begin{aligned}
\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} &= \frac{1}{\cosh^2 x}\\
\Rightarrow 1 - \tanh^2 x &= \mathrm{sech}^2 x.
\end{aligned}$$

3. Dividing (i) by $\sinh^2 x$, we have

$$\begin{aligned}
\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} &= \frac{1}{\sinh^2 x}\\
\coth^2 x - 1 &= \mathrm{csch}^2 x\\
\text{or } 1 + \mathrm{csch}^2 x &= \coth^2 x.
\end{aligned}$$

Theorem 5.7. Addition Theorems

1. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
2. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$ and $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x + -1 = 2\sinh x + 1$
3. $\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$ and $\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$
4. $\sinh x \pm \sinh y = 2\sinh\left(\frac{x \pm y}{2}\right)\cosh\left(\frac{x \mp y}{2}\right)$
5. $\cosh x + \cosh y = 2\cosh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$ and $\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$

Proof. By definition, □

1.

$$\begin{aligned}
\sinh(x+y) &= \tfrac{1}{2}\left(e^{x+y} + e^{-(x+y)}\right) \\
&= \tfrac{1}{2}\left(e^x e^y + e^{-x}e^{-y}\right) \\
&= \tfrac{1}{2}\left[(\cosh x + \sinh x)(\cosh y + \sinh y) - (\cosh x - \sinh x)(\cosh y - \sinh y)\right] \\
&= \tfrac{1}{2}(\cosh x \cosh y + \cosh x \sinh x + \sinh x \cosh y + \sinh x \sinh y \\
&= -(\cosh x \cosh y - \cosh x \sinh x - \sinh x \cosh y + \sinh x \sinh y)) \\
&= \tfrac{1}{2}(2\cosh x \sinh y + 2\sinh x \cosh y) \\
&= \cosh x \sinh y + \sinh x \cosh y
\end{aligned}$$

Theorem 5.8. Osborn's Rule
For $i^2 = -1$,

$$\begin{aligned}
\cos &\to \cosh \\
\sin &\to i\sinh.
\end{aligned}$$

Proof. Euler's definition of the exponential function states that, for $i^2 = -1$

$$e^{ix} = \cos x + i\sin x. \tag{5.1}$$

Therefore, we can write

$$\cos x = \tfrac{1}{2}\left(e^{ix} + e^{-ix}\right) = \cosh(ix)$$

$$\begin{aligned}
\sinh x &= \tfrac{1}{2}\left(e^x - e^{-x}\right) \\
&= \tfrac{1}{2}(\cos x + i\sin x - (\cos(-x) + i\sin(-x))) \\
&= \tfrac{1}{2}(\cos x + i\sin x - \cos x + i\sin x) \\
&= i\sin x.
\end{aligned}$$

.

.

.

.

.

□

Example 5.9. To convert the trigonometric identity $\cos^2 x + \sin^2 x = 1$ into an analogous hyperbolic identity, one makes the substitutions $\cos \to \cosh$ and $\sin \to i \sinh$ to obtain

$$\begin{aligned} 1 &= \cosh^2 x + (i \sinh x)^2 \\ &= \cosh^2 x + i^2 \sinh^2 x \\ &= \cosh^2 x - \sinh^2 x, \quad \because i^2 = -1. \end{aligned}$$

5.3 Graphs of the Hyperbolic Functions

Graph of the Hyperbolic Cosine Function

Let $f(x) = \cosh x = \frac{1}{2}(e^x + e^{-x})$. When $x = 0$, we have $f(0) = \frac{1}{2}(e^0 + e^0) = 1$, thus, the point $(0, 1)$ is the y-intercept. However, there is no x-intercept since $f(x) = \frac{1}{2}(e^x + e^{-x}) > 0 \ \forall x \in \mathbb{R}$, that is, the graph of $f(x)$ does not cross the x-axis.

Now,

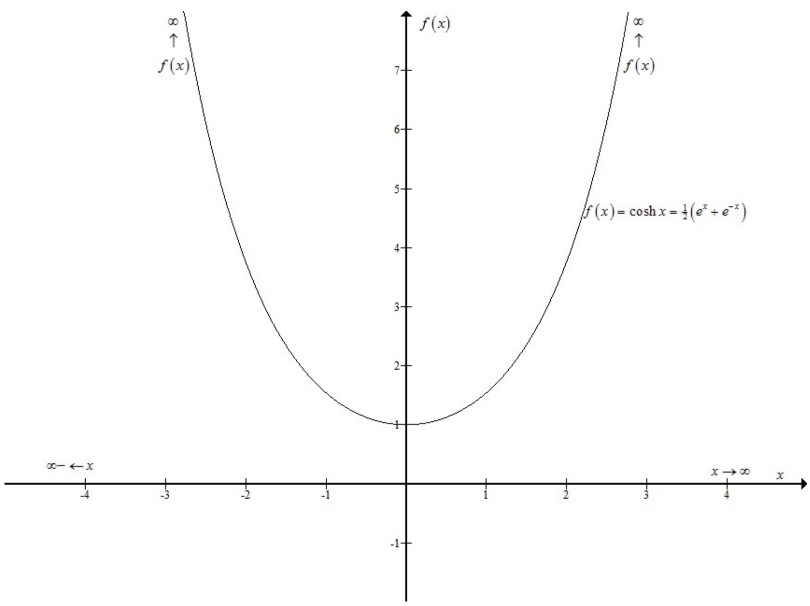
$$\begin{aligned} f'(x) = \frac{d}{dx}(\cosh x) &= \sinh x = \tfrac{1}{2}\left(e^x - e^{-x}\right) \\ &= 0 \Leftrightarrow e^{2x} = 1 \Rightarrow x = 0 \end{aligned}$$

and so the turning point also occurs at $(0, 1)$. In addition,

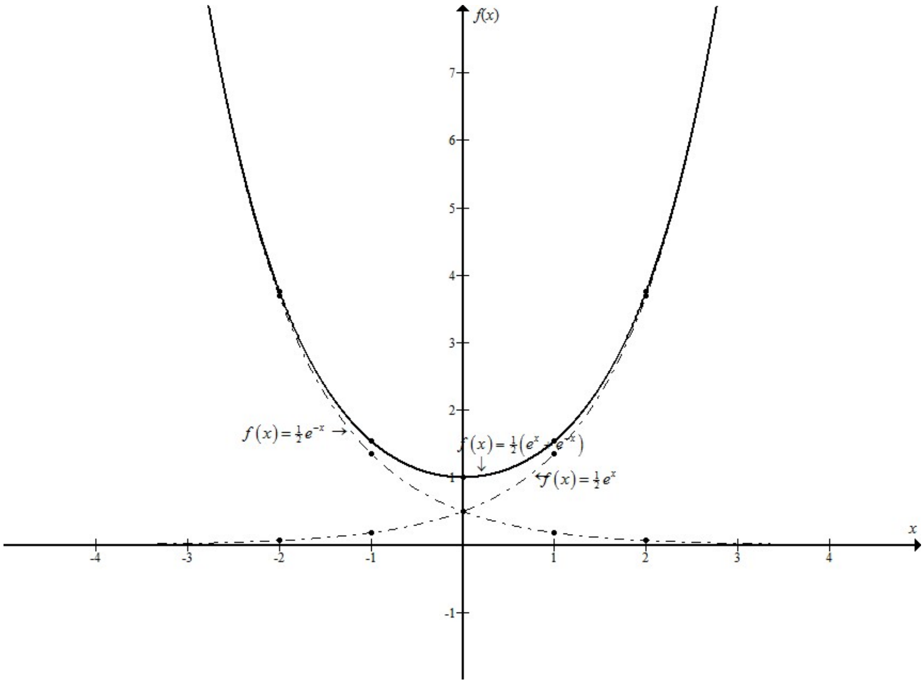
$$\begin{aligned} f''(x) = \frac{d^2}{dx^2}(\cosh x) &= \cosh x \\ &> 0, \ \forall x \in \mathbb{R}. \end{aligned}$$

Therefore, the graph is concave upward and there is no point of inflexion.

Also, as $x \to \pm\infty$, $f(x) \to \frac{1}{2}\left(\frac{1+e^{-2x}}{e^{-x}}\right)\Big|_{x=\infty} = \frac{1}{2}\left(\frac{1+0}{0}\right) = \infty$. Hence, the graph of $f(x) = \cosh x$ is shown below:



Alternatively, one can obtain the graph of $f(x) = \cosh x = \frac{1}{2}(e^x + e^{-x})$ as a pointwise sum of the graphs of the functions $f(x) = \frac{1}{2}e^x$ and $f(x) = \frac{1}{2}e^{-x}$ as shown in the diagram below:



Graph of the Hyperbolic Sine Function

Let $g(x) = \sinh x = \frac{1}{2}(e^x - e^{-x})$. Then when $x = 0$, we have $g(0) = \frac{1}{2}(e^0 - e^0) = 0$ and when $g(x) = 0$, we get $e^x - e^{-x} = 0 \Rightarrow e^{2x} = 1 \Leftrightarrow x = 0$. Thus, the point $(0, 0)$ is the intercept on both axes.

To determine the turning point(s), we see that

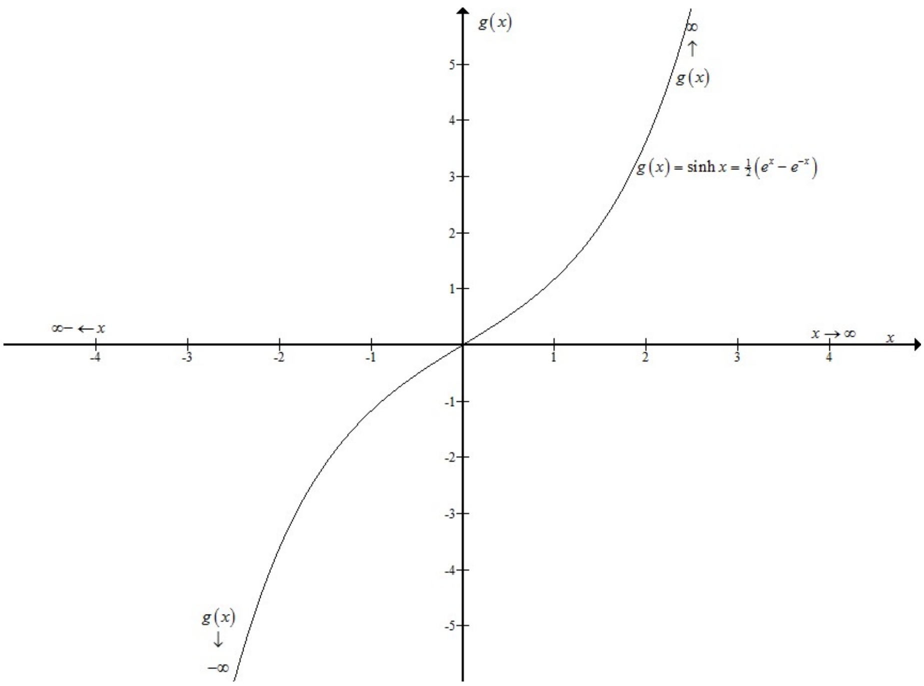
$$g'(x) = \frac{d}{dx}(\sinh x) = \cosh x > 0 \ \forall x \in \mathbb{R}$$

and so there is no turning point. However,

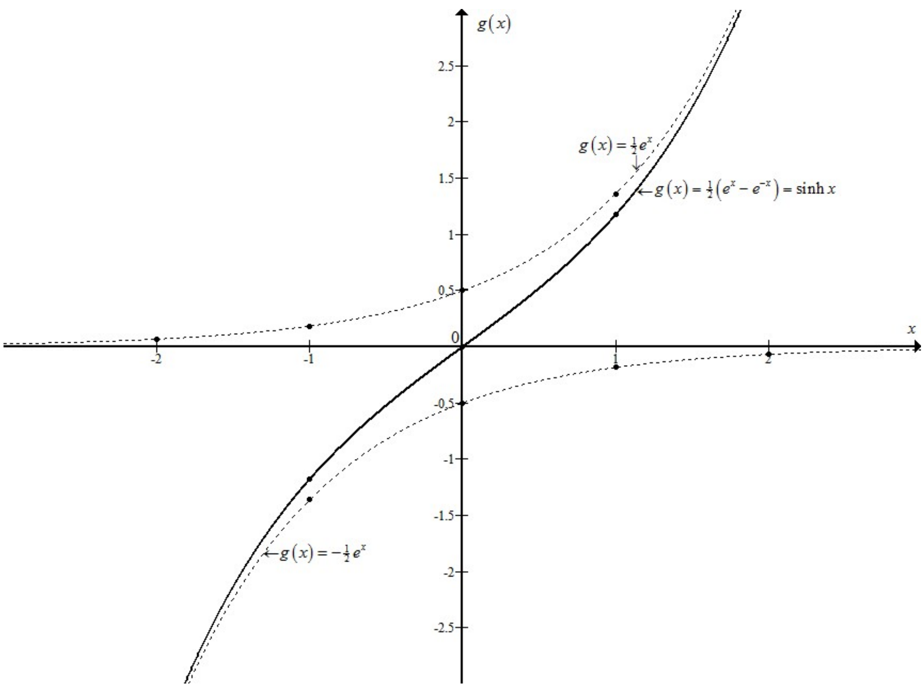
$$\begin{aligned} g''(x) = \frac{d^2}{dx^2}(\sinh x) &= \sinh x \\ &= 0 \Leftrightarrow e^{2x} = 1 \Rightarrow x = 0, \end{aligned}$$

thus, $(0, 0)$ is a point of inflexion, and the graph is concave upward for $x > 0$ and concave downward for $x < 0$ since $g''(x) > 0$ on $(0, \infty)$ and $g''(x) < 0$ on $(-\infty, 0)$.

Now, as $x \to \infty$, $f(x) \to \frac{1}{2}\left(\frac{1-e^{-2x}}{e^{-x}}\right)\Big|_{x=\infty} = \frac{1}{2}\left(\frac{1-0}{0}\right) = \infty$ and as $x \to -\infty$, $f(x) \to \frac{1}{2}\left(\frac{e^{2x}-1}{e^x}\right)\Big|_{x=-\infty} = \frac{1}{2}\left(\frac{0-1}{0}\right) = -\infty$. Hence, the graph of $f(x) = \sinh x$ is shown below:



As in the case of the graph of the hyperbolic cosine function, we can *alternatively* obtain the graph of $g(x) = \sinh x = \frac{1}{2}(e^x - e^{-x})$ as a pointwise sum of the graphs of the functions $g(x) = \frac{1}{2}e^x$ and $g(x) = -\frac{1}{2}e^{-x}$ as shown in the diagram below:



Graph of the Hyperbolic Tangent Function

Let $f(x) = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$. When $x = 0$, we have $f(0) = \frac{e^0 - e^0}{e^0 + e^0} = 0$, and when $f(x) = 0$, we get $\frac{e^x - e^{-x}}{e^x + e^{-x}} = 0 \Leftrightarrow e^{2x} = 1 \Rightarrow x = 0$. Thus, the point $(0, 0)$ is the intercept on both axes.

Now,

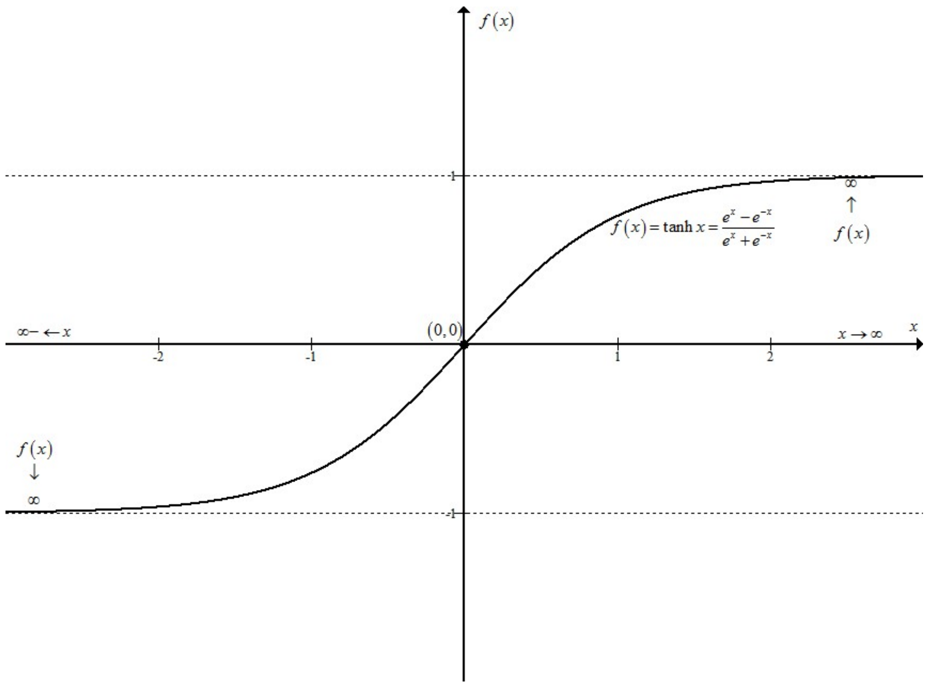
$$f'(x) = \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$
$$= \frac{2e^x}{e^{2x}+1}$$
$$> 0 \quad \forall x \in \mathbb{R},$$

and so there is no turning point. But since,

$$f''(x) = \frac{d^2}{dx^2}(\tanh x) = \frac{\left(e^{2x}+1\right)e^x - 2e^{2x}\left(e^x\right)}{\left(e^{2x}+1\right)^2}$$
$$= 0 \Leftrightarrow e^{2x} = 1 \Rightarrow x = 0,$$

the point $(0,0)$ is also a point of inflexion.

Also, as $x \to \infty$, $f(x) \to \frac{1-e^{-2x}}{1+e^{-2x}}\Big|_{x=\infty} = \frac{1-0}{1+0} = 1$, and as $x \to -\infty$, $f(x) \to \frac{e^{2x}-1}{e^{2x}+1}\Big|_{x=-\infty} \frac{0-1}{0+1} = -1$. That is, the lines $f(x) = \pm 1$ are horizontal asymptotes.



Graph of the Hyperbolic Cotangent Function

Let $f(x) = \coth x = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1} = \frac{1+e^{-2x}}{1-e^{-2x}}$, $x \neq 0$. Notice that as $x \to 0$, $f(x) \to \infty$ or in other words $f(0) = \frac{e^0+e^0}{e^0-e^0} = \frac{2}{0} = \infty$, meaning that there is no y-intercept but the line $x = 0$ is a vertical asymptote. And when $f(x) = 0$, we have $\frac{e^x+e^{-x}}{e^x-e^{-x}} = 0 \Leftrightarrow e^{2x} = -1$, Thus, there is no x-intercept too since $\nexists x \in \mathbb{R}$ such that $e^{2x} = -1$. Moreover, as $x \to \infty$,

$f(x) \to \frac{1+e^{-2x}}{1-e^{-2x}}\Big|_{x=\infty} = \frac{1+0}{1-0} = 1$, and as $x \to -\infty$, $f(x) \to \frac{e^{2x}+1}{e^{2x}-1}\Big|_{x=-\infty} \frac{0+1}{0-1} = -1$. Hence, the lines $f(x) = \pm 1$ are horizontal asymptotes.

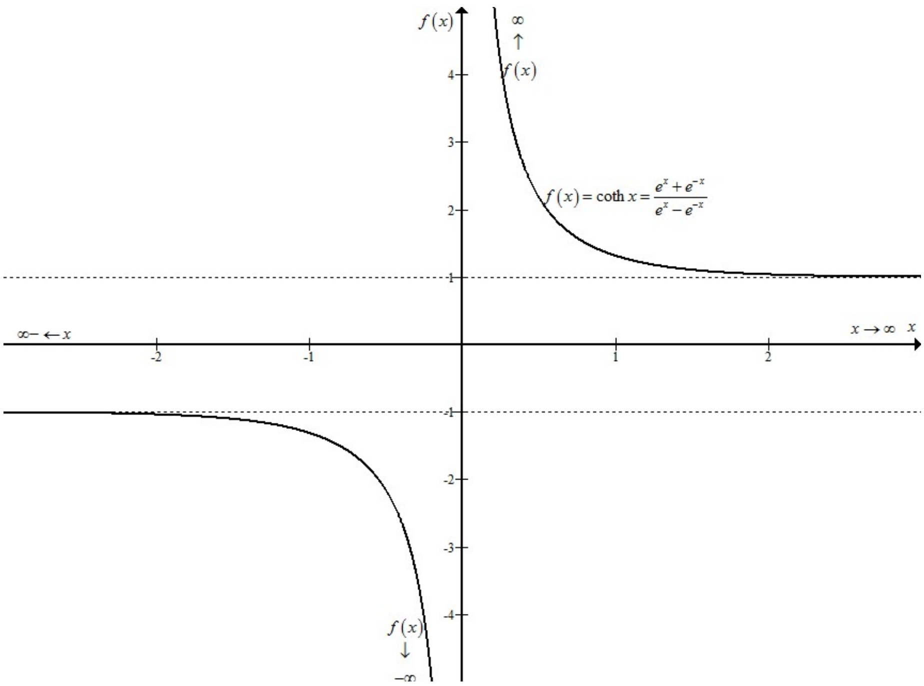
Now,

$$\begin{aligned} f'(x) = \frac{d}{dx}(\coth x) &= -\text{cosech}^2 x \\ &= -\frac{4e^{2x}}{(e^{2x}-1)^2} \\ &< 0 \quad \forall x \in \mathbb{R} \setminus \{0\}, \end{aligned}$$

and so there is no turning point. However,

$$\begin{aligned} f''(x) = \frac{d^2}{dx^2}(\coth x) &= -\frac{d}{dx}\left(\frac{4e^{2x}}{(e^{2x}-1)^2}\right) \\ &= -\frac{8e^{4x} - 8e^{2x} - 16e^{4x}}{(e^{2x}-1)^3} \\ &= \frac{8e^{2x}(e^{2x}+1)}{(e^{2x}-1)^3}. \end{aligned}$$

Thus, $f''(x) > 0$ when $x > 0$ and $f''(x) < 0$ when $x < 0$. Hence, $f(x)$ is concave upward for $(0, \infty)$ and concave downward for $(-\infty, 0)$. There is no point of inflexion since $f''(x) \neq 0$ for any real value of x.



Graph of the Hyperbolic Secant Function

Let $f(x) = \text{sech}x = \frac{2}{e^x+e^{-x}} = \frac{2e^x}{e^{2x}+1} = \frac{2e^{-x}}{1+e^{-2x}}$. When $x = 0$, we have $f(0) = \frac{2}{e^0+e^0} = 1$. But $f(x) = \frac{2e^{-x}}{1+e^{-2x}} \neq 0$ for any real x, so the only intercept is $(0, 1)$.

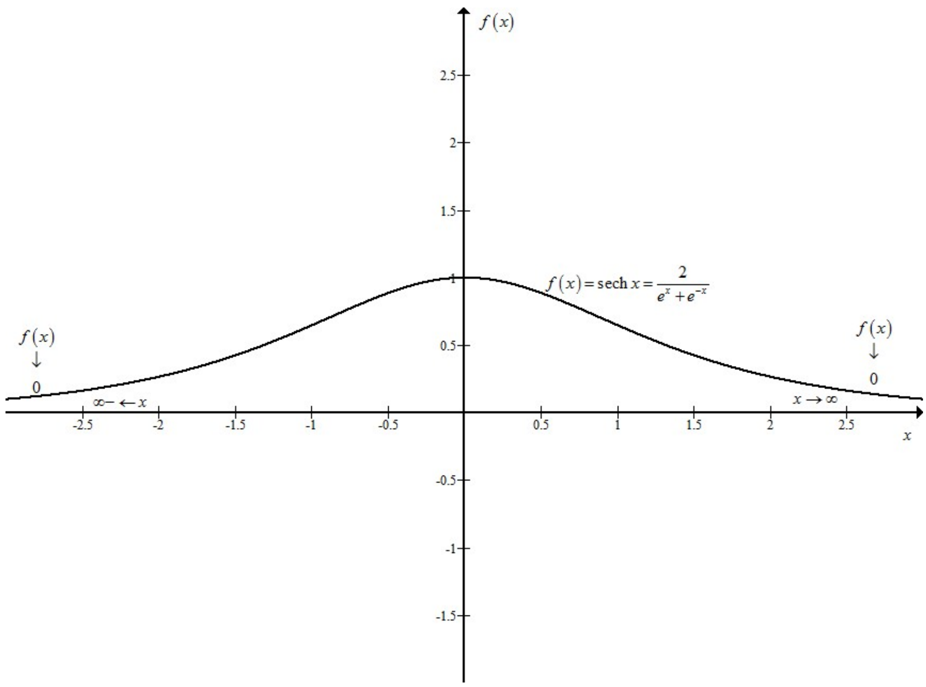
Notice also that, as $x \to \infty$, $f(x) \to \frac{2e^{-x}}{1+e^{-2x}}\Big|_{x=\infty} = \frac{2(0)}{1+0} = 0^+$ and as $x \to -\infty$, $f(x) \to \frac{2e^{x}}{e^{2x}+1}\Big|_{x=-\infty} = \frac{2(0)}{0+1} = 0^+$. Thus, the line $f(x) = 0$ is a horizontal asymptote.
To find our turning points, we have

$$\begin{aligned} f'(x) = \frac{d}{dx}(\text{sech}x) &= -\text{sech}x \tanh x \\ &= -\frac{2e^{-x}}{1+e^{-2x}} \cdot \frac{1-e^{-2x}}{1+e^{-2x}} \\ &= \frac{2e^{-x}\left(e^{-2x}-1\right)}{\left(1+e^{-2x}\right)^2} \\ &= 0 \Leftrightarrow 2e^{-x}\left(e^{-2x}-1\right) = 0 \Rightarrow x = 0. \end{aligned}$$

Therefore, the turning point of the curve is $(0, 1)$. Also,

$$\begin{aligned} f''(x) = \frac{d^2}{dx^2}(\text{sech}x) &= \frac{d}{dx}(-\text{sech}x \tanh x) \\ &= -\left(-\text{sech}x \tanh x \cdot \tanh x + \text{sech}x \cdot \text{sech}^2 x\right) \\ &= \text{sech}x \tanh^2 x - \text{sech}^3 x \\ &< 0 \ \forall x \in \mathbb{R}, \end{aligned}$$

and so the the curve is concave downwards.



Graph of the Hyperbolic Cosecant Function

Let $f(x) = \text{csch}x = \frac{2}{e^x - e^{-x}} = \frac{2e^x}{e^{2x}-1} = \frac{2e^{-x}}{1-e^{-2x}}$, $x \neq 0$. Here also, as $x \to 0$, $f(x) \to \frac{e^0+e^0}{e^0-e^0} = \frac{2}{0} = \infty$, thus, the line $x = 0$ is a vertical asymptote. Also, $f(x) = 0 \Leftrightarrow e^{2x} = 0$, and since there is no real number x such that $e^{2x} = 0$, we conclude that there is no x-intercept too but rather the line $f(x) = 0$ a horizontal asymptote, since as $x \to \infty$, $f(x) \to \left.\frac{2e^{-x}}{1-e^{-2x}}\right|_{x=\infty} = \frac{2(0)}{1-0} = 0^+$, and as $x \to -\infty$, $f(x) \to \left.\frac{2e^{x}}{e^{2x}-1}\right|_{x=-\infty} \frac{2(0)}{0-1} = 0^-$.

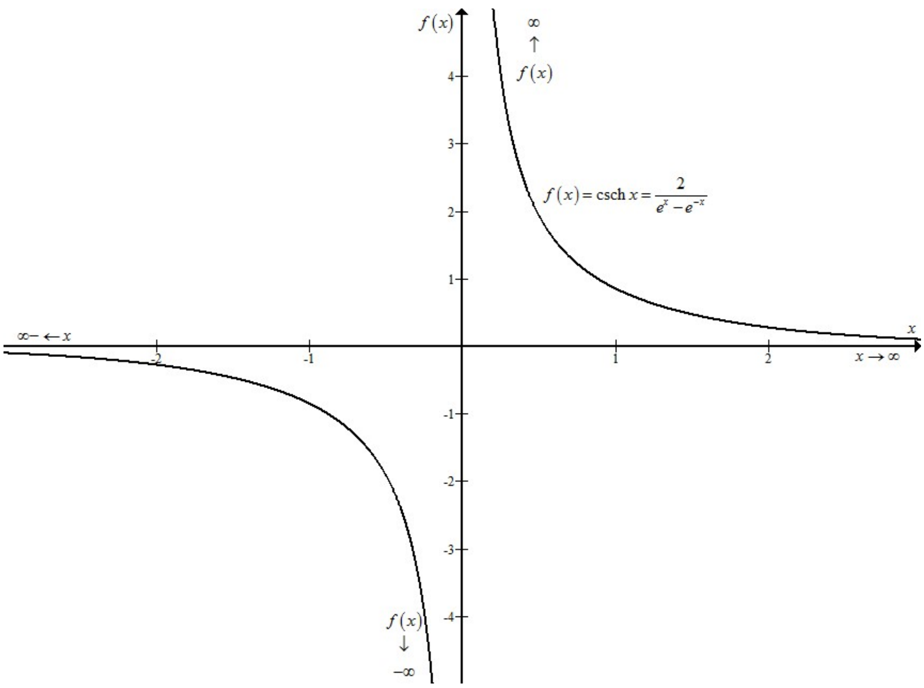
Also,

$$\begin{aligned} f'(x) = \frac{d}{dx}(\text{csch}x) &= -\text{csch}x \coth x \\ &= -\frac{e^x}{e^{2x}-1} \cdot \frac{e^{2x}+1}{e^{2x}-1} \\ &= -\frac{e^x\left(e^{2x}+1\right)}{\left(e^{2x}-1\right)^2} \\ &< 0 \quad \forall x \in \mathbb{R} \setminus \{0\}, \end{aligned}$$

and so there is no turning point. In addition,

$$\begin{aligned} f''(x) = \frac{d^2}{dx^2}(\text{csch}x) &= -\frac{d}{dx}(\text{csch}x \coth x) \\ &= -\left(-\text{csch}x \coth x \cdot \coth x - \text{csch}x \cdot \text{csch}^2 x\right) \\ &= \text{csch}x \coth^2 x + \text{csch}^3 x \end{aligned}$$

which is positive when $x > 0$ and negative when $x < 0$. Hence, $f(x)$ is concave upward for $(0, \infty)$ and concave downward for $(-\infty, 0)$. There is no point of inflexion since $f''(x) \neq 0$ for any real value of $x \neq 0$.



5.4 Inverse Hyperbolic Functions, Their Graphs and Derivatives

The Inverse Hyperbolic Cosine ($\cosh^{-1}$) Function

Notice that $f(x) = \cosh x$ is not $1-1$ since

$$f'(x) = \sinh x \begin{cases} \geq 0 & \text{on } [0, \infty) \\ < 0 & \text{on } (\infty, 0). \end{cases}$$

That is, $f(x) = \cosh x$ increases for $x \geq 0$ and decreases for $x < 0$, and so the inverse does not exist on $(-\infty, \infty)$. However, if we restrict the domain of $f(x) = \cosh x$ to either $[0, \infty)$ or $(-\infty, 0)$, then we can define its inverse. By convention, we choose the former and define the inverse hyperbolic cosine function

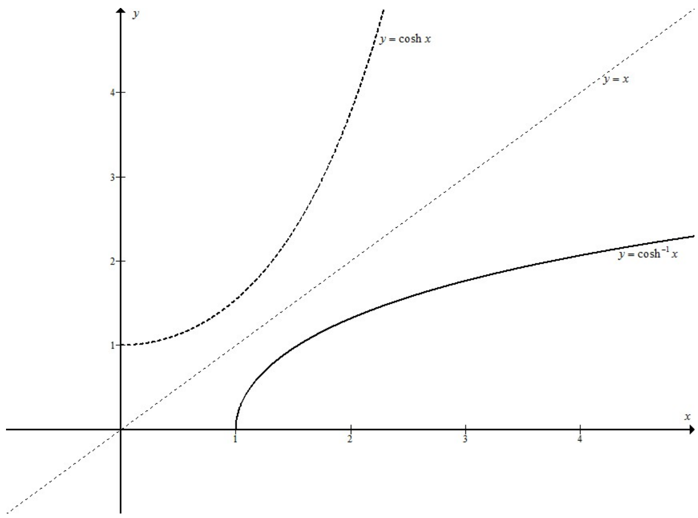
$$\cosh^{-1} : [1, \infty) \to [0, \infty)$$

by

$$y = \cosh^{-1} x \Leftrightarrow \cosh y = x, \quad x \geq 0 \tag{5.2}$$

Graph of $y = \cosh^{-1} x$.

We obtain the graph of the $y = \cosh^{-1} x, \;\; x \geq 1$ by reflecting the graph of $y = \cosh x$ in the line $y = x$ for $x \geq 0$.



The Derivative of $\cosh^{-1} x$

From (5.2), differentiating implicitly with respect to x, we have

$$\sinh y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y}.$$

But $\cosh^2 y - \sinh^2 y = 1 \Rightarrow \sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$.

$$\therefore \quad \frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2 - 1}}.$$

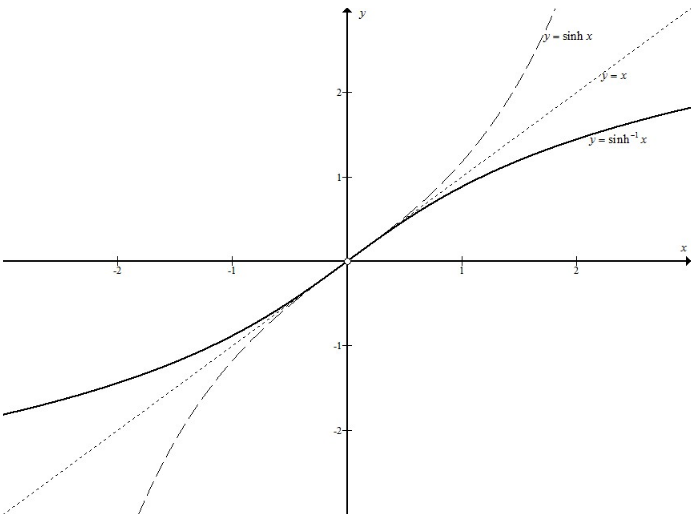
The Inverse Hyperbolic Sine $(\sinh^{-1})$ Function

Let $f(x) = \sinh x, \; x \in \mathbb{R}$. Then since $f'(x) = \cosh x > 0 \; \forall x \in \mathbb{R}$, $f(x) = \sinh x$ is strictly increasing $\mathbb{R}$ and so the inverse $f^{-1} :\equiv \sinh^{-1} : \mathbb{R} \to \mathbb{R}$ exists, and is defined by

$$y = \sinh^{-1} x \Leftrightarrow \sinh y = x, \;\; x \in \mathbb{R} \tag{5.3}$$

Graph of $y = \sinh^{-1} x$.

Again, we obtain the graph of the $y = \sinh^{-1} x, \;\; x \in \mathbb{R}$ by reflecting the graph of $y = \sinh x$ in the line $y = x$ for $x \in \mathbb{R}$.



The Derivative of $\sinh^{-1} x$.

From (5.3), we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\cosh y} \\ &= \frac{1}{\sqrt{\sinh^2 y + 1}} \\ \text{i.e.} \quad \frac{d}{dx}\left(\sinh^{-1} x\right) &= \frac{1}{\sqrt{x^2 + 1}}. \end{aligned}$$

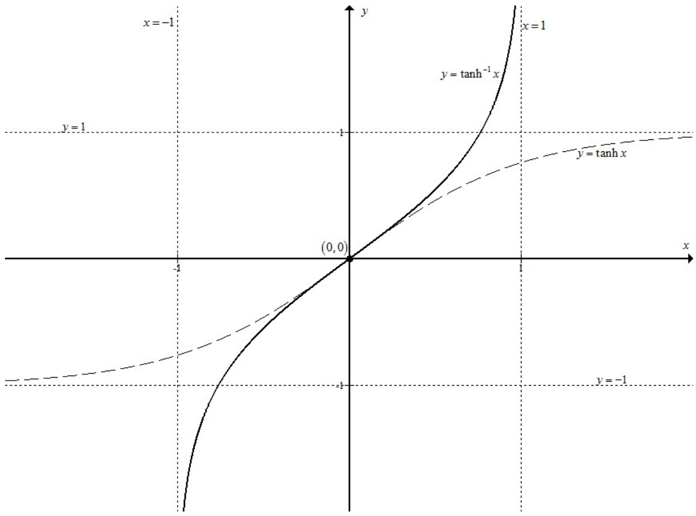
The Inverse Hyperbolic Tangent $(\tanh^{-1})$ Function

Let $f(x) = \tanh x,\ x \in \mathbb{R}$. Then $f'(x) = \text{sech}^2 x > 0\ \forall x \in \mathbb{R}$. Therefore, $f(x) = \tanh x$ strictly increases for all x in $\mathbb{R}$ and thus has an inverse $f^{-1} :\equiv \tanh^{-1} : (-1,1) \to \mathbb{R}$. That is,

$$y = \tanh^{-1} x \Leftrightarrow \tanh y = x, \quad x \in (-1,1) \tag{5.4}$$

Graph of $y = \tanh^{-1} x$.

Again, we obtain the graph of the $y = \tanh^{-1} x,\ x \in \mathbb{R}$ by reflecting the graph of $y = \tanh x$ in the line $y = x$ for $x \geq 0$.



The Derivative of $\tanh^{-1} x$.

From (5.4), we get

$$\text{sech}^2 y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\text{sech}^2 y}.$$

But since $\text{sech}^2 y = 1 - \tanh^2 y$, we get

$$\begin{aligned} \frac{d}{dx}\left(\tanh^{-1} x\right) &= \frac{1}{1 - \tanh^2 y} \\ &= \frac{1}{1 - x^2}. \end{aligned}$$

The Inverse Hyperbolic Cotangent $(\coth^{-1})$ Function

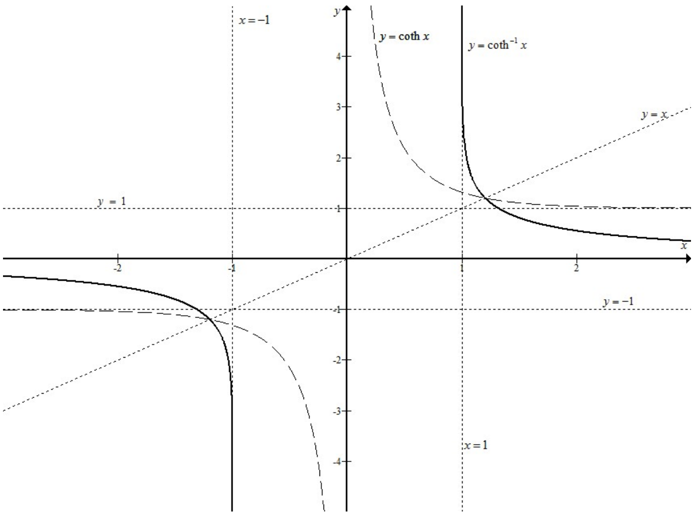
Here, $f(x) = \coth x \Rightarrow f'(x) = -\text{csch}^2 x < 0\ \forall x \in \mathbb{R} \setminus \{0\}$ .That is, $f(x) = \coth x$ decreases for $x \geq 0$ and decreases for $\forall x \in \mathbb{R} \setminus \{0\}$, and so has an inverse

$$\coth^{-1} : \mathbb{R} \setminus [-1,1] \to \mathbb{R} \setminus \{0\}$$

defined as

$$y = \coth^{-1} x \Leftrightarrow \coth y = x, \;\; \forall x \in \mathbb{R} \setminus [-1, 1]. \tag{5.5}$$

Graph of $y = \coth^{-1} x$.



The Derivative of $\coth^{-1} x$.

$$\begin{aligned} \frac{d}{dx}\left(\coth^{-1} x\right) &= \frac{1}{-\text{csch}^2 y} \\ &= \frac{1}{\coth^2 y - 1} \\ &= \frac{1}{x^2 - 1}. \end{aligned}$$

The Inverse Hyperbolic Secant (sech^{-1}) Function

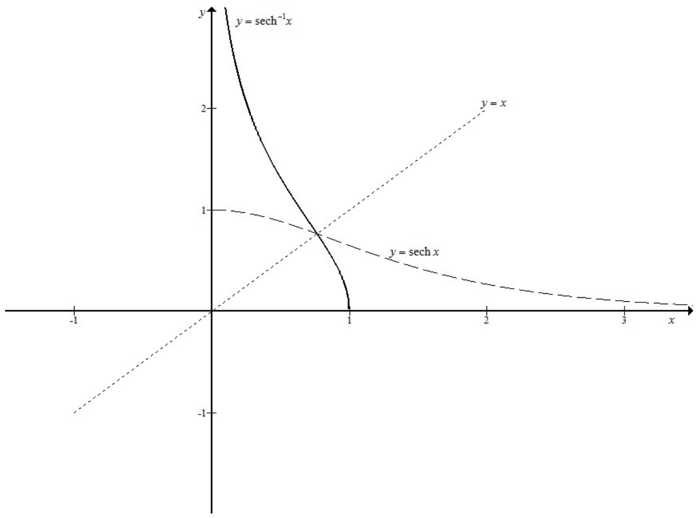
$f(x) = \text{sech}x \Rightarrow f'(x) = -\text{sech}x \tanh x$ and so $f'(x) > 0$ on $[0, \infty)$ and $f'(x) > 0$ on $(-\infty, 0)$. Hence, $f(x) = \text{sech}x$ increases for $x < 0$ and decreases for $x \geq 0$. It means $f(x) = \text{sech}x$ is not $1 - 1$ on $(-\infty, \infty)$, and so by convention, we restrict the domain to $[0, \infty)$ so that $f(x) = \text{sech}x$ has the inverse

$$\text{sech}^{-1} : [1, \infty) \to \mathbb{R}$$

defined by

$$y = \text{sech}^{-1} x \Leftrightarrow \text{sech}y = x, \;\; \forall x \in \mathbb{R}. \tag{5.6}$$

Graph of $y = \mathbf{sech}^{-1}x$.



The Derivative of $\mathbf{sech}^{-1}x$.

From (5.6), we have

$$\begin{aligned}\frac{d}{dx}\left(\operatorname{sech}^{-1}x\right) &= \frac{1}{-\operatorname{sech}y\tanh y} \\ &= \end{aligned}$$

The Inverse Hyperbolic Cosecant $(\mathbf{csch}^{-1})$ Function

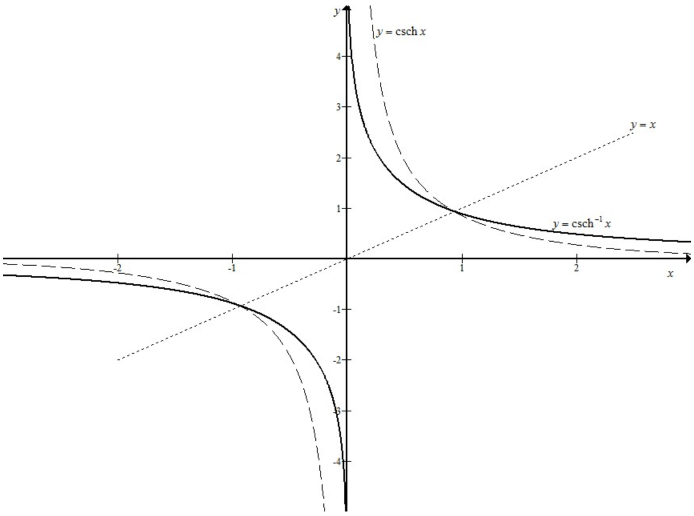
$f(x) = \operatorname{csch}x \Rightarrow f'(x) = -\operatorname{csch}x\coth x < 0\ \forall x \in \mathbb{R}\setminus\{0\}$ .That is, $f(x) = \coth x$ decreases for $\forall x \in \mathbb{R}\setminus\{0\}$, and so has an inverse

$$\operatorname{csch}^{-1} : \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\},$$

defined by

$$y = \operatorname{csch}^{-1}x \Leftrightarrow \operatorname{csch}y = x,\ \ \forall x \in \mathbb{R}\setminus\{0\}. \tag{5.7}$$

Graph of $y = \mathbf{csch}^{-1}x$.



The Derivative of $\operatorname{csch}^{-1}x$.

From (5.7), we have

5.5 Logarithmic Equivalence of the Inverse Hyperbolic Functions

Theorem 5.10. *We can write the inverse hyperbolic functions in terms of the logarithmic functions as follows*

1. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1.$
2. $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right), \quad x \in \mathbb{R}.$
3. $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \ln\sqrt{\frac{1+x}{1-x}}, \quad |x| < 1.$
4. $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \quad x \in (0, 1]\,.$

5. $\coth^{-1} x = \dfrac{1}{2} \ln \left(\frac{x+1}{x-1}\right) = \ln \sqrt{\frac{x+1}{x-1}}, \quad |x| > 1.$

6. $\operatorname{csch}^{-1} x = \ln \left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right), \quad x \neq 0.$

Proof.

i. Let $y = \cosh^{-1} x$. Then $\cosh y = x$, by definition. That is,

$$\begin{aligned}
\tfrac{1}{2}\left(e^y + e^{-y}\right) &= x \\
\Rightarrow e^y + e^{-y} &= 2x \\
\Rightarrow e^{2y} + 1 &= 2xe^y
\end{aligned}$$

or

$$e^{2y} - 2xe^y + 1 = 0.$$

This is a quadratic equation in e^y, and has solution given by

$$\begin{aligned}
e^y &= \frac{2x \pm \sqrt{(2x)^2 - 4(1)(1)}}{2(1)} \\
&= \frac{2x \pm \sqrt{4x^2 - 4}}{2} \\
&= \frac{2x \pm 2\sqrt{x^2 - 1}}{2} \\
&= x \pm \sqrt{x^2 - 1}.
\end{aligned}$$

Thus

$$e^y = x + \sqrt{x^2 - 1}$$

or

$$\begin{aligned}
e^y &= x - \sqrt{x^2 - 1} \\
&= \frac{\left(x - \sqrt{x^2 - 1}\right)\left(x + \sqrt{x^2 - 1}\right)}{\left(x + \sqrt{x^2 - 1}\right)} \\
&= \frac{x^2 - \left(\sqrt{x^2 - 1}\right)^2}{x + \sqrt{x^2 - 1}} \\
&= \frac{1}{x + \sqrt{x^2 - 1}} \\
&= \left(x + \sqrt{x^2 - 1}\right)^{-1}.
\end{aligned}$$

Hence, we can write

$$e^y = \left(x + \sqrt{x^2 - 1}\right)^{\pm 1},$$

and so,

$$y = \pm \ln \left(x + \sqrt{x^2 - 1}\right).$$

But by definition, $y = \cosh^{-1} x > 0$ for all $x \geq 1$. Hence, we discard $y = -\ln\left(x + \sqrt{x^2 - 1}\right)$, and conclude that

$$\cosh^{-1} x = \ln \left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1.$$

ii.

iii.

iv.

v.

vi. ☐

Example 5.11. Show that

$$\lim_{x\to\infty} \tanh \ln x = 1.$$

Proof. By definition

$$\begin{aligned}\tanh \ln x &= \frac{e^{\ln x} - e^{-\ln x}}{e^{\ln x} + e^{-\ln x}}\\ &= \frac{e^{\ln x} - e^{\ln\left(\frac{1}{x}\right)}}{e^{\ln x} + e^{\ln\left(\frac{1}{x}\right)}}\\ &= \frac{x - \frac{1}{x}}{x + \frac{1}{x}} = \frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x^2}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{x\to\infty} \tanh \ln x &= \lim_{x\to\infty}\left(\frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x^2}}\right)\\ &= \frac{1-0}{1+0} = 1.\end{aligned}$$

☐

Example 5.12. Simplify the expression

$$\frac{\cosh \ln x + \sinh \ln x}{\cosh \ln x - \sinh \ln x}.$$

Solution.

$$\begin{aligned}\frac{\cosh \ln x + \sinh \ln x}{\cosh \ln x - \sinh \ln x} &= \frac{\frac{1}{2}\left(e^{\ln x} + e^{-\ln x}\right) + \frac{1}{2}\left(e^{\ln x} - e^{-\ln x}\right)}{\frac{1}{2}\left(e^{\ln x} + e^{-\ln x}\right) - \frac{1}{2}\left(e^{\ln x} - e^{-\ln x}\right)}\\ &= \frac{x + \frac{1}{x} + x - \frac{1}{x}}{x + \frac{1}{x} - \left(x - \frac{1}{x}\right)}\\ &= \frac{2x}{\frac{2}{x}}\\ &= x^2.\end{aligned}$$

Example 5.13. Find the derivatives of $y = \tanh^{-1}(\sinh x)$, $f(x) = x\sqrt{1+x^2} + \sinh^{-1} x$, $g(x) = x\sqrt{x^2 - a^2} - a^2 \cosh^{-1}\left(\frac{x}{a}\right)$ and $h(x) = x^{\sinh x}$.

Solution. The derivative of $y = \tanh^{-1}(\sinh x)$ is given by

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\tanh^{-1}(\sinh x)\right) \\
&= \frac{d}{d\sinh x}\left(\tanh^{-1}(\sinh x)\right)\cdot\frac{d}{dx}(\sinh x) \\
&= \frac{1}{1-\sinh^2 x}\cdot\cosh x \\
&= \frac{\cosh x}{1-\left(\cosh^2 x-1\right)} \\
&= \frac{\cosh x}{-\cosh^2 x} \\
&= \frac{1}{-\cosh x} = -\mathrm{sech} x.
\end{aligned}$$

Similarly,

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(x\sqrt{1+x^2}+\sinh^{-1} x\right) \\
&= \sqrt{1+x^2}+x\left(\tfrac{1}{2}(2x)\left(1+x^2\right)^{-\frac{1}{2}}\right)+\frac{1}{\sqrt{1+x^2}} \\
&= \sqrt{1+x^2}+\frac{x^2}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}} \\
&= \frac{\left(\sqrt{1+x^2}\right)^2+x^2+1}{\sqrt{1+x^2}} \\
&= \frac{2\left(x^2+1\right)}{\sqrt{1+x^2}} \\
&= \frac{2\left(\sqrt{x^2+1}\right)^2}{\sqrt{1+x^2}} \\
&= 2\sqrt{x^2+1};
\end{aligned}$$

and

$$\begin{aligned}
g'(x) &= \frac{d}{dx}\left(x\sqrt{x^2-a^2}-a^2\cosh^{-1}\left(\frac{x}{a}\right)\right) \\
&= \sqrt{x^2-a^2}+\frac{x^2}{\sqrt{x^2-a^2}}-\frac{a^2}{\sqrt{\left(\frac{x}{a}\right)^2-1}} \\
&= \sqrt{x^2-a^2}+\frac{x^2}{\sqrt{x^2-a^2}}-\frac{a^3}{\sqrt{x^2-a^2}} \\
&= \frac{x^2-a^2+x^2-a^3}{\sqrt{x^2-a^2}} \\
&= \frac{2x^2-a^2-a^3}{\sqrt{x^2-a^2}}.
\end{aligned}$$

To differentiate $h(x)=x^{\sinh x}$, however, we use logarithmic differentiation. Since

$$\ln h(x)=\sinh x\ln x,$$

we have

$$\begin{aligned}
\frac{h'(x)}{h(x)} &= \sinh x \cdot \frac{1}{x} + \cosh x \ln x \\
\Rightarrow h'(x) &= h(x)\left(\frac{1}{x}\sinh x + \cosh x \ln x\right) \\
&= x^{\sinh x}\left(\frac{1}{x}\sinh x + \cosh x \ln x\right) \\
&= x^{\sinh x}\left(\frac{1}{x}\sinh x \ln e + \cosh x \ln x\right) \\
&= x^{\sinh x}\left(\ln e^{\frac{\sinh x}{x}} + \ln x^{\cosh x}\right) \\
&= x^{\sinh x}\left(\ln\left(x^{\cosh x} e^{\frac{\sinh x}{x}}\right)\right).
\end{aligned}$$

Example 5.14. Prove that

$$\tanh^{-1} x - \tanh^{-1} y = \tanh^{-1}\left(\frac{x-y}{1-xy}\right).$$

Proof. Let $\tanh^{-1} x = c$ and $\tanh^{-1} y = k$. Then $\tanh c = x$ and $\tanh y = k$. Now,

$$\begin{aligned}
\tanh(c-k) &= \frac{\tanh c - \tanh k}{1 - \tanh c \tanh k} \\
&= \frac{x-y}{1-xy}. \\
\therefore \quad c-k &= \tanh^{-1}\left(\frac{x-y}{1-xy}\right) \\
\text{i.e. } \tanh^{-1} x - \tanh^{-1} y &= \tanh^{-1}\left(\frac{x-y}{1-xy}\right).
\end{aligned}$$

□

Example 5.15. Show that if $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $y > 0$ and $\cos x \cosh y = 1$, then

$$y = \ln(\sec x + \tan x),$$

$$\frac{dy}{dx} = \sec x$$

and

$$\frac{dx}{dy} = \text{sech} y.$$

Proof. Given that $\cos x \cosh y = 1$, then

$$\begin{aligned}
\cosh y &= \frac{1}{\cos x} = \sec x, \quad -\tfrac{\pi}{2} < x < \tfrac{\pi}{2}, y > 0. \\
\text{i.e. } \tfrac{1}{2}\left(e^{y} + e^{-y}\right) &= \sec x \\
\Rightarrow \quad \left(e^{2y} + 1\right) &= 2\sec x e^{y} \\
\text{or } e^{2y} - 2\sec x e^{y} + 1 &= 0. \\
\therefore \quad e^{y} &= \frac{2\sec x \pm \sqrt{4\sec^{2} x - 4}}{2} \\
&= \sec x \pm \sqrt{\sec^{2} x - 1} \\
&= \sec x \pm \sqrt{\tan^{2} x} = \sec x \pm \tan x. \\
\text{i.e. } \quad e^{y} &= \sec x + \tan x, \qquad \text{or} \\
&= \sec x - \tan x = \frac{(\sec x - \tan x)(\sec x + \tan x)}{\sec x + \tan x} \\
&\qquad = \frac{\sec^{2} x - \tan^{2} x}{\sec x + \tan x} = \frac{1}{\sec x + \tan x} \\
&\qquad = (\sec x + \tan x)^{-1} \\
\therefore \quad e^{y} &= (\sec x + \tan x)^{\pm 1} \\
\Rightarrow y &= \pm \ln(\sec x + \tan x) \\
&= \ln(\sec x + \tan x)
\end{aligned}$$

since $y > 0$. Now,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\ln(\sec x + \tan x)\right) \\
&= \frac{\sec x \tan x + \sec^{2} x}{\sec x + \tan x} \\
&= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} \\
&= \sec x,
\end{aligned}$$

and

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\sec x} = \cos x.$$

But as we were given, $\cos x = \frac{1}{\cosh y} = \text{sech} y$. Hence,

$$\frac{dx}{dy} = \text{sech} y.$$

□

Example 5.16. Prove that

$$16 \sinh^{2} x \cosh^{3} x = \cosh 5x + \cosh 3x - 2\cosh x.$$

Hence, or otherwise, deduce that

$$\int_{0}^{1} 16 \sinh^{2} x \cosh^{3} x dx = \frac{1}{5}\sinh 5 + \frac{1}{3}\sinh 3 - 2\sinh 1.$$

Proof. Since $\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$ and $\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$, we have

$$\begin{aligned}
16\sinh^2 x\cosh^3 x &= 16\left[\left(\frac{1}{2}\left(e^{x} - e^{-x}\right)\right)^2\left(\frac{1}{2}\left(e^{x} + e^{-x}\right)\right)^3\right] \\
&= 16\left[\frac{1}{4}\left(e^{2x} - 2e^{x}e^{-x} + e^{-2x}\right)\left(\frac{1}{8}\left(e^{3x} + 3e^{2x}e^{-x} + 3e^{x}e^{-2x} + e^{-3x}\right)\right)\right] \\
&= \frac{1}{2}\left(e^{2x} - 2e^{x}e^{-x} + e^{-2x}\right)\left(e^{3x} + 3e^{2x}e^{-x} + 3e^{x}e^{-2x} + e^{-3x}\right) \\
&= \frac{1}{2}\left(e^{5x} + 3e^{3x} + 3e^{x} + e^{-x} - 2e^{3x} - 6e^{x} - 6e^{-x} - 2e^{-3x} + e^{x} + 3e^{-x} + 3e^{-3x} + e\right. \\
&= \frac{1}{2}\left(e^{5x} + e^{-5x}\right) + \frac{1}{2}\left(e^{3x} + e^{-3x}\right) + \frac{1}{2}\left(-2e^{x} - 2e^{-x}\right) \\
&= \cosh 5x + \cosh 3x - \cosh x.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^1 16\sinh^2 x\cosh^3 x dx &= \int_0^1 \cosh 5x + \cosh 3x - \cosh x dx \\
&= \left[\frac{1}{5}\sinh 5x + \frac{1}{3}\sinh 3x - 2\sinh x\right]_0^1 \\
&= \frac{1}{5}\sinh 5 + \frac{1}{3}\sinh 3 - 2\sinh 1.
\end{aligned}$$

□

Example 5.17. Show that

$$\sinh u + \sinh(u+v) + \sinh(u+2v) + \cdots + \sinh(u+(n-1)v) = \sinh\left(u + \tfrac{n-1}{2}v\right)\sinh\tfrac{nv}{2}\operatorname{csch}\tfrac{v}{2}.$$

Hence, evaluate

$$\sum_{k=0}^{10}\sinh kx.$$

Proof. Let us denote by S the sum in question; that is,

$$S = \sinh u + \sinh(u+v) + \sinh(u+2v) + \cdots + \sinh(u+(n-1)v) = \sum_{k=1}^{n}\sinh\left[u + (k-1)v\right].$$

Also, let

$$C = \sum_{k=1}^{n}\cosh\left[u + (k-1)v\right].$$

Then

$$\begin{aligned}
C+S &= \sum_{k=1}^{n}\cosh\left[u+(k-1)\,v\right]+\sum_{k=1}^{n}\sinh\left[u+(k-1)\,v\right]\\
&= \sum_{k=1}^{n}\left(\cosh\left[u+(k-1)\,v\right]+\sinh\left[u+(k-1)\,v\right]\right).\\
&= \sum_{k=1}^{n}e^{u+(k-1)v}=\sum_{k=1}^{n}e^{u}\cdot e^{(k-1)v}\\
&= e^{u}\sum_{k=1}^{n}e^{(k-1)v}=e^{u}\left(1+e^{v}+e^{2v}+\cdots+e^{(n-1)v}\right)\\
&= e^{u}\left(\frac{1-e^{nv}}{1-e^{v}}\right). \qquad (5.8)
\end{aligned}$$

Similarly,

$$\begin{aligned}
C-S &= \sum_{k=1}^{n}e^{-(u+(k-1)v)}\\
&= e^{-u}\left(\frac{1-e^{-(n-1)v}}{1-e^{v}}\right). \qquad (5.9)
\end{aligned}$$

Hence, from (5.8) and (5.9), we have that

$$\begin{aligned}
2S &= e^{u}\left(\frac{1-e^{nv}}{1-e^{v}}\right)-e^{-u}\left(\frac{1-e^{-nv}}{1-e^{-v}}\right)\\
&= e^{u}\left(\frac{e^{\frac{nv}{2}}e^{-\frac{nv}{2}}-e^{\frac{nv}{2}}e^{\frac{nv}{2}}}{e^{\frac{v}{2}}e^{-\frac{v}{2}}-e^{\frac{v}{2}}e^{\frac{v}{2}}}\right)-e^{-u}\left(\frac{e^{\frac{nv}{2}}e^{-\frac{nv}{2}}-e^{-\frac{nv}{2}}e^{-\frac{nv}{2}}}{e^{\frac{v}{2}}e^{-\frac{v}{2}}-e^{-\frac{v}{2}}e^{-\frac{v}{2}}}\right)\\
&= e^{u}\left(\frac{e^{\frac{nv}{2}}}{e^{\frac{v}{2}}}\right)\left(\frac{e^{-\frac{nv}{2}}-e^{\frac{nv}{2}}}{e^{-\frac{v}{2}}-e^{\frac{v}{2}}}\right)-e^{-u}\left(\frac{e^{-\frac{nv}{2}}}{e^{-\frac{v}{2}}}\right)\left(\frac{e^{\frac{nv}{2}}-e^{-\frac{nv}{2}}}{e^{\frac{v}{2}}-e^{-\frac{v}{2}}}\right)\\
&= \left(\frac{e^{u+\frac{nv}{2}}}{e^{\frac{v}{2}}}\right)\left(\frac{e^{-\frac{nv}{2}}-e^{\frac{nv}{2}}}{e^{-\frac{v}{2}}-e^{\frac{v}{2}}}\right)-\left(\frac{e^{-\left(u+\frac{nv}{2}\right)}}{e^{-\frac{v}{2}}}\right)\left(\frac{e^{\frac{nv}{2}}-e^{-\frac{nv}{2}}}{e^{\frac{v}{2}}-e^{-\frac{v}{2}}}\right)\\
&= \left(\frac{e^{u+\frac{nv}{2}}}{e^{\frac{v}{2}}}-\frac{e^{-\left(u+\frac{nv}{2}\right)}}{e^{-\frac{v}{2}}}\right)\left(\frac{e^{\frac{nv}{2}}-e^{-\frac{nv}{2}}}{e^{\frac{v}{2}}-e^{-\frac{v}{2}}}\right)\\
&= \left(e^{u+\frac{nv}{2}}\cdot e^{-\frac{v}{2}}-e^{-\left(u+\frac{nv}{2}\right)}\cdot e^{\frac{v}{2}}\right)\left(\frac{e^{\frac{nv}{2}}-e^{-\frac{nv}{2}}}{e^{\frac{v}{2}}-e^{-\frac{v}{2}}}\right)\\
&= \left(e^{u+\frac{1}{2}(n-1)v}-e^{-u+\frac{1}{2}(n-1)v}\right)\left(\frac{e^{\frac{nv}{2}}-e^{-\frac{nv}{2}}}{e^{\frac{v}{2}}-e^{-\frac{v}{2}}}\right)\\
&= \sinh\left[u+\tfrac{1}{2}(n-1)\,v\right]\frac{\sinh\frac{nv}{2}}{\sinh\frac{v}{2}}\\
&= \sinh\left[u+\tfrac{1}{2}(n-1)\,v\right]\sinh\tfrac{nv}{2}\operatorname{csch}\tfrac{v}{2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{k=0}^{10} \sinh kx &= \sinh(0) + \sum_{k=1}^{10} \sinh kx \\
&= \sum_{k=1}^{11} \sinh(0 + (k-1)x) \\
&= \sinh\left[\tfrac{1}{2}(11-1)x\right] \sinh \tfrac{11x}{2} \operatorname{csch} \tfrac{x}{2} \\
&= \sinh 5x \sinh \tfrac{11x}{2} \operatorname{csch} \tfrac{x}{2}.
\end{aligned}$$

□

Solving Equations Involving Hyperbolic and Inverse Hyperbolic Functions

Example 5.18. Solve the equation

$$7 \sinh x + 3 \cosh x = 9.$$

Solution. We solve this equation by first writing the hyperbolic functions in terms of the exponential function; that is,

$$\begin{aligned}
7 \cosh x + 3 \sinh x &= 9. \\
\Rightarrow 7\left[\tfrac{1}{2}\left(e^{x} - e^{-x}\right)\right] + 3\left[\tfrac{1}{2}\left(e^{x} + e^{-x}\right)\right] &= 9 \\
\Rightarrow 7\left(e^{2x} - 1\right) + 3\left(e^{2x} + 1\right) &= 18e^{x} \\
\Rightarrow 10e^{2x} + 4 &= 18e \\
\text{i.e.} \quad 5e^{2x} - 9e^{x} - 2 &= 0 \\
\text{or} \quad (5e^{x} + 1)(e^{x} - 2) &= 0 \\
\Rightarrow e^{x} &= 2. \\
\\
\therefore \quad x &= \ln 2.
\end{aligned}$$

Example 5.19. Solve the equation

$$\operatorname{csch}^{-1} x + \ln x = 3.$$

Solution. We have seen that, for $x \neq 0$, $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$. Therefore, the equation $\operatorname{csch}^{-1} x + \ln x = 3$ becomes

$$\begin{aligned}
\ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right) + \ln x &= 3 \\
\Rightarrow 3 &= \ln\left(x\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)\right) \\
&= \ln\left(1 + \frac{x\sqrt{1+x^2}}{|x|}\right)
\end{aligned}$$

and so

$$1 + \frac{x\sqrt{1+x^2}}{|x|} \quad = \quad e^3$$

or

$$\frac{x\sqrt{1+x^2}}{|x|} = e^3 - 1.$$

Squaring both sides, we have

$$\begin{aligned}
\frac{x^2\left(1+x^2\right)}{x^2} &= \left(e^3-1\right)^2 \\
\Rightarrow \quad 1+x^2 &= e^6 - 2e^3 + 1 \\
\Rightarrow \quad x^2 &= e^6 - 2e^3 = e^3\left(e^3-2\right) \\
\therefore \quad x &= \pm\sqrt{e^3\left(e^3-2\right)}.
\end{aligned}$$

Chapter 6

Integration

6.1 Integration as a Sum: The Riemann Sum

Definition 6.1. Partition of a Set

Let P be a finite sequence of ordered points between the numbers a and b on a real line, say

$$P = \{x_0, x_1, \cdots, x_{n-1}, x_n\} = \{[x_{i-1}, x_i]\}_{i=1}^{n},$$

where $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. This set P is called a ***partition*** of the interval $[a, b]$.

The partition $[x_{i-1}, x_i]$, $i = 1, \cdots, n$ divides an interval $[a, b]$ into n *subintervals*; and the *length* of the ith subinterval is

$$\Delta x_i = x_i - x_{i-1}, \quad 1 \leq i \leq n.$$

Thus, a partition is a collection of subintervals

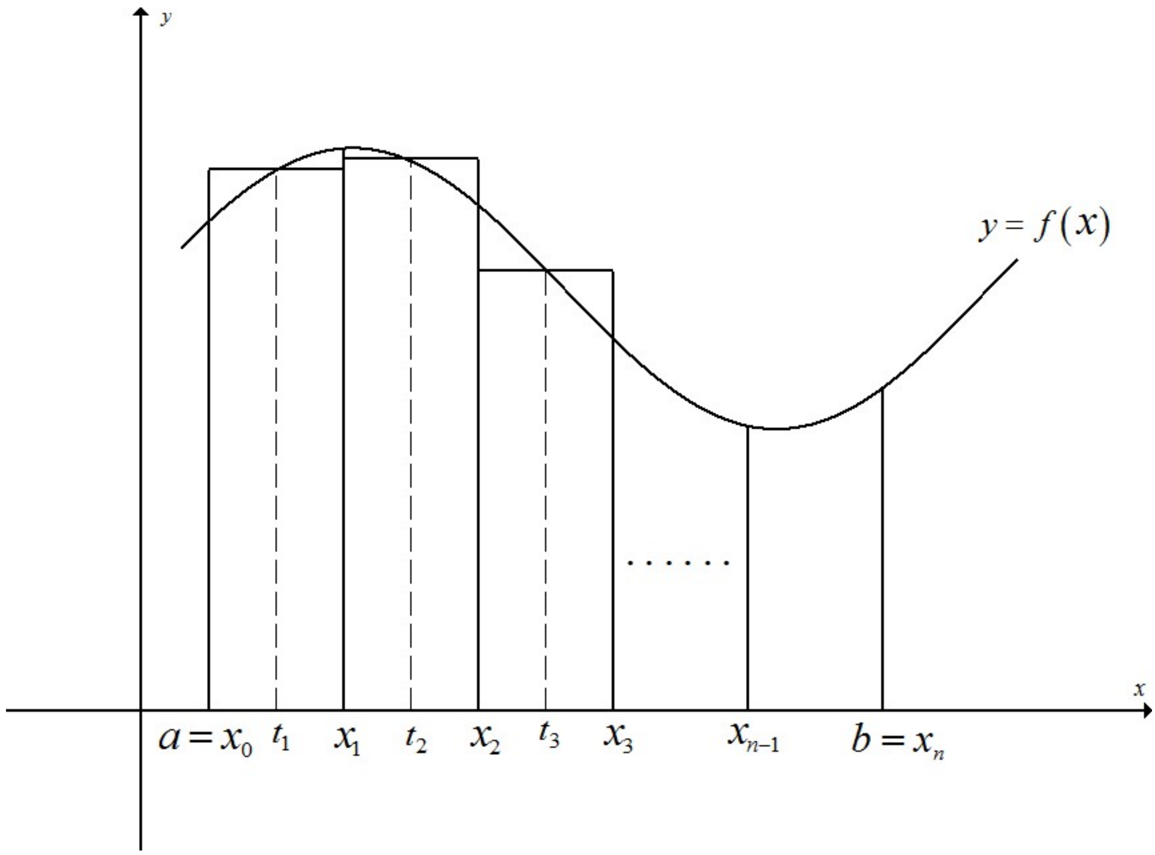
$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \cdots, [x_{n-1}, x_n].$$

Definition 6.2. The Mesh of a Partition

The mesh of a partition P, denoted $|P|$ is the largest of the lengths $\Delta x_i = x_i - x_{i-1}$ of the subintervals; that is,

$$|P| = \max\{\Delta x_i\}.$$

Let t_i (called a *selection*) be a point in the ith subinterval; i.e. $t_i \in [x_{i-1}, x_i]$, $i = 1, \cdots, n$ and consider the diagram below.



Then the height of each rectangle is given by $f(t_i)$ and the width $x_i - x_{i-1}$. Hence, the area of each rectangle is

$$f(t_i)(x_i - x_{i-1}).$$

Therefore, the sum of the areas of all these rectangles is

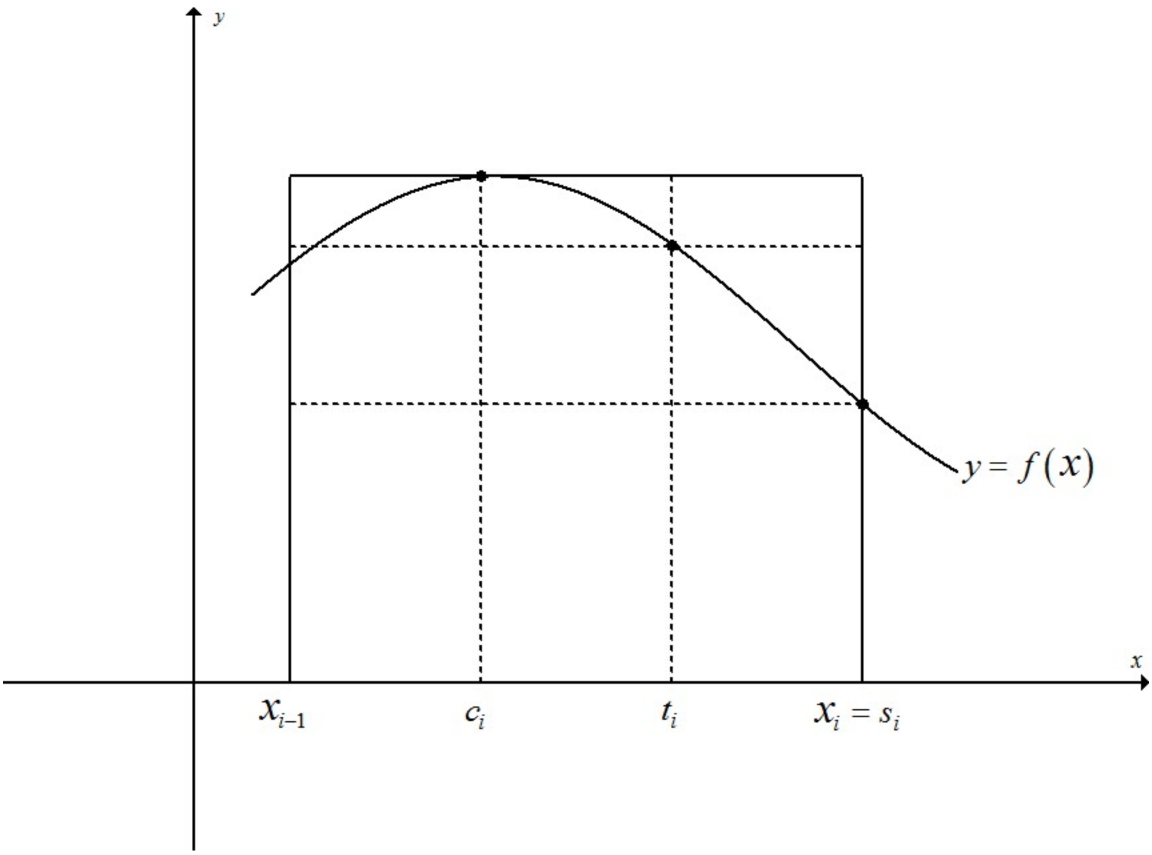
$$\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \equiv \sum_{i=1}^{n} f(t_i)\Delta x_i.$$

Definition 6.3. Riemann Sum

The number $R(f,P) = \sum_{i=1}^{n} f(t_i)\Delta x_i$ is called the *Riemann Sum* of the function with respect to the partition P.

Since f is continuous on $[a,b]$, we have that each subinterval $[x_{i-1}, x_i]$ closed and bounded. Hence, by the the Maximum-Minimum Theorem (Theorem 1.1), we have that there exist some points $c_i, s_i \in [x_{i-1}, x_i]$ such that

$$f(s_i) \leq f(x) \leq f(c_i), \quad \forall x \in [x_{i-1}, x_i]_{i=1}^{n}.$$



Consider the diagram above for the subinterval $[x_{i-1}, x_i]$. Then

$$f(s_i) \le f(t_i) \le f(c_i), \quad \forall t_i \in [x_{i-1}, x_i]_{i=1}^{n},$$

and so,

$$\sum_{i=i}^{n} f(s_i)\,\Delta x_i \le \sum_{i=i}^{n} f(t_i)\,\Delta x_i \le \sum_{i=i}^{n} f(c_i)\,\Delta x_i,$$

since $\Delta x_i > 0$ for all $i = 1, \cdots, n$.

Definition 6.4. Lower and Upper Riemann Sums

The numbers $L_R(f, P) = \sum_{i=i}^{n} f(s_i)\,\Delta x_i$ and $U_R(f, P) = \sum_{i=i}^{n} f(c_i)\,\Delta x_i$ are called the *lower* and the *upper Riemann sums*, respectively for the function f with partition P.

If the mesh $|P| = \max\{\Delta x_i\}$ of the rectangles is sufficiently small, that is, if $|P| \to 0$, then the Riemann sum $R(f, P)$ will closely approximate the area from a to b under the graph of $y = f(x)$ and above the x-axis. Thus, this area is equal to

$$\lim_{|P|\to 0} \sum_{i=i}^{n} f(t_i)\,\Delta x_i = \lim_{n\to\infty} \sum_{i=i}^{n} f(t_i)\,\Delta x_i,$$

provided the limit exists, and the function f is said to be *Riemann integrable* on $[a, b]$.

Definition 6.5. The Definite Integral

The area

$$\lim_{n\to\infty} \sum_{i=i}^{n} f(t_i)\,\Delta x_i$$

between the the graph of $y = f(x) \geq 0$, the x-axis, the lines $x = a$ and $x = b$ is called the *definite integral of* f from a to b and is denoted by $\int_a^b f(x)\,dx$. That is,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=i}^{n} f(t_i)\,\Delta x_i \tag{6.1}$$

for $f(x) \geq 0$. The function $f(x)$ is called the *integrand* and $[a, b]$ is called the *interval of integration*, with a called the *lower endpoint* and b called the *upper endpoint* of the integration.

Computing Riemann Sums

Definition 6.6. Regular Partitions

A partition $P = \{[x_{i-1}, x_i]\}_{i=1}^{n}$ of an interval $[a, b]$ is said to be *regular* if the subintervals have the same length; that is, if

$$\Delta x_i = x_i - x_{i-1} = \frac{b-a}{n} = \Delta x \quad \forall\ 1 \leq i \leq n.$$

Thus, the interval has been partitioned into equal subintervals

$$\Delta x = \Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \frac{b-a}{n},$$

so that the Riemann sum becomes

$$R(f, P) = \sum_{i=1}^{n} f(t_i)\,\Delta x.$$

Hence, the Riemann sum for the function $f : [a, b] \to \mathbb{R}$ with a regular partition of length Δx is given as

$$R(f, P) \;=\; \sum_{i=1}^{n} f(a + i\Delta x)\,\Delta x, \tag{6.2}$$

where the t_i is conveniently chosen to be the right endpoint of the ith subinterval and is given by

$$t_i = x_i = b = a + i\Delta x,$$

and so

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n}. \tag{6.3}$$

Note the following summation formulae well:

1.

$$\sum_{i=1}^{n} k \;=\; \underbrace{k + k + \cdots + k}_{n \text{ times}} = k\left(\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}}\right) = k\sum_{i=1}^{n} 1 = kn.$$

2.

$$\begin{aligned}\sum_{i=1}^{n} i &= 1+2+\cdots+n-1+n = \tfrac{1}{2}n\left(n+1\right) \\ &= \frac{1}{2}n^2+\frac{1}{2}n.\end{aligned}$$

3.

$$\begin{aligned}\sum_{i=1}^{n} i^2 &= 1^2+2^2+\cdots+n^2 = \tfrac{1}{6}n\left(n+1\right)\left(2n+1\right) \\ &= \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n.\end{aligned}$$

4.

$$\begin{aligned}\sum_{i=1}^{n} i^3 &= 1^3+2^3+\cdots+n^3 = \left(\tfrac{1}{2}n\left(n+1\right)\right)^2 = \tfrac{1}{4}n^2\left(n+1\right)^2 = \left(\sum_{i=1}^{n} i\right)^2 \\ &= \frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2.\end{aligned}$$

5.

$$\begin{aligned}\sum_{i=1}^{n} i^4 &= 1^4+2^4+\cdots+n^4 = \tfrac{1}{2}n\left(n+1\right)\left(2n+1\right)\left(3n^2+3n-1\right) \\ &= \frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3+\frac{1}{30}n.\end{aligned}$$

6. In general,

$$\sum_{i=1}^{n} i^k = 1^k+\cdots+n^k = \frac{1}{n+1}n^{k+1}+\frac{1}{2}n^k+p\left(n\right),$$

where $p\left(n\right)$ is a polynomial in n with $\deg\left(p\left(n\right)\right)<k$.

7.

$$\sum_{i=1}^{n} r^{i-1} = 1+r+r^2+\cdots+r^{n-1} = \frac{r^n-1}{r-1},\ \ r\neq 1$$

is the sum of the first n terms of a geometric series with first term 1 and common ration r.

Example 6.7. Use Riemann summation to compute the integral

$$\int_0^5 4x dx.$$

Solution. We subdivide the interval $[0,5]$ into n subintervals of equal length Δx; that is, for $f\left(x\right)=4x$, $a=0$ and $b=5$, we have

$$\Delta x = \frac{b-a}{n} = \frac{5-0}{n} = \frac{5}{n}.$$

Next, we compute

$$t_i = a + i\Delta x = 0 + i\frac{5}{n} = \frac{5i}{n}$$

and

$$f\left(t_i\right) = f\left(\frac{5i}{n}\right) = 4\left(\frac{5i}{n}\right) = \frac{20i}{n}.$$

Hence,

$$\begin{aligned}
\int_0^5 4x dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(t_i\right)\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{20i}{n}\cdot\frac{5}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{20i}{n^2} \\
&= \lim_{n\to\infty}\frac{20}{n^2}\sum_{i=1}^{n} i \\
&= \lim_{n\to\infty}\frac{100}{n^2}\left(\frac{n\left(n+1\right)}{2}\right) \\
&= 50\lim_{n\to\infty}\left(\frac{n^2+n}{n^2}\right) \\
&= 50\lim_{n\to\infty}\left(1+\frac{1}{n}\right) \\
&= 50.
\end{aligned}$$

Example 6.8. Use the definition of the definite integral to evaluate

$$\int_0^4 \left(2x^2+3\right) dx.$$

Solution. Here, $f\left(x\right) = 2x^2+3$, $a = 0$ and $b = 4$, and so

$$\Delta x = \frac{4-0}{n} = \frac{4}{n}, \quad t_i = a + i\Delta x = 0 + i\frac{4}{n} = \frac{4i}{n}$$

and

$$f\left(t_i\right) = f\left(\frac{4i}{n}\right) = 2\left(\frac{4i}{n}\right)^2 + 3 = \frac{32i^2}{n^2} + 3.$$

Therefore,

$$\begin{aligned}
\int_0^4 \left(2x^2+3\right) dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{32i^2}{n^2}+3\right)\frac{4}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{128i^2}{n^3}+\frac{12}{n}\right) \\
&= \lim_{n\to\infty}\left(\frac{128}{n^3}\sum_{i=1}^{n} i^2+\frac{12}{n}\sum_{i=1}^{n}1\right) \\
&= \lim_{n\to\infty}\left(\frac{128}{n^3}\left(\frac{n\left(n+1\right)\left(2n+1\right)}{6}\right)+\frac{12}{n}\left(n\right)\right) \\
&= \lim_{n\to\infty}\left(\frac{128}{6}\left(\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\right)+12\right) \\
&= \lim_{n\to\infty}\left(\frac{64}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+12\right) \\
&= \frac{64}{3}\left(1\right)\left(2\right)+12 \\
&= \frac{164}{3}.
\end{aligned}$$

Example 6.9. Evaluate

$$\int_{-1}^{2} x^3 dx$$

using Riemann sums.

Solution. $f\left(x\right)=x^3$, $a=-1$ and $b=2$, and so

$$\Delta x=\frac{2-\left(-1\right)}{n}=\frac{3}{n}, \quad t_i=-1+i\frac{3}{n}=\frac{3i}{n}-1$$

and

$$f\left(t_i\right)=f\left(\frac{3i}{n}-1\right)=\left(\frac{3i}{n}-1\right)^3=\frac{27i^3}{n^3}-\frac{27i^2}{n^2}+\frac{9i}{n}-1.$$

Therefore,

$$\begin{aligned}
\int_0^4 x^3 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{27i^3}{n^3} - \frac{27i^2}{n^2} + \frac{9i}{n} - 1 \right) \frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{81i^3}{n^4} - \frac{81i^2}{n^3} + \frac{27i}{n^2} - \frac{3}{n} \right) \\
&= \lim_{n\to\infty} \left(\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{81}{n^3} \sum_{i=1}^{n} i^2 + \frac{27}{n^2} \sum_{i=1}^{n} i - \frac{3}{n} \sum_{i=1}^{n} 1 \right) \\
&= \lim_{n\to\infty} \left(\frac{81}{n^4} \left(\frac{n^2 (n+1)^2}{4} \right) - \frac{81}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right) + \frac{27}{n^2} \left(\frac{n(n+1)}{2} \right) - \frac{3}{n} (n) \right) \\
&= \lim_{n\to\infty} \left(\frac{81}{4} \left(\frac{n^2}{n^2} \right) \left(\frac{n^2+2n+1}{n^2} \right) - \frac{81}{6} \left(\frac{n}{n} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} \right) + \frac{27}{2} \left(\frac{n^2}{n^2} + \frac{n}{n^2} \right) - 3 \right) \\
&= \lim_{n\to\infty} \left(\frac{81}{4} \left(1 + \frac{2}{n} + \frac{1}{n^2} \right) - \frac{81}{6} \left(\left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \right) + \frac{27}{2} \left(1 + \frac{1}{n} \right) - 3 \right) \\
&= \left(\frac{81}{4} (1) - \frac{81}{6} ((1)(2)) + \frac{27}{2} (1) - 3 \right) \\
&= \frac{15}{4}.
\end{aligned}$$

Example 6.10. Find the limit as $n \to \infty$ of the sum

$$\frac{1}{n} + \frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \cdots + \frac{n}{(2n-1)^2}.$$

Solution. Let

$$\begin{aligned}
S_n &= \frac{1}{n} + \frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \cdots + \frac{n}{(2n-1)^2} \\
&= \frac{n}{n^2} + \frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \cdots + \frac{n}{(n+(n-1))^2} \\
&= \sum_{i=0}^{n-1} \frac{n}{(n+i)^2} = \sum_{i=0}^{n-1} \frac{n}{\left(1+\frac{i}{n}\right)^2} \cdot \frac{1}{n^2} \\
&= \sum_{i=0}^{n-1} \frac{1}{\left(1+\frac{i}{n}\right)^2} \cdot \frac{1}{n} \\
&= \sum_{i=0}^{n-1} f(t_i) \Delta x
\end{aligned}$$

where $t_i = \frac{i}{n} = 0 + i\left(\frac{1-0}{n}\right)$, and so $f(t_i) = \frac{1}{(1+t_i)^2}$, $a = 0$ and $b = 1$. Hence, by the definition definition of the definite integral, we have

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i) \Delta x \\
&= \int_0^1 \frac{1}{(1+x)^2} dx \\
&= -\frac{1}{1+x}\Bigg|_0^1 = \frac{1}{2}.
\end{aligned}$$

Note that the choice of the selection t_i and for that matter the limit points in this kind of problems is not unique; for example, one may choose $t_i = 1 + \frac{i}{n} = 1 + i\left(\frac{2-1}{n}\right)$, so that $f(t_i) = \frac{1}{t_i^2}$, $a = 1$, $b = 2$, and

$$\begin{aligned}\lim_{n\to\infty} S_n &= \int_1^2 \frac{1}{x^2}dx \\ &= -\frac{1}{x}\bigg|_1^2 = \frac{1}{2}.\end{aligned}$$

Example 6.11. By interpreting each of the following limits as definite integrals, show that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n+i} = \ln 2$$

and

$$\lim_{n\to\infty} \frac{1}{n}\left((n+1)(n+2)\cdots(2n)\right)^{\frac{1}{n}} = \frac{4}{e}.$$

Solution. Let

$$\begin{aligned}S_n &= \sum_{i=1}^{n} \frac{1}{n+i} = S_n = \sum_{i=1}^{n} \frac{1}{\left(1+\frac{i}{n}\right)} \cdot \frac{1}{n} \\ &= \sum_{i=1}^{n} f(t_i)\,\Delta x,\end{aligned}$$

where $t_i = 1 + \frac{i}{n} = 1 + i\frac{(2-1)}{n}$, $f(t_i) = \frac{1}{t_i}$, $a = 1$, $b = 2$ and $\Delta x = \frac{1}{n} = \frac{2-1}{n}$.

$$\begin{aligned}\therefore\ \lim_{n\to\infty} S_n &= \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\,\Delta x \\ &= \int_1^2 \frac{1}{x}dx \\ &= [\ln x]_1^2 \\ &= \ln 2.\end{aligned}$$

Also, let

$$\begin{aligned}
S_n &= \frac{1}{n}\left((n+1)(n+2)\cdots(2n)\right)^{\frac{1}{n}} \\
&= \frac{1}{n}\left(n^n\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots(2)\right)^{\frac{1}{n}} \\
&= \frac{1}{n}\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots(2)\right) \\
\Rightarrow \ln S_n &= \frac{1}{n}\left(\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\cdots+\ln 2\right) \\
&= \frac{1}{n}\left(\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\cdots+\ln\left(1+\frac{n}{n}\right)\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\ln\left(1+\frac{i}{n}\right) \\
& \sum_{i=1}^{n}\ln\left(1+\frac{i}{n}\right)\cdot\frac{1}{n} \\
&= \sum_{i=1}^{n} f(t_i)\,\Delta x,
\end{aligned}$$

where $t_i = 1+\frac{i}{n} = 1+i\frac{(2-1)}{n}$, $f(t_i) = \ln(t_i)$, $a=1$, $b=2$ and $\Delta x = \frac{1}{n} = \frac{2-1}{n}$.

$$\begin{aligned}
\therefore \quad \lim_{n\to\infty}\ln S_n &= \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\,\Delta x \\
&= \int_1^2 \ln x dx \\
&= [x\ln x - x]_1^2 \\
&= 2\ln 2 - 1 = \ln 4 - \ln e = \ln\left(\frac{4}{e}\right).
\end{aligned}$$

But since ln is a continuous function, we have that

$$\lim_{n\to\infty}\ln S_n = \ln \lim_{n\to\infty} S_n$$

and so

$$\lim_{n\to\infty} S_n = \frac{4}{e}.$$

Example 6.12. Show that

$$\lim_{n\to\infty}\frac{1^p+2^p+3^p+\cdots+n^p}{n^{p+1}} = \frac{1}{p+1}.$$

Solution. As usual, let

$$\begin{aligned}
S_n &= \frac{1^p + 2^p + 3^p + \cdots + n^p}{n^{p+1}} \\
&= \frac{1}{n}\left(\frac{1^p + 2^p + 3^p + \cdots + n^p}{n^p}\right) \\
&= \frac{1}{n}\left(\left(\frac{1}{n}\right)^p + \left(\frac{2}{n}\right)^p + \left(\frac{3}{n}\right)^p + \cdots + \left(\frac{n}{n}\right)^p\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^p \\
&= \sum_{i=1}^{n}\left(\frac{i}{n}\right)^p \cdot \frac{1}{n} \\
&= \sum_{i=1}^{n} f(t_i)\,\Delta x
\end{aligned}$$

where $t_i = \frac{i}{n} = 0 + i\frac{(1-0)}{n}$, $f(t_i) = t_i^p$, $a = 0$, $b = 1$ and $\Delta x = \frac{1}{n} = \frac{1-0}{n}$.

$$\begin{aligned}
\therefore \lim_{n\to\infty} S_n &= \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\,\Delta x \\
&= \int_0^1 x^p dx \\
&= \left[\frac{x^{p+1}}{p+1}\right]_0^1 \\
&= \frac{1}{p+1} - \frac{0}{p+1} \\
&= \frac{1}{p+1}.
\end{aligned}$$

Example 6.13. m

Solution .

Theorem 6.14. Properties of the Definite Integral

Let $f(x)$ and $g(x)$ be a real-valued functions defined on $[a,b]$. Then:

1. An integral over an interval of zero length is zero.

$$\int_a^a f(x)\,dx = 0. \tag{6.4}$$

2.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx. \tag{6.5}$$

3. If r and s are any constants, then

$$\int_a^b (rf(x) \pm sg(x))\,dx = r\int_a^b f(x)\,dx \pm s\int_a^b g(x)\,dx. \tag{6.6}$$

4. If f is continuous on $[a,b]$ and $c \in [a,b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{6.7}$$

5. If $f(x) = k$ on $[a,b]$ where k is an arbitrary constant, then

$$\int_a^b f(x)\,dx = k(b-a). \tag{6.8}$$

6. If $f(x) \geq 0$ for every $x \in [a,b]$, then

$$\int_a^b f(x)\,dx \geq 0. \tag{6.9}$$

7. If $f(x) \leq g(x)$ for every $x \in [a,b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx. \tag{6.10}$$

8. For any given f,

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx \quad \forall x \in [a,b]. \tag{6.11}$$

9. $$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx. \tag{6.12}$$

10. $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx. \tag{6.13}$$

11. $$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^{2a} f(2a-x)\,dx. \tag{6.14}$$

12. $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \text{ i.e. even} \\ 0, & \text{if } f(2a-x) = -f(x) \text{ i.e. odd.} \end{cases} \tag{6.15}$$

13. $$\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f \text{ is even} \\ 0, & \text{if } f \text{ is. odd.} \end{cases} \tag{6.16}$$

Proof. By definition of the definite integral,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\left(\frac{b-a}{n}\right),$$

where t_i is any tag. Therefore,

i.

$$\begin{aligned}\int_a^a f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\left(\frac{a-a}{n}\right)\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)(0) = \lim_{n\to\infty}\sum_{i=1}^{n}(0)\\ &= \lim_{n\to\infty}(n(0)) = \lim_{n\to\infty} 0 = 0.\end{aligned}$$

ii.

$$\begin{aligned}\int_b^a f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\left(\frac{a-b}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\left(-\left(\frac{b-a}{n}\right)\right)\\ &= \lim_{n\to\infty}\left(-\sum_{i=1}^{n} f(t_i)\left(\frac{b-a}{n}\right)\right) = -\lim_{n\to\infty}\left(\sum_{i=1}^{n} f(t_i)\left(\frac{b-a}{n}\right)\right)\\ &= -\int_a^b f(x)\,dx.\end{aligned}$$

iii. Let r and s be some constants. Then

$$\begin{aligned}\int_a^b (rf(x)\pm sg(x))\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}(rf(t_i)\pm sg(t_i))\left(\frac{b-a}{n}\right)\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(rf(t_i)\left(\frac{b-a}{n}\right)\pm sg(t_i)\left(\frac{b-a}{n}\right)\right)\\ &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} rf(t_i)\left(\frac{b-a}{n}\right)\pm\sum_{i=1}^{n} sg(t_i)\left(\frac{b-a}{n}\right)\right)\\ &= \lim_{n\to\infty}\left(r\sum_{i=1}^{n} f(t_i)\left(\frac{b-a}{n}\right)\pm s\sum_{i=1}^{n} g(t_i)\left(\frac{b-a}{n}\right)\right)\\ &= \left(\lim_{n\to\infty} r\sum_{i=1}^{n} f(t_i)\left(\frac{b-a}{n}\right)\pm\lim_{n\to\infty} s\sum_{i=1}^{n} g(t_i)\left(\frac{b-a}{n}\right)\right)\\ &= \left(r\lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\left(\frac{b-a}{n}\right)\pm s\lim_{n\to\infty}\sum_{i=1}^{n} g(t_i)\left(\frac{b-a}{n}\right)\right)\\ &= r\int_a^b f(x)\,dx \pm s\int_a^b g(x)\,dx.\end{aligned}$$

iv. Suppose $a\le c\le b$ and let Q_n be a partition of $[a,c]$ into n subintervals with endpoints

$$a = u_0 < u_1 < \cdots < u_{n-1} < u_n = c.$$

Then

$$\int_a^c f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(u_i)\,\Delta u_i.$$

Also, let R_n be a partition of $[c,b]$ into n subintervals with endpoints

$$c = v_0 < v_1 < \cdots < v_{n-1} < v_n = b$$

so that

$$\int_c^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(v_i)\,\Delta v_i.$$

Then, the points

$$a = x_0 = u_0, x_1 = u_1, \cdots, x_{-1} = u_{n-1}, x_n = u_n, x_{n+1} = v_0, x_{n+2} = v_1, \cdots, x_{2n-1} = v_{n-1}, x_{2n} = v_n = b$$

form a partition $P_n = Q_n \cup R_n$ of $[a,b]$ into $2n$ subintervals of lengths $\Delta x_i = \Delta u_i$ and $\Delta x_{n+i} = \Delta v_i$, for $i = 1, 2, \cdots, n$. Hence, for each i, we have

$$\begin{aligned}
\int_a^b f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{2n} f(t_i)\,\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x_i + \lim_{n\to\infty}\sum_{n+i=1}^{2n} f(x_{n+i})\,\Delta x_{n+i} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f(u_i)\,\Delta u_i + \lim_{n\to\infty}\sum_{n+i=1}^{2n} f(v_i)\,\Delta v_i \\
&= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.
\end{aligned}$$

Alternatively, we can prove this by letting $F(x)$ be an antiderivative of $f(x)$. Then, we have for any $c \in [a,b]$, by the FTC II that

$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \int_a^c f(x)\,dx = F(c) - F(a), \quad \text{and} \quad \int_c^b f(x)\,dx = F(b) - F(c).$$

Therefore,

$$\begin{aligned}
\int_a^c f(x)\,dx + \int_c^b f(x)\,dx &= F(c) - F(a) + F(b) - F(c) \\
&= F(b) - F(a) \\
&= \int_a^b f(x)\,dx.
\end{aligned}$$

v.

vi.

vii.

viii.

ix.

x.

xi.

xii.

xiii. □

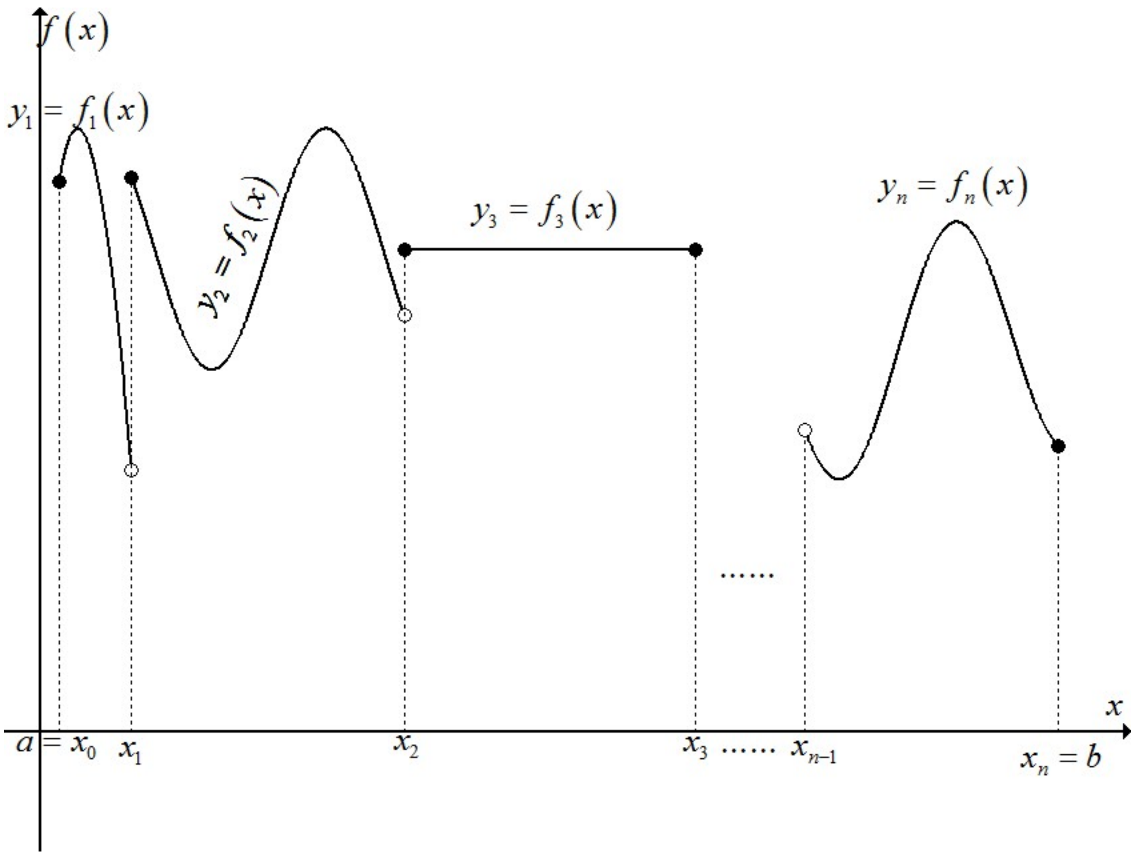
Definition 6.15. Piecewise Continuous Functions

Let $a = x_0, x_1, \cdots, x_n = b$ be a finite sequence of ordered points in $\mathbb{R}$. A function f defined on $[a, b]$ except possibly at some points x_i for $i = 0, 1, \cdots, n$, is called a **piecewise continuous function** on $[a, b]$ if $\forall i \in (0, 1, \cdots, n)$, there exists a continuous function f_i on the *closed* interval $[x_{i-1}, x_i]$ such that $f(x) = f_i(x)$ on the *open* interval (x_{i-1}, x_i).

Theorem 6.16. Integration of Piecewise Continuous Functions

Let f be a piecewise continuous function on $[a, b]$ with $a = x_0 < x_1 < \cdots < x_n = b$, that is, there are continuous functions f_i on (x_{i-1}, x_i) such that $f(x) = f_i(x)$ $\forall i \in (0, 1, \cdots, n)$. Then

$$\int_a^b f(x)\, dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f_i(x)\, dx.$$



Proof. First of all, notice that since $a = x_0 < x_1 < \cdots < x_n = b$, we have

$$\begin{aligned}
\frac{b-a}{n} &= \frac{x_n - x_0}{n} \\
&= \frac{x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - \cdots + x_1 - x_0}{n} \\
&= \frac{(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + (x_{n-2} - x_{n-3}) - \cdots + (x_1 - x_0)}{n} \\
&= \sum_{i=1}^{n} \frac{x_i - x_{i-1}}{n}.
\end{aligned}$$

Similarly,

$$f(x) = \sum_{i=1}^{n} f_i(x).$$

Now, let $t_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \cdots, n$ be a selection. Then by definition,

$$\begin{aligned}
\int_a^b f(x)\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i) \frac{b-a}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\sum_{i=1}^{n} f_i(t_i) \right) \sum_{i=1}^{n} \frac{x_i - x_{i-1}}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\sum_{i=1}^{n} f_i(t_i) \frac{x_i - x_{i-1}}{n} \right) \\
&= \sum_{i=1}^{n} \left(\lim_{n\to\infty} \sum_{i=1}^{n} f_i(t_i) \frac{x_i - x_{i-1}}{n} \right) \\
&= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f_i(x)\, dx.
\end{aligned}$$

$\square$

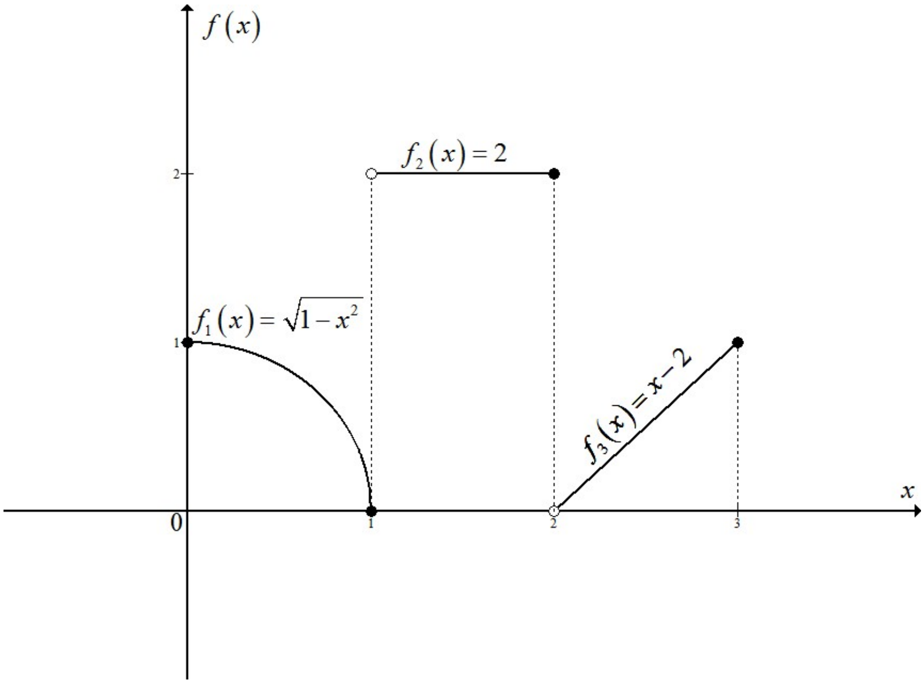
Example 6.17. Find

$$\int_0^3 f(x)\, dx$$

where

$$f(x) = \begin{cases} \sqrt{1-x^2}, & \text{if } 0 \le x \le 1 \\ 2, & \text{if } 1 < x \le 2 \\ x-2, & \text{if } 2 < x \le 3. \end{cases}$$

Solution. The function f is sketched in the diagram below:



Thus,

$$
\begin{aligned}
\int_0^3 f(x)\,dx &= \sum_{i=1}^{3} \int_{x_{i-1}}^{x_i} f_i(x)\,dx \\
&= \int_0^1 f_1(x)\,dx + \int_1^2 f_2(x)\,dx + \int_2^3 f_3(x)\,dx \\
&= \int_0^1 \sqrt{1-x^2}dx + \int_1^2 2dx + \int_2^3 (x-2)\,dx \\
&= \left(\tfrac{1}{4} \times \pi \times 1^2\right) + (2 \times 1) + \left(\tfrac{1}{2} \times 1 \times 1\right) \\
&= \frac{\pi + 10}{4}.
\end{aligned}
$$

Example 6.18. mm

Solution. m

Example 6.19. mm

Solution. m

Problem 6.1

1. Use the definition of the definite integral (Riemann Sum) to evaluate the following

   (a)

$$\int_0^2 \left(x^2 + 10\right) dx$$

(b)

$$\int_0^3 \left(5x^2 - 2x\right) dx$$

(c)

$$\int_0^4 \left(-3x^2 + 5x - 1\right) dx$$

(d)

$$\int_4^6 2x^2 dx$$

(e)

$$\int_1^5 \left(x - 4x^2\right) dx$$

(f)

$$\int_{-2}^0 \left(3x^2 + 2x\right) dx$$

(g)

$$\int_{-3}^0 \left(4x^2 - 5x - 1\right) dx$$

(h)

$$\int_{-8}^{-3} \left(x - 4x^2\right) dx$$

(i)

$$\int_{-1}^{-5} \left(x^2 + 3x + 5\right) dx$$

(j)

$$\int_{-2}^3 \left(1 - 5x^2\right) dx.$$

2. By interpreting each as a definite integral, evaluate each of the following limits.

(a)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n} - 1\right) \frac{1}{n}$$

(b)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3}$$

(c)

$$\lim_{n\to\infty} \frac{1 + 2 + 3 + \cdots + n}{n^2}$$

(d)

$$\lim_{n\to\infty} \frac{1^3 + 2^3 + 3^3 + \cdots + n^3}{n^4}$$

(e)

$$\lim_{n\to\infty} \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n\sqrt{n}}$$

(f)

$$\lim_{n\to\infty} \left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\cdots+\sin\pi\right).$$

(g)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\tan^{-1}\left(\frac{2i-1}{n}\right)$$

(h)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n^2}{n^2+i^2}$$

(i)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\sqrt{\frac{i}{n}}$$

(j)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\sqrt{\frac{i-1}{n}}$$

(k)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\ln\left(1+\frac{2i}{n}\right)$$

(l)

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^5}}\sum_{i=1}^{n} \sqrt{i^3}$$

(m)

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{\sqrt{n^2-i^2}}$$

3. Evaluate the following limits

(a)

$$\lim_{n\to\infty} \frac{1}{n}\left(\left(1+\frac{1}{n}\right)^5+\left(1+\frac{2}{n}\right)^5+\left(1+\frac{3}{n}\right)^5+\cdots+2^5\right).$$

(b)

$$\lim_{n\to\infty} \frac{\pi}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\cdots+\sin\pi\right).$$

(c)

$$\lim_{n\to\infty} \left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\frac{n}{n^2+9}+\cdots+\frac{n}{2n^2}\right).$$

(d)

$$\lim_{n\to\infty} \frac{1}{n^{\frac{3}{2}}}\left(1+\sqrt{2}+\cdots+\sqrt{n-1}\right).$$

(e)
$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right).$$

(f)
$$\lim_{n\to\infty}\left(\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n^{\frac{3}{2}}}\right).$$

(g)
$$\lim_{n\to\infty}\left(\frac{1^5+2^5+3^5+\cdots+n^5}{n^6}\right).$$

(h)
$$\lim_{n\to\infty}\frac{1}{n^{11}}\left(1^{10}+2^{10}+\cdots+n^{10}\right).$$

(i)
$$\lim_{n\to\infty}\sum_{i=1}^{n}\sin\left(\frac{\pi i}{2}\right).$$

(j)
$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{i+1}\right).$$

4. Show that
$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\cos\left(\frac{i\pi}{2n}\right)=\frac{\pi}{2}.$$

5. Let $t_i=\sqrt{x_{i-1}x_i}$ for $i=1,2,\cdots,n$ on $[a,b]$. Show that
$$\sum_{i=1}^{n}\frac{\Delta x_i}{t_i^2}=\frac{1}{b}-\frac{1}{a}.$$

6. Let
$$(t_i)^{\frac{1}{2}}\left(x_i-x_{i-1}\right)=\frac{2}{3}\left[(x_i)^{\frac{3}{2}}-(x_{i-1})^{\frac{3}{2}}\right],\quad i=1,2,\cdots,n.$$
Show that
$$x_i<t_i<x_{i-1}.$$
Show also that, using this selection for the given partition of $[a,b]$,
$$\int_a^b\sqrt{x}dx=\frac{2}{3}\left[b^{\frac{3}{2}}-a^{\frac{3}{2}}\right].$$

7. m
8. m
9. m
10. m

6.2 The Fundamental Theorem of Calculus (FTC)

Theorem 6.20. The MVT for Integrals

If f is continuous on $[a,b]$, then $\exists c \in (a,b)$ such that

$$\int_a^b f(t)\,dt = (b-a)\,f(c)\,. \tag{6.17}$$

Proof. From equation (1.2) of Theorem 1.1 (the Maximum-Minimum Theorem), we have

$$\min f \leq \frac{1}{(b-a)} \int_a^b f(x)\,dx \leq \max f.$$

Since f is continuous on $[a,b]$, the Intermediate Value Theorem says that f must assume every value between $\min f$ and $\max f$. Therefore, f must assume the value $\frac{1}{(b-a)} \int_a^b f(x)\,dx$ at some point $c \in [a,b]$. Hence,

$$\frac{1}{(b-a)} \int_a^b f(x)\,dx = f(c) \text{ for some } c \in [a,b].$$

□

Theorem 6.21. The FTC I

If f is continuous on $[a,b]$, then the function $F(x) = \int_a^x f(t)\,dt$ has a derivative at every point of $[a,b]$ and

$$\frac{dF}{dx} = \frac{d}{dx} \int_a^x f(t)\,dt = f(x)\,, \quad a \leq x \leq b. \tag{6.18}$$

Intuitively, the FTC I states that

- every continuous function f is the derivative of some other function, namely $\int_a^x f(t)\,dt$, or
- every continuous function $F(x) = \int_a^x f(t)\,dt$ has an antiderivative, namely f, or
- the process of differentiation and integration are inverse processes of one another.

Proof. By the definition of the derivative, we have

$$\begin{aligned}
\frac{dF}{dx} &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \right) \\
&= \lim_{h\to 0} \frac{1}{h} \left(\int_x^{x+h} f(t)\,dt \right) \\
&= \lim_{h\to 0} \frac{1}{h} \left(hf(c) \right), \qquad c \in [x, x+h] \text{ by the MVT for integtrals} \\
&= \lim_{c\to x} f(c) \qquad \text{since as } h \to 0,\ c \to x \\
&= f(x)\,.
\end{aligned}$$

□

Corollary 6.22. *For all differentible functions $g(x)$ and $h(x)$, we have*

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\frac{d}{dx}g(x)$$

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))\frac{d}{dx}g(x) - f(h(x))\frac{d}{dx}h(x).$$

Proof. By the Chain Rule, we let $u = g(x)$ and $v = h(x)$ .... □

Example 6.23. Find $\frac{dy}{dx}$ of the following functions.

i.	$y = \int_{-\pi}^{x} \cos t dt$	iv.	$y = \int_1^{\sin x} 3t^2 dt$
ii.	$y = \int_1^{x^2} \sin t dt$	v.	$y = \int_1^{\tan x} \sec^2 t dt$
iii.	$y = \int_1^{x^4} \sec t dt$	vi.	$y = \int_1^{\sin x} \frac{1}{\sqrt{1-t^2}} dt$

Solution:

1. $y = \int_{-\pi}^{x} \cos t dt \implies f(t) = \cos t$. Therefore, $\frac{dy}{dx} = \frac{d}{dx}\int_{-\pi}^{x} \cos t dt = \cos x$.

2. For $y = \int_1^{x^2} \sin t dt$, we have $f(t) = \sin t$. Let $u = x^2$ so that $y = \int_1^{u} \sin t dt$, $\frac{du}{dx} = 2x$ and $\frac{dy}{du} = \sin u$. Then by the chain rule, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= \sin u \cdot 2x\\ &= 2x\sin x^2.\end{aligned}$$

3. Similarly, we have

$$\begin{aligned}\frac{d}{dx}\int_1^{x^4} \sec t dt &= \sec x^4 \frac{d}{dx}\left(x^4\right)\\ &= 4x^3 \sec x^4.\end{aligned}$$

4.

$$\begin{aligned}\frac{d}{dx}\int_1^{\sin x} 3t^2 dt &= 3x^2 \frac{d}{dx}(\sin x)\\ &= 3\cos x \sin^2 x.\end{aligned}$$

5.

$$\begin{aligned}\frac{d}{dx}\int_1^{\tan x} \sec^2 t dt &= \sec^2(\tan x)\frac{d}{dx}(\tan x)\\ &= \sec^2(\tan x)\sec^2 x.\end{aligned}$$

6.

$$\begin{aligned} \frac{d}{dx}\int_1^{\sin x} \frac{1}{\sqrt{1-t^2}}dt &= \frac{1}{\sqrt{1-\cos^2}}\frac{d}{dx}(\sin x) \\ &= \frac{\cos x}{\sin x} \\ &= \cot x. \end{aligned}$$

Theorem 6.24. The FTC II

If f is continuous at every point of $[a,b]$ and F is any antiderivative of f on $[a,b]$, that is, $F'(x) = f(x)\ \forall x \in [a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{6.19}$$

That is, the FTC II states that to evaluate the definite integral of a continuous function f from a to b, all we need to do is to find the antiderivative F of f and calculate the number $F(b) - F(a)$.

Proof. Suppose

$$G(x) = \int_a^x f(t)\,dt.$$

Then by FTC I, we have $G'(x) = f(x)$. If F is any other derivative of f on $[a,b]$, then by Corollary 1.7, $F(x) = G(x) + C\ \forall x \in [a,b]$ and for some constant C. Hence,

$$\begin{aligned} F(b) - F(a) &= (G(b) + C) - (G(a) + C) \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt - 0 \\ &= \int_a^b f(x)\,dx. \end{aligned}$$

□

Definition 6.25. The Evaluation Symbol

We define the evaluation symbol as

$$F(x)\Bigg|_a^b \equiv \Bigg[F(x)\Bigg]_a^b := F(b) - F(a).$$

Example 6.26. If $f(x) = x^n$ with $n \neq -1$, then an antiderivative of f is

$$F(x) = \frac{x^{n+1}}{n+1},$$

and so,

$$\begin{aligned} \int_a^b x^n dx &= F(b) - F(a) = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} \\ &= \frac{b^{n+1} - a^{n+1}}{n+1}. \end{aligned}$$

Example 6.27. Evaluate the following integrals:

i.	$\int_{-1}^{2}\left(x^2-3x+2\right)dx$	iv.	$y=\int_1^{\sin x}3t^2dt$
ii.	$\int_0^{\pi}\sin x dx$	v.	$y=\int_1^{\tan x}\sec^2 tdt$
iii.	$y=\int_1^{x^4}\sec tdt$	vi.	$y=\int_1^{\sin x}\frac{1}{\sqrt{1-t^2}}dt$

Solution:

Problem 6.28. Solve the integral equation $f\left(x\right)=2+3\int_4^x f\left(t\right)dt$.

Proof. First of all, differentiate both sides of the equation to get

$$\begin{aligned} f'\left(x\right) &= 3f\left(x\right) \\ \implies \frac{f'\left(x\right)}{f\left(x\right)} &= 3 \\ \frac{d}{dx}\left(\ln f\left(x\right)\right) &= 3 \\ \ln f\left(x\right) &= 3x+k \\ f\left(x\right) &= e^{3x+k}=Ke^{3x},\ \ K=e^k. \end{aligned}$$

Now, when $x=4$, we have $f\left(4\right)=2=Ke^{12}\implies K=2e^{-12}$. Hence,

$$f\left(x\right)=2e^{3x-12}.$$

□

Definition 6.29. Average (Mean) Value of a Function

If f is integrable on $[a,b]$, then the *average (mean) value* of f on $[a,b]$, denoted by $\bar{f}$, is given by

$$\bar{f}=\frac{1}{b-a}\int_a^b f\left(x\right)dx. \tag{6.20}$$

Example 6.30. mm

Solution. m

Example 6.31. Find the average value of $f\left(x\right)=e^{-x}+\cos x$ on the interval $\left[-\frac{\pi}{2},0\right]$.

Solution. The average value if

$$\begin{aligned} \bar{f} &= \frac{1}{0-\left(-\frac{\pi}{2}\right)}\int_{-\frac{\pi}{2}}^{0}\left(e^{-x}+\cos x\right)dx \\ &= \frac{\pi}{2}\ \left(-e^{-x}+\sin x\right)\Bigg|_{-\frac{\pi}{2}}^{0} \\ &= \frac{\pi}{2}\left(-1+0+e^{\frac{\pi}{2}}-\left(-1\right)\right) \\ &= \frac{\pi}{2}e^{\frac{\pi}{2}}. \end{aligned}$$

Example 6.32. Find the mean value of $\sin^2\theta$ for $0 \leq \theta \leq \pi$.

The quantities v, s, and t are connected by the relations

$$v = n\sqrt{a^2 - s^2}, \qquad s = a\sin nt.$$

Show that the mean value of v considered as a function of t between $t = 0$ and $t = \frac{\pi}{2n}$ is $\frac{2as}{\pi}$.

Solution. m

Example 6.33. mm

Solution. m

Example 6.34. mm

Solution. m

Theorem 6.35. The Integral as a Function of the Upper Limit: The Indefinite Integral

(See TK's Book) mm

Proof. mm □

Problem 6.2

6.3 Techniques of Integration

6.3.1 Method of Substitution

The process of integration by substitution can be compared to the Chain Rule in differentiation. That is, the method of substitution is used to reverse the Chain Rule when used to differentiate a function.

Theorem 6.36. *Let u be a real-valued function which has a derivative on $[a, b]$. Let I be an open interval which contains the image of $[a, b]$ under u. Given that f is a real-valued function that is continuous on I, if F is an antiderivative of f, then*

$$\begin{aligned} \int f(u(x))\,u'(x)\,dx &= F(u) + C \\ &= \int f(u(x))\,du. \end{aligned} \tag{6.21}$$

Proof. Let F be an antiderivative of f. Then $F(u) = \int f(u)\,du$ or $F'(u) = f(u) \quad \forall u \in [a,b]$. Hence, by the Chain Rule, we have

$$\frac{d}{dx}(F(u)) = F'(u) \cdot u'(x).$$

Therefore,

$$\begin{aligned} \int \frac{d}{dx}(F(u))\,dx &= \int F'(u) \cdot u'(x)\,dx \\ \Rightarrow \int F'(u) \cdot u'(x)\,dx &= F(u) + C \\ \Rightarrow \int f(u)\,u'(x)\,dx &= F(u) + C \\ &= \int f(u)\,du. \end{aligned}$$

$\square$

Example 6.37. Evaluate the following integrals.

$$\int \frac{x}{x^2+1}dx, \quad \int e^x\sqrt{1+e^x}dx, \quad \int \sec x\sqrt{\sec x + \tan x}dx,$$

$$\int \frac{\sin 2x - \cos 2x}{\sin 2x + \cos 2x}dx, \quad \textit{and} \quad \int \frac{\sin x + \cos x}{e^{-x} + \sin x}dx.$$

Solution. Let us use the method of substitution to evaluate all these integrals.

For $\int \frac{x}{x^2+1}dx$, let $u = x^2 + 1$. Then $du = 2xdx$ or $xdx = \frac{1}{2}du$. Therefore,

$$\begin{aligned} \int \frac{x}{x^2+1}dx &= \int \frac{1}{x^2+1}xdx \\ &= \int \frac{1}{u}\frac{1}{2}du = \frac{1}{2}\int \frac{1}{u}du \\ &= \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln\left|x^2+1\right| + C \\ &= \ln\sqrt{x^2+1} + C. \end{aligned}$$

For$\int e^x\sqrt{1+e^x}dx$, let $u = 1 + e^x$. Then $du = e^x dx$.

$$\begin{aligned} \int e^x\sqrt{1+e^x}dx &= \int \sqrt{1+e^x}\,(e^x dx) \\ &= \int \sqrt{u}du \\ &= \frac{3}{2}u^{\frac{3}{2}} + C \\ &= \frac{3}{2}(1+e^x)^{\frac{3}{2}} + C \\ &= \frac{3}{2}\sqrt{(1+e^x)^3} + C. \end{aligned}$$

For $\int \sec x\sqrt{\sec x + \tan x}dx\int \frac{x}{x^2+1}dx$, we can let $u = \sqrt{\sec x + \tan x} = (\sec x + \tan x)^{\frac{1}{2}}$ so that

$$\begin{aligned} du &= \tfrac{1}{2}(\sec x + \tan x)^{-\frac{1}{2}}\left(\sec x \tan x + \sec^2 x\right)dx \\ &= \tfrac{1}{2}(\sec x + \tan x)^{-\frac{1}{2}}\sec x(\tan x + \sec x)dx \\ &= \tfrac{1}{2}\sec x(\sec x + \tan x)^{-\frac{1}{2}}(\tan x + \sec x)dx \\ &= \tfrac{1}{2}\sec x(\sec x + \tan x)^{-\frac{1}{2}+1}dx = \tfrac{1}{2}\sec x(\sec x + \tan x)^{\frac{1}{2}}dx \\ \Rightarrow 2du &= \sec x\sqrt{\sec x + \tan x}dx. \end{aligned}$$

$$\begin{aligned} \therefore \quad \int \sec x\sqrt{\sec x + \tan x}dx &= \int 2du \\ &= 2u + C \\ &= 2\sqrt{\sec x + \tan x} + C \end{aligned}$$

For $\int \frac{\sin 2x - \cos 2x}{\sin 2x + \cos 2x}dx$, let $u = \sin 2x + \cos 2x$. Then $du = (2\cos 2x - 2\sin 2x)dx = -2udx$, or $dx = -\frac{1}{2u}du$. Hence,

$$\begin{aligned} \int \frac{\sin 2x - \cos 2x}{\sin 2x + \cos 2x}dx &= \int \left(-\frac{1}{2u}\right)du \\ &= -\tfrac{1}{2}\int \frac{1}{u}du \\ &= -\tfrac{1}{2}\ln|u| + C \\ &= \tfrac{1}{2}\ln|\sin 2x + \cos 2x| + C. \end{aligned}$$

For $\int \frac{\sin x + \cos x}{e^{-x} + \sin x}dx$, notice that

$$\begin{aligned} \int \frac{\sin x + \cos x}{e^{-x} + \sin x}dx &= \int \frac{(\sin x + \cos x)e^x}{(e^{-x} + \sin x)e^x}dx \\ &= \int \frac{e^x \sin x + e^x \cos x}{1 + e^x \sin x}dx. \end{aligned}$$

Now, let $u = 1 + e^x \sin x$ so that $du = (e^x \sin x + e^x \cos x)dx$. Therefore,

$$\begin{aligned} \int \frac{\sin x + \cos x}{e^{-x} + \sin x}dx &= \int \frac{e^x \sin x + e^x \cos x}{1 + e^x \sin x}dx \\ &= \int \frac{1}{u}du \\ &= \ln|u| + C \\ & \quad \ln|1 + e^x \sin x| + C. \end{aligned}$$

6.3.1.1 Evaluating Integrals of the Forms

$$\int p(x)\sqrt[n]{ax+b}dx, \quad \int \frac{p(x)}{\sqrt[n]{ax+b}}dx, \quad \int p(x)\sqrt[n]{\frac{ax+b}{cx+d}}dx.$$

To evaluate integrals of these forms, we make use of the substitutions $u^n = ax+b$ in the first two, and $u^n = \frac{ax+b}{cx+d}$ in the last one.

Example 6.38. Integrate the following functions with respect to x:

$$x^2\sqrt{x-2}, \quad \frac{x}{\sqrt[3]{x+1}}, \quad \text{and} \quad \sqrt{\frac{1+x}{1-x}}.$$

Solution.

- Let $u^2 = x-2 \Rightarrow u = \sqrt{x-2}$ and $2udu = dx$. Hence,

$$\begin{aligned}
\int x^2\sqrt{x-2}dx &= \int \left(u^2+2\right)^2 u\,(2udu) \\
&= 2\int \left(u^4+4u^2+4\right)u^2du \\
&= 2\int \left(u^6+4u^4+4u^2\right)du \\
&= 2\left(\tfrac{1}{7}u^7+\tfrac{4}{5}u^5+\tfrac{1}{3}u^3\right)+C \\
&= \tfrac{2}{7}\left(\sqrt{x-2}\right)^7-\tfrac{8}{5}\left(\sqrt{x-2}\right)^5+\tfrac{2}{3}\left(\sqrt{x-2}\right)^3+C \\
&= \tfrac{2}{7}\sqrt{(x-2)^7}-\tfrac{8}{5}\sqrt{(x-2)^5}+\tfrac{2}{3}\sqrt{(x-2)^3}+C \\
&= \tfrac{2}{7}(x-2)^{\frac{7}{2}}-\tfrac{8}{5}(x-2)^{\frac{5}{2}}+\tfrac{2}{3}(x-2)^{\frac{3}{2}}+C.
\end{aligned}$$

- Let $u^3 = x+1 \Rightarrow u = (x+1)^{\frac{1}{3}}$ and $3u^2du = dx$ and so,

$$\begin{aligned}
\int \frac{x}{\sqrt[3]{x+1}}dx &= \int \left(\frac{u^3-1}{u}\right)\left(3u^2du\right) \\
&= 3\int \left(u^4-u\right)du \\
&= \tfrac{3}{5}u^5-\tfrac{1}{2}u^2+C \\
&= \tfrac{3}{5}(x+1)^{\frac{5}{3}}-\tfrac{1}{2}(x+1)^{\frac{2}{3}}+C.
\end{aligned}$$

- Let $u^2 = \frac{1+x}{1-x} \Rightarrow u = \sqrt{x-2}$ and $2udu = dx$. Hence,

Alternatively, since

$$\begin{aligned}
\sqrt{\frac{1+x}{1-x}} &= \sqrt{\left(\frac{1+x}{1-x}\right)\left(\frac{1+x}{1+x}\right)} \\
&= \sqrt{\frac{(1+x)^2}{1-x^2}} = \frac{1+x}{\sqrt{1-x^2}},
\end{aligned}$$

we have,

$$\int \sqrt{\frac{1+x}{1-x}}dx = \int \frac{1+x}{\sqrt{1-x^2}}dx$$

and let $x = \sin\theta$ for $0 < \theta < \frac{\pi}{2}$. Then $dx = \cos\theta d\theta$ and the integral becomes

$$\begin{aligned}\int \frac{1+x}{\sqrt{1-x^2}}dx &= \int \frac{1+\sin\theta}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta \\ &= \int (1+\sin\theta)\, d\theta \\ &= \theta - \cos\theta + C \\ &= \sin^{-1} x + \sqrt{1-x^2} + C.\end{aligned}$$

Give more Examples in full note

6.3.1.2 Trigonometric and Hyperbolic Substitutions

For integrals involving squares and square roots such as $a^2 \pm b^2x^2$ and $\sqrt{a^2 \pm b^2x^2}$ as denominators, we can make trigonometric and hyperbolic substitutions to integrate them to obtain results in terms of the inverse trigonometric hyperbolic functions.

Theorem 6.39. *For $0 \leq \theta \leq \frac{\pi}{2}$, we have*

1.

$$\int \frac{1}{\sqrt{a^2-b^2x^2}}dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + C;$$

2.

$$\int \frac{1}{\sqrt{a^2+b^2x^2}}dx = \frac{1}{b}\sinh^{-1}\left(\frac{bx}{a}\right) + C;$$

3.

$$\begin{aligned}\int \frac{1}{\sqrt{b^2x^2-a^2}}dx &= \frac{1}{b}\cosh^{-1}\left(\frac{bx}{a}\right) + C; \\ &= \frac{1}{b}\sec^{-1}\left(\frac{bx}{a}\right) + C\end{aligned}$$

4.

$$\int \frac{1}{a^2+b^2x^2}dx = \frac{1}{b}\tan^{-1}\left(\frac{bx}{a}\right) + C;$$

5.

$$\int \frac{1}{\sqrt{a^2-b^2x^2}}dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + C;$$

Proof. n □

Example 6.40. Evaluate the integral

i. $\int \frac{1}{\sqrt{1-x^2}}dx$ iii. $\int \frac{\sqrt{9-x^2}}{x^2}dx$ v. $\int \frac{1}{(a^2x^2-b^2)^{\frac{3}{2}}}dx$

ii. $\int \sqrt{1-4x^2}dx$ iv. $\int \frac{1}{x^2\sqrt{x^2-25}}dx$ vi. $\int x\sqrt{x^2+4}dx$.

Solution. *Suppose* $0 \le t \le \frac{\pi}{2}$.

i. *Let* $x = \sin t$, *then* $dx = \cos t dt$ *and* $t = \sin^{-1} x$. *Then*

$$\begin{aligned}
\int \frac{1}{\sqrt{1-x^2}}dx &= \int \frac{1}{\sqrt{1-\sin^2 t}}\cos t dt \\
&= \int \frac{\cos t}{\sqrt{\cos^2 t}}dt \\
&= \int \frac{\cos t}{\cos t}dt = \int dt \\
&= t + C \\
&= \sin^{-1} x + C
\end{aligned}$$

ii. *Let* $x = 3\sin t$, *then* $dx = 3\cos t dt$ *and* $t = \sin^{-1}\left(\frac{x}{3}\right)$. *Then*

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}dx &= \int \frac{\sqrt{9-9\sin^2}}{9\sin^2 t}3\cos t dt \\
&= \int \frac{9\cos^2 t}{9\sin^2 t}dt = \int \cot^2 t dt \\
&= \int \left(\csc^2 t - 1\right) dt \\
&= -\cot t - t + C \\
&= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C
\end{aligned}$$

iii.

iv.

v.

vi.

Example 6.41. By using an appropriate trigonometric substitution, show that, for some constant k,

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(kx + k\sqrt{x^2+a^2}\right).$$

Solution. *Let* $x = a\sinh t$. *Then* $dx = a\cosh t dt$ *and* $x^2 + a^2 = a^2\sinh^2 t + a^2 = a^2\cosh^2 t$. *Hence,*

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+a^2}} &= \int \frac{a\cosh t dt}{\sqrt{a^2\cosh^2 t}} \\
&= \int dt \\
&= t + C,
\end{aligned}$$

where C is a constant. Now, $x = a\sinh t \Rightarrow t = \sinh^{-1}\left(\frac{x}{a}\right)$. That is,

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+a^2}} &= \sinh^{-1}\left(\frac{x}{a}\right) + C \\
&= \ln\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 + 1}\right) + C \\
&= \ln\left(\frac{x}{a} + \sqrt{\frac{x^2+a^2}{a^2}}\right) + C \\
&= \ln\left(\frac{x + \sqrt{x^2+a^2}}{a}\right) + C \\
&= \ln\left(x + \sqrt{x^2+a^2}\right) - \ln a + C \\
&= \ln\left(x + \sqrt{x^2+a^2}\right) + \ln k, \qquad \textit{where } \ln k = C - \ln a \\
&= \ln\left(k\left(x + \sqrt{x^2+a^2}\right)\right) \\
&= \ln\left(kx + k\sqrt{x^2+a^2}\right).
\end{aligned}$$

Integrals containing quadratic terms *such as ax^2+bx+c can transformed into those types $az^2 \pm m^2$ or $m^2 - az^2$ by completing the squares; thus, for example,*

$$\begin{aligned}
ax^2+bx+c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) \\
&= a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right),
\end{aligned}$$

where $z = x + \frac{b}{2a}$ and $m = \frac{4ac-b^2}{4a}$, and the appropriate trigonometric or hyperbolic substitutions can be made.

Example 6.42. Evaluate the following integrals

$$\int \sqrt{2x-x^2}dx,\ \int \frac{1}{\sqrt{9x^2+6x-8}}dx,\ \int \frac{x}{\sqrt{3-2x-x^2}}dx,\ \int \frac{dx}{(x^2+2x+2)^2},\ \int \frac{dx}{\sqrt{5+4x-x^2}}.$$

Solution. *For $\int \sqrt{2x-x^2}dx$, notice that $2x - x^2 = -(x^2-2x) = 1 - (x^2-2x+1) = 1-(x-1)^2$. Hence,*

$$\int \sqrt{2x-x^2}dx = \int \sqrt{1-(x-1)^2}dx.$$

So let $x-1=\sin t$ *for* $t\in\left[0,\frac{\pi}{2}\right]$*, then* $dx=\cos t dt$ *and* $t=\sin^{-1}(x-1)$*, so that*

$$\begin{aligned}
\int\sqrt{2x-x^2}dx &= \int\sqrt{1-(x-1)^2}dx\\
&= \int\sqrt{1-\sin^2 t}\cos t dt\\
&= \int\cos^2 t dt=\tfrac{1}{2}\int(1+\cos 2t)\,dt\\
&= \tfrac{1}{2}\left(t+\tfrac{1}{2}\sin 2t\right)+C\\
&= \tfrac{1}{2}t+\tfrac{1}{2}\sin t\cos t+C\\
&= \tfrac{1}{2}\sin^{-1}(x-1)+\tfrac{1}{2}(x-1)\sqrt{1-(x-1)^2}+C.
\end{aligned}$$

For $\int\frac{1}{\sqrt{9x^2+6x-8}}dx$*,*

$$\int\frac{1}{\sqrt{9x^2+6x-8}}dx=\int\frac{1}{\sqrt{(3x+1)^2-9}}dx, \tag{6.22}$$

so let $3x+1=3\cosh t,\ t\in[0,\infty)$ *so that* $t=\cosh^{-1}\left(\frac{3x+1}{3}\right)$ *and* $dx=\sinh t dt$*. Hence,*

$$\begin{aligned}
\int\frac{1}{\sqrt{9x^2+6x-8}}dx &= \int\frac{1}{\sqrt{9\cosh^2 t-9}}\sinh t dt\\
&= \tfrac{1}{3}\int\frac{1}{\sqrt{\sinh^2 t}}\sinh t dt=\tfrac{1}{3}\int dt\\
&= \tfrac{1}{3}t+C\\
&= \tfrac{1}{3}\cosh^{-1}\left(\frac{3x+1}{3}\right)+C.
\end{aligned}$$

Alternatively, *one can let, say,* $3x+1=3\sec t,$ *in (6.22) so that* $dx=\sec t\tan t dt$*, and*

$$\begin{aligned}
\int\frac{1}{\sqrt{9x^2+6x-8}}dx &= \int\frac{1}{\sqrt{(3x+1)^2-9}}dx\\
&= \tfrac{1}{3}\int\frac{\sec t\tan t}{\sqrt{\sec^2 t-1}}dt=\tfrac{1}{3}\int\frac{\sec t\tan t}{\sqrt{\tan^2 t}}dt\\
&= \tfrac{1}{3}\int\sec t dt\\
&= \tfrac{1}{3}\ln|\sec t+\tan t|+C\\
&= \tfrac{1}{3}\ln\left|\frac{3x+1}{3}+\frac{\sqrt{(3x+1)^2-9}}{3}\right|+C\\
&= \tfrac{1}{3}\ln\left|\frac{3x+1}{3}+\sqrt{\frac{(3x+1)^2-9}{9}}\right|+C\\
&= \tfrac{1}{3}\ln\left|\frac{3x+1}{3}+\sqrt{\left(\frac{3x+1}{3}\right)^2-1}\right|+C\\
&= \tfrac{1}{3}\cosh^{-1}\left(\frac{3x+1}{3}\right)+C.
\end{aligned}$$

Give more Examples in full note

Problem 6.3

6.4 Integration by use of Partial Frations: Integration of Rational Functions

To integrate integrals of the form
$$\int \frac{f(x)}{g(x)}dx,$$
where both $f(x)$ and $g(x)$ are polynomials in x, we either, where appropriate, *use partial fractions*, *split the numerator* we use or both.

The rational function of the form $R(x) = \frac{f(x)}{g(x)}$ is said to be *proper* if $\deg(f(x)) < \deg(g(x))$, otherwise, it is *improper*. If $R(x)$ is improper, then we can express it as
$$R(x) = q(x) + \frac{r(x)}{g(x)},$$
where $q(x)$, called the *quotient*, is a polynomial, and $\frac{r(x)}{g(x)}$, with $r(x)$ called the *remainder*, is a proper polynomial. In this case, the integral becomes
$$\begin{aligned}\int \frac{f(x)}{g(x)}dx &= \int \left(q(x) + \frac{r(x)}{g(x)}\right)dx \\ &= \int q(x)\,dx + \int \frac{r(x)}{g(x)}dx.\end{aligned}$$

Example 6.43. Evaluate
$$\int \frac{x^2+1}{x+1}dx.$$

Solution. Notice that this integral is improper since $\deg(x^2+1) = 2 > 1 = \deg(x+1)$, so we divide (**long division**) before we integrate. Now, since
$$\frac{x^2+1}{x+1} = x - 1 + \frac{2}{x+1},$$
we have
$$\begin{aligned}\int \frac{x^2+1}{x+1}dx &= \int \left(x - 1 + \frac{2}{x+1}\right)dx \\ &= \int (x-1)\,dx + \int \frac{2}{x+1}dx \\ &= \int (x-1)\,dx + 2\int \frac{\frac{d}{dx}(x+1)}{x+1}dx \\ &= \tfrac{1}{2}x^2 - x + 2\ln(x+1) + C.\end{aligned}$$

m

Example 6.44. Evaluate
$$\int \frac{x^3+2x^2}{x^2+1}dx.$$

Solution. Here too, $\deg\left(x^3+2x^2\right) > \deg\left(x^2+1\right)$ so the integrand is improper and so long division is required. Thus, since, by long division,
$$\frac{x^3+2x^2}{x^2+1} = x+3-\frac{x+3}{x^2+1},$$
we have
$$\begin{aligned}
\int \frac{x^3+2x^2}{x^2+1}dx &= \int \left(x+3-\frac{x+3}{x^2+1}\right)dx \\
&= \int (x+3)\,dx - \int \frac{x+3}{x^2+1}dx \\
&= \int (x+3)\,dx - \int \frac{x}{x^2+1}dx - \int \frac{3}{x^2+1}dx, \\
&\qquad \text{(Where we } \textit{split the nemerator} \text{ in the second integral)} \\
&= \int (x+3)\,dx - \tfrac{1}{2}\int \frac{\frac{d}{dx}\left(x^2+1\right)}{x^2+1}dx - 3\int \frac{1}{x^2+1}dx \\
&= \tfrac{1}{2}x^2+3x-\tfrac{1}{2}\ln\left(x^2+1\right)-3\tan^{-1}x+C.
\end{aligned}$$

The basic problem, however, is to concentrate on proper rational functions, that is when $\deg\left(f(x)\right) < \deg\left(g(x)\right)$.

- If $\deg\left(g(x)\right) = 1$, that is when the denominator is linear, i.e. $g(x) = ax+b,\ \ a \neq 0$, then $\frac{f(x)}{g(x)}$ takes the form $\frac{k}{ax+b}$, whence the substitution $u = ax+b$ leads to
$$\begin{aligned}
\int \frac{k}{ax+b}dx &= \frac{k}{a}\int \frac{1}{u}du \\
&= \frac{k}{a}\ln|u|+C \\
&= \frac{k}{a}\ln|ax+b|+C.
\end{aligned}$$

- If the denominator is quadratic, i.e. $\deg\left(g(x)\right) = 2$, then $\frac{f(x)}{g(x)}$ will take the form $\frac{k}{ax^2+bx+c}$ or $\frac{kx+l}{ax^2+bx+c}$. The former can be easily evaluated by completing the squares in the denominator and using trigonometric/hyperbolic substitutions. The latter, however, can best be evaluated best by use of partial fractions.

  Example 6.45. Evaluate
$$\int \frac{x+4}{x^2-5x+6}dx.$$

Solution. We first resolve $\frac{x+4}{x^2-5x+6}$ into partial fractions as follows:

$$\begin{aligned}
\frac{x+4}{x^2-5x+6} &= \frac{x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} \\
&= \frac{A(x-3)+B(x-2)}{(x-2)(x-3)} \\
&\Leftrightarrow A = -6, \quad B = 7.
\end{aligned}$$

Hence, we can write

$$\frac{x+4}{x^2-5x+6} = \frac{7}{x-3} - \frac{6}{x-2},$$

and so

$$\begin{aligned}
\int \frac{x+4}{x^2-5x+6} dx &= \int \left(\frac{7}{x-3} - \frac{6}{x-2} \right) dx \\
&= \int \frac{7}{x-3} dx - \int \frac{6}{x-2} dx \\
&= 7 \ln(x-3) - 6 \ln(x-2) + C \\
&= \ln \left(\frac{(x-3)^7}{(x-2)^6} \right) + C.
\end{aligned}$$

Factorising the denominator is very essential in integrating functions by use of partial fractions, and the following theorem may be helpful.

Theorem 6.46. *Every polynomial with real constants can be factorised into linear and quadratic factors; and more importantly,*

$$\begin{aligned}
x^n - 1 &= \begin{cases} \left(x^{\frac{n}{2}} - 1\right)\left(x^{\frac{n}{2}} + 1\right) & \text{if } n \text{is even} \\ (x-1)\left(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + 1\right) & \text{if } n \text{ is odd;} \end{cases} \\
x^n + 1 &= \begin{cases} \left(x^{\frac{n}{2}} + 1\right)^2 - 2x^{\frac{n}{2}} = \left(x^{\frac{n}{2}} + \sqrt{2}x^{\frac{n}{4}} + 1\right)\left(x^{\frac{n}{2}} - \sqrt{2}x^{\frac{n}{4}} + 1\right) & \text{if } n \text{is even} \\ (x+1)\left(x^{n-1} - x^{n-2} + x^{n-3} - x^{n-4} + \cdots - 1\right) & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Example 6.47.

$$\begin{aligned}
x^3 - 1 &= (x-1)\left(x^2 + x + 1\right) \\
x^3 + 1 &= (x+1)\left(x^2 - x + 1\right) \\
x^4 - 1 &= \left(x^2 - 1\right)\left(x^2 + 1\right) \\
&= (x-1)(x+1)\left(x^2 + 1\right) \\
x^4 + 1 &= \left(x^2 + 1\right)^2 - x^2 \\
&= \left(x^2 - \sqrt{2}x + 1\right)\left(x^2 + \sqrt{2}x + 1\right).
\end{aligned}$$

Solution.

Example 6.48. m

Solution.

Example 6.49. m

Solution.

Example 6.50. m

Solution.

Give more Examples in full note

Problem 6.4

6.5 Integration of Irrational Functions

- **Intergrating Integrals of the Form**

$$\int \frac{px+q}{ax^2+bx+c}dx, \quad \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx, \quad \int (px+q)\sqrt{ax^2+bx+c}dx,$$

where a, b, c, p, q are real numbers and ax^2+bx+c is not factorisable into linear terms.

To evaluate these types of integrals, we suppose the numerator can be written as a derivative of the denominator plus a constant, that is,

$$\begin{aligned} px+q &= \lambda \frac{d}{dx}\left(ax^2+bx+c\right)+\mu \\ &= \lambda(2ax+b)+\mu \end{aligned}$$

where λ and μ are scalars to be determined. In that case the integral becomes, for example,

$$\begin{aligned} \int \frac{px+q}{ax^2+bx+c}dx &= \int \frac{\lambda \frac{d}{dx}(ax^2+bx+c)+\mu}{ax^2+bx+c}dx \\ &= \lambda \int \frac{2ax+b}{ax^2+bx+c}dx + \int \frac{\mu}{ax^2+bx+c}dx \\ &= \lambda \ln\left|ax^2+bx+c\right| + \mu \int \frac{1}{ax^2+bx+c}dx, \end{aligned}$$

where the last integral can be possibly evaluated by making appropriate trigonometric/hyperbolic substitutions.

Note also that

$$\begin{aligned} \int \frac{f'(x)}{f(x)}dx &= \ln|f(x)|+C \\ \int \frac{f'(x)}{\sqrt{f(x)}}dx &= 2\sqrt{f(x)}+C \\ \int f'(x)\sqrt{f(x)}dx &= \tfrac{2}{3}(f(x))^{\frac{3}{2}}+C \\ &= \tfrac{2}{3}\sqrt{(f(x))^3}+C. \end{aligned}$$

Example 6.51. Evaluate the following integrals:

$$\int \frac{x^3+7x^2-3x+4}{x^4+x^2+1}dx, \quad \int \frac{x+2}{\sqrt{1+2x-x^2}}dx, \quad \int ......\sqrt{....}dx.$$

Solution. For,

$$\int \frac{x^3+7x^2-3x+4}{x^4+x^2+1}dx,$$

notice that

$$\begin{aligned}\frac{x^3+7x^2-3x+4}{x^4+x^2+1} &= \frac{x^3+7x^2-3x+4}{(x^2+1)^2-x^2}\\ &= \frac{x^3+7x^2-3x+4}{(x^2+x+1)(x^2-x+1)}.\end{aligned}$$

Let

$$\begin{aligned}\frac{x^3+7x^2-3x+4}{(x^2+x+1)(x^2-x+1)} &= \frac{Ax+B}{(x^2+x+1)}+\frac{Cx+D}{(x^2-x+1)}\\ \text{i.e. } x^3+7x^2-3x+4 &= (Ax+B)\left(x^2-x+1\right)+(Cx+D)\left(x^2+x+1\right);\end{aligned}$$

so that

$$x=0 \Rightarrow 4=A+D \tag{6.23}$$
$$x=1 \Rightarrow 9=A+B+3C+3D \tag{6.24}$$
$$x=-1 \Rightarrow 13=-3A+3B-C+D \tag{6.25}$$
$$x=2 \Rightarrow 34=6A+3B+14C+7D. \tag{6.26}$$

Solving (6.23), (6.24), (6.25) and (6.26) simultaneously, we obtain

$$A=-1,\ B=4,\ C=2,\ D=0.$$

Thus,

$$\begin{aligned}\int \frac{x^3+7x^2-3x+4}{x^4+x^2+1}dx &= \int \left(\frac{4-x}{(x^2+x+1)}+\frac{2x}{(x^2+x+1)}\right)dx\\ &= \int \frac{4-x}{x^2+x+1}dx+\int \frac{2x}{x^2-x+1}dx.\end{aligned}$$

For $\int \frac{4-x}{x^2+x+1}dx$, we let

$$\begin{aligned}4-x &= \lambda\frac{d}{dx}\left(x^2+x+1\right)+\mu\\ &= \lambda(2x+1)+\mu=2\lambda x+\lambda+\mu.\end{aligned}$$

Comparing coefficients, we obtain

$$\begin{aligned}-1 &= 2\lambda \Rightarrow \lambda=-\tfrac{1}{2}, \text{ and}\\ 4 &= \lambda+\mu \Rightarrow \mu=4+\tfrac{1}{2}=\tfrac{9}{2}.\end{aligned}$$

Hence,

$$\begin{aligned}
\int \frac{4-x}{x^2+x+1}dx &= \int \frac{\frac{9}{2}-\frac{1}{2}(2x+1)}{x^2+x+1}dx \\
&= \tfrac{9}{2}\int \frac{1}{x^2+x+1}dx - \tfrac{1}{2}\int \frac{2x+1}{x^2+x+1}dx \\
&= \tfrac{9}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx - \tfrac{1}{2}\int \frac{2x+1}{x^2+x+1}dx \\
&= \tfrac{9}{2}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \tfrac{1}{2}\ln\left|x^2+x+1\right| + C_1.
\end{aligned}$$

Similarly for $\int \frac{2x}{x^2-x+1}dx$, we let

$$\begin{aligned}
2x &= \lambda \frac{d}{dx}\left(x^2-x+1\right)+\mu \\
&= \lambda(2x-1)+\mu = 2\lambda x - \lambda + \mu
\end{aligned}$$

and so

$$\begin{aligned}
2 &= 2\lambda \Rightarrow \lambda = 1, \text{ and} \\
0 &= -\lambda + \mu \Rightarrow \mu = 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int \frac{2x}{x^2-x+1}dx &= \int \frac{(2x-1)+1}{x^2-x+1}dx \\
&= \int \frac{2x-1}{x^2-x+1}dx + \int \frac{1}{x^2-x+1}dx \\
&= \int \frac{2x-1}{x^2-x+1}dx + \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx \\
&= \ln\left|x^2-x+1\right| + \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C_2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int \frac{x^3+7x^2-3x+4}{x^4+x^2+1}dx &= \tfrac{9}{2}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \tfrac{1}{2}\ln\left|x^2+x+1\right| + C_1 \\
&\quad + \ln\left|x^2-x+1\right| + \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C_2 \\
&= \tfrac{9}{2}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) \\
&\quad + \ln\left|x^2-x+1\right| - \tfrac{1}{2}\ln\left|x^2+x+1\right| + C_1 + C_2 \\
&= \tfrac{9}{2}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + + \ln\left|\frac{x^2-x+1}{\sqrt{x^2+x+1}}\right| + C
\end{aligned}$$

where $C = C_1 + C_2$.

Example 6.52. m

Solution.

m

Problem 6.5

6.6 Integrals of Trigonometric and Hyperbolic Functions

6.6.1 The Basic Integrals

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \tan x dx = \ln|\sec x| + C = -\ln|\cos x| + C$$

$$\int \cot x dx = \ln|\sin x| + C = -\ln|\csc x| + C$$

$$\int \sec x dx \quad = \quad \ln|\sec x + \tan x| + C = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$$

$$\int \csc x dx \quad = \quad \ln|\csc x - \cot x| + C = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc \cot x dx = -\csc x + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \tan(ax+b)\,dx = -\frac{1}{a}\ln|\cos(ax+b)| + C$$

$$\int \cot(ax+b)\,dx = \frac{1}{a}\ln|\sin(ax+b)| + C$$

$$\int \sec(ax+b)\,dx = \frac{1}{a}\ln|\sec(ax+b) + \tan(ax+b)| + C$$

$$\int \csc(ax+b)\,dx = -\frac{1}{a}\ln|\csc(ax+b) + \cot(ax+b)| + C$$

6.6.2 Integrating Products of Trigonometric Functions

To evaluate integrals of the forms

$$\int \sin mx \cos nx dx, \quad \int \sin mx \sin nx dx, \quad \int \cos mx \cos nx dx,$$

where $m \neq n$, we make use of the following trigonometric identities:

$$\begin{aligned} \sin x \cos y &= \tfrac{1}{2}\left(\sin(x+y) + \sin(x-y)\right) & (6.27) \\ \sin x \sin y &= \tfrac{1}{2}\left(\cos(x-y) - \cos(x+y)\right) & (6.28) \\ \cos x \cos y &= \tfrac{1}{2}\left(\cos(x-y) + \cos(x+y)\right). & (6.29) \end{aligned}$$

Example 6.53. Find

$$\int \sin 3x \cos 5x dx, \quad \int \sin x \sin 2x dx, \quad \int \cos 15x \cos 4x dx.$$

Solution.

- $\sin 3x \cos 5x = \frac{1}{2}(\sin 8x + \sin 2x)$.

$$\begin{aligned} \therefore \quad \int \sin 3x \cos 5x dx &= \int \tfrac{1}{2}(\sin 8x + \sin 2x)\, dx \\ &= \tfrac{1}{2}\left(-\tfrac{1}{8}\cos 8x - \tfrac{1}{2}\cos 2x\right) + C \\ &= -\tfrac{1}{16}\cos 8x - \tfrac{1}{4}\cos 2x + C. \end{aligned}$$

- $\sin x \sin 2x = \frac{1}{2}(\cos x - \cos 3x)$.

$$\begin{aligned} \therefore \quad \int \sin x \sin 2x dx &= \tfrac{1}{2}\int (\cos x - \cos 3x)\, dx \\ &= \tfrac{1}{2}\left(\sin x - \tfrac{1}{3}\sin 3x\right) + C \\ &= \tfrac{1}{2}\sin x - \tfrac{1}{6}\sin 3x + C. \end{aligned}$$

- $\cos 15x \cos 4x = \frac{1}{2}(\cos 11x + \cos 19x)$.

$$\begin{aligned} \therefore \quad \int \cos 15x \cos 4x dx &= \tfrac{1}{2}\int (\cos 11x + \cos 19x)\, dx \\ &= \tfrac{1}{2}\left(\tfrac{1}{11}\sin 11x + \tfrac{1}{19}\sin 19x\right) + C \\ &= \tfrac{1}{22}\cos 11x + \tfrac{1}{38}\cos 19x + C. \end{aligned}$$

6.6.3 Integrating Powers of Trigonometric Functions.

6.6.3.1 Integrals of Powers of Sines and Cosines.

Consider integrals of the form

$$\int \sin^m x \cos^n x dx.$$

Case I: Either m or n is Odd or Both are Odd.

Suppose n is odd, that is $n = 2k+1$, $k \in \mathbb{N}$. Then

$$\begin{aligned}
\cos^n x &= \cos^{2k} x \cos x \\
&= \left(\cos^2\right)^k \cos x \\
&= \left(1 - \sin^2\right)^k \cos x,
\end{aligned}$$

so that

$$\int \sin^m x \cos^n x dx = \int \sin^m x \left(1 - \sin^2\right)^k \cos x dx.$$

Let $u = \sin x$. Then $du = \cos x dx$ and the integral becomes

$$\int \sin^m x \cos^n x dx = \int u^m \left(1 - u^2\right)^k du.$$

The integrand in the last integral is a polynomial in u and so can be easily evaluated.

Similarly, if m is odd, i.e. $m = 2k+1$, then

$$\int \sin^m x \cos^n x dx = -\int \left(1 - u^2\right)^k u^n du, \quad u = \cos x.$$

Case II: Both m and n are Even.

If both m and n are even, that is $m = 2k$ and $n = 2r$, $k, r \in \mathbb{N}$. Then as before, we can make use of the following identities:

$$\sin^2 x = \tfrac{1}{2}\left(1 - \cos 2x\right) \tag{6.30}$$

$$\cos^2 x = \tfrac{1}{2}\left(1 + \cos 2x\right), \tag{6.31}$$

and

$$\begin{aligned}
\int \sin^m x \cos^n x dx &= \int \sin^{2k} x \cos^{2r} x dx, \quad k, r \in \mathbb{N} \\
&= \int \left(\sin^2 x\right)^k \left(\cos^2 x\right)^r x dx \\
&= \int \left(\tfrac{1}{2}\left(1 - \cos 2x\right)\right)^k \left(\tfrac{1}{2}\left(1 + \cos 2x\right)\right)^r x dx \\
&= \frac{1}{2^k} \cdot \frac{1}{2^r} \int \left(1 - \cos 2x\right)^k \left(1 + \cos 2x\right)^r dx \\
&= \frac{1}{2^{k+r}} \int \left(1 - \cos 2x\right)^k \left(1 + \cos 2x\right)^r dx.
\end{aligned} \tag{6.32}$$

We shall be relying on binomial theorem to expand the factors $(1 - \cos 2x)^k$ and $(1 + \cos 2x)^r$.

Example 6.54. Evaluate the following:

$$\int \sin^3 x \cos^2 x dx, \quad \int \sin^2 x \cos^2 x dx, \quad \int \frac{\sin^7 x}{\cos^4 x} dx, \quad \int \sin^4 x dx, \quad \int \sin^n x \cos x dx, \quad \int \cos^n x \sin x dx.$$

- $$\begin{aligned} \int \sin^3 x \cos^2 x dx &= \int \sin x \sin^2 x \cos^2 x dx \\ &= \int \left(1 - \cos^2 x\right) \cos^2 x \sin x dx. \end{aligned}$$

  Let $u = \cos x$ so that $du = -\sin x dx$.

  $$\begin{aligned} \therefore \quad \int \sin^3 x \cos^2 x dx &= -\int \left(1 - u^2\right) u^2 du \\ &= -\int \left(u^2 - u^4\right) du \\ &= -\tfrac{1}{3}u^3 + \tfrac{1}{5}u^5 + C \\ &= -\tfrac{1}{3}\cos^3 x + \tfrac{1}{5}\cos^5 x + C. \end{aligned}$$

- $$\begin{aligned} \int \sin^2 x \cos^2 x dx &= \int \left(\frac{1}{2}(1 - \cos 2x) \cdot \frac{1}{2}(1 + \cos 2x)\right) dx \\ &= \frac{1}{4}\int \left(1 - \cos^2 2x\right) dx \\ &= \frac{1}{4}\int \sin^2 2x dx \\ &= \frac{1}{4}\int \frac{1}{2}(1 - \cos 4x) dx \\ &= \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + C. \end{aligned}$$

- $$\begin{aligned} \int \frac{\sin^7 x}{\cos^4 x} dx &= \int \frac{\sin^6 x}{\cos^4 x} \sin x dx \\ &= \int \frac{\left(1 - \cos^2 x\right)^3}{\cos^4 x} \sin x dx. \end{aligned}$$

Let $u = \cos x$. Then $du = -\sin x dx$ and so

$$\begin{aligned}
\int \frac{\sin^7 x}{\cos^4 x} dx &= \int \frac{\left(1-\cos^2 x\right)^3}{\cos^4 x} \sin x dx \\
&= -\int \frac{\left(1-u^2\right)^3}{u^4} du \\
&= -\int \frac{1-3u^2+3u^4-u^6}{u^4} du, \quad , \quad \text{by the Binomial Theorem} \\
&= -\int \left(\frac{1}{u^4} - \frac{3}{u^2} + 3 - u^2\right) du \\
&= -\left(-\frac{1}{3u^3} + \frac{3}{u} + 3u - \frac{u^3}{3}\right) + C \\
&= \left(\frac{u^3}{3} + \frac{1}{3u^3} - \frac{3}{u} - 3u\right) + C \\
&= \frac{1}{3}\left(\cos^3 x + \frac{1}{\cos^3 x}\right) - 3\left(\cos x + \frac{1}{\cos x}\right) + C \\
&= \frac{1}{3}\left(\cos^3 x + \sec^3 x\right) - 3\left(\cos x + \sec x\right) + C.
\end{aligned}$$

- $$\begin{aligned}
\int \sin^4 x dx &= \int \left(\sin^2 x\right)^2 dx \\
&= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx \\
&= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) dx, \quad \text{using Binomial expansion} \\
&= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right) dx \\
&= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx \\
&= \frac{1}{4}\left(\frac{3}{2}x - 2\cdot\frac{1}{2}\sin 2x + \frac{1}{2}\cdot\frac{1}{4}\sin 4x\right) + C \\
&= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.
\end{aligned}$$

- Let $u = \sin x \Rightarrow du = \cos x dx$.

$$\begin{aligned}
\int \sin^n x \cos x dx &= \int u^n du \\
&= \frac{1}{n+1} u^{n+1} + C \\
&= \frac{1}{n+1} \sin^{n+1} x + C.
\end{aligned}$$

- Let $u = \cos x \Rightarrow du = -\sin x dx$.

$$\begin{aligned}
\int \cos^n x \sin x dx &= -\int u^n du \\
&= -\frac{1}{n+1} u^{n+1} + C \\
&= -\frac{1}{n+1} \cos^{n+1} x + C.
\end{aligned}$$

6.6.3.2 Integrals of Powers of Tangents and Secants.

Consider integrals of the form

$$\int \tan^m x \sec^n x dx.$$

Case I: n is Even.

If $n = 2k$, $k \in \mathbb{N}$, then we can use the identity

$$\sec^2 x = 1 + \tan^2 x$$

and make the substitution $u = \tan x$. Then $du = \sec^2 x dx$ and

$$\begin{aligned}
\int \tan^m x \sec^n x dx &= \int \tan^m x \sec^{n-2} x \sec^2 x dx \\
&= \int \tan^m x \sec^{2k-2} x \sec^2 x dx, \quad n = 2k; \; k \in \mathbb{N} \\
&= \int \tan^m x \left(\sec^2 x\right)^{k-1} \sec^2 x dx \\
&= \int \tan^m x \left(1 + \tan^2 x\right)^{k-1} \sec^2 x dx \\
&= \int u^m \left(1 + u^2\right)^{k-1} du, \quad u = \tan x
\end{aligned}$$

which again reduces to a polynomial in u and can be easily evaluated.

Case II: m is Odd or Both m and n are Odd.

Suppose $m = 2k + 1$, $k \in \mathbb{N}$, then we can make the substitution $u = \sec x \Rightarrow du = \sec x \tan x dx$ and use the identity

$$\tan^2 x = \sec^2 x - 1.$$

Hence,

$$\begin{aligned}
\int \tan^m x \sec^n x dx &= \int \tan^{2k+1} x \sec^n x dx \\
&= \int \tan^{2k} x \sec^{n-1} x \sec x \tan x dx, \quad n = 2k; \; k \in \mathbb{N} \\
&= \int \left(\sec^2 x - 1\right)^k \sec^{n-1} x du \\
&= \int \left(u^2 - 1\right)^k u^{n-1} du, \quad u = \sec x
\end{aligned}$$

which also having a polynomial integrand in u.

Case III: m is Even and n is Odd.

If m is even and n is odd, use the identity

$$\tan^2 x = \sec^2 x - 1$$

and use integration by parts (see Section 6.7) or a reduction formula (see Chapter 7) to integrate $\sec^k x$, $k \in \mathbb{N}$.

Example 6.55. Evaluate the following integrals:

$$\int \sec^9 x \tan^5 x dx, \quad \int \sec^4 x \tan^6 x dx, \quad \int \tan^3 x dx, \quad \int \sec^2 x \tan^n x dx.$$

- Since both $m = 5$ and $n = 9$ are odd, we can write

$$\begin{aligned} \int \sec^9 x \tan^5 x dx &= \int \sec^8 x \tan^4 x \sec x \tan x dx \\ &= \int \sec^8 x \left(\sec^2 x - 1\right)^2 \sec x \tan x dx \end{aligned}$$

and let $u = \sec x \Rightarrow \sec x \tan x dx$ so that

$$\begin{aligned} \int \sec^9 x \tan^5 x dx &= \int u^8 \left(u^2 - 1\right)^2 du \\ &= \int u^8 \left(u^4 - 2u^2 + 1\right) du \\ &= \int \left(u^{12} - 2u^{10} + u^8\right) du \\ &= \frac{1}{13} u^{13} - \frac{2}{11} u^{11} + \frac{1}{9} u^9 + C \\ &= \frac{1}{13} \sec^{13} x - \frac{2}{11} \sec^{11} x + \frac{1}{9} \sec^9 x + C. \end{aligned}$$

-

$$\begin{aligned} \int \sec^4 x \tan^6 x dx &= \int \sec^2 x \tan^6 x \sec^2 x dx \\ &= \int \left(1 + \tan^2 x\right) \tan^6 x \sec^2 x dx. \end{aligned}$$

Let $u = \tan x \Rightarrow \sec^2 x dx$ so that

$$\begin{aligned} \int \sec^4 x \tan^6 x dx &= \int \left(1 + u^2\right) u^6 du \\ &= \int \left(u^2 + u^8\right) du \\ &= \frac{1}{3} u^3 + \frac{1}{9} u^9 + C \\ &= \frac{1}{3} \tan^3 x + \frac{1}{9} \tan^9 x + C. \end{aligned}$$

- $$\begin{aligned}\int \tan^3 x dx &= \int \tan^2 x \tan x dx \\ &= \int \left(\sec^2 x - 1\right) \tan x dx \\ &= \int \tan x \sec^2 x dx - \int \tan x dx.\end{aligned}$$

  For $\int \tan x \sec^2 x dx$, let $u = \tan x \Rightarrow \sec^2 x dx$ and $\int \tan x dx$, we write $\tan x$ as $\frac{\sin x}{\cos x}$. Therefore,

  $$\begin{aligned}\int \tan^3 x dx &= \int u du + \int \frac{-\sin x}{\cos x} dx \\ &= \frac{1}{2}u^2 + \ln|\cos x| + C \\ &= \frac{1}{2}\tan^2 x + \ln|\cos x| + C.\end{aligned}$$

  Alternatively, we can write $\int \tan x \sec^2 x dx = \int \sec x \sec x \tan x dx$ and let $u = \sec x \Rightarrow du = \sec x \tan x dx$

  $$\begin{aligned}\therefore \quad \int \tan x \sec^2 x dx &= \int u du \\ &= \frac{1}{2}\sec^2 x + C.\end{aligned}$$

- $$\int \tan^n x \sec^2 x dx = \frac{1}{n+1}\tan^{n+1} x + C.$$

6.6.3.3 Integrals of Powers of Cotangents and Cosecants.

Integrals of the form

$$\int \cot^m x \csc^n x dx$$

are handled similarly to those of the form

$$\int \tan^m x \sec^n x dx.$$

Hence, we have the following scenarios.

Case I: n is Even.

If n is even, make the substitution $u = \cot x$, where $du = -\csc^2 x dx$ and use the identity

$$\csc^2 x = \cot^2 x + 1.$$

That is,

$$\begin{aligned}
\int \cot^m x \csc^n x dx &= \int \cot^m x \csc^{2k} x dx, \quad k \in \mathbb{N} \\
&= \int \cot^m x \csc^{2k-2} x \csc^2 x dx \\
&= \int \cot^m x \left(\cot^2 x + 1\right)^{k-1} \csc^2 x dx \\
&= -\int u^m \left(u^2+1\right)^{k-1} du, \quad u = \cot u.
\end{aligned}$$

Case II: m is Odd.

If m is odd, make the substitution $u = \csc x$, where $du = -\csc x \cot x dx$ and use the identity

$$\cot^2 x = \csc^2 x - 1.$$

That is,

$$\begin{aligned}
\int \cot^m x \csc^n x dx &= \int \cot^{2k+1} x \csc^n x dx, \quad k \in \mathbb{N} \\
&= \int \cot^{2k} x \csc^{n-1} x \cot x \csc x dx \\
&= \int \left(\csc^2 x - 1\right)^k \csc^{n-1} x \cot x \csc x dx \\
&= -\int \left(u^2-1\right)^k u^{n-1} du, \quad u = \csc u.
\end{aligned}$$

Example 6.56. Find

$$\int \cot^3 x \csc^7 x dx, \quad \ldots\ldots\ldots\ldots\ldots\ldots \int \cot x \csc^n x dx, \quad \int \cot^m x \csc^2 x dx.$$

6.6.4 Integrating Rational Functions of the Trigonometric and Hyperbolic Functions: The t-Substitution.

Here, we seek to evaluate integrals of the form,

$$\int \frac{a\sin x + b\cos x + c}{a_1 \sin x + a_1 \cos x + c_1} dx, \quad \int \frac{\sin^2 x + \cos x}{\cos^2 x + \sin x} dx, \quad \int \frac{a\sinh x + b\cosh x + c}{a_1 \sinh x + a_1 \cosh x + c_1} dx.$$

Any integral of rational expressions of trigonometric functions, such as

$$\int R(\sin x, \cos x)\, dx$$

can always be reduced to integrating rational functions of polynomials by making the so called "universal trigonometric substitution" otherwise known as the t-substitution

$$t = \tan\frac{x}{2}, \quad -\pi < x < \pi.$$

Then $x = 2\tan^{-1} t$, and

$$\begin{aligned}
\sin x \equiv \sin\left(\frac{x}{2} + \frac{x}{2}\right) &= 2\sin\frac{x}{2}\cos\frac{x}{2} \\
&= \frac{2\sin\frac{x}{2}}{\sec\frac{x}{2}}, \quad \because \cos\frac{x}{2} = \frac{1}{\sec\frac{x}{2}} \\
&= \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}}, \quad \because \tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} \Rightarrow \sin\frac{x}{2} = \tan\frac{x}{2}\cos\frac{x}{2} = \frac{\tan\frac{x}{2}}{\sec\frac{x}{2}} \\
&= \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}} \\
&= \frac{2t}{1+t^2}.
\end{aligned}$$

Thus, if $t = \tan\frac{x}{2}$ then

$$\sin x = \frac{2t}{1+t^2}. \tag{6.33}$$

Similarly,

$$\begin{aligned}
\cos x &= \frac{1-t^2}{1+t^2} && (6.34) \\
\tan x &= \frac{2t}{1-t^2} && (6.35) \\
\sec x &= \frac{1+t^2}{1-t^2} && (6.36) \\
\cot x &= \frac{1-t^2}{2t} && (6.37) \\
\csc x &= \frac{1+t^2}{2t}. && (6.38)
\end{aligned}$$

Finally, since $x = 2\tan^{-1} t$, we have

$$dx = \frac{2}{1+t^2}dt. \tag{6.39}$$

Hence, by making the substitution $t = \tan\frac{x}{2}$, we have that

$$\int R(\sin x, \cos x)\, dx \to \int R(t)\, dt,$$

where R denotes a collection of rational functions.

Special Cases:

- To evaluate integrals of the form

$$\int R(\sin x)\cos x dx,$$

that is, a function rational only in $\sin x$, make the substitution $t = \sin x \Rightarrow dt = \cos x dx$. If however, both $\sin x$ and $\cos x$ have even powers, make the substitution $t = \tan x$, so that

$$\cos^2 x = \frac{1}{\sec^2 x} = \frac{1}{1+\tan^2 x} = \frac{1}{1+t^2},$$

$$\sin^2 x = \tan^2 x \cos^2 x = \frac{\tan^2 x}{\sec^2 x} = \frac{\tan^2 x}{1+\tan^2 x} = \frac{t^2}{1+t^2},$$

and

$$x = \tan^{-1} t \Rightarrow dx = \frac{1}{1+t^2} dt.$$

- Similarly, for an integral of the form

$$\int R(\cos x) \sin x dx$$

can be evaluated by using the substitution $t = \cos x \Rightarrow dt = -\sin x dx$ and so

$$\int R(\cos x) \sin x dx \to -\int R(t)\, dt.$$

Example 6.57. Evaluate the following:

$$\int \frac{dx}{1+\sin x}, \quad \int \frac{dx}{1+\cos \frac{x}{2}}, \quad \int \frac{dx}{\cos x + \sin x}, \quad \int \frac{dx}{\sec x + 1},$$

Solution.

- Let $t = \tan \frac{x}{2}$ for $-\pi < x < \pi$, so that $x = 2\tan^{-1} t$, $\sin x = \frac{2t}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$. Then

$$\begin{aligned}
\int \frac{dx}{1+\sin x} &= \int \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}} = \int \frac{\frac{2}{1+t^2}}{\frac{1+t^2+2t}{1+t^2}} dt \\
&= \int \frac{2}{t^2+2t+1} dt = \int \frac{2}{(t+1)^2} dt \\
&= -\frac{2}{t+1} + C \\
&= -\frac{2}{\tan \frac{x}{2} + 1} + C.
\end{aligned}$$

- Let $t = \tan \frac{x}{4}$, $x \in (-2\pi, 2\pi)$. Then $x = 4\tan^{-1} t$, $\cos \frac{x}{2} = \frac{1-t^2}{1+t^2}$ and $d\left(\frac{x}{2}\right) = \frac{2dt}{1+t^2} \Rightarrow dx = \frac{4dt}{1+t^2}$. Hence

$$\begin{aligned}
\int \frac{dx}{1+\cos \frac{x}{2}} &= \int \frac{\frac{4dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} = 4\int \frac{\frac{1}{1+t^2}}{\frac{1+t^2+1-t^2}{1+t^2}} dt \\
&= 4\int \frac{1}{1+t^2+1-t^2} dt = 2\int dt \\
&= 2t + C \\
&= 2\tan \frac{x}{4} + C.
\end{aligned}$$

- Let $t = \tan\frac{x}{2}$. Then $x = 2\tan^{-1} t$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$ $\forall x \in (-\pi, \pi)$. Therefore

$$\begin{aligned}
\int \frac{dx}{\sin x + \cos x} &= \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} = \int \frac{\frac{2}{1+t^2}}{\frac{2t+1-t^2}{1+t^2}} dt \\
&= 2\int \frac{1}{1+2t-t^2} dt = 2\int \frac{1}{1-(t^2-2t)} dt \\
&= 2\int \frac{1}{2-(t-1)^2} dt = 2\int \frac{1}{\left(\sqrt{2}\right)^2 - (t-1)^2} dt \\
&= 2\int \frac{1}{\left(\sqrt{2}+(t-1)\right)\left(\sqrt{2}-(t-1)\right)} dt.
\end{aligned}$$

Resolving the integrand in the last integral into partial fractions, we have

$$\begin{aligned}
\frac{1}{\left(\sqrt{2}+(t-1)\right)\left(\sqrt{2}-(t-1)\right)} &= \frac{A}{\left(\sqrt{2}+(t-1)\right)} + \frac{B}{\left(\sqrt{2}-(t-1)\right)} \\
&= \frac{A\left(\sqrt{2}-(t-1)\right) + B\left(\sqrt{2}+(t-1)\right)}{\left(\sqrt{2}+(t-1)\right)\left(\sqrt{2}-(t-1)\right)} \\
\text{i.e. } 1 &= A\left(\sqrt{2}-(t-1)\right) + B\left(\sqrt{2}+(t-1)\right) \\
\therefore \text{ when } t-1=\sqrt{2} &\Rightarrow 1 = 2\sqrt{2}B \Rightarrow B = \frac{1}{2\sqrt{2}}; \\
\text{and when } t-1=-\sqrt{2} &\Rightarrow 1 = 2\sqrt{2}A \Rightarrow A = \frac{1}{2\sqrt{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int \frac{dx}{\sin x + \cos x} &= 2\int \frac{1}{\left(\sqrt{2}+(t-1)\right)\left(\sqrt{2}-(t-1)\right)} dt \\
&= 2\int \left(\frac{1}{2\sqrt{2}}\left(\frac{1}{\left(\sqrt{2}+(t-1)\right)} + \frac{1}{\left(\sqrt{2}-(t-1)\right)}\right)\right) dt \\
&= \frac{1}{\sqrt{2}}\left(\ln\left|\sqrt{2}+(t-1)\right| - \ln\left|\sqrt{2}-(t-1)\right|\right) + C \\
&= \frac{1}{\sqrt{2}} \ln\left|\frac{\sqrt{2}+(t-1)}{\sqrt{2}-(t-1)}\right| + C \\
&= \frac{1}{\sqrt{2}} \ln\left|\frac{\sqrt{2}+\tan\frac{x}{2}-1}{\sqrt{2}-\tan\frac{x}{2}+1}\right| + C.
\end{aligned}$$

- Let $t = \tan\frac{x}{2}$ for $-\pi < x < \pi$, so that $x = 2\tan^{-1} t$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$.

Then

$$\begin{aligned}
\int \frac{dx}{\sec x + 1} &= \int \frac{dx}{\frac{1}{\cos x} + 1} = \int \frac{\cos x}{1 + \cos x} dx \\
&= \int \frac{\frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} = 2\int \frac{\frac{1-t^2}{(1+t^2)^2}}{\frac{1+t^2+1-t^2}{1+t^2}} dt \\
&= 2\int \frac{\frac{1-t^2}{(1+t^2)}}{1+1} dt = \int \frac{1-t^2}{1+t^2} dt \\
&= \int \frac{-\left(1+t^2-2\right)}{1+t^2} dt = -\int \frac{1+t^2}{1+t^2} dt + \int \frac{2}{1+t^2} dt \\
&= -\int dt + \int \frac{2}{1+t^2} dt \\
&= -t + 2\tan^{-1} t + C \\
&= -\tan\frac{x}{2} + 2\tan^{-1}\left(\tan\frac{x}{2}\right) + C \\
&= -\tan\frac{x}{2} + x + C.
\end{aligned}$$

Example 6.58. Show that

$$\sin^4 x + \cos^4 x = 1 - \tfrac{1}{2}\sin^2 2x \quad \forall x \in \mathbb{R}.$$

Hence or otherwise, evaluate of the integral

$$\int \frac{dx}{\sin^4 x + \cos^4 x}.$$

Solution. Notice that, since

$$\begin{aligned}
1 &= \sin^2 x + \cos^2 x \quad \forall x \in \mathbb{R} \\
\Rightarrow 1 &= \left(\sin^2 x + \cos^2 x\right)^2 = \sin^4 x + \cos^4 x + 2\sin^2 x \cos^2 x \\
\Rightarrow \sin^4 x + \cos^4 x &= 1 - 2\sin^2 x \cos^2 x \\
&= 1 - \tfrac{1}{2}\sin^2 2x.
\end{aligned}$$

$$\begin{aligned}
\therefore \int \frac{dx}{\sin^4 x + \cos^4 x} &= \int \frac{dx}{1 - \frac{1}{2}\sin^2 2x} \\
&= \int \frac{2dx}{2 - \sin^2 2x}.
\end{aligned}$$

Now, let $t = \tan 2x$, $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \Rightarrow x = \frac{1}{2}\tan^{-1} t$, $\sin 2x = \frac{\tan^2 2x}{1+\tan^2 2x} = \frac{t^2}{1+t^2}$ and $dx = \frac{1}{2}\frac{dt}{1+t^2}$.That is,

$$\begin{aligned}
\therefore \int \frac{dx}{\sin^4 x + \cos^4 x} &= \int \frac{2dx}{2 - \sin^2 2x} \\
&= \int \frac{2 \cdot \frac{1}{2}\frac{dt}{1+t^2}}{2 - \frac{t^2}{1+t^2}} = \int \frac{dt}{2\left(1+t^2\right) - t^2} \\
&= \int \frac{dt}{2+t^2} = \frac{1}{\sqrt{2}} \int \frac{1}{1 + \left(\frac{t}{\sqrt{2}}\right)^2} dt \\
&= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C \\
&= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{\sqrt{2}}\tan 2x\right) + C.
\end{aligned}$$

6.6.4.1 Alternative Ways of Evaluating Some Integrals of the Form

$$\int \frac{a\sin x + b\cos x}{a_1 \sin x + a_1 \cos x} dx.$$

What we seek to do here is to *write the numerator as a derivative and a constant multiple of the denominator.* That is, we shall introduce and then determine non-zero constants λ and μ such that

$$\begin{aligned} a\sin x + b\cos x &= \lambda \frac{d}{dx}(a_1 \sin x + b_1 \cos x) + \mu (a_1 \sin x + b_1 \cos x) \\ &= \lambda (a_1 \cos x - b_1 \sin x) + \mu (a_1 \sin x + b_1 \cos x) \\ &= (-b_1 \lambda + a_1 \mu)\sin x + (a_1 \lambda + b_1 \mu)\cos x. \end{aligned}$$

Comparing coefficients, we get

$$a = -b_1\lambda + a_1\mu \tag{6.40}$$

$$b = a_1\lambda + b_1\mu. \tag{6.41}$$

Solving for λ and μ simultaneously, we obtain

$$\lambda = \frac{a_1 b - a b_1}{a_1^2 + b_1^2}, \qquad \mu = \frac{a a_1 + b b_1}{a_1^2 + b_1^2}.$$

Hence,

$$\begin{aligned} \int \frac{a\sin x + b\cos x}{a_1 \sin x + a_1 \cos x} dx &= \int \frac{\lambda (a_1 \cos x - b_1 \sin x) + \mu (a_1 \sin x + b_1 \cos x)}{a_1 \sin x + a_1 \cos x} dx \\ &= \lambda \int \frac{a_1 \cos x - b_1 \sin x}{a_1 \sin x + a_1 \cos x} dx + \mu \int \frac{a_1 \sin x + b_1 \cos x}{a_1 \sin x + a_1 \cos x} dx \\ &= \lambda \int \frac{a_1 \cos x - b_1 \sin x}{a_1 \sin x + a_1 \cos x} dx + \mu \int dx \\ &= \lambda \ln |a_1 \sin x + a_1 \cos x| + \mu x + C \\ &= \frac{a_1 b - a b_1}{a_1^2 + b_1^2} \ln |a_1 \sin x + a_1 \cos x| + \frac{a a_1 + b b_1}{a_1^2 + b_1^2} x + C. \end{aligned}$$

Note that this technique may not work or may be difficult when applied to trigonometric rationals where the numerator is a constant (as we have seen in Examples 6.57 and 6.58) since it is almost always difficult if not impossible to obtain a constant as a derivative of a trigonometric function. In those cases, the universal t-substitution is most suitable.

Example 6.59. Evaluate

$$\int \frac{\cos x + 3\sin x}{\sin x + 3\cos x} dx.$$

Solution. Let

$$\begin{aligned} \cos x + 3\sin x &= \lambda \frac{d}{dx}(\sin x + 3\cos x) + \mu (\sin x + 3\cos x) \\ &= \lambda (\cos x - 3\sin x) + \mu (\sin x + 3\cos x) \\ &= (\mu - 3\lambda)\sin x + (\lambda + 3\mu)\cos x \\ \Leftrightarrow 3 = \mu - 3\lambda \quad &, \quad 1 = \lambda + 3\mu \\ \Rightarrow \lambda = -\frac{4}{5} \quad &, \quad \mu = \frac{3}{5}. \end{aligned}$$

Hence,

$$\begin{aligned}\int \frac{\cos x + 3\sin x}{\sin x + 3\cos x}dx &= -\frac{4}{5}\int \frac{\cos x - 3\sin x}{\sin x + 3\cos x}dx + \frac{3}{5}\int \frac{\sin x + 3\cos x}{\sin x + 3\cos x}dx \\ &= -\frac{4}{5}\ln|\sin x + 3\cos x| + \frac{3}{5}x + C.\end{aligned}$$

Example 6.60. Evaluate the following:

$$\int \frac{2\cos x + 3\sin x}{\cos x + \sin x}dx, \quad \int \frac{dx}{3\sin 2x + 4\cos x}.$$

Solution. m

6.6.4.2 Evaluating Integrals of the Form

$$\int \frac{a\sinh x + b\cosh x}{a_1\sinh x + a_1\cosh x}dx.$$

The techniques for evaluating rational functions of hyperbolic functions are similar to those of their trigonometric counterparts. Hence, when we make the substitution

$$t = \tanh\frac{x}{2}, \tag{6.42}$$

then

$$\int R(\sinh x, \cosh x)\, dx \to \int R(t)\, dt;$$

and when we introduce the constants λ and μ such that

$$a\sinh x + b\cosh x = \lambda\frac{d}{dx}(a_1\sinh x + b_1\cosh x) + \mu(a_1\sinh x + b_1\cosh x),$$

then

$$\lambda = \frac{a_1b - ab_1}{a_1^2 - b_1^2}, \quad \mu = \frac{aa_1 - bb_1}{a_1^2 - b_1^2}, \ \ a_1 \neq \pm b_1.$$

and so

$$\int \frac{a\sinh x + b\cosh x}{a_1\sinh x + a_1\cosh x}dx = \frac{a_1b - ab_1}{a_1^2 - b_1^2}\ln|a_1\sinh x + a_1\cosh x| + \frac{aa_1 - bb_1}{a_1^2 - b_1^2}x + C.$$

Example 6.61. m

Solution. m

Example 6.62. m

Solution. m

Problem 6.6

1. m
2. m
3. m
4. m
5. m
6. Integrate the following functions

    (a) $\frac{\cos^3 x}{1-\sin x}$

7. Evaluate the following integrals

$$\int \frac{dx}{1+\tan x} = \frac{1}{2}\ln|\tan x + 1| - \frac{1}{4}\ln\left(\tan^2 x + x + 1\right) + \frac{x}{2} + C;\ \ t = \tan x.$$

$$\int \frac{1}{a\sin x + b\cos x}dx = \frac{1}{\sqrt{a^2+b^2}}\ln\left|\frac{\sqrt{a^2+b^2} + b\tan\frac{x}{2} - a}{\sqrt{a^2+b^2} - b\tan\frac{x}{2} + a}\right| + C;\quad t = \tan\frac{x}{2}.$$

8. Show that

$$\int \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{b^2-a^2}}\ln\left|\frac{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}}\right| + C$$

and

$$\int \frac{dx}{a+b\sin x + c\cos x} = \frac{2}{\sqrt{a^2-b^2-c^2}}\tan^{-1}\left(\frac{(a-c)\tan\frac{x}{2}+b}{\sqrt{a^2-b^2-c^2}}\right) + C.$$

6.7 Integration by Parts.

Recall the product rule of differentiating two functions $f(x)$ and $g(x)$ with respect to x; that is

$$\frac{d}{dx}\left(f(x)\,g(x)\right) = f(x)\,g'(x) + f'(x)\,g(x).$$

If we integrate both sides with respect to x, we shall obtain

$$\int \frac{d}{dx}\left(f(x)\,g(x)\right)dx = \int f(x)\,g'(x)\,dx + \int f'(x)\,g(x)\,dx;$$

that is,

$$f(x)\,g(x) = \int f(x)\,g'(x)\,dx + \int f'(x)\,g(x)\,dx$$

or

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx.$$

Let us denote $u = f(x)$ and $v = g(x)$ so that $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Then we obtain the compact formula

$$\int u dv = uv - \int v du \tag{6.43}$$

called *integration by parts*. The formula assumes that it is relatively difficult to evaluate $\int u dv$ than $\int v du$. So we ought to develop the skills to choose u and v appropriate.

The trick is to choose the function which is easier to integrate as our dv and the other one as u. If both functions have the same degree of difficulty as far as differention and integration is concerned, choose the one which may varnish after a finite number of differentiation, as u.

Example 6.63. Evaluate the following:

$$\int xe^x dx, \quad \int x \sin x dx, \quad \int x^2 e^x dx, \quad \int x^7 \ln x dx.$$

Solution.

- Let $u = x$ and $dv = e^x dx$. Then $du = 1dx = dx$ and $v = \int e^x dx = e^x$.

$$\begin{aligned} \therefore \int xe^x dx = \int u dv &= uv - \int v du \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + C \\ &= (x-1)\,e^x + C. \end{aligned}$$

- Let $u = x$ and $dv = \sin x dx$. Then $du = dx$ and $v = \int \sin x dx = -\cos x$.

$$\begin{aligned} \therefore \int x \sin x dx &= uv - \int v du \\ &= -x\cos x - \int (-\cos x)\,dx \\ &= -x\cos x + \int \cos x dx \\ &= -x\cos x + \sin x + C. \end{aligned}$$

- Let $u = x^2$ and $dv = e^x dx \Rightarrow du = 2x dx$, $v = \int e^x dx = e^x$.

$$\begin{aligned} \therefore \int x^2 e^x dx &= uv - \int v du \\ &= x^2 e^x - 2\int xe^x dx. \end{aligned}$$

  Applying integration by parts again to the last term, we have

$$\int xe^x dx = (x-1)\,e^x + C.$$

  Hence,

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - 2\int xe^x dx \\ &= x^2 e^x - 2\,(x-1)\,e^x + C \\ &= \left(x^2 - 2x + 1\right) e^x + C. \end{aligned}$$

- Let $u = \ln x$ and $dv = x^7 dx$. Then $du = \frac{1}{x}dx$ and $v = \int x^7 dx = \frac{1}{8}x^8$. Thus,

$$\begin{aligned}
\int x^7 \ln x dx &= \tfrac{1}{8}x^8 \ln x - \tfrac{1}{8}\int x^8 \cdot \frac{1}{x} dx \\
&= \tfrac{1}{8}x^8 \ln x - \tfrac{1}{8}\int x^7 dx \\
&= \tfrac{1}{8}x^8 \ln x - \tfrac{1}{64}x^8 + C.
\end{aligned}$$

Example 6.64. Evaluate the following:

$$\int \frac{xe^x}{(x+1)^2} dx, \quad \int \ln x dx, \quad \int x^3 e^{x^2} dx, \quad \int (3x+5)\cos \tfrac{x}{4} dx.$$

Solution.

- Let $u = xe^x$ and $dv = \frac{1}{(x+1)^2}dx \Rightarrow du = (xe^x + e^x)\,dx = (x+1)\,e^x dx$, $v = \int \frac{1}{(x+1)^2}dx = -\frac{1}{x+1}$. Then

$$\begin{aligned}
\int \frac{xe^x}{(x+1)^2} dx &= -\frac{xe^x}{x+1} + \int \frac{1}{x+1}(x+1)\,e^x dx \\
&= -\frac{xe^x}{x+1} + \int e^x dx \\
&= -\frac{xe^x}{x+1} + e^x + C \\
&= \frac{(x+1)\,e^x - xe^x}{x+1} + C \\
&= \frac{e^x}{x+1} + C.
\end{aligned}$$

- Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}dx$ and $v = \int dx = x$.

$$\begin{aligned}
\therefore \quad \int \ln x dx &= uv - \int v du \\
&= x \ln x - \int x \cdot \frac{1}{x} dx \\
&= x \ln x - \int dx \\
&= x \ln x - x + C.
\end{aligned}$$

- First of all, let us make the substitution $t = x^2$ so that $dt = 2xdx$. Then the integral reduces to

$$\int x^3 e^{x^2} dx = \int x^2 x e^{x^2} dx = \tfrac{1}{2}\int te^t dt.$$

Now, we have already seen that

$$\int te^t dt = te^t - e^t + C.$$

Therefore,

$$\begin{aligned}
\int x^3 e^{x^2} dx &= \tfrac{1}{2}\left(te^t - e^t\right) + C \\
&= \tfrac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C.
\end{aligned}$$

- Let $u = 3x + 5$ and $dv = \cos\frac{x}{4}dx \Rightarrow du = 3dx$, $v = \int \cos\frac{x}{4}dx = 4\sin\frac{x}{4}$.

$$\begin{aligned} \therefore \int (3x+5)\cos\tfrac{x}{4}dx &= 4(3x+5)\sin\tfrac{x}{4} - 12\int \sin\tfrac{x}{4}dx \\ &= 4(3x+5)\sin\tfrac{x}{4} + 48\cos\tfrac{x}{4} + C. \end{aligned}$$

Example 6.65. Evaluate the following:

$$\int e^x \sin x dx, \quad \int \sec^3 x dx.$$

Solution.

- Let $u = e^x$ and $dv = \sin x dx$. Then $du = e^x dx$, $v = \int \sin x dx = -\cos x$.

$$\begin{aligned} \therefore \int e^x \sin x dx &= uv - \int v du \\ &= -e^x \cos x + \int e^x \cos x dx. \end{aligned}$$

  Again, we evaluate $\int e^x \cos x dx$ using integration by parts as follows:
  let $s = e^x \Rightarrow ds = e^x dx$ and $dt = \cos x dx \Rightarrow t = \sin x$. Then

$$\begin{aligned} \int e^x \cos x dx &= st - \int t ds \\ &= e^x \sin x - \int e^x \sin x dx. \end{aligned}$$

  Therefore, the original integral becomes

$$\begin{aligned} \therefore \int e^x \sin x dx &= -e^x \cos x + \int e^x \cos x dx \\ &= -e^x \cos x + e^x \sin x - \int e^x \sin x dx, \end{aligned}$$

  where the given integral has resurfaced again. Let's now solved for the integral of interest, i.e.

$$\begin{aligned} \int e^x \sin x dx &= -e^x \cos x + e^x \sin x - \int e^x \sin x dx \\ \Rightarrow 2\int e^x \sin x dx &= -e^x \cos x + e^x \sin x \\ \therefore \int e^x \sin x dx &= \tfrac{1}{2}(\sin x - \cos x)e^x + C. \end{aligned}$$

- Notice that $\sec^3 x = \sec x \sec^2 x$. Let $u = \sec x \Rightarrow du = \sec x \tan x dx$ and $dv =$

$\sec^2 x dx \Rightarrow v = \tan x$. Therefore,

$$\begin{aligned}
\therefore \int \sec^3 x dx &= \sec x \tan x - \int \sec x \tan^2 x dx \\
&= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) dx \\
&= \sec x \tan x + \int \sec x dx - \int \sec^3 x dx \\
\Rightarrow 2\int \sec^3 dx &= \sec x \tan x + \ln|\sec x + \tan x| + C \\
\therefore \int \sec^3 dx &= \tfrac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C.
\end{aligned}$$

Problem 6.7

See Calculus of Single Variable Larson, Edwards pg 318 - 322.

1. Evaluate the following:
   (a)
   $$\int x^2 \ln x dx.$$
   (b)
   $$\int \ln^2 x dx.$$
   (c)
   $$\int \frac{x}{\sqrt{5+4x}} dx.$$
   (d)
   $$\int x \sec x \tan x dx.$$
   (e)
   $$\int 2x\sqrt{2x-2} dx.$$
   (f) m
   (g) m
   (h) m
   (i) m
   (j) m
2. Evaluate the following:
   (a)
   $$\int e^{-x} \cos 2x dx, \quad \int x \tan^{-1} x dx.$$
   (b) m

(c) m

(d) m

(e) m

(f) m

(g) m

(h) m

(i) m

(j) m

(k) m

3. Prove the following:

(a)

$$\int x^n \ln x = \frac{x^{n+1}}{(n+1)^2}\left((n+1)\ln x - 1\right) + C, \quad n \neq -1.$$

(b)

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2}\left(a \sin bx - b \cos bx\right) + C.$$

(c) m

(d) m

(e) m

4. S

5. m

6. m

7. m

8. m

9. m

10. m

6.8 Integration Techniques for Definite Integrals

6.8.1 The Method of Substitution for Definite Integrals

Theorem 6.66. *Suppose that g is a differentiable function of x on $[a, b]$. Suppose also that f is a continuous function on the range of g. Then*

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Proof. Let F be an antiderivative of f, i.e.

$$F'(u) = f(u) \quad \text{or } F(u) = \int_a^b f(u)\, du.$$

Then

$$\begin{aligned} \frac{d}{dx} F(g(x)) &= F'(g(x))\, g'(x) \\ &= f(g(x))\, g'(x). \end{aligned}$$

Thus

$$\begin{aligned} \int_a^b f(g(x))\, g'(x) &= \int_a^b \frac{d}{dx} F(g(x))\, dx \\ &= \Bigg[F(g(x)) \Bigg]_a^b \\ &= F(g(b)) - F(g(a)) \\ &= \int_{g(a)}^{g(b)} f(u)\, du. \end{aligned}$$

$\square$

Example 6.67. Evaluate the following integrals:

$$\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}} dx, \quad \int_0^\pi \left(2 + \sin\tfrac{x}{2}\right)^2 \cos\tfrac{x}{2} dx, \quad \int_e^{e^2} \frac{dt}{t \ln t}, \quad \int_0^1 \frac{dx}{(2-x)\sqrt{4-x}}.$$

Solution.

- Let $u = \sqrt{x+1}$. Then $du = \frac{1}{2\sqrt{x+1}} dx \Rightarrow dx = 2u du$. Now, when $x = 0$, $u = \sqrt{0+1} = 1$ and when $x = 8$, $u = \sqrt{8+1} = 3$. Therefore,

$$\begin{aligned} \int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}} dx &= \int_1^3 \frac{\cos u}{u} 2u du \\ &= 2 \int_1^3 \cos u du \\ &= 2 [\sin u]_1^3 \\ &= 2 (\sin 3 - \sin 1). \end{aligned}$$

- Let $u = 2 + \sin\frac{x}{2}$. Then $du = \frac{1}{2}\cos\frac{x}{2} dx$, and $x = 0 \Rightarrow u = 2 + \sin 0 = 2$; $x = \pi \Rightarrow u = 2 + \sin\frac{\pi}{2} = 3$.

$$\begin{aligned} \int_0^\pi \left(2 + \sin\tfrac{x}{2}\right)^2 \cos\tfrac{x}{2} dx &= 2 \int_2^3 u^2 du \\ &= 2 \left[\frac{u^3}{3}\right]_2^3 \\ &= \frac{2}{3}(27 - 8) = \frac{38}{3}. \end{aligned}$$

- Let $u = \ln t \Rightarrow du = \frac{1}{t}dt$; $t = e \Rightarrow u = \ln e = 1$, and $t = e^2 \Rightarrow u = \ln e^2 = 2$.

$$\begin{aligned}
\int_e^{e^2} \frac{1}{t\ln t}dt &= \int_e^{e^2} \frac{1}{\ln t}\cdot\frac{1}{t}dt \\
&= \int_1^2 \frac{1}{u}du \\
&= [\ln u]_1^2 \\
&= \ln 2 - \ln 1 = \ln 2.
\end{aligned}$$

- Let $u = \frac{1}{2-x}$. Then $x = 2 - \frac{1}{u}$ and so $dx = \frac{1}{u^2}du$. Also, when $x = 0$, $u = \frac{1}{2}$ and when $x = 1$, $u = 1$. Hence,

$$\begin{aligned}
\int_0^1 \frac{dx}{(2-x)\sqrt{4-x^2}}dx &= \int_0^1 \frac{\frac{1}{u^2}du}{u\sqrt{4-\left(2-\frac{1}{u}\right)^2}} \\
&= \int_{\frac{1}{2}}^1 \frac{\frac{1}{u^2}du}{u\sqrt{4-\left(2-\frac{1}{u}\right)^2}} \\
&= \int_{\frac{1}{2}}^1 \frac{\frac{1}{u^2}du}{u\sqrt{4-4+\frac{4}{u^2}-\frac{1}{u^2}}} = \int_{\frac{1}{2}}^1 \frac{4}{\sqrt{4u-1}}du \\
&= \frac{1}{4}\left[2\sqrt{4u-1}\right]_{\frac{1}{2}}^1 = \frac{1}{2}\left(\sqrt{3}-1\right).
\end{aligned}$$

6.8.2 Integration by Parts for Definite Integrals

To apply the technique of integrating by parts to definite integrals, the same formula is applied only that each term is evaluated at the endpoints; that is

$$\begin{aligned}
\int_a^b udv &= \Bigg[\, uv \,\Bigg]_a^b - \int_a^b vdu \\
&= u(b)\,v(b) - u(a)\,v(a) - \int_a^b vdu.
\end{aligned}$$

Example 6.68. Evaluate the following:

$$\int_0^1 \tan^{-1} xdx, \quad \int_1^2 x^2e^{-x}dx, \quad \int_0^1 \sin^{-1} xdx, \quad \int_2^3 \ln^2 xdx, \quad \int_1^e \frac{\ln x}{x^2}dx.$$

Solution.

- Let $u = \tan^{-1} x$ and $dv = dx$. Then $du = \frac{1}{1+x^2}dx$ and $v = \int dx = x$.

$$\begin{aligned}
\therefore \int_0^1 \tan^{-1} xdx &= x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}dx \\
&= 1\cdot\tan^{-1} 1 - 0\cdot\tan^{-1} 0 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}dx \\
&= \frac{\pi}{4} - \frac{1}{2}\left[\ln\left(1+x^2\right)\right]_0^1 \\
&= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - \ln 1) \\
&= \frac{\pi}{4} - \frac{1}{2}\ln 2.
\end{aligned}$$

- Let $u = x^2$ and $dv = e^{-x}dx$. Then $du = 2xdx$ and $v = \int e^{-x}dx = -e^{-x}$.

$$\begin{aligned} \therefore \int_1^2 x^2 e^{-x} dx &= -x^2 e^{-x}\Big|_1^2 + 2\int_1^2 xe^{-x}dx \\ &= -4e^{-2} + e^{-1} + 2\int_1^2 xe^{-x}dx \\ &= -4e^{-2} + e^{-1} + 2\left(-xe^{-x}\Big|_1^2 + \int_1^2 e^{-x}dx\right) \\ &= -4e^{-2} + e^{-1} + 2\left(-2e^{-2} + e^{-1} - e^{-2} + e^{-1}\right) \\ &= \frac{5}{e}\left(1 - \frac{2}{e}\right). \end{aligned}$$

- Let
- Let
- Let

6.8.3 Using Some Properties of Definite Integrals To Evaluate Definite Integrals

Recall the following important properties of the definite integral:

1. $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx;$$

2. $$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^{2a} f(2a-x)\,dx.$$

We can use these and other properties of the definite integral to facilitate the easy evaluation of definite integrals.

Example 6.69. By using the result

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx,$$

evaluate the following integrals:

$$\int_0^{\pi} \frac{x}{1+\sin x}dx, \quad \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx, \quad \int_0^{\frac{\pi}{2}} \frac{\cot x}{1+\cot x}dx.$$

Solution.

-

$$
\begin{aligned}
\int_0^{\pi} \frac{x}{1+\sin x}dx &= \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}dx \\
&= \int_0^{\pi} \frac{\pi - x}{1+\sin x}dx, \quad \because \ \sin(\pi - x) = \sin x \\
&= \int_0^{\pi} \frac{\pi}{1+\sin x}dx - \int_0^{\pi} \frac{x}{1+\sin x}dx \\
\Rightarrow \ 2\int_0^{\pi} \frac{x}{1+\sin x}dx &= \int_0^{\pi} \frac{\pi}{1+\sin x}dx \\
\text{i.e. } \int_0^{\pi} \frac{x}{1+\sin x}dx &= \frac{\pi}{2}\int_0^{\pi} \frac{1-\sin x}{(1+\sin x)(1-\sin x)}dx \\
&= \frac{\pi}{2}\int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}dx \\
&= \frac{\pi}{2}\int_0^{\pi} \frac{1-\sin x}{\cos^2 x}dx \\
&= \frac{\pi}{2}\int_0^{\pi} \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right)dx \\
&= \frac{\pi}{2}\int_0^{\pi} \left(\sec^2 x - \sec x \tan x\right)dx \\
&= \frac{\pi}{2}\left[\tan x - \sec x\right]_0^{\pi} \\
&= \frac{\pi}{2}\left((0 - -1) - (0 - 1)\right) \\
&= \pi.
\end{aligned}
$$

-

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx &= \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4} - x\right)\right)dx \\
&= \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4} - x\right)\right)dx \\
&= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{\tan\frac{\pi}{4} - \tan x}{1+\tan\frac{\pi}{4}\tan x}\right)dx \\
&= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right)dx \\
&= \int_0^{\frac{\pi}{4}} \ln\left(\frac{1+\tan x + 1 - \tan x}{1+\tan x}\right)dx \\
&= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx \\
&= \int_0^{\frac{\pi}{4}} \ln(2)\,dx - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx \\
\Rightarrow \ 2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx &= \int_0^{\frac{\pi}{4}} \ln(2)\,dx = \ln 2\,[x]_0^{\frac{\pi}{4}} \\
&= \ln 2\left(\frac{\pi}{4} - 0\right) \\
\therefore \ \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx &= \frac{\pi}{8}\ln 2.
\end{aligned}
$$

- $$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\cot x}{1+\cot x}dx &= \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\tan x}}{1+\frac{1}{\tan x}}dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\tan x}}{1+\frac{1}{\tan x}}dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\tan x}}{\frac{\tan x+1}{\tan x}}dx = \int_0^{\frac{\pi}{2}} \frac{1}{\tan x+1}dx \\
&= \int_0^{\frac{\pi}{2}} \frac{1+\tan x-\tan x}{\tan x+1}dx \\
&= \int_0^{\frac{\pi}{2}} dx - \int_0^{\frac{\pi}{2}} \frac{\tan x}{\tan x+1}dx \\
&= [x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\cot x}}{\frac{1}{\cot x}+1}dx \\
\int_0^{\frac{\pi}{2}} \frac{\cot x}{1+\cot x}dx &= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \frac{\cot x}{1+\cot x}dx \\
\therefore \int_0^{\frac{\pi}{2}} \frac{\cot x}{1+\cot x}dx &= \frac{\pi}{4}.
\end{aligned}$$

Problem 6.8

1. m
2. m
3. m
4. m
5. m
6. m
7. m
8. By using the properties

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx;$$

or

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^{2a} f(2a-x)\,dx.$$

of the definite integral, evaluate the following

$$\int_0^{\frac{\pi}{2}} \frac{\tan x}{1+\tan x}dx, \ldots\ldots\ldots\ldots\ldots\ldots$$

9. Show that
$$\int_0^1 \ln^n x dx = (-1)^n n! \quad \forall n.$$

10. Show that
$$\int_{-1}^1 \left(1 - x^2\right) dx = \frac{2^{2n+1} (n!)^2}{(2n+1)!}.$$
Hence, evaluate
$$\int_{-1}^1 \left(1 - x^2\right)^n dx.$$

Chapter 7

The Reduction Formulae

Definition 7.1. Reduction Formula

A *reduction formula* is an integral that gives an integral with a higher power (of some part of it) in terms of a very similar integral with a lower (or reduced) power of the same part.

Examples of reduction formulae include

1. $$\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx$$

2. $$\int \cos^n x dx = \frac{1}{n}\sin x\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2} x dx.$$

Almost all reduction formulae are proved or obtained by using integration by parts; and in most cases, n or the power is almost always a non-negative integer.

7.1 Reduction Formulae for Indefinite Integrals

Example 7.2. Let

$$I_n = \int x^n e^x dx.$$

Show that

$$I_n = x^n e^x - nI_{n-1}, \quad n \geq 1.$$

Hence, find I_3.

Solution. Let $u = x^n$ and $dv = e^x dx$. Then $du = nx^{n-1}dx$ and $v = \int e^x dx = e^x$.

$$\begin{aligned} \therefore\ I_n = \int x^n e^x dx &= uv - \int v du \\ &= x^n e^x - \int e^x \left(nx^{n-1}dx\right) \\ &= x^n e^x - n\int x^{n-1}e^x dx \\ &= x^n e^x - nI_{n-1}, \quad n \geq 1. \end{aligned}$$

Hence,

$$\begin{aligned}
I_3 &= \int x^3 e^x dx \\
&= x^3 e^x - 3I_2; \\
I_2 &= \int x^2 e^x dx \\
&= x^2 e^x - 2I_1; \\
I_1 &= \int x e^x dx \\
&= xe^x - \int e^x dx \\
& \quad xe^x - e^x + C. \\
\therefore \; I_3 &= x^3 e^x - 3\left(x^2 e^x - 2\left(xe^x - e^x\right)\right) + C \\
& \quad \left(x^3 e^x - 3x^2 + 6x - 6\right) e^x + C.
\end{aligned}$$

Example 7.3. By writing

$$\begin{aligned}
I_n &= \int \frac{dx}{\left(x^2+a^2\right)^n} \\
&= \frac{1}{a^2}\int \frac{dx}{\left(x^2+a^2\right)^{n-1}} - \frac{1}{a^2}\int \frac{xdx}{\left(x^2+a^2\right)^n}
\end{aligned}$$

and integrating the last integral by parts, using $u = x$, obtain a reduction for I_n. Use this formula to find I_3.

7.1.1 Reduction Formula for Powers Trigonometric and Hyperbolic Functions.

Here, we seek to obtain reduction formulae for integrals of the form

$$\int \cos^n xdx, \quad \int \sin^n xdx, \quad \int \tan^n xdx, \quad \int \sec^n xdx, \quad \int \cot^n xdx, \quad \int \csc^n xdx$$

and

$$\int \cosh^n xdx, \quad \int \sinh^n xdx, \quad \int \tanh^n xdx, \quad \int \mathrm{sech}^n xdx, \quad \int \coth^n xdx, \quad \int \mathrm{csch}^n xdx, \quad n \in \mathbb{N}.$$

Let us consider, as an example, the integral

$$I_n = \int \cos^n xdx = \int \cos^{n-1} x \cos xdx$$

and let $u = \cos^{n-1} x$ and $dv = \cos xdx$ so that $du = -(n-1)\cos^{n-2} x \sin xdx$ and $v = \int \cos xdx = \sin x$.

$$
\begin{aligned}
\therefore\;\; I_n = \int \cos^n x dx &= \int \cos^{n-1} x \cos x dx \\
&= uv - \int v du \\
&= \cos^{n-1} x \sin x - \int \sin x \left(-(n-1)\cos^{n-2} x \sin x dx\right) \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \left(1-\cos^2 x\right) dx \\
& \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^{n-2} x \cos^2 x dx \\
& \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^{n} x dx \\
\text{i.e. } I_n &= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n \\
\Rightarrow\;\; (1+n-1) I_n &= \cos^{n-1} x \sin x + (n-1) I_{n-2} \\
n I_n &= \cos^{n-1} x \sin x + (n-1) I_{n-2} \\
\therefore\;\; I_n &= \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}, \;\; n \geq 2.
\end{aligned}
$$

Hence, the reduction formula for $I_n = \int \cos^n x dx$ is given by

$$
I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2} \tag{7.1}
$$

or

$$
\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx, \;\; n \geq 2. \tag{7.2}
$$

Example 7.4. For $n = 2$, we have

$$
\begin{aligned}
\int \cos^2 x dx &= \frac{1}{2}\cos^{2-1} x \sin x + \frac{2-1}{2}\int dx \\
&= \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C \\
&= \frac{1}{2}\left(x + \cos x \sin x\right) + C \\
&= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C
\end{aligned}
$$

as we have seen before.

Also, for $n = 6$ we have

$$
\begin{aligned}
\int \cos^6 x dx &= \frac{1}{6}\cos^{6-1} x \sin x + \frac{6-1}{6}\int \cos^{6-2} x dx = \frac{1}{6}\cos^5 x \sin x + \frac{5}{6}\int \cos^4 x dx \\
&= \frac{1}{6}\cos^5 x \sin x + \frac{5}{6}\left(\frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x dx\right) \\
&= \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{15}{24}\left(\frac{1}{2}\left(x + \cos x \sin x\right)\right) + C \\
&= \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{5}{16}\cos x \sin x + \frac{5}{16}x + C.
\end{aligned}
$$

Let us also find a reduction formula for

$$\begin{aligned} I_n = \int \tanh^n x dx &= \int \tanh^{n-2} x \tanh^2 x dx \\ &= \int \tanh^{n-2} x \left(1 - \text{sech}^2 x\right) dx \\ &= \int \tanh^{n-2} x dx - \int \tanh^{n-2} x \text{sech}^2 x dx. \end{aligned}$$

let $u = \tanh x \;\Rightarrow\; du = \text{sech}^2 x dx$.

$$\begin{aligned} \therefore\; I_n = \int \tanh^n x dx &= \int \tanh^{n-2} x dx - \int \tanh^{n-2} x \text{sech}^2 x dx \\ &= \int \tanh^{n-2} x dx - \int u^{n-2} du \\ & \int \tanh^{n-2} x dx - \frac{u^{n-1}}{n-1} \\ \text{i.e. } I_n &= I_{n-2} - \frac{1}{n-1} \tanh^{n-1} x \end{aligned}$$

or

$$\int \tanh^n x dx = \int \tanh^{n-2} x dx - \frac{1}{n-1} \tanh^{n-1} x. \tag{7.3}$$

Similarly, we have the following reduction formulae:

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx, \quad n \geq 2. \tag{7.4}$$

$$\int \tan^n x dx = \frac{1}{n} \tan^{n-1} x - \int \tan^{n-2} x dx, \quad n \geq 2. \tag{7.5}$$

$$\int \sec^n x dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x dx, \quad n \geq 2. \tag{7.6}$$

$$\int \cot^n x dx = \tag{7.7}$$

$$\int \csc^n x dx = \tag{7.8}$$

$$\int \cosh^n x dx = \frac{1}{n} \cosh^{n-1} x \sinh x + \frac{n-1}{n} \int \cosh^{n-2} x dx, \quad n \geq 2. \tag{7.9}$$

$$\int \sinh^n x dx = \frac{1}{n} \sinh^{n-1} x \cosh x - \frac{n-1}{n} \int \sinh^{n-2} x dx, \quad n \geq 2. \tag{7.10}$$

$$\int \text{sech}^n x dx = \tag{7.11}$$

$$\int \coth^n x dx = \tag{7.12}$$

$$\int \text{csch}^n x dx = \tag{7.13}$$

Consider also a reduction formula for the integral

$$I_n = \int \sin^{-n} x dx, \;\; n \in \mathbb{N}.$$

$$\vdots$$

$$\therefore \;\; \int \sin^{-n} x dx \;\; = \;\; x \sin^{-n} x + n\sqrt{1-x^2} \sin^{-n+1} x - n\,(n-1) \int \sin^{-n+2} x dx$$

or

$$I_n = x \sin^{-n} x + n\sqrt{1-x^2} \sin^{-n+1} x - n\,(n-1)\, I_{n+2}.$$

7.1.2 Reduction for Formulae for Products of Powers of Trigonometric and Hyperbolic Functions

Consider integrals of the forms

$$\int \sin^m x \cos^n x dx, \;\; \int \tan^m x \sec^n x dx, \;\; \int \cot^m x \csc^n x dx;$$

and

$$\int \sin^m x \cos nx dx, \;\; \int \sin^m x \sin nx dx, \;\; \int \cos^m x \cos nx dx, \;\; \int \cos^m x \sin nx dx,$$

etc, where $m, n \in \mathbb{N}$.

Let

$$I_{m,n} = \int \sin^m x \cos^n x dx \;\; = \;\; \int \sin^m x \cos^{n-1} x \cos x dx,$$

and let $u = \cos^{n-1} x$ and $dv = \sin^m x \cos x dx$. Then $du = -\,(n-1) \cos^{n-2} x \sin x dx$ and

$v=\int \sin^m x \cos x dx = \frac{1}{m+1}\sin^{m+1} x.$

$$\begin{aligned}
\therefore\ I_{m,n} &= \frac{1}{m+1}\sin^{m+1} x\cos^{n-1} x - \int \frac{-(n-1)}{m+1}\sin^{m+1} x\cos^{n-2} x\sin x dx\\
&= \frac{1}{m+1}\sin^{m+1} x\cos^{n-1} x + \frac{n-1}{m+1}\int \sin^{m+2} x\cos^{n-2} x dx\\
&= \frac{1}{m+1}\sin^{m+1} x\cos^{n-1} x + \frac{n-1}{m+1}\int \sin^{m} x\sin^2 x\cos^{n-2} x dx\\
&= \frac{1}{m+1}\sin^{m+1} x\cos^{n-1} x + \frac{n-1}{m+1}\int \sin^{m} x\left(1-\cos^2 x\right)\cos^{n-2} x dx\\
&= \frac{1}{m+1}\sin^{m+1} x\cos^{n-1} x + \frac{n-1}{m+1}\int \sin^{m} x\cos^{n-2} x dx\\
&\quad -\frac{n-1}{m+1}\int \sin^{m} x\cos^{n} x dx\\
&= \frac{1}{m+1}\sin^{m+1} x\cos^{n-1} x + \frac{n-1}{m+1}I_{m,n-2}\\
\Rightarrow \left(1+\frac{n-1}{m+1}\right) I_{m,n} &= \frac{1}{m+1}\sin^{m+1} x\cos^{n-1} x + \frac{n-1}{m+1}I_{m,n-2}\\
\Rightarrow \frac{m+n}{m+1}I_{m,n} &= \frac{1}{m+1}\sin^{m+1} x\cos^{n-1} x + \frac{n-1}{m+1}I_{m,n-2}\\
\therefore I_{m,n} &= \frac{m+1}{m+n}\left(\frac{1}{m+1}\sin^{m+1} x\cos^{n-1} x + \frac{n-1}{m+1}I_{m,n-2}\right)\\
&= \frac{1}{m+n}\sin^{m+1} x\cos^{n-1} x + \frac{n-1}{m+n}I_{m,n-2}
\end{aligned}$$

or

$$\int \sin^m x\cos^n x dx = \frac{1}{m+n}\sin^{m+1} x\cos^{n-1} x + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2} x dx.$$

We can also do this by writing

$$\int \sin^m x\cos^n x dx = \int \sin^{m-1} x\cos^n x\sin x dx$$

and let $u=\sin^{m-1} x$ and $dv=\int \cos^n x\sin x dx$.

Example 7.5. We have already seen that

$$\int \sin^3 x\cos^2 x dx = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos x + C.$$

Let apply the above reduction formula to verify this result; with $m=3$ and $n=2$, we have

$$\begin{aligned}
\int \sin^3 x\cos^2 x dx = I_{3,2} &= \frac{1}{3+2}\left(\sin^{3+1} x\cos^{2-1} x + (2-1)\, I_{3,2-2}\right)\\
&= \frac{1}{5}\left(\sin^4 x\cos x + I_{3,0}\right).
\end{aligned}$$

Now,

$$\begin{aligned}
I_{3,0} &= \int \sin^3 x dx = \int \left(1-\cos^2 x\right)\sin x dx\\
&= \int \sin x dx - \int \cos^2 x\sin x dx\\
&= -\cos x + \frac{1}{3}\cos^3 x + C.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\therefore \quad \int \sin^3 x \cos^2 x dx &= \frac{1}{5}\left(\sin^4 x \cos x + I_{3,0}\right) \\
&= \frac{1}{5}\left(\sin^4 x \cos x - \cos x + \frac{1}{3}\cos^3 x\right) + C \\
&= \frac{1}{5}\left(\left(1-\cos^2 x\right)^2 \cos x - \cos x + \frac{1}{3}\cos^3 x\right) + C \\
&= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.
\end{aligned}
$$

Similarly, let

$$I_{m,n} = \int \cos^m x \cos nx dx$$

with $u = \cos^m x$ and $dv = \cos nx dx$ so that $du = -m\cos^{m-1} x \sin x$ and $v = \frac{1}{n}\sin nx$. Hence,

$$I_{m,n} \quad = \quad \frac{1}{n}\cos^m x \sin nx + \frac{m}{n}\int \cos^{m-1} x \sin x \sin nx dx.$$

Now, since

$$
\begin{aligned}
\cos(n-1)x &= \cos(nx - \cos x) \\
&= \cos nx \cos x + \sin nx \sin x,
\end{aligned}
$$

we have that

$$\sin nx \sin x = \cos(n-1)x - \cos nx \cos x.$$

$$
\begin{aligned}
\therefore \quad I_{m,n} \quad = \quad & \frac{1}{n}\cos^m x \sin nx + \frac{m}{n}\int \cos^{m-1} x \left[\cos(n-1)x - \cos nx \cos x\right] dx. \\
& \frac{1}{n}\cos^m x \sin nx + \frac{m}{n}\int \cos^{m-1} x \cos(n-1) x dx - \frac{m}{n}\int \cos^{m-1} x \cos nx \cos x dx \\
& \frac{1}{n}\cos^m x \sin nx + \frac{m}{n}\int \cos^{m-1} x \cos(n-1) x dx - \frac{m}{n}\int \cos^{m} x \cos nx dx
\end{aligned}
$$

$$
\begin{aligned}
\text{i.e. } I_{m,n} &= \frac{1}{n}\cos^m x \sin nx + \frac{m}{n} I_{m-1,n-1} - \frac{m}{n} I_{m,n} \\
\Rightarrow \left(1 + \frac{m}{n}\right) I_{m,n} &= \frac{1}{n}\cos^m x \sin nx + \frac{m}{n} I_{m-1,n-1} \\
\Rightarrow \frac{m+n}{n} I_{m,n} &= \frac{1}{n}\cos^m x \sin nx + \frac{m}{n} I_{m-1,n-1} \\
\therefore \; I_{m,n} &= \frac{1}{m+n}\cos^m x \sin nx + \frac{m}{m+n} I_{m-1,n-1}
\end{aligned}
$$

i.e.

$$\int \cos^m x \cos nx dx = \frac{1}{m+n}\cos^m x \sin nx + \frac{m}{m+n}\int \cos^{m-1} x \cos(n-1) x dx.$$

Problem 7.1

1. Obtain the reduction formula
$$\int \ln^n x dx = x \ln^n x - n \int \ln^{n-1} x dx$$
where $\ln^n x \equiv (\ln x)^n$.

2. Show that
$$\int x^n e^{-x} dx = -x^n e^{-x} + n \int x^{n-1} e^{-x} dx.$$
Hence or otherwise, deduce a reduction formula for
$$I_n = \int x^{-n} e^{-x} dx, \quad n \geq 1.$$

3. Construct reduction formulae for the integrals
$$I_n = \int \left(ax^2 + bx + c\right)^n dx$$
and
$$J_n = \int p^m (x) dx$$
where a, b, c are real constants with $a \neq 0$ and p is an n-degree polynomial, i.e. $p^m (x) = (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)^m$.

4. m

5. m

6. m

7. m

8. m

9. m

10. m

7.2 Reduction Formulae for Definite Integrals

Example 7.6. Let
$$I_n = \int_0^{\frac{\pi}{2}} x \sin^n x dx.$$
Then we can write
$$\int_0^{\frac{\pi}{2}} x \sin^n x dx = \int_0^{\frac{\pi}{2}} x \sin^{n-1} x \sin x dx$$

and let $u = x\sin^{n-1} x$ and $dv = \sin x dx$ from which we get $du = (n-1)\left(x\sin^{n-2} x\cos x + \sin^{n-1} x\right) dx$ and $v = -\cos x$, respectively. Hence

$$\begin{aligned}
I_n &= \underbrace{-x\sin^{n-1} x\cos x\Big|_0^{\frac{\pi}{2}}}_{0} + \int_0^{\frac{\pi}{2}} (n-1)\left(x\sin^{n-2} x\cos x + \sin^{n-1} x\right)\cos x dx \\
&= \int_0^{\frac{\pi}{2}} (n-1)\left(x\sin^{n-2} x\cos^2 x + \sin^{n-1} x\cos x\right) dx \\
&= (n-1)\int_0^{\frac{\pi}{2}} x\sin^{n-2} x\left(1-\sin^2 x\right) dx + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-1} x\cos x dx \\
& \quad (n-1)\int_0^{\frac{\pi}{2}} x\sin^{n-2} x dx - (n-1)\int_0^{\frac{\pi}{2}} x\sin^n x dx + \frac{n-1}{n}\sin^n x\Big|_0^{\frac{\pi}{2}} \\
\Rightarrow (1+n-1) I_n &= (n-1) I_{n-2} + \frac{n-1}{n} \\
\therefore\ I_n &= \frac{n-1}{n} I_{n-2} + \frac{n-1}{n^2}
\end{aligned}$$

or

$$\int_0^{\frac{\pi}{2}} x\sin^n x dx = \frac{n-1}{n^2} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} x\sin^{n-2} x dx, \quad n \geq 2.$$

Example 7.7. Obtain a reduction formula for the integral

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx.$$

Solution. We write

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx = \int_0^{\frac{\pi}{2}} \cos^{n-1} x\cos x dx$$

and let $u = \cos^{n-1} x \Rightarrow du = -(n-1)\cos^{n-2} x\sin x dx$ and $dv = \cos x dx \Rightarrow v = \sin x$. Therefore,

$$\begin{aligned}
I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx &= \underbrace{\sin x\cos^{n-1} x\Big|_0^{\frac{\pi}{2}}}_{0} + (n-1)\int_0^{\frac{\pi}{2}} \sin x\cos^{n-2} x\sin x dx \\
&= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x\sin^2 x dx \\
&= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x\left(1-\cos^2 x\right) dx \\
&= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x dx - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x dx \\
\text{i.e. } nI_n &= (n-1) I_{n-2} \\
\therefore\ I_n &= \frac{n-1}{n} I_{n-2}, \quad n \geq 2
\end{aligned}$$

or

$$\int_0^{\frac{\pi}{2}} \cos^n x dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x dx, \quad n \geq 2.$$

Example 7.8. Deduce a reduction formula for the integral

$$\int_0^1 \left(x^2+a^2\right)^n dx.$$

Solution. Let

$$I_n = \int_0^1 \left(x^2+a^2\right)^n dx,$$

$u = \left(x^2+a^2\right)^n$ and $dv = dx$. Then $du = 2nx\left(x^2+a^2\right)^{n-1} dx$ and $v = x$. Thus,

$$\begin{aligned}
I_n = \int_0^1 \left(x^2+a^2\right)^n dx &= x\left(x^2+a^2\right)^n\Big|_0^1 - 2n\int_0^1 x^2\left(x^2+a^2\right)^{n-1} dx \\
&= \left(1+a^2\right)^n - 2n\int_0^1 \left(x^2+a^2-a^2\right)\left(x^2+a^2\right)^{n-1} dx \\
&= \left(1+a^2\right)^n - 2n\int_0^1 \left(x^2+a^2\right)^n dx - 2na^2\int_0^1 \left(x^2+a^2\right)^{n-1} dx \\
\text{i.e. } I_n &= \left(1+a^2\right)^n - 2nI_n - 2a^2nI_{n-1} \\
\Rightarrow (2n+1)\, I_n &= \left(1+a^2\right)^n - 2a^2nI_{n-1} \\
\therefore\ I_n &= \frac{1}{2n+1}\left(\left(1+a^2\right)^n - 2a^2nI_{n-1}\right)
\end{aligned}$$

or

$$\int_0^1 \left(x^2+a^2\right)^n dx = \frac{\left(1+a^2\right)^n}{2n+1} - \frac{2a^2n}{2n+1}\int_0^1 \left(x^2+a^2\right)^{n-1} dx, \quad n \geq 1.$$

Example 7.9. For any $n \in \mathbb{Z}^+$,let

$$I_n = \int_1^e x\ln^n x dx.$$

Prove that, for $n \geq 1$,

$$I_n = \frac{1}{2}e^2 - \frac{1}{2}nI_{n-1}.$$

Hence, find the value of I_3.

Solution.

Example 7.10. If

$$f_n(x) = \frac{1}{n!}\int_0^x t^n e^{-t} dt,$$

show that

$$f_{n-1}(x) - f_n(x) = \frac{1}{n!}f_n'(x).$$

Hence, or otherwise, evaluate the integral

$$\int_1^2 t^3 e^{-t} dt.$$

Solution.

Example 7.11.

Problem 7.2

1. m
2. Obtain a reduction formula for the integral

$$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx.$$

3. If

$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos nx dx,$$

show that if $n > 1$ then

$$I_n = \left(\frac{\pi}{2}\right)^n - n\,(n-1)\, I_{n-2}.$$

Hence, evaluate I_4.

4. Find

$$\int x\sqrt{1-x^2}dx.$$

If

$$\int_0^1 x^n\sqrt{1-x^2}dx,$$

where $n \in \mathbb{Z}^+$, show, by writing the integrand as $\left(x\sqrt{1-x^2}\right) x^{n-1}$, show that for $n \geq 2$,

$$I_n = \frac{n-1}{n+2} I_{n-2}$$

and hence evaluate I_5.

5. m
6. m
7. m
8. m
9. m
10. m

See

1. Q31-38 on pg 337 Adams, Essex on Wallis Product
2. Pg 565-...... Larson, Edwardson, Single Variable Calc
3. Pg 52 Chapter V...... Edwood - Elements of Integral Calculus

TK's Book

Chapter 8

Improper Integrals

Definition 8.1. Improper Integral.

An integral $\int f(x)\,dx$ over an interval $[a,b]$ is said to be *improper* if

1. the integrand, $f(x)$, has an infinite discontinuity at some point(s) in the closed interval $[a,b]$, or
2. one, or both of the limits of integration, a or b, is infinite.

In either case, the improper integral is evaluated by calculating the limit of a *proper* definite integral; if the limit exists, the improper integral exists, or converges, or is finite, otherwise it does not exist, or diverges, or is infinite.

Case I: The Integrand is Discontinuous at the Endpoint(s).

Suppose f is discontinuous at a, that is f is continuous on $(a,b]$. Then f is continuous on $[a+\varepsilon,b]$ for some $\varepsilon>0$ and $a+\varepsilon<b$ and

$$\int_{a+\varepsilon}^{b} f(x)\,dx$$

exists. Given that F is an antiderivative of f on $[a,b]$, then

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{\varepsilon\to 0^+}\int_{a+\varepsilon}^{b} f(x)\,dx && (8.1)\\ &= \lim_{\varepsilon\to 0^+}\left[F(b)-F(a+\varepsilon)\right]\\ &= F(b)-\lim_{\varepsilon\to 0^+}F(a+\varepsilon). && (8.2)\end{aligned}$$

Alternatively, since as $\varepsilon\to 0^+$, $\varepsilon+a\to a^+$, we can write

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{\varepsilon\to a^+}\int_{\varepsilon}^{b} f(x)\,dx && (8.3)\\ &= F(b)-\lim_{\varepsilon\to a^+}F(\varepsilon).\end{aligned}$$

Similarly, if f is not continuous at b, then

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{\varepsilon\to 0^+}\int_a^{b-\varepsilon} f(x)\,dx &(8.4)\\ &= \lim_{\varepsilon\to 0^+}\left[F(b-\varepsilon)-F(a)\right]\\ &= F(b-\varepsilon)-\lim_{\varepsilon\to 0^+} F(a). &(8.5)\end{aligned}$$

or

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{\varepsilon\to b^-}\int_a^{\varepsilon} f(x)\,dx &(8.6)\\ &= \lim_{\varepsilon\to b^-} F(\varepsilon)-F(a).\end{aligned}$$

Example 8.2. Evaluate the following integrals:

$$\int_0^1 \frac{1}{\sqrt{x}}dx,\quad \int_1^2 \frac{1}{x^3-1}dx,\quad \int_3^4 \frac{dx}{(x-3)^{\frac{3}{2}}},\quad \int_2^4 \frac{1}{x\sqrt{x^2-4}}dx,\quad \int_0^1 x\ln x dx.$$

Solution.

- $\int_0^1 \frac{1}{\sqrt{x}}dx$. Clearly, $f(x)=\frac{1}{\sqrt{x}}$ is not continuous (or not defined) at $x=0$; so for some $\varepsilon>0$, we have

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}dx &= \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{1}{\sqrt{x}}dx\\ &= \lim_{\varepsilon\to 0^+}\left[2\sqrt{x}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0^+}\left[2-2\sqrt{\varepsilon}\right]\\ &= 2-0=2.\end{aligned}$$

- $\int_1^2 \frac{1}{x^3-1}dx$. Here too, $f(x)=\frac{1}{x^3-1}=\frac{1}{(x-1)(x^2+x+1)}=\frac{1}{3}\left(\frac{1}{x-1}-\frac{x+2}{x^2+x+1}\right)$, (resolve into partial fractions to aid integration) is not defined at $x=1$. Let $\varepsilon>0$, then

$$\begin{aligned}\int_1^2 \frac{1}{x^3-1}dx &= \lim_{\varepsilon\to 1^+}\int_\varepsilon^2 \frac{1}{x^3-1}dx\\ &= \lim_{\varepsilon\to 1^+}\int_\varepsilon^2 \frac{1}{3}\left(\frac{1}{x-1}-\frac{x+2}{x^2+x+1}\right)dx\\ &= \frac{1}{3}\lim_{\varepsilon\to 1^+}\int_\varepsilon^2\left(\frac{1}{x-1}-\frac{1}{2}\left(\frac{2x+2}{x^2+x+1}\right)-\frac{1}{x^2+x+1}\right)dx\\ &= \frac{1}{3}\lim_{\varepsilon\to 1^+}\left[\ln(x-1)-\frac{1}{2}\ln\left(x^2+x+1\right)-\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right]_\varepsilon^2\\ &= \frac{1}{3}\lim_{\varepsilon\to 1^+}\left[\ln(1)-\frac{1}{2}\ln(7)-\tan^{-1}\left(\frac{5}{\sqrt{3}}\right)\right.\\ &= \left.-\ln(\varepsilon-1)-\frac{1}{2}\ln\left(\varepsilon^2+\varepsilon+1\right)-\tan^{-1}\left(\frac{2\varepsilon+1}{\sqrt{3}}\right)\right]\\ &= \frac{1}{3}\left[\ln\left(\frac{1}{5}\sqrt{\frac{3}{7}}\right)-\tan^{-1}\left(\frac{3}{\sqrt{3}}\right)-\left(\ln 0-\frac{1}{2}\ln 3-\tan^{-1}\left(\frac{3}{\sqrt{3}}\right)\right)\right]\\ &= -\infty,\quad \because\ \ln 0=\infty.\end{aligned}$$

  Thus the improper integral $\int_1^2 \frac{1}{x^3-1}dx$ does not exist.

- $\int_3^4 \frac{dx}{(x-3)^{\frac{3}{2}}}$
- $\int_2^4 \frac{1}{x\sqrt{x^2-4}}dx$
- $\int_0^1 x \ln x dx$

Example 8.3. Prove the following:

$$\int_0^a \frac{dx}{\sqrt{ax - x^2}} = \pi, \qquad \int_1^2 \frac{dx}{x\sqrt{x^2-1}} = \frac{\pi}{3}.$$

Solution.

Example 8.4. E

Solution. m

Case II: The Integrand is Discontinuous at some Point(s) $c \in (a, b)$.

Recall that for $a < c < b$,

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Suppose f has a discontinuous at some point $x = c \in (a, b)$. Then f is continuous on $[a, c) \cup (c, b]$ or $[a, c-\varepsilon] \cup [c+\varepsilon, b]$ for some $\varepsilon > 0$ and $a + \varepsilon < c < b - \varepsilon$. Therefore,

$$\begin{aligned} \int_a^b f(x)\, dx &= \lim_{\varepsilon \to 0^+} \left(\int_a^{c-\varepsilon} f(x)\, dx + \int_{c+\varepsilon}^b f(x)\, dx \right) && (8.7) \\ &= \lim_{\varepsilon \to c^-} \int_a^{\varepsilon} f(x)\, dx + \lim_{\varepsilon \to c^+} \int_{\varepsilon}^b f(x)\, dx. && (8.8) \end{aligned}$$

Also, if the discontinuity exists at n points $c_i \in (a, b)$ for $i = 1, 2, \cdots, n$, then we have

$$\begin{aligned} \int_a^b f(x)\, dx &= \lim_{\varepsilon \to 0^+} \sum_{i=1}^{n} \left(\int_a^{c_i-\varepsilon} f(x)\, dx + \int_{c_i+\varepsilon}^b f(x)\, dx \right) && (8.9) \\ &= \sum_{i=1}^{n} \left(\lim_{\varepsilon \to c_i^-} \int_a^{\varepsilon} f(x)\, dx + \lim_{\varepsilon \to c_i^+} \int_{\varepsilon}^b f(x)\, dx \right). && (8.10) \end{aligned}$$

Example 8.5. Find

$$\int_{-1}^2 \frac{1}{x^3} dx, \quad \int_0^2 \frac{dx}{(x-1)^{\frac{3}{2}}}, \quad \int_0^3 \frac{1}{(x-1)^2} dx, \quad \int_{-1}^1 \frac{1}{\sqrt{x}} dx.$$

Solution.

- For $\int_{-1}^{2} \frac{1}{x^3} dx$, $f(x) = \frac{1}{x^3}$ is not continuous at $x = 0 \in (-1, 2)$. Let $\varepsilon > 0$. Then

$$
\begin{aligned}
\int_{-1}^{2} \frac{1}{x^3} dx &= \lim_{\varepsilon \to 0^+} \left(\int_{-1}^{0-\varepsilon} \frac{1}{x^3} dx + \int_{0+\varepsilon}^{2} \frac{1}{x^3} dx \right) \\
&= \lim_{\varepsilon \to 0^+} \left(\int_{-1}^{-\varepsilon} \frac{1}{x^3} dx + \int_{\varepsilon}^{2} \frac{1}{x^3} dx \right) \\
&= \lim_{\varepsilon \to 0^+} \left(\left[-\frac{1}{2x^2} \right]_{-1}^{-\varepsilon} + \left[-\frac{1}{2x^2} \right]_{\varepsilon}^{2} \right) \\
&= \lim_{\varepsilon \to 0^+} \left(-\frac{1}{2\varepsilon^2} + \frac{1}{2} - \frac{1}{8} + \frac{1}{2\varepsilon^2} \right) \\
&= \lim_{\varepsilon \to 0^+} \left(\frac{1}{2} - \frac{1}{8} \right) = \frac{3}{8}.
\end{aligned}
$$

- For $\int_{0}^{2} \frac{dx}{(x-1)^{\frac{3}{2}}}$

- For $\int_{0}^{3} \frac{1}{(x-1)^2} dx$

- For $\int_{-1}^{1} \frac{1}{\sqrt{x}} dx$

Example 8.6. m

Solution.

Example 8.7. m

Solution.

Case III: One of the Endpoints of Integration is Infinite.

Suppose the upper endpoint of the integral is infinite, that is $b \to \infty$. Then we can write

$$
\begin{aligned}
\int_{a}^{\infty} f(x)\, dx &= \lim_{b \to \infty} \int_{a}^{b} f(x)\, dx \qquad (8.11) \\
&= \lim_{b \to \infty} \left[F(x) \right]_{a}^{b} \\
&= \lim_{b \to \infty} F(b) - F(a).
\end{aligned}
$$

Similarly, if the lower endpoint is infinite, i.e. $a \to -\infty$, then

$$
\begin{aligned}
\int_{-\infty}^{b} f(x)\, dx &= \lim_{a \to -\infty} \int_{a}^{b} f(x)\, dx \qquad (8.12) \\
&= \lim_{a \to -\infty} \left[F(x) \right]_{a}^{b} \\
&= F(b) - \lim_{a \to -\infty} F(a),
\end{aligned}
$$

where F is an antiderivative of f.

Example 8.8. Evaluate the following:

$$\int_1^{\infty} \frac{1}{x^2}dx, \quad \int_{-\infty}^{1} \frac{1}{1+x^2}dx, \quad \int_1^{\infty} \frac{1}{x}dx, \quad \int_0^{\infty} \frac{1}{1+x^2}dx.$$

Solution.

- For $\int_1^{\infty} \frac{1}{x^2}dx$, we have

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2}dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}dx \\
&= \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b \\
&= \lim_{b\to\infty} \left(-\frac{1}{b}+1\right) = 0+1 = 1. \\
\therefore \int_1^{\infty} \frac{1}{x^2}dx &= 1.
\end{aligned}$$

- For $\int_{-\infty}^{1} \frac{1}{1+x^2}dx$, we have

$$\begin{aligned}
\int_{-\infty}^{1} \frac{1}{1+x^2}dx &= \lim_{a\to-\infty} \int_a^1 \frac{1}{1+x^2}dx \\
&= \lim_{a\to-\infty} \left[\tan^{-1} x\right]_a^1 = \lim_{a\to-\infty} \left(\tan^{-1} 1 - \tan^{-1} a\right) \\
&= \tan^{-1} 1 - \tan^{-1}(-\infty) \\
&= \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}.
\end{aligned}$$

- For $\int_1^{\infty} \frac{1}{x}dx$, we have

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x}dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x}dx \\
&= \lim_{b\to\infty} [\ln x]_1^b \\
&= \lim_{b\to\infty} (\ln b - \ln 1) \\
&= \ln\infty - 0 \\
&= \infty.
\end{aligned}$$

- For $\int_0^{\infty} \frac{1}{1+x^2}dx$, we have

$$\begin{aligned}
\int_0^{\infty} \frac{1}{1+x^2}dx &= \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}dx \\
&= \lim_{b\to\infty} \left[\tan^{-1} x\right]_0^b \\
&= \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 0\right) \\
&= \tan^{-1}\infty - 0 \\
&= \frac{\pi}{2}.
\end{aligned}$$

Example 8.9. Show that, for all $a \neq 0$,

$$\int_0^\infty \frac{dx}{a^2+x^2} = \frac{\pi}{2a}.$$

Solution.

Example 8.10. Given that $c > 0$, find the value of

$$\int_0^\infty e^{-cx} \cos kx dx.$$

Solution.

Case IV: Both Endpoints of Integration are Infinite.

Suppose both endpoints of the integral are infinite, that is $a \to -\infty$ and $b \to \infty$. Then since

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^\infty f(x)\,dx,$$

we have

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to\infty}\int_0^b f(x)\,dx & (8.13)\\
&= \lim_{a\to-\infty}[F(x)]_a^0 + \lim_{b\to\infty}[F(x)]_0^b\\
&= \lim_{a\to-\infty}(F(0)-F(a)) + \lim_{b\to\infty}(F(b)-F(0))\\
&= F(0) - \lim_{a\to-\infty}F(a) + \lim_{b\to\infty}F(b) - F(0)\\
&= \lim_{b\to\infty}F(b) - \lim_{a\to-\infty}F(a), & (8.14)
\end{aligned}$$

for some antiderivative F of f.

Note that both $\int_{-\infty}^{0} f(x)\,dx$ and $\int_0^\infty f(x)\,dx$ must exist independently of each other before $\int_{-\infty}^{\infty} f(x)\,dx$ will exist.

Example 8.11. Evaluate the following:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution.

-

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx &= \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}dx + \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}dx\\
&= \lim_{a\to-\infty}\left[\tan^{-1}x\right]_a^0 + \lim_{b\to\infty}\left[\tan^{-1}x\right]_0^b\\
&= \lim_{a\to-\infty}\left(\tan^{-1}0 - \tan^{-1}a\right) + \lim_{b\to\infty}\left(\tan^{-1}b - \tan^{-1}0\right)\\
&= \tan^{-1}0 - \lim_{a\to-\infty}\left(\tan^{-1}a\right) + \lim_{b\to\infty}\left(\tan^{-1}b\right) - \tan^{-1}0\\
&= \lim_{b\to\infty}\left(\tan^{-1}b\right) - \lim_{a\to-\infty}\left(\tan^{-1}a\right)\\
&= \tan^{-1}(\infty) - \tan^{-1}(-\infty)\\
&= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.
\end{aligned}$$

- m
- F
- F

Example 8.12. m

Solution.

Example 8.13. m

Solution.

Theorem 8.14. *Let $k \in \mathbb{R}$. Then*

$$\int_1^\infty \frac{1}{x^k} dx = \begin{cases} \frac{1}{k-1}, & \text{if } k > 1 \\ \infty, & \text{if } k \leq 1. \end{cases}$$

Proof. See Adams pg 363. □

Problem 8.1

1. m
2. Find
$$\int_a^b \frac{dx}{\sqrt{(x-a)(b-a)}}$$
by using the substitution $x = a + (b-a)t^2, \ \ t \geq 0$.
3. By making the substitution $x - 1 = t^2$, $t > 0$, prove that
$$\int_1^\infty \frac{dx}{(x+3)\sqrt{x-1}} = \frac{1}{2}\pi.$$
4. m
5. m
6. m
7. m
8. m
9. m
10. m

Comparison Test for Improper Integrals

See Paul Dawkins' Calculus II pg 66 - .....;

Adams pg 364

and Calculus and Analytic Geometry McGraw Hill pg 506

Applications of the Theory of Improper Integrals

The Gamma Function

mm

Mechanics: Escape Velocity

See Integration Techniques in Calculus Lecture Notes Folder

Probability Deensity Functions

Problem 8.2

Check Elements of Integral Calculus pg 78 ....

Chapter 9

Applications of Integration

Chapter 10

Introduction to Ordinary Differential Equations

10.1 Review of First Order Ordinary Differential Equations

Definition 10.1. Differential equations

A differential equation is an equation involving an unknown function and one or more of its derivatives. The study of differential equations helps us to:

1. discover the (differential) equations that describes a specified physical situation; and

2. find the appropriate solution of that equation.

10.1.1 Classifications of differential equations

We can classify differential equations by type, order, or linearity:

- **By type:** Classification by type concerns the type of derivatives contained in the equation. A differential equation is called an *ordinary differential equation* (ode) if the unknown function(s) (or dependent variables) depend(s) only on a single independent variable, so that ordinary derivatives appear in the equation, eg

$$\frac{dy}{dx} - 13y = \ln x,$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - \cos y = e^x.$$

  A differential equation is called a *partial differential equation* (pde) if dependent variable(s) is/are a function of two or more independent variables, so that partial derivatives appear in the equation, eg

$$\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial y},$$

$$\frac{\partial^2 f}{\partial^2 x} = \frac{1}{k}\frac{\partial f}{\partial t}.$$

- **By order:** The order of a differential equation is the order of the highest derivative that appears in it. For example the differential equation $\frac{dy}{dx} - 13y = \ln x$ is a first order ode, whiles $\frac{\partial^2 f}{\partial^2 x} = \frac{1}{k}\frac{\partial f}{\partial t}$ is a second order pde.

  The general nth order ode with independent variable x and unknown function $y = y(x)$ can be represented as

$$f\left(x, y, y', y'', \ldots, y^{(n-1)}, y^{(n)}\right) = 0, \tag{10.1}$$

  where $y^{(n)} \equiv \frac{d^n y}{dx^n}$ and f is a specific real-valued function of $n+2$ variables.

- **By linearity:** An nth order ordinary differential equation is said to be *linear* if it can be written in the form

$$\sum_{i=n}^{0} a_i(x)\frac{d^i y}{dx^i} = g(x), \tag{10.2}$$

  that is, the unknown function y and all of its derivatives $\frac{d^i y}{dx^i}, i = n, n-1, \ldots, 1, 0$ are of degree 1; the coefficients of all these terms are functions of the independent variable only; and the equation does not contain any transcendental functions nor their derivatives of the dependent variable, such as $\sin y$, $\ln y$, etc. An ordinary differential equation which is not linear is said to be nonlinear.

10.1.2 Some terminologies associated with differential equations

- **The solution of a differential equation:** The function $y = u(x)$ is said to be a solution of the differential equation

$$f\left(x, y, y', \ldots, y^{(n)}\right) = 0 \tag{10.3}$$

  on the interval I provided the derivatives $u', u'', \ldots, u^{(n)}$ exist on I and

$$f\left(x, u, u', \ldots, u^{(n)}\right) = 0 \ \ \forall x \in I. \tag{10.4}$$

  We also say $y = u(x)$ ***satisfies*** (10.3).

  Example 10.2. The function

$$y(x) = A\cos x + B\sin x \tag{10.5}$$

  is a solution of the differential equation $y'' + 9y = 0$ for all real x since

$$\begin{aligned} y'(x) &= -3A\sin x + 3B\cos x \\ y''(x) &= -9A\cos x - 9B\sin x \\ &= -9y(x) \ \ \forall x \in \mathbb{R}. \end{aligned}$$

  Thus, the function (10.5) satisfies $y'' + 9y = 0$.

- **General solution:** A solution of a differential equation is called a *general solution* if it contains arbitrary constants (or parameters), and for every choice of these constants we obtain a solution. An nth order differential equation may have a solution which contains n constants, and called an *n-parameter family of solutions*. For example, the function (10.5) is a 2-parameter family of solution of $y'' + 9y = 0$ since it contains two arbitrary constants A and B.

- **Particular solution:** A solution of a differential equation is called a *particular solution* if it does not contain arbitrary constants. For example, the function $y(x) = \cos x + \sin x$ is a particular solution of $y'' + 9y = 0$ if $A = B = 1$.

- **System of differential equations:** A system of differential equations arises when there are two or more unknown functions dependent on a single independent variable. An example is

$$\begin{aligned} \frac{df}{dx} &= af - mfg, \\ \frac{dg}{dx} &= -bg + nfg, \end{aligned}$$

  where a, b, m, n are constants.

- **Initial-value problems (IVPs) and Boundary-value problems (BVPs):** An initial-value problem is a system consisting of the differential equation together with some auxiliary conditions imposed on the dependent variable and/or its derivatives at a single point x_0 of the independent variable in some interval of interest I, i.e., the problem of finding the solution of the equation $f\left(x, y, y', \ldots, y^{(n)}\right) = 0$ given that $y^{(i)}(x_0) = y_i, i = 0, 1, \ldots, n-1$.

  Examples are

  1. $y' - y = 0; \quad y(0) = 2$
  2. $y'' + 9y = 0; \;\; y'(0) = 3, \; y(0) = 1.$

  When the conditions are placed at more that one point on the interval of interest, then the ensuing problem is referred to as boundary-value problem. Some examples include

  1. $y'' + y = x; \;\; y(0) = 2, \; y\left(\frac{\pi}{2}\right) = 1$ for $0 \leq x \leq \frac{\pi}{2}$.
  2. $y'' - 4\lambda y' + 4\lambda^2 y = 0; \;\; y(0\,()) + y'(0) = 0, \; y(1) - y'(1) = 0.$

10.1.3 Formation of differential equations

We shall form differential equations by elimination of arbitrary constants

Example 10.3. Form the differential equation satisfied by the function $y = Ax + Be^x$.

We can eliminate the arbitrary constants by, when appropriate, making them free of the variables before each stage of differentiation. Hence, we rewrite $y = Ax + Be^x$ as

$$ye^{-x} = Axe^{-x} + B.$$

Differentiating both sides wrt x, we have

$$y'e^{-x} - ye^{-x} = Ae^{-x} - Axe^{-x} \Rightarrow y' - y = A(1 - x)$$

or

$$\frac{y' - y}{1 - x} = A.$$

Differentiating again, we obtain the second order ode

$$(1 - x)\, y'' + y' - y = 0.$$

Example 10.4. Find the differential equation having its general solution

$$y = Ae^{-2x} + 3x - 4$$

where A and B are arbitrary constants.

Example 10.5. Prove the following:

If $xy = Ax^2 + B + \frac{1}{2}x^2 \ln x$, where A and B are constants, show that

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = x + y;$$

and if

$$y = \frac{\sin x}{x^2},$$

show that

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + \left(x^2 + 2\right) y = 0$$

10.1.4 First Order Differential Equations

The general form of a first order ode is given by

$$\frac{dy}{dx} = f(x, y)\,. \tag{10.6}$$

- If f is independent of y, i.e. $f(x, y) = g(x)$, then (10.6) becomes $\frac{dy}{dx} = g(x)$ or $dy = g(x)\,dx$, so that $\int dy = \int g(x)\,dx$ or $y = \int g(x)\,dx + c$.
- If on the other hand f is independent on x for which $f(x, y) = h(y)$, (10.6) becomes $\frac{dy}{dx} = h(y)$ or $\frac{dy}{h(y)} = dx$, and $\int \frac{dy}{h(y)} = \int dx$ or $x = \int \frac{dy}{h(y)} + c$.

10.1.5 Separable first order differential equations

A separable equation is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be written in the form

$$\frac{dy}{dx} = f(x)\,g(y) \tag{10.7}$$

To solve (10.7), we rewrite it in the form

$$\frac{dy}{g(y)} = f(x)\,dx$$

and integrate both sides to get the required solution:

$$\int \frac{1}{g(y)} dy = \int f(x)\,dx.$$

Example 10.6. Solve the initial value problem

$$\frac{dy}{dx} = -6xy; \qquad y(0) = 7.$$

Solution: We rewrite the equation as

$$\frac{dy}{y} = -6x dx.$$

Hence,

$$\begin{aligned} \int \frac{dy}{y} &= -\int 6x dx; \\ \ln|y| &= -3x^2 + c. \end{aligned}$$

From the initial condition, $y(x) > 0$ near $x = 0$, so we may ignore the absolute value symbol:

$$\begin{aligned} \ln y &= -3x^2 + c, \\ \therefore y(x) &= e^{-3x^2+c} \\ &= e^c e^{-3x^2} = Ae^{-3x^2}; \;\; A = e^c. \end{aligned}$$

Hence, when $y(0) = 0$, we get $A = 7$, so the desired solution is

$$y(x) = 7e^{-3x^2}.$$

Example 10.7. Solve the equation

$$(1+x)\,dy - ydx = 0.$$

Solution:

$$\begin{aligned} (1+x)\,dy - ydx &= 0 \\ \Rightarrow (1+x)\,dy &= ydx \\ \Leftrightarrow \frac{dy}{y} &= \frac{dx}{1+x}. \end{aligned}$$

Therefore, integrating both sides, we write

$$\begin{aligned} \int \frac{dy}{y} &= \int \frac{dx}{1+x} \\ \Rightarrow \ln y &= \ln(1+x) + \ln c \\ &= \ln(c(1+x)) \\ \Leftrightarrow y &= c(1+x). \end{aligned}$$

Problem 10.8. Find the general/particular solutions of the following equations.

1. $y^3 \frac{dy}{dx} = (y^4+1)\cos x$
2. $\frac{dy}{dx} = 3\sqrt{xy}$
3. $\frac{dy}{dx} = ye^x; \quad y(0) = 2e$
4. $\frac{dy}{dx} = 4x^3y - y; \quad y(1) = 1$
5. $y' \tan x = y; \;\; y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

10.1.6 Linear First Order Differential Equations

Consider the first order linear ode

$$\frac{dy}{dx} + p(x)\, y = q(x) \tag{10.8}$$

where $p(x)$ and $q(x)$ are continuous functions on a given interval I. The equation (10.8) is said to be *nonhomogeneous* since $q(x) \neq 0$, and *homogeneous* when $q(x) \equiv 0$.

The homogeneous first order linear ode is clearly separable and so can be solved by the procedure above.

To solve a nonhomogeneous first order linear ode,

1. Calculate the ***integrating factor***, a function (which we shall denote here by $\mu(x)$) such that multiplication of each side of a differential equation by it yields an equation in which each side can be recognised as a derivative. The integrating factor of (10.8) is calculated as
$$\mu(x) = e^{\int p(x)dx}. \tag{10.9}$$

2. Multiply each side of the differential equation by $\mu(x)$.
$$\mu(x)\frac{dy}{dx} + p(x)\,\mu(x)\, y = \mu(x)\, q(x)$$

3. Recognise and rewrite the LHS of the resulting equation as the derivative of a product, that is
$$\frac{d}{dx}\left[\mu(x)\, y(x)\right] \;=\; \mu(x)\, q(x)\,.$$

4. Finally, integrate this equation to get
$$\mu(x)\, y(x) = \int \mu(x)\, q(x)\, dx + c,$$
and solve for $y(x)$ to obtain the general solution of the original differential equation, i.e.
$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)\, q(x)\, dx + c\right).$$

Example 10.9. Solve the initial value problem

$$\frac{dy}{dx} - 3y = e^{2x}; \quad y(0) = 3.$$

Solution: Here, we have $p(x) = -3$ and $q(x) = e^{2x}$, so the integrating factor is

$$\mu(x) = e^{-\int 3dx} = e^{-3x}.$$

Multiplying both sides of the given equation, we have

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{-x}$$

which we can rewrite as

$$\frac{d}{dx}\left(ye^{-3x}\right) = e^{-x}.$$

Hence, integration wrt x gives

$$ye^{-3x} = -e^{-x} + c,$$

or

$$y(x) = ce^{3x} - e^{2x}.$$

Now, when $x = 0$ and $y = 3$, we have $c = 4$. Therefore, the desired particular solution is

$$y(x) = 4e^{3x} - e^{2x}.$$

Problem 10.10. Solve the following equations

1. $y' = 1 + x + y + xy$
2. $\left(x^2 + 4\right) y' + 3xy = x$
3. $x^2 y' + 2xy = \ln x; \quad y(1) = 2$
4. $\left(x^2 + 1\right) y' + 3x(y - 1) = 0; \quad y(0) = 2$

10.1.7 Some Applications of First Order Ordinary Differential Equations

Example 10.11. Inductance-Resistance Electrical Circuits

In an simple electric circuit system the electromotive force produces a voltage of $E(t)$ volts (V) and a current of $I(t)$ ampères (A) at time t. If the resistance and the inductance in the circuit are R ohms (Ω) and L henries (H) respectively, one of Kirchhoff's laws states that the sum of the voltage drops of RI due to the resistor and $L\frac{dI}{dt}$ due to the inductor is equal to the supplied voltage $E(t)$; that is,

$$L\frac{dI}{dt} + RI = E(t).$$

Given that the current at time $t = 0$ is zero,

1. determine the current at time t.
2. What is the limiting value of the current?
3. Suppose that a resistor of 3 Ω and an inductor of 9 H are in the circuit together with with a constant voltage of 60 V, obtain $I(t)$, the current after 10 s and the limiting value of the current.

Solution:

$$L\frac{dI}{dt}+RI=E \implies \frac{dI}{dt}+\frac{R}{L}I=\frac{E}{L} \tag{10.10}$$

which is in the form of (10.8), i.e. $p(t)=\frac{R}{L}$. Hence, the integrating factor is

$$\mu(t)=e^{\int\frac{R}{L}dt}=e^{\frac{R}{L}t}.$$

Therefore, we can write (10.10) as

$$e^{\frac{R}{L}t}\frac{dI}{dt}+\frac{R}{L}e^{\frac{R}{L}t}I=\frac{E}{L}e^{\frac{R}{L}t}$$

or

$$\frac{d}{dt}\left(e^{\frac{R}{L}t}I\right)=\frac{E}{L}e^{\frac{R}{L}t},$$

i.e.

$$\begin{aligned} e^{\frac{R}{L}t}I &= \frac{E}{L}\int e^{\frac{R}{L}t}dt \\ &= \frac{E}{R}e^{\frac{R}{L}t}+k \\ \therefore I(t) &= \frac{E}{R}+ke^{-\frac{R}{L}t}. \end{aligned}$$

Now, when $I(0)=0 \implies k=-E/R$.

$$\therefore I(t) \therefore= \frac{E}{R}-\frac{E}{R}e^{-\frac{R}{L}t}=\frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right).$$

(b)

$$\begin{aligned} \lim_{t\to\infty} I(t) &= \lim_{t\to\infty}\frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right) \\ &= \frac{E}{R}\lim_{t\to\infty}\left(1-e^{-\frac{R}{L}t}\right) \\ &= \frac{E}{R}. \end{aligned}$$

(c)

Example 10.12. Motion of a Particle Under Gravity With Air Resistance

Suppose an object of mass m kg falling near the surface of the earth is retarded by a air resistance proportional to its velocity $v=v(t)$ at any given time t. Then Newton's second law of motion states that

$$m\frac{dv}{dt}=mg-kv, \tag{10.11}$$

where g is the acceleration due to gravity near the surface of the earth and k is a constant. If the object starts from rest, find the velocity for any $t>0$ and show that $v(t)$ approaches a limiting value as $t\to\infty$.

Solution:

$$m\frac{dv}{dt} = mg - kv \implies \frac{dv}{dt} + \frac{k}{m}v = g,$$

so that

$$\mu(t) = e^{\int \frac{k}{m}dt} = e^{\frac{kt}{m}}.$$

Therefore,

$$\begin{aligned} ve^{\frac{kt}{m}} &= g\int e^{\frac{kt}{m}}dt = \frac{mg}{k}e^{\frac{kt}{m}} + c \\ \implies v(t) &= \frac{mg}{k} + ce^{-\frac{kt}{m}}. \end{aligned}$$

Hence, since $v(0) = 0$, we have that $c = -\dfrac{mg}{k}$.

$$\therefore v(t) = \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right),$$

and

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right) = \frac{mg}{k}.$$

10.1.8 Population Growth/Decay

The *Law of Exponential Exchange* is one of the ways to model population growth/decay.

Let $P(t)$ be the population or number of the species or objects at time t. Then for the population to be *conserved*, the rate of exchange of the population is the difference in the rate of population increase. Note that the rate of change in the dynamics of the population at any given time is proportion to the population at that time; that is

$$\frac{dP}{dt} \propto P.$$

Hence, if r_i and r_f denote the initial and final rates of change in the population

$$\frac{dP}{dt} = r_i P$$

and

$$\frac{dP}{dt} = r_f P.$$

Therefore, the rate of population increase is given by

$$\begin{aligned} \frac{dP}{dt} &= (r_f - r_i)P \\ &= kP, \quad k = r_f - r_i. \end{aligned} \tag{10.12}$$

Then we have

$$\begin{aligned} \frac{dP}{P} &= kdt \\ \Rightarrow \int \frac{1}{P}dP &= \int kdt \\ \Rightarrow \ln P &= kt + C \\ P &= e^{kt+C} \\ &= e^C e^{kt} = P_0 e^{kt}, \end{aligned}$$

$$\text{i.e. } P(t) = P_0 e^{kt}, \tag{10.13}$$

where $P = P(t)$ is the population at time t and $P_0 = P(0)$ is the population at time $t = 0$, otherwise known as the *initial population.*

When $k > 0$, then the population increases and that *growth* and when $k < 0$, the population decreases and that is *decay.*

Example 10.13. mm

Solution. m

Example 10.14. mm

Solution. m

Example 10.15. A town initially has P_0 number of people and after $t = 1$ the population is one and half times its initial amount. If the rate of growth is proportional to the number $P(t)$ of people present, show that

$$P(t) = \left(\frac{3}{2}\right)^t P_0.$$

Solution.

$$\begin{aligned}
\frac{dP}{dt} &\propto P(t) \Rightarrow \frac{dP}{dt} = kP(t) \\
&\Rightarrow \frac{dP}{P(t)} = kdt \Rightarrow \ln P(t) = kt + C \\
&\therefore P(t) = P_0 e^{kt}, \;\; P_0 = e^C.
\end{aligned}$$

Now, $P(1) = \frac{3}{2} P_0$ and so

$$\begin{aligned}
P(1) = \frac{3}{2} P_0 &= P_0 e^{k(1)} \\
\Rightarrow e^k &= \frac{3}{2}.
\end{aligned}$$

$$\therefore \;\; P(t) = \left(\frac{3}{2}\right)^t P_0.$$

Example 10.16. A rumour sparked off by a quarrel between two bakers is spreading in the community. It is conjectured that if h is a fraction of those who heard the rumour at time t, then

$$\frac{dh}{dt} = kh\left(1 - h^2\right),$$

where k is a constant. If initially 10% of the community heard the rumour, find an expression for h in terms of t and k.

Solution. m

Example 10.17. mm

Solution. m

Example 10.18. mm

Solution. m

Example 10.19. Predator-prey Systems

mmm

10.2 Second Order Differential Equations

The general form of the second order linear ordinary differential equation is

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = F(x), \tag{10.14}$$

where the coefficients $A(x)$, $B(x)$, $C(x)$ and $F(x)$ are continuous functions of x on the open interval I and $A(x) \neq 0$ at any point of I. Hence, dividing through by $A(x)$ gives an equation of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x), \tag{10.15}$$

where $p(x) = \frac{B(x)}{A(x)}$, $q(x) = \frac{C(x)}{A(x)}$ and $f(x) = \frac{F(x)}{A(x)}$. We can write also (10.15) compactly as

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \tag{10.16}$$

or

$$f(x, y, y', y'') = 0$$

We shall seek, in this course, to solve equations of the form (10.15).

10.2.1 Solution of Second Order Differential Equations By Reduction of Order (to First Order) Techniques.

Consider the second order differential equation (10.16), i.e.

$$f(x, y, y', y'') = 0$$

and suppose *either the dependent variable y or the independent variable x is missing in the equation*, then we can reduce this equation from second order to first order by making some substitutions. In this case, we can use first order techniques to solve the "new" equation in terms of the introduced variable. We consider the two cases below:

Case I: The Dependent Variable y is Missing.

Suppose the variable y is missing in the equation. Then (10.16) becomes

$$f(x, y', y'') = 0. \tag{10.17}$$

Let us make the substitution $z = \frac{dy}{dx} = y'$. Then $z' = \frac{dz}{dx} = \frac{d^2y}{dx^2} = y''$ and so (10.17) becomes

$$f(x, z, z') = 0,$$

a first order equation which can easily solved by first order techniques.

Example 10.20. Solve the equation

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 1.$$

Solution. Clearly, the variable y is absent from the equation and so we can make the substitution $z = \frac{dy}{dx}$ so that $\frac{dz}{dx} = \frac{d^2y}{dx^2}$. Hence, the given equation becomes

$$\frac{dz}{dx} + \frac{2}{x}z = 1,$$

which is a first order differential equation in the form of (10.8). We solve the new equation for z by finding the integrating factor; i.e.

$$\mu(x) = e^{\int p(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2.$$

$$\begin{aligned}
\therefore \quad \frac{dz}{dx} + \frac{2}{x}z = 1 \quad &\Rightarrow \quad x^2\frac{dz}{dx} + 2xz = x^2 \\
&\Rightarrow \quad \frac{d}{dx}\left(zx^2\right) = x^2 \\
&\Rightarrow \quad zx^2 = \int x^2 dx = \tfrac{1}{3}x^3 + C_1 \\
&\therefore \quad z = \tfrac{1}{3}x + \frac{C_1}{x^2}.
\end{aligned}$$

That is,

$$\begin{aligned}
\frac{dy}{dx} &= \tfrac{1}{3}x + \frac{C_1}{x^2} \\
\Rightarrow y &= \int \left(\tfrac{1}{3}x + \frac{C_1}{x^2}\right) dx = \tfrac{1}{6}x^2 - \frac{C_1}{x} + C_2.
\end{aligned}$$

Example 10.21. Solve the equation

$$x^2y'' = x^2 + 1$$

given that $y(1) = 1$ and $y'(1) = 0$.

Solution. Let $z = y'$. Then $z' = y''$. More so, $z(1) = 0$, and

$$\begin{aligned}
x^2y'' = x^2 + 1 \quad &\Rightarrow \quad x^2z' = x^2 + 1 \\
&\Rightarrow \quad z' = 1 + \frac{1}{x^2} \\
&\Rightarrow \quad z = \int \left(1 + \frac{1}{x^2}\right) dx \\
&\qquad\quad = x - \frac{1}{x} + C.
\end{aligned}$$

But when $x = 1$, $y' = z = 0$, that is,

$$0 = 1 - \frac{1}{1} + C \Rightarrow C = 0,$$

hence,

$$z = y' = x - \frac{1}{x}.$$

and so

$$\begin{aligned} y &= \int \left(1 - \frac{1}{x}\right) dx \\ &= \tfrac{1}{2}x^2 - \ln x + C_1. \end{aligned}$$

Now, when $x = 1$, $y = 1$ and so

$$1 = \tfrac{1}{2} - \ln 1 + C_1 \Rightarrow C_1 = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

Therefore,

$$\begin{aligned} y &= \tfrac{1}{2}x^2 - \ln x + \tfrac{1}{2} \\ &= \tfrac{1}{2}\left(x^2 + 1\right) - \ln x. \end{aligned}$$

Example 10.22. Find the general solution of the equation

$$y'' + xy' = x.$$

Solution.

Example 10.23. Solve the equation

$$y'' = x\left(y'\right)^2; \quad y\left(0\right) = 1,\ y'\left(0\right) - 2.$$

Solution.

Example 10.24. Find the function y for which

$$xy''' + y'' = 1.$$

Solution.

Case II: The Independent Variable x is Missing

Suppose the variable x is missing in the equation (10.16), that is, the equation takes the form

$$f(y, y', y'') = 0. \tag{10.18}$$

Again, let $z = \frac{dy}{dx} = y'$ so that $y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dz}{dx} = \frac{dy}{dx} \cdot \frac{dz}{dy} = z\frac{dz}{dy} = z'$, and so (10.18) becomes

$$f\left(y, z, z\frac{dz}{dy}\right) = 0,$$

also a first order differential equation that can be easily solved.

Example 10.25. Solve the equation

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1.$$

Solution. It is obvious that the independent variable x is missing in the equation. Let $z = \frac{dy}{dx}$. Then $\frac{d^2y}{dx^2} = \frac{dz}{dx} = \frac{dy}{dx} \cdot \frac{dz}{dy} = z\frac{dz}{dy}$.

$$\begin{aligned}
\therefore\ y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1 \quad &\Rightarrow \quad yz\frac{dz}{dy} + z^2 = 1 \\
&\Rightarrow \quad yzdz = \left(1 - z^2\right) dy \\
&\Rightarrow \quad \frac{z}{1 - z^2}dz = \frac{1}{y}dy \\
&\Rightarrow \quad \int \frac{z}{1 - z^2}dz = \int \frac{1}{y}dy \\
&\Rightarrow \quad -\tfrac{1}{2}\int \frac{-2z}{1 - z^2}dz = \int \frac{1}{y}dy \\
&\Rightarrow \quad -\tfrac{1}{2}\ln\left|1 - z^2\right| = \ln y + \ln C \\
&\Rightarrow \quad \ln\left|1 - z^2\right| = -2\ln y - 2\ln C \\
&\Rightarrow \quad \ln\left|1 - z^2\right| = \ln\left(\frac{1}{C^2y^2}\right) \\
&\Rightarrow \quad 1 - z^2 = \frac{1}{C^2y^2} \\
&\text{or} \quad z^2 = 1 - \frac{1}{C^2y^2} = \frac{C^2y^2 - 1}{C^2y^2} \\
&\Rightarrow \quad z = \frac{dy}{dx} = \pm\sqrt{\frac{C^2y^2 - 1}{C^2y^2}} \\
&\Leftrightarrow \quad \frac{Cy}{\sqrt{C^2y^2 - 1}}dy = \pm dx.
\end{aligned}$$

Integrating both sides, we have

$$\int \frac{Cy}{\sqrt{C^2y^2 - 1}}dy \quad = \quad \pm\int dx.$$

Now, let $w = C^2y^2 - 1 \Rightarrow dw = 2C^2ydy \Rightarrow ydy = \frac{1}{2C^2}dw$ so that the above integral becomes

$$\begin{aligned}
\frac{C}{2C^2}\int \frac{1}{\sqrt{w}}dw &= \pm\int dx \\
\Rightarrow \frac{1}{2C}\int w^{-\frac{1}{2}}dw &= \pm\int dx \\
\Rightarrow \frac{1}{2C}\sqrt{w} &= \pm x + A \\
\Rightarrow \frac{1}{2C^2}w &= (\pm x + A)^2 \\
\text{i.e. } = C^2y^2 - 1 &= C^2(\pm x + A)^2 \\
\text{or } y &= \sqrt{\frac{1 + C^2(\pm x + A)^2}{C^2}}.
\end{aligned}$$

Example 10.26. mmt

Solution. .

Example 10.27. mm

Solution. .

Example 10.28. mm

Solution. .

10.2.2 Second Order Differential Equations With Constant Coefficients

These are differential equations of the form

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x) \tag{10.19}$$

where a_i, $i = 0, 1, 2$ are constants. Again, (10.19) is *homogeneous* when $f(x) = 0$ and *nonhomogeneous* when $f(x) \neq 0$.

Definition 10.29. The Complementary Solution/Function, y_h or y_c.

The *complementary solution* of (10.19) is the solution of the associated homogeneous equation; thus the function y_c such that

$$a_0\frac{d^2y_c}{dx^2} + a_1\frac{dy_c}{dx} + a_2y_c = 0. \tag{10.20}$$

Definition 10.30. The Particular Solution/Integral, y_p.

The *particular solution* of (10.19) is the solution of the nonhomogeneous equation; that is the function y_p which satisfies the equation

$$\begin{aligned} a_0\frac{d^2y_p}{dx^2}+a_1\frac{dy_p}{dx}+a_2y_p &= f(x)\\ \Rightarrow\ \left(a_0\frac{d^2}{dx^2}+a_1\frac{d}{dx}+a_2\right)y_p &= f(x)\\ \Rightarrow\ y_p &= \frac{f(x)}{a_0\frac{d^2}{dx^2}+a_1\frac{d}{dx}+a_2}. \end{aligned} \tag{10.21}$$

Definition 10.31. The Complete/General Solution

The *complete* or *general solution* of the differential equation (10.19) is the function

$$y = y_c + y_p. \tag{10.22}$$

10.2.3 Finding the Complementary Solution: Solving the Homogeneous Differential Equation

To find the solution of (10.20), that is,

$$a_0\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=0,$$

we suppose that, for some constants A and r, the function

$$y = Ae^{rx}$$

is a solution. Then we have that

$$\frac{dy}{dx}=y'=Are^{rx}$$

and

$$\frac{d^2y}{dx^2}=y''=Ar^2e^{rx}.$$

Therefore, substituting for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the above homogeneous equation, we have

$$a_0Ar^2e^{rx}+a_1Are^{rx}+a_2Ae^{rx}=0;$$

and upon dividing through by $Ae^{rx}\neq 0$, we get the equation

$$a_0r^2+a_1r+a_2=0, \tag{10.23}$$

called the *characteristic* or *auxiliary equation* of (10.20), which is a quadratic in r. Solving (10.23) for using the quadratic formula, we have that

$$r=\frac{-a_1\pm\sqrt{a_1^2-4a_0a_2}}{2a_0}. \tag{10.24}$$

Suppose that the roots (10.24) are $r = r_1$ and $r = r_2$. The solution of (10.20) is given by

$$y_c = \begin{cases} Ae^{r_1 x} + Be^{r_2 x}, & \text{if } a_1^2 - 4a_0a_2 > 0; \text{ i.e. there are real and distinct roots } r_1,\, r_2. \\ (Ax + B)\, e^{rx}, & \text{if } a_1^2 - 4a_0a_2 = 0; \text{ i.e. there are real but equal roots } r = r_1 = r_2. \\ e^{\mu x}\left(A\cos\lambda x + B\sin\lambda x\right), & \text{if } a_1^2 - 4a_0a_2 < 0; \text{ i.e. there are no realroots, and} \\ & r_1 = \mu + i\lambda,\ r_2 = \mu - i\lambda \text{ are complex conjugate roots.} \end{cases}$$

Example 10.32. Find the general solution of the differential equation

$$2y'' - 7y' + 3y' = 0.$$

Solution. Notice that the given equation is only homogeneous and so the general solution is the same as the complementary solution. Now, let $y = Ke^{rx}$ where K, r are constants. Then $y' = Kre^{rx}$ and $y'' = Kr^2e^{rx}$ and so the given equation becomes

$$\begin{aligned} 2Kr^2e^{rx} - 7Kre^{rx} + 3Ke^{rx} &= 0 \\ \Rightarrow 2r^2 - 7r + 3 &= 0 \\ \Rightarrow (2r-1)(r-3) &= 0 \\ \Rightarrow r = r_1 = \tfrac{1}{2} &\text{ or } r = r_2 = 3 \end{aligned}$$

which are real and distinct roots. Hence,

$$y_c = C_1e^{\frac{1}{2}x} + C_2e^{3x}$$

where C_1 and C_2 are constants.

Example 10.33. F

Solution. N

Example 10.34. F

Solution. N

10.2.4 Differential Operators

Definition 10.35. The D-operator.

Let $D \equiv \frac{d}{dx}$. Then the symbol D is called the *differential operator* or simply the *D-operator.*

Thus,

$$\frac{dy}{dx} = \left(\frac{d}{dx}\right) y = Dy.$$

Also,

$$D^2 = \frac{d^2}{dx^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = D\,(Dy) = D^2y,$$

and so in general,

$$\frac{d^n}{dx^n} \equiv D^n.$$

Theorem 10.36. Algebraic Properties of the D-operator.

Let $f = f(x)$, $g = g(x)$ and $f_i = f_i(x)$, $i \in \mathbb{N}$ be differentiable functions and m, n be positive integers. Then for all $k, k_i \in \mathbb{F}$, the D-operator satisfies the following algebraic laws:

1. $$D(f + kg) = Df + kDg$$
   and in general
   $$D\left(\sum_{i=1}^{n} k_i f_i\right) = \sum_{i=1}^{n} k_i D f_i.$$

2. $$\begin{aligned} D(fg) &= gDf + fDg \\ &\neq fDg. \end{aligned}$$

3. $$(D^m + D^n) f = D^m f + D^n f$$
   and
   $$\left(\sum_{i=1}^{n} D^i\right) f = \sum_{i=1}^{n} \left(D^i f\right).$$

4. $$(D^m \cdot D^n) f = D^m (D^n f) = D^{m+n} f.$$

5. m

6. m

7. m

8. m

Proof. mm □

10.2.4.1 The Inverse Differential Operator, D^{-1}.

Let

$$Dy = f(x).$$

Then we can write

$$\begin{aligned} D^{-1}(Dy) &= D^{-1}[f(x)] \\ \Leftrightarrow y &= D^{-1}[f(x)]. \end{aligned}$$

Now, since $D \equiv \frac{d}{dx}$, we must have that $D^{-1} \equiv \int$ since $\frac{d}{dx}$ and $\int$ are inverse processes. Thus,

$$\begin{aligned} y &= D^{-1}[f(x)]. \\ &= \int f(x)\, dx. \end{aligned}$$

Therefore,

$$Dy = f(x) \Leftrightarrow y = \int f(x)\,dx; \tag{10.25}$$

and hence,

$$\begin{aligned} D^{-2}f(x) &= D^{-1}\left[D^{-1}f(x)\right] \\ &= D^{-1}\left[\int f(x)\,dx\right] \\ &= \int\left[\int f(x)\,dx\right]dx \\ &= \int\int f(x)\,dx\cdot dx \equiv \int\int f(x)\,d^2x. \end{aligned}$$

In general,

$$D^{-n}f(x) = \underbrace{\int\cdots\int}_{n\,times} f(x)\,d^nx.$$

Example 10.37. Find $D^{-1}f(x)$ and $D^{-3}f(x)$ if $f(x) = 6x$.

Solution. Given that $f(x) = 6x$, we have

$$\begin{aligned} D^{-1}f(x) &= D^{-1}(6x) = \int 6xdx \\ &= 3x^2 + A, \end{aligned}$$

and

$$\begin{aligned} D^{-3}f(x) &= D^{-3}(6x) = D^{-2}\left[D^{-1}(6x)\right] \\ &= D^{-2}\left[\int 6xdx\right] = D^{-2}\left(3x^2+A\right) = D^{-1}\left[D^{-1}\left(3x^2+A\right)\right] \\ &= D^{-1}\left[\int\left(3x^2+A\right)dx\right] = D^{-1}\left(x^3+Ax+B\right) \\ &= \int\left(x^3+Ax+B\right)dx \\ & \tfrac{1}{4}x^4 + \tfrac{1}{2}Ax^2 + Bx + C, \end{aligned}$$

where A, B, C are arbitrary constants.

Example 10.38. m

Solution. m

Theorem 10.39. The Substitution Rule.
Let us defined a polynomial in the operator D by

$$P(D) = a_0D^2 + a_1D + a_2. \tag{10.26}$$

Then for some constant a, we have that

$$P(D)\,e^{ax} = P(a)\,e^{ax}. \tag{10.27}$$

The result is also generally true for all n such that

$$P(D) = \sum_{i=0}^{n} a_iD^{n-i} = a_0D^n + a_1D^{n-1} + \cdots + a_n.$$

Proof. We prove first the case when $P(D) = a_0D^2 + a_1D + a_2$. Then by repeated differentiation, we have

$$\begin{aligned} P(D)e^{ax} &= \left(a_0D^2 + a_1D + a_2\right)e^{ax} \\ &= a_0D^2e^{ax} + a_1De^{ax} + a_2e^{ax} \\ &= a_0a^2e^{ax} + a_1ae^{ax} + a_2e^{ax} \\ &= \left(a_0a^2 + a_1a + a_2\right)e^{ax} \\ &= P(a)e^{ax}. \end{aligned}$$

To prove the general case, we have by the same approach that

$$\begin{aligned} P(D)e^{ax} &= \left(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n\right)e^{ax} \\ &= a_0D^ne^{ax} + a_1D^{n-1}e^{ax} + a_2D^{n-2}e^{ax} + \cdots + a_ne^{ax} \\ &= a_0a^ne^{ax} + a_1a^{n-1}e^{ax} + a_2a^{n-2}e^{ax} + \cdots + a_ne^{ax} \\ &= \left(a_0a^n + a_1a^{n-1} + a_2a^{n-2} + \cdots + a_n\right)e^{ax} \\ &= P(a)e^{ax}. \end{aligned}$$

□

Theorem 10.40. The Shift Theorem.

Let $P(D) = a_0D^2 + a_1D + a_2$ and a an arbitrary constant. Then for all differentiable functions $f(x)$, we have

1. $$P(D)\left[e^{ax}f(x)\right] = e^{ax}\left[P(D+a)\right]f(x) \tag{10.28}$$

2. $$e^{-ax}P^2(D)\left[e^{ax}f(x)\right] = P^2(D+a)f(x), \tag{10.29}$$

   where $P^2(D) \equiv \left[P(D)\right]^2$, etc.

Proof. □

1. Let $P(D) = a_0D^2 + a_1D + a_2$. Then we have by repeated differentiation and the product rule that

$$\begin{aligned} P(D)\left[e^{ax}f(x)\right] &= e^{ax}P(D)f(x) + f(x)P(D)e^{ax} \\ &= e^{ax}\left(a_0D^2 + a_1D + a_2\right)f(x) + f(x)\left(a_0D^2 + a_1D + a_2\right)e^{ax} \\ &= e^{ax}\left(a_0D^2 + a_1D + a_2\right)f(x) + f(x)\left(a_0a^2e^{ax} + a_1ae^{ax} + a_2e^{ax}\right) \\ &= e^{ax}\left(a_0D^2 + a_1D + a_2\right)f(x) + f(x)\left(a_0a^2 + a_1a + a_2\right)e^{ax} \\ &= e^{ax}\left(a_0D^2 + a_1D + a_2 + a_0a^2 + a_1a + a_2\right)f(x) \\ &= e^{ax}\left(a_0\left(D^2 + a^2\right) + a_1(D+a) + (a_2 + a_2)\right)f(x) \\ &= e^{ax}\left[P(D+a)\right]f(x). \end{aligned}$$

   Check the last proof again

2. Similarly, since

$$P(D)\left[e^{ax}f(x)\right] = e^{ax}\left[P(D+a)\right]f(x),$$

we have that for $f(x) = 1$,

$$P(D)\,e^{ax} = e^{ax}P(D+a)$$

which is just an extension of (10.27). Therefore,

$$P(D+a) = e^{-ax}P(D)\,e^{ax} \tag{10.30}$$

or

$$P(D+a)\left[f(x)\right] = e^{-ax}P(D)\left[e^{ax}f(x)\right] \tag{10.31}$$

Squaring both sides of (10.30), we have

$$\begin{aligned}
(P(D+a))^2 &= \left(e^{-ax}P(D)\,e^{ax}\right)^2 \\
\Rightarrow P^2(D+a) &= \left(e^{-ax}P(D)\,e^{ax}\right)\cdot\left(e^{-ax}P(D)\,e^{ax}\right) \\
&= e^{-ax}P(D)\,e^{ax}\cdot e^{-ax}P(D)\,e^{ax} \\
&= e^{-ax}P(D)\,P(D)\,e^{ax} \\
&= e^{-ax}P^2(D)\,e^{ax}.
\end{aligned}$$

Therefore, when we right-multiply both sides by any differentiable function $f(x)$, we obtain the desired result:

$$P^2(D+a)\,f(x) = e^{-ax}P^2(D)\left[e^{ax}f(x)\right].$$

The main theme of the Shift Theorem is to reduce the burden of differentiating products that contain the exponential function to differentiating only the non-exponential function. That is, the exponential function has been "shifted" from the right of the operator as part of the product to the left as a factor.

Example 10.41. If we apply (10.28) to $(D+3)\,e^{2x}y$, then $P(D) = D+3$, $a = 2$ and $f(x) = y$. Thus

$$\begin{aligned}
(D+3)\,e^{2x}y &= e^{2x}(D+3+2)\,e^{2x}y \\
&= e^{2x}(D+5)\,y.
\end{aligned}$$

To verify this, we use the normal product rule as follows: with $D \equiv \frac{d}{dx}$,we have

$$\begin{aligned}
(D+3)\,e^{2x}y &= \left(\frac{d}{dx}+3\right)e^{2x}y \\
&= \frac{d}{dx}\left(e^{2x}y\right) + 3e^{2x}y \\
&= e^{2x}\frac{d}{dx}(y) + y\frac{d}{dx}\left(e^{2x}\right) + 3e^{2x}y \\
&= e^{2x}\frac{d}{dx}(y) + 2e^{2x}y + 3e^{2x}y \\
&= e^{2x}\frac{d}{dx}(y) + 5e^{2x}y \\
&= e^{2x}\left(\frac{d}{dx}(y) + 5y\right) \\
&= e^{2x}(Dy + 5y) \\
&= e^{2x}(D+5)\,y.
\end{aligned}$$

Corollary 10.42. Deductions from the Shift Theorem.

1. Generalising (10.29), we have that for all $n \in \mathbb{N}$,
$$e^{-ax}P^n(D)\left[e^{ax}f(x)\right] = P^n(D+a)f(x). \tag{10.32}$$

2. More specifically, combining (10.30) and (10.32), we get
$$D^n\left[e^{ax}f(x)\right] = e^{ax}(D+a)^n f(x) \tag{10.33}$$
or
$$e^{-ax}D^n\left[e^{ax}f(x)\right] = (D+a)^n f(x) \tag{10.34}$$

3. When $f(x) = k$, a constant, then (10.30) becomes
$$P^n(D)\left[e^{ax}f(x)\right] = kP^n(a)e^{ax}. \tag{10.35}$$
Particularly,
$$P(D)\left[ke^{ax}\right] = kP(a)e^{ax}. \tag{10.36}$$

4. If $P^{-1}(D)$ is the inverse of $P(D)$, then
$$P^{-1}(D)\left[e^{ax}f(x)\right] = e^{ax}P^{-1}(D+a)f(x) \tag{10.37}$$

Example 10.43. Evaluate $D^3e^{-x}\sin x$.

Solution. Here, $a = -1$ and $f(x) = \sin x$. We shall use (10.28) again to write

$$\begin{aligned}
D^3e^{-x}\sin x &= e^{-x}(D-1)^3\sin x \\
&= e^{-x}\left(D^3 - 3D^2 + 3D - 1\right)\sin x \\
&= e^{-x}\left(D^3(\sin x) - 3D^2(\sin x) + 3D(\sin x) - \sin x\right) \\
&= e^{-x}(-\cos x + 3\sin x + 3\cos x - \sin x) \\
&= 2e^{-x}(\sin x + \cos x).
\end{aligned}$$

Example 10.44. Find $(D-2)^2\left(e^{4x}\cdot 5\right)$.

Solution. Here, direct differentiation of the function will be difficult since we have to deal with the square of the differential operator. However, a careful observation shows that the given operation takes the form of the right-hand side of (10.35) with $P(D) = D - 2$, $a = 4$ and $f(x) = 5$. Hence, we can write

$$\begin{aligned}
(D-2)^2\left(e^{4x}\cdot 5\right) &= 5(D-2)^2\left(e^{4x}\right) \\
e^{-ax}D^n\left[e^{ax}f(x)\right] = (D+a)^n f(x) &= 5(4-2)^2\left[e^{4x}\right] \\
&= 5\left(2^2\right)\left[e^{4x}\right] \\
&= 20e^{4x}.
\end{aligned}$$

Example 10.45. Solve the differential equation

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0.$$

Solution. Rewriting the given equation using operators, we have

$$\left(D^3 - 3D^2 + 3D - 1\right) y = 0$$

or

$$(D-1)^3 y = 0.$$

Multiplying through by e^{-x}, we get

$$e^{-x} (D-1)^3 y = 0,$$

which now takes the form of the right-hand side of (10.33) with $a = -1$, $n = 3$ and $f(x) = y$. Thus,

$$e^{-x} (D-1)^3 y = D^3 \left(e^{-x} y\right) = 0.$$

Therefore,

$$\begin{aligned} e^{-x} y &= D^{-3}(0) \\ \Rightarrow y &= e^x D^{-3}(0) \\ &= e^x \iiint (0)\, dxdxdx \\ &= e^x \iint A dxdx \\ &= e^x \int (Ax + B)\, dx \\ &= e^x \left(\tfrac{1}{2} A x^2 + Bx + C\right), \end{aligned}$$

where A, B, C are constants.

Example 10.46. m

Solution. m

10.2.4.2 The Operator $(D-a)^{-1}$ for some Constant a.

Given the nohomogeneous first order ordinary differential equation of the form

$$\frac{dy}{dx} - ay = f(x)$$

or

$$(D-a)\, y = f(x), \tag{10.38}$$

the integrating factor is given by

$$\mu(x) = e^{-\int a dx} = e^{-ax}.$$

Hence, multiplying through by this factor, we have

$$\begin{aligned} e^{-ax} (D-a)\, y &= e^{-ax} f(x) \\ \Rightarrow D\left(e^{-ax} y\right) &= e^{-ax} f(x) \\ \Rightarrow e^{-ax} y &= D^{-1}\left(e^{-ax} f(x)\right) \\ &= \int e^{-ax} f(x)\, dx \end{aligned}$$

and so

$$y = e^{ax} \int e^{-ax} f(x)\, dx \tag{10.39}$$

Alternatively, let $(D-a)^{-1}$ be the inverse operator for the operator $(D-a)$. Then "composing" both sides of (10.38) with $(D-a)^{-1}$, we have

$$\begin{aligned} (D-a)^{-1} \circ (D-a)\, y &= (D-a)^{-1} f(x) \\ \Leftrightarrow y &= (D-a)^{-1} f(x). \end{aligned} \tag{10.40}$$

Hence, in view of (10.45) and (10.43), we have that

Theorem 10.47. *For all constants a and all differentiable functions $f(x)$,*

$$(D-a)^{-1} f(x) = e^{ax} \int e^{-ax} f(x)\, dx, \tag{10.41}$$

and hence,

$$(D-a)\left(e^{ax} \int e^{-ax} f(x)\, dx\right) = f(x). \tag{10.42}$$

Example 10.48. Evaluate $\frac{x^2}{D-3}$.

Solution. $\frac{x^2}{D-3} = \frac{1}{D-3}x^2 = (D-3)^{-1} x^2$. Now,

$$\begin{aligned} (D-3)^{-1} x^2 &= e^{3x} \int e^{-3x} x^2 dx \\ &= e^{3x} \left(-\tfrac{1}{3}x^2 e^{-3x} + \tfrac{2}{3} \int x e^{-3x} dx\right) \\ & e^{3x} \left(-\tfrac{1}{3}x^2 e^{-3x} + \tfrac{2}{3} \left(-\tfrac{1}{3} x e^{-3x} + \tfrac{1}{3} \int e^{-3x} dx\right)\right) \\ & e^{3x} \left(-\tfrac{1}{3}x^2 e^{-3x} + \tfrac{2}{3} \left(-\tfrac{1}{3} x e^{-3x} + \tfrac{1}{3} \left(-\tfrac{1}{3} e^{-3x}\right)\right)\right) + C \\ & -\tfrac{1}{3}x^2 + \tfrac{2}{3}\left(-\tfrac{1}{3}x + \tfrac{1}{3}\left(-\tfrac{1}{3}\right)\right) + C \\ & -\tfrac{1}{3}x^2 - \tfrac{2}{9}x - \tfrac{2}{27} + C. \end{aligned}$$

What we have just done (finding the function which the operation $\frac{x^2}{D-3}$ evaluates to) is similar to solving the differential equation $y' - 3y = x^2$, that is when we suppose that $\frac{x^2}{D-3} = y$; whence $(D-3)\, y = x^2$ or $Dy - 3y \equiv y' - 3y = x^2$, whose integrating factor is $\mu(x) = e^{\int (-3) dx} = e^{-3x}$ and so $\frac{d}{dx}\left(y e^{-3x}\right) = x^2 e^{-3x}$ or $y = e^{3x} \int e^{-3x} x^2 dx$.

Corollary 10.49. Consequences of Theorem 10.47

1. $$P^{-1}(D)\left[e^{ax} f(x)\right] = e^{ax} P^{-1}(D+a)\, f(x).$$

2. For all constants a_i, $i = 1, 2, \cdots, n$, we have

$$[(D-a_1)(D-a_2)\cdots(D-a_n)]^{-1} = e^{a_n x}\int e^{(a_{n-1}-a_n)x}\int e^{(a_{n-2}-a_{n-1})x}\cdots\int e^{(a_1-a_2)x}\int e^{-a_1 x}d^n x$$

3. Define a polynomial in the operator D by $P^{-1}(D) = a_n D^n + a_{n-1}D^{n-1} + \cdots a_1 D + a_0$. Then

$$P^{-1}(D)(e^{ax}) = \begin{cases} P^{-1}(a)\, e^{ax} & \text{if } L(a) \neq 0 \\ \frac{x^n e^{ax}}{L^{(n)}} & \text{if } L(a) = L'(a) = \cdots = L^{(n-1)}(a) = 0 \text{ and } L^{(n)}(a) \neq 0, \end{cases}$$

where $L^{(n)}(a) = \left.\frac{d^n L}{dx^n}\right|_{x=a}$.

4. By applying binomial expansion to $P^{-1}(D)$, that is if we can write

$$P^{-1}(D) \equiv \left(a_n D^n + a_{n-1}D^{n-1} + \cdots a_1 D + a_0\right), \tag{10.43}$$

a polynomial, then

$$P^{-1}(D)\, x^m = \left(a_r D^r + a_{r-1}D^{r-1} + \cdots a_1 D + a_0\right) x^m$$

where $r = \min\{m, n\}$.

In particular, we have the following:

1. If $f(x) = e^{bx}$ with $b \neq a$, then

$$\begin{aligned} (D-a)^{-1} e^{bx} &= e^{ax}\int e^{-ax}e^{bx}dx \\ &= e^{ax}\int e^{(b-a)x}dx \\ &= \frac{e^{ax}}{b-a}, \quad b \neq a. \end{aligned} \tag{10.44}$$

2. If $f(x)$ is a polynomial of degree n, that is $f(x) = \sum_{i=0}^{n} p_i x^i$, then we can write

$$\begin{aligned} (D-a)^{-1} f(x) &= -\tfrac{1}{a}\left(1 - \frac{D}{a}\right)^{-1} f(x) \\ &= -\tfrac{1}{a}\sum_{r=0}^{n}\begin{pmatrix} n \\ r \end{pmatrix}\left(\frac{D}{a}\right)^r f(x) \\ &= -\frac{1}{a}\left(1 + \frac{D}{a} + \frac{D^2}{a^2} + \frac{D^3}{a^3} + \cdots + \frac{D^n}{a^n}\right) f(x) \end{aligned} \tag{10.45}$$

3. If $f(x) = \cos\lambda x$ or $f(x) = \sin\lambda x$, then

$D(\cos\lambda x) = -\lambda\sin\lambda x$ and $D^2(\cos\lambda x) = D(-\lambda\sin\lambda x) = D(\cos\lambda x) = -\lambda^2\cos\lambda x$;

that is

$$D^2(\cos\lambda x) = -\lambda^2\cos\lambda x$$

or

$$D^2 = -\lambda^2 \tag{10.46}$$

Similarly,

$$\begin{aligned} D^2 (\sin \lambda x) &= D(\lambda \cos \lambda x) = -\lambda^2 \sin \lambda x \\ \Rightarrow D^2 &= -\lambda^2. \end{aligned}$$

Also, since from (10.43) or (10.45), we can write

$$\left(D^2 - a\right)^{-1} \equiv \frac{1}{D^2 - a} = p_n D^n + \cdots + p_2 D^2 + p_1 D + p_0,$$

for some constants p_i, we have

$$\frac{1}{D^2 - a}(\sin \lambda x) = \frac{1}{D^2 - a}(\cos \lambda x) = p_n D^n + \cdots + p_2\left(-\lambda^2\right) + p_1 D + p_0 \tag{10.47}$$

Example 10.50. When $f(x) = x$, a first degree polynomial, then

$$\begin{aligned} (D - a)^{-1} x &= -\frac{1}{a}\left(1 - \frac{D}{a}\right) x \\ &= -\frac{1}{a}\left(x - \frac{1}{a} D(x)\right) \\ &= -\frac{1}{a}\left(x - \frac{1}{a}\right) \\ &= -\frac{x}{a} + \frac{1}{a^2}. \end{aligned}$$

When $f(x) = \sin 2x$ and $P(D) = D^2 + 2D + 1$, then

$$\begin{aligned} P^{-1}(D) f(x) &= \left(D^2 + 2D + 1\right)^{-1} \sin 2x \\ &= \left(-2^2 + 2D + 1\right)^{-1} \sin 2x, \quad \because D^2 = -2^2 \\ &= (2D - 3)^{-1} \sin 2x \\ &= \frac{1}{2D - 3}(\sin 2x) \\ &= \frac{1}{2D - 3} \times \frac{2D + 3}{2D + 3}(\sin 2x) \\ &= \frac{2D + 3}{(2D - 3)(2D + 3)}(\sin 2x) \\ &= \frac{2D + 3}{4D^2 - 9}(\sin 2x) \\ &= \frac{2D + 3}{4\left(-2^2\right) - 9}(\sin 2x), \quad \because D^2 = -2^2 \\ &= \frac{2D + 3}{-16 - 9}(\sin 2x) \\ &= -\tfrac{1}{25}(2D + 3)(\sin 2x) \\ &= -\tfrac{1}{25}(4\cos 2x + 3\sin 2x). \end{aligned}$$

Example 10.51. Solve the equation

$$\frac{dy}{dx} - 6y = e^{5x}.$$

Solution. We can rewrite the above equation as

$$\begin{aligned}
(D-6)\,y &= e^{5x} \\
\Leftrightarrow y &= (D-6)^{-1}\,e^{5x} \\
&= e^{6x}\int e^{-6x}\cdot e^{5x}dx \\
&= e^{6x}\int e^{-x}dx \\
&= e^{6x}\left(-e^{-x}+C\right) \\
&= -e^{5x}+Ce^{6x}.
\end{aligned}$$

10.2.5 Finding the Particular Solution, y_p: Solving the Nonhomogeneous Differential Equation

Given the differential equation

$$P(D)\,y \equiv \left(a_0D^2+a_1D+a_2\right)y = f(x),$$

the particular solution y_p is given by

$$\begin{aligned}
y &= P^{-1}(D)\,f(x) \\
&= \left(a_0D^2+a_1D+a_2\right)^{-1}f(x) \qquad (10.48)
\end{aligned}$$

Example 10.52. Determine the particular solution of

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 2e^{3x}.$$

Hence find the general solution of the equation.

Solution. We write the given equation using operators as

$$\left(D^2-4D+3\right)y = 2e^{3x}.$$

Therefore, the particular solution y_p is then given by

$$\begin{aligned}
y_p &= \frac{1}{D^2-4D+3}2e^{3x} = \frac{2}{D^2-4D+3}e^{3x} \\
&= \frac{2}{(D-1)(D-3)}e^{3x} \\
&= 2\left(\frac{1}{2}\left(\frac{1}{D-3}-\frac{1}{D-3}\right)\right)e^{3x}, \quad \text{(resolve into partial fractions)} \\
&= (D-3)^{-1}e^{3x}-(D-1)^{-1}e^{3x} \\
&= e^{3x}\int e^{-3x}\cdot e^{3x}dx - e^{x}\int e^{-x}\cdot e^{3x}dx \\
&= e^{3x}\int dx - e^{x}\int e^{2x}dx \\
&= xe^{3x}-e^{x}\cdot\frac{1}{2}e^{2x}+C_1 \\
&= \left(x-\frac{1}{2}\right)e^{3x}+C_1.
\end{aligned}$$

Also, the associated homogeneous equation is

$$\left(D^2-4D+3\right)y=0.$$

Let $y=Ae^{rx}$ so that $Dy=Are^{rx}$ and $D^2y=Ar^2e^{rx}$. Then the characteristic equation is

$$\begin{aligned}
r^2-4r+3 &= 0 \\
(r-1)(r-3) &= 0 \\
&\Rightarrow r=1 \text{ or } r=3.
\end{aligned}$$

Therefore, the complementary solution is

$$y_c=C_2e^{x}+C_3e^{3x}.$$

Hence, the general solution of

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=2e^{3x}$$

is

$$\begin{aligned}
y &= y_c+y_p \\
&= C_2e^{x}+C_3e^{3x}+\left(x-\frac{1}{2}\right)e^{3x}+C_1.
\end{aligned}$$

Example 10.53. Find the complete solution of the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}=8e^{3x}+4\sin x.$$

Solution. We write the given equation as

$$\left(D^2 - 3D\right) y = 8e^{3x} + 4\sin x$$

and hence, the associated homogeneous equation is

$$\left(D^2 - 3D\right) y = 0.$$

Let $y = Ae^{rx}$ so that $Dy = Are^{rx}$ and $D^2 y = Ar^2 e^{rx}$. Then the characteristic equation is

$$\begin{aligned} r^2 - 3r &= r(r-3) = 0 \\ &\Rightarrow r = 0 \text{ or } r = 3. \end{aligned}$$

Therefore, the complementary solution is

$$y_c = C_1 e^{0x} + C_2 e^{3x} = C_1 + C_2 e^{3x}.$$

The particular solution y_p is then given by

$$\begin{aligned} y_p &= \frac{1}{D^2 - 3D}\left(8e^{3x} + 4\sin x\right) \\ &= \frac{8}{D^2 - 3D} e^{3x} + \frac{4}{D^2 - 3D}\sin x \\ &= \frac{8}{D(D-3)} e^{3x} + \frac{4}{-1^2 - 3D}\sin x, \quad \because D^2(\sin x) = -1^2 \\ &= -\frac{8}{3}\left(\frac{1}{D} - \frac{1}{D-3}\right) e^{3x} - \frac{4}{3D+1}\sin x, \quad \text{(resolve the first term into partial fractions)} \\ &= -\frac{8}{3}\left(D^{-1} e^{3x} - (D-3)^{-1} e^{3x}\right) - \left(\frac{4(3D-1)}{9D^2 - 1}\right)\sin x \\ &= -\frac{8}{3}\left(D^{-1} e^{3x} - (D-3)^{-1} e^{3x}\right) - \left(\frac{4(3D-1)}{9(-1^2) - 1}\right)\sin x, \quad \because D^2(\sin x) = -1^2 \\ &= -\frac{8}{3}\left(\int e^{3x} dx - e^{3x}\int e^{-3x} \cdot e^{3x} dx\right) - \left(-\frac{4}{10}(3D-1)\right)\sin x \\ &= -\frac{8}{3}\left(\frac{1}{3}e^{3x} - e^{3x}\int dx\right) + \frac{2}{5}(3D-1)\sin x \\ &= -\frac{8}{3}\left(\frac{1}{3}e^{3x} - e^{3x}\int dx\right) + \frac{2}{5}(3D(\sin x) - \sin x) \\ &= -\frac{8}{3}\left(\frac{1}{3}e^{3x} - xe^{3x}\right) + \frac{2}{5}(3\cos x - \sin x) \\ &= \frac{2}{5}(3\cos x - \sin x) + \frac{8}{3}\left(x - \frac{1}{3}\right) e^{3x}. \end{aligned}$$

Hence, the complete solution of

$$\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} = 8e^{3x} + 4\sin x$$

is

$$\begin{aligned} y &= y_c + y_p \\ &= C_1 + C_2 e^{3x} + \frac{2}{5}(3\cos x - \sin x) + \frac{8}{3}\left(x - \frac{1}{3}\right) e^{3x} \\ & \frac{2}{5}(3\cos x - \sin x) + \frac{8}{3}K\left(x - \frac{1}{3}\right) e^{3x} + C, \quad K = C_2 + \frac{8}{3}, \ C = C_1. \end{aligned}$$

Example 10.54. Solve the equation $y'' + y' - 2y = 4x$.

Solution. $y'' + y' - y = 4x \equiv (D^2 + D - 2)\, y = 4x$. The associated homogeneous equation $(D^2 + D - 2)\, y = 0$. If $y = Ae^{rx}$, then auxiliary equation is

$$\begin{aligned} r^2 + r - 2 &= 0 \\ \Rightarrow (r-1)(r+2) &= 0 \\ \Rightarrow r = 1 \quad &\text{or} \quad r = -2. \end{aligned}$$

Hence, the complementary solution is

$$y_c = A_1 e^{x} + A_2 e^{-2x}.$$

Now, the particular solution is given by

$$\begin{aligned} y_p &= \frac{1}{D^2 + D - 2} 4x = \frac{1}{(D-1)(D+2)} (4x) \\ &= \left(\frac{1}{D+2} - \frac{1}{D-1} \right) (4x) \\ &= (D+2)^{-1} (x) - (D-1)^{-1} (4x) \\ &= e^{-2x} \int 4xe^{2x} dx - e^{x} \int 4xe^{-x} dx \\ &= 4e^{-2x} \left(\frac{1}{2} xe^{2x} - \frac{1}{4} e^{2x} \right) - 4e^{x} \left(-xe^{-x} - e^{-x} \right) + A_3 \\ &= 2x - 1 + 4x + 4 + A_3 = 6x + 3 + A_3. \end{aligned}$$

Hence, the general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= A_1 e^{x} + A_2 e^{-2x} + 6x + 3 + A_3 \\ &= A_1 e^{x} + A_2 e^{-2x} + 6x + A_3, \quad A_3 \to 3 + A_3. \end{aligned}$$

Example 10.55. The current I in a certain electrical circuit varies with time t in accordance with the relation

$$\frac{d^2 I}{dt^2} + 4\frac{dI}{dt} + 13I = 120 \cos 3t.$$

Find the current at time t given that $I = 0$ and $\frac{dI}{dt} = 0$ when $t = 0$.

Solution. Let $D \equiv \frac{d}{dt}$. Then we can write the given equation as

$$\left(D^2 + 4D + 13\right) I = 120 \sin 3t,$$

and so the associated homogeneous equation is given by

$$\left(D^2 + 4D + 13\right) I = 0$$

whose auxiliary equation is

$$a^2 + 4a + 13 = 0,$$

a quadratic in a, where $y = Ce^{ax}$ with a and C being constants. Solving for a, we have

$$\begin{aligned} a &= \frac{-4 \pm \sqrt{16 - 4\,(13)}}{2} = \frac{-4 \pm 2\sqrt{4 - 13}}{2} \\ &= -2 \pm \sqrt{-9} = -2 \pm 3i \\ \Rightarrow a_1 = -2 + 3i &\quad \text{or} \quad a_2 = -2 - 3i, \end{aligned}$$

which are complex conjugates. Therefore, the complementary solution is

$$I_c = e^{-2t}\left(A \cos 3t + B \sin 3t\right).$$

Now,

$$\begin{aligned} I_p &= \frac{1}{D^2 + 4D + 13}\left(120 \sin 3t\right) \\ &= \frac{120}{D^2 + 4D + 13}\left(\sin 3t\right) \\ &= \frac{120}{-3^2 + 4D + 13}\left(\sin 3t\right), \quad \because D^2 = -3^2 \\ &= \frac{120}{4D + 4}\left(\sin 3t\right) \\ &= \frac{30}{D + 1}\left(\sin 3t\right) \\ &= \frac{30\,(D - 1)}{D^2 - 1}\left(\sin 3t\right) \\ &= \frac{30\,(D - 1)}{-3^2 - 1}\left(\sin 3t\right), \quad \because D^2 = -3^2 \\ &= -3\,(D - 1)\left(\sin 3t\right) \\ &= -3\left(D\left(\sin 3t\right) - \sin 3t\right) \\ &= -3\left(3 \cos 3t - \sin 3t\right). \end{aligned}$$

Hence, the current at any time t is given by

$$I = e^{-2t}\left(A \cos 3t + B \sin 3t\right) - 3\left(3 \cos 3t - \sin 3t\right),$$

where A and B are constants to be determined. Now when $t = 0$, $I = 0$ and $DI = 0$.

$$\therefore\ 0 = A - 3\,(3) \Rightarrow A = 9,$$

and since

$$\begin{aligned} DI &= e^{-2t}\left(-3A \sin 3t + 3B \cos 3t\right) - 2e^{-2t}\left(A \cos 3t + B \sin 3t\right) + 3\left(9 \sin 3t + 3 \cos 3t\right) \\ \Rightarrow 0 &= 3B - 2A + 3\,(3) \Rightarrow 0 = 3B - 18 + 9 \Rightarrow B = 3. \end{aligned}$$

Therefore, the required solution is

$$\begin{aligned} I &= e^{-2t}\left(9 \cos 3t + 3 \sin 3t\right) - 3\left(3 \cos 3t - \sin 3t\right) \\ &= 3e^{-2t}\left(3 \cos 3t + \sin 3t\right) - 3\left(3 \cos 3t - \sin 3t\right). \end{aligned}$$

Problems 10.........

1. Use induction to prove that

$$P(D)\,e^{ax} = P(a)\,e^{ax}$$

where

$$P(D) = \sum_{i=0}^{n} a_i D^{n-i}.$$

Hence ....

2. Prove Corollary 10.42.

3. Show that

$$(D-a)\left(e^{ax}\int e^{-ax} f(x)\,dx\right) = f(x)$$

for all constants a and for all differentiable functions $f(x)$.

4. m

5. m

6. m

7. m

8. m

9. Solve the following differential equations.

   (a)
   $$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 3e^{-2x}.$$

   (b) m

   (c) m

   (d) m

   (e) m

10. A system vibrates according to the equation

$$8\frac{d^2s}{dt^2} + 4\frac{ds}{dt} + s = \sin t - 2\cos t.$$

Use the D-operator method to solve for s in terms of t and hence find the steady state of the displacement s.

Chapter 11

Solutions to End-of-Topic Problems