UNIVERSITY OF GHANA

DEPARTMENT OF MATHEMATICS

MATH 223: CALCULUS II (3 credits)

PRE IA EXERCISES - 2

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1. Given that $f(x) = b^x$ for $b > 0$, determine the values of b that will make the function f an increasing or a decreasing function.

2. Given that $f(x) = \log_b x$ for $b > 0$, determine the values of b that will make the function f an increasing or a decreasing function.

3. Explain with graphs why the coordinate (a, b) is on the graph of f whenever the coordinate (b, a) is on the graph of f^{-1}.

4. Evaluate the following limits

   (a) $\lim\limits_{x\to 0^+} x \ln x$

   (b) $\lim\limits_{x\to\infty} x^{-3}e^x$

   (c) $\lim\limits_{x\to 0}(1+2h)^{\frac{1}{h}}$

   (d) $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x}$

   (e) $\lim\limits_{u\to\frac{\pi}{4}} \dfrac{\tan u - \cot u}{u - \frac{\pi}{4}}$

   (f) $\lim\limits_{x\to 0^+} (\sin x)\sqrt{\dfrac{1-x}{x}}$

   (g) $\lim\limits_{x\to 0^+} \cot x - \dfrac{1}{x}$

   (h) $\lim\limits_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^{\ln x}$

5. Find the derivatives of the following:

   (a) $y = \log_8(|\tan x|)$

   (b) $y = \log_2(\log_2 x)$

   (c) $f(x) = (\sin x)^{\tan x}$ for $0 < x < \pi$ and $x \neq \frac{\pi}{2}$

   (d) $f(x) = \dfrac{e^{2x}x(\sin x)^{\cos x}}{x \cot x}$ for $x \neq \frac{\pi}{2}$.

6. Suppose f and g are differentiable functions of x and show, by logarithimic differentiation, that

$$\frac{d}{dx}(f^g) = g\left(f^{g-1}\right)\frac{df}{dx} + (f^g \ln f)\frac{dg}{dx}$$

7. For each of the following functions below, find $\left(f^{-1}\right)'(a)$ where possible.
   (a) $f(x) = x + 5$ for $a = 5$
   (b) $f(x) = 2x^2 + 3$ for $a = 1$
   (c) $f(x) = x^3$ for $a = 27$
   (d) $f(x) = 16 - x^2$ for $0 \leq x \leq 3$ and $a = 15$
   (e) $f(x) = x^3 + 3\sin x + 2\sin x$ for $a = 2$.
   (f) $f(x) = 2x^3 + 3x^2 + 7x + 4$ for $a = 4$
8. Consider the following functions defined on the entire real line. Find the inverse if it exists. If the inverse does not exist, provide a restriction to the domain so that you can find an inverse to the function.
   (a) $f(x) = x^2 - 2x$
   (b) $g(x) = \frac{1}{x}$
   (c) $h(x) = 10 - 3x$
   (d) $f(x) = |x|$
   (e) $h(x) = 1 + \cos x$
9. Do all linear functions have inverses? If yes, provide a proof. If no, provide a proof for those that have inverses and a proof for those that do not have inverses.
10. Determine the intervals for which $f(x) = x^2 - 1$ has an inverse function.
11. If a function f is one-to-one, does it mean that it has an inverse? If yes, show why. If no, what can you do to make it possible for the function to have an inverse?
12. Prove that $\dfrac{2}{\pi} < \dfrac{\sin x}{x} < 1$ whenever $0 < x < \frac{\pi}{2}$.
13. Suppose $3 \leq f'(x) \leq 5$ for all x, show that $18 \leq f(8) - f(2) \leq 30$.
14. Without evaluating derivatives, which of the following functions have the same derivatives?
$$f(x) = \ln x, \quad g(x) = \ln 2x, \quad h(x) = \ln x^2, \quad k(x) = \ln 10x^2$$
15. Find the values of k for which the function $f(x) = -4x^2 + (4k - 1)x - k^2 + 4$ is negative for all values of x.
16. Show that for real values of x, $f(x) = \dfrac{3\sin x}{2 + \cos x}$ cannot have a value greater than $\sqrt{3}$ or a value less than $-\sqrt{3}$.
17. If $\epsilon \in \mathbb{R}$ and the function $f(x) = \epsilon x - \dfrac{x^3}{1 + x^2}$ is increasing $\forall x \in \mathbb{R}$, show also that $\epsilon \geq \frac{9}{8}$.

18. Find the interval of increase and decrease of the following functions;
    (a) $f(x) = x^3 + 4x + 1$
    (b) $f(x) = x^3(5 - x)^2$
    (c) $f(x) = x + \sin x$
    (d) $f(x) = (x^2 - 4)^2$
19. Prove that $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ whenever $0 < x < \frac{\pi}{2}$.
20. Use the Mean Value Theorem (MVT) to establish the following inequalities:
    (a) $|\sin x - \sin y| \leq |x - y| \ \forall x, y \in \mathbb{R}$.
    (b) $\frac{x}{1+x} < \ln(1 + x) < x$ for $-1 < x < 0$, $x > 0$
21. Show that the real valued function defined by $f(x) = x^3 - 3x^2 + 3x + 1$ increases for all $x \in \mathbb{R}$.
22. Consider the function $f(x) = x^3 - 7x^2 + 10x$
    (a) Find an interval on which Rolles' Theorem applies to f.
    (b) Find all points in the interval for which $f'(x) = 0$
23. The lapse rate is the rate at which the temperature T decreases in the atmosphere with respect to increasing altitude z. It is typically reported in units of $°C/\text{km}$ and is defined by $\gamma = \dfrac{dT}{dz}$. When the lapse rate rises above $7°C/\text{km}$ in a certain layer of the atmosphere, it indicates favourable conditions for thunderstorm and tornado formation, provided other atmospheric conditions are also present. Assume the following:
    - the temperature function is continuous and differentiable at all altitudes of interest;
    - the temperature at $z = 2.9\text{km}$ is $T = 7.6°C$
    - the temperature at $z = 5.26\text{km}$ is $T = -14.3°C$.

    What can a meteorologist conclude from this information?