UNIVERSITY OF GHANA

DEPARTMENT OF MATHEMATICS

MATH 223: CALCULUS II (3 credits)

EXERCISE 5 (Integration

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1. Use the Riemann sums to evaluate the following:
   (a) $\int_0^4 (2x^2+3)dx$
   (b) $\int_{-1}^3 x^3 dx$
   (c) $\int_1^5 (x-4x^2)dx$
   (d) $\int_{-5}^{-1} (x^2+3x+5)dx$
   (e) $\int_0^4 (-3x^2+5x-1)$
   (f) $\int_1^3 x^3\,dx$
   (g) $\int_3^8 (x-4x^2)\,dx$
2. Find the limit as $n \to \infty$ of the sum $\frac{1}{n} + \frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \cdots + \frac{n}{(2n-1)^2}$.
3. By interpreting the following limit as a definite integral, show that $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n+i} = \ln 2$.
4. Show that $\lim_{n\to\infty} \frac{1^p+2^p+3^p+\cdots+n^p}{n^{p+1}} = \frac{1}{p+1}$.
5. Find $\int_0^3 f(x)dx$ where $f(x) = \begin{cases} \sqrt{1-x^2} & 0 \le x \le 1 \\ 2 & 1 < x \le 2 \\ x-2 & 2 < x \le 3 \end{cases}$
6. Find the derivatives of the following;
   (a) $y = \int_{-\pi}^{x} \cos t\,dt$
   (b) $y = \int_1^{x^2} \sin t\,dt$
   (c) $y = \int_1^{x^4} \sec t\,dt$
   (d) $y = \int_1^{\sin x} 3t^2\,dt$
   (e) $y = \int_1^{\tan x} \sec^2 t\,dt$
   (f) $y = \int_1^{\sin x} \frac{dt}{\sqrt{1-t^2}}$

$$\left[\textit{Note: } \frac{d}{dx}\int_a^{g(x)} f(t) \;= f(g(x))g'(x) \text{ and } \frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))g'(x) - f(h(x))h'(x)\right]$$

7. Evaluate the following;

    (a) $y = \int_1^{x^4} \sec t\, dt$

    (b) $y = \int_1^{\sin x} 3t^2\, dt$

    (c) $y = \int_1^{\tan x} \sec^2 t\, dt$

    (d) $y = \int_1^{\sin x} \frac{dt}{\sqrt{1-t^2}}$

    (e) $\int_0^{\pi} \sin x\, dx$

    (f) $\int \frac{x}{x^2+1} dx$

    (g) $\int e^x \sqrt{1+e^x} dx$

    (h) $\int \sec x \sqrt{\sec x + \tan x} dx$

    (i) $\int \frac{\sin x + \cos x}{e^{-x}+\sin x} dx$

    (j) $\int_2^4 (2x+3) dx$

    (k) $\int_1^6 (2x-6) dx$

    (l) $\int_{-1}^2 (x^2 - 3x + 2) dx$

    (m) $\int \frac{\sin 2x - \cos 2x}{\sin 2x + \cos 2x} dx$

8. Find the following integrals:

    (a) $\int x^2 \sqrt{x-2} dx$

    (b) $\int \frac{x}{\sqrt[3]{x+1}}$

[For integrals of the form $\displaystyle\int p(x)\sqrt[n]{ax+b}\,dx$ *and* $\displaystyle\int \frac{p(x)}{\sqrt[n]{ax+b}}dx$ *we make a substitution* $u^n = ax+b$*. For integrals of the form* $\displaystyle\int p(x)\sqrt[n]{\frac{ax+b}{cx+d}}dx$ *we make a substitition* $u^n = \dfrac{ax+b}{cx+d}$*].*

9. Solve the integral equation $f(x) = 2 + 3\int_4^x f(t)\, dt$.

10. Prove the following;

    (a) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$

    (b) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$

    (c) $\int \tan(ax+b)dx = -\frac{1}{a}\ln|\cos(ax+b)| + C$

    (d) $\int \cot(ax+b)dx = \frac{1}{a}\ln|\sin(ax+b)| + C$

    (e) $\int \sec(ax+b)dx = \frac{1}{a}\ln|\sec(ax+b) + \tan(ax+b)|C$

    (f) $\int \csc(ax+b)dx = \frac{1}{a}\ln|\csc(ax+b) + \cot(ax+b)|C$

11. Find the following integrals;

(a) $\int \cos^n x \sin x dx$ Let $u = \cos x$

(b) $\int \sin^n x \cos x dx$ Let $u = \sin x$

(c) $\int \cos^5 dx$

(d) $\int \sin^3 x dx$

(e) $\int \cos^7 x dx$

(f) $\int \sin^9 x dx$ $\int \sin^3 x \cos^{-2} x dx$

(g) $\int \frac{\sin^7 x}{\cos^4} dx$

(h) $\int \sin^3 x \cos^2 x dx$

Integrals of the form $\int \sin^{2k+1} x dx$ *and* $\int \cos^{2k+1} x dx$
Let $\int \sin^{2k+1} x dx = \int \sin^{2k} x \sin x dx = \int (\sin^2 x)^k \sin x dx = \int (1 - \cos^2 x)^k \sin x dx$.
Let $u = \cos x$, *then* $\int \sin^{2k+1} x dx = -\int (1 - u^2)^k du]$.

Let $\int \cos^{2k+1} x dx = \int \cos^{2k} x \cos x dx = \int (\cos^2 x)^k \cos x dx = \int (1 - \sin^2 x)^k \cos x dx$.
Let $u = \sin x$, *then* $\int \cos^{2k+1} x dx = \int (1 - u^2)^k du]$.

(i) $\int \sin^4 x dx$

(j) $\int \cos^2 x dx$

(k) $\int \sin^2 x dx$

(l) $\int \cos^4 x dx$

(m) $\int \sin^4 x \cos^2 x dx$

(n) $\int \sin^2 x \cos^2 x dx$

Integrals of the form $\int \sin^{2k} x dx$ *and* $\int \cos^{2k} x dx$
Let $\int \sin^{2k} x dx = \int \frac{(1-\cos 2x)^k}{2} dx$.
Let $\int \cos^{2k} x dx = \int (\frac{(1+\cos 2x)^k}{2} dx$.

12. Find the following integrals;

(a) $\int \tan^4 x dx$

(b) $\int \tan^3 x \sec^4 x dx$

(c) $\int \sec^9 x \tan^5 x dx$

(d) $\int \sec^4 x \tan^6 x dx$

(e) $\int \tan^3 x dx$

(f) $\int \sec^2 x \tan^n x dx$

Integrals of the form $\int \tan^m x \sec^n x dx$
If n *is even, then* $\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{2k} x dx = \int \tan^m x (\sec^2 x)^{k-1} (\sec^2 x) dx$
So $\int \tan^m x \sec^n x dx = \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x dx$ and let $u = \tan x$

If m *is odd, then* $\int \tan^m x \sec^n x dx = \int \tan^{2k+1} x \sec^n x dx = \int \tan^{2k} x \tan x \sec^n x dx$
So $\int \tan^m x \sec^n x dx = \int (\sec^2 -1)^k x \tan x \sec^n x dx$ and let $u = \sec x$

13. Find the following integrals;

    (a) $\int \cot^3 x \csc^7 x dx$

    (b) $\int \cot x \csc^n x dx$

    (c) $\int \cot^m x \csc^2 x dx$

Integrals of the form $\int \cot^m x \csc^n x dx$
If n *is even, then* $\int \cot^m x \csc^{2k} x dx = \int \cot^m x \csc^{2k-2} x \csc^2 x dx = \int \cot^m x (\cot^2 x + 1)^k \csc^2 x dx$ and let $u = \cot x$

If m *is odd, then* $\int \cot^{2k+1} x \csc^n x dx = \int \cot^{2k} x \cot x \csc^{n-1} x \csc x dx$
So $\int \cot^{2k+1} \csc^n x dx = \int (\csc^2 x - 1)^k \csc^{n-1} x \cot x \csc x dx$ and let $u = \csc x$