UNIVERSITY OF GHANA

DEPARTMENT OF MATHEMATICS

MATH 223: CALCULUS II (3 credits)
CHAPTER 6: Sequences

At the end of the chapter, learners are expected to be able to determine:

1. Define sequences
2. Determine if a sequence convergences and find its limit
3. Find sums of terms of geometric and arithmetic sequences

LESSON HIGHLIGHTS

Definition

A sequence $\{a_n\}$ is a function whose domain is the set of positive integers. The functional values $a_1, a_2, \cdots, a_n; \cdots$ are the **terms** of the sequence, and the term a_n is called the $n^{\mathbf{th}}$ **term** of the sequence.

Note that

1. The sequence $\{a_n\}$ is also denoted by $\{a_n\}_{n=1}^{\infty}$
2. Sometimes it is convenient to begin a sequence with a_k. In this case, the sequence is $\{a_n\}_{n=k}^{\infty}$, and its terms are $a_k, a_{k+1}, \cdots, a_n, \cdots$

We may define a sequence recursively by specifying the first term or the first few terms of the sequence and a rule for calculating other terms of the sequence from the preceding term(s).

Definition of the Limit of a Sequence

A sequence $\{a_n\}$ has a limit L, written as $\lim_{n\to\infty} a_n = L$ if a_n can be made as close to L as we please by taking n sufficiently large.

Precise definition of the limit of a sequence

A sequence $\{a_n\}$ converges and has a limit L, written as $\lim_{n\to\infty} a_n = L$ if for every $\epsilon > 0$, $\exists$ a positve integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.

Theorem

If $\lim\limits_{x\to\infty} f(x) = L$ and $\{a_n\}$ is a sequence defined by $a_n = f(n)$, where n is a positive integer, then $\lim\limits_{n\to\infty} a_n = L$

Limit Laws for sequences

Suppose that $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} b_n = M$ and c is a constant. Then we have the following:

1. $\lim\limits_{n\to\infty} ca_n = c\lim\limits_{n\to\infty} a_n = cL$
2. $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = L \pm M$
3. $\lim\limits_{n\to\infty} a_n b_n = \lim\limits_{n\to\infty} a_n \lim\limits_{n\to\infty} b_n = LM$
4. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \dfrac{L}{M}$ provided $b_n \neq 0$ and $M \neq 0$
5. $\lim\limits_{n\to\infty} a_n^p = \left(\lim\limits_{n\to\infty} a_n\right)^p = L^p$ if $p > 0$ and $a_n > 0$

Squeeze Theorem for Sequences

If there exists some integer N such that $a_n \leq b_n \leq c_n$, $\forall\, n \geq N$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

Theorem

If $\lim\limits_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim\limits_{n\to\infty} f(a_n) = f\left(\lim\limits_{n\to\infty} a_n\right) = f(L)$.

Definition (Monotonic Sequence)

A sequence $\{a_n\}$ is increasing if $a_1 < a_2 < a_3 < \cdots < a_n < a_{n+1} < \cdots$ and decreasing if $a_1 > a_2 > a_3 > \cdots > a_n > a_{n+1} > \cdots$

A sequence is monotonic if it is either increasing or decreasing.

Definition of Bounded Sequences

- A sequence $\{a_n\}$ is bounded above if $\exists\, M \in \mathbb{R}$ such that $a_n \leq M$, $\forall n \geq 1$.
- A sequence $\{a_n\}$ is bounded below if $\exists\, m \in \mathbb{R}$ such that $m \leq a_n$, $\forall n \geq 1$.
- A sequence is bounded if it is both bounded above and bounded below.

Monotone Convergence Theorem for Sequences

Every bounded, monotonic sequence is convergent

Properties of the Sequence $\{r^n\}$

The sequence $\{r^n\}$ converges if $-1 < r \leq 1$ and

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad -1 < r \leq 1 \\ 1 & \text{if} \quad r = 1 \end{cases}$$

It diverges for all other values of r.

Arithmetic and Geometric Sequences

An arithmetic sequence is a sequence in which consecutive terms differ by some constant d.

$$a_n = a_1 + (n-1)d$$

A geometric sequence is a sequence in which consecutive terms differ by some constant r.

$$a_n = a_1 r^{n-1}$$

Sum of Sequences

The sum of the first n terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}(a_1 + a_n)$$

The sum of the first n terms of a geometric sequence is given by

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

IMPORTANT THINGS TO NOTE

- Spend time trying the exercises on your own. This would give you an idea of what you truly understand and what you need to work on.
- Revise your notes before class and make an effort to read ahead of each class.
- Seek help before it is too late.