UNIVERSITY OF GHANA

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DEPARTMENT OF MATHEMATICS

MATH 223: CALCULUS II (3 credits)
CHAPTER 9: Integration

At the end of the chapter, learners are expected to be able to determine:

1. Apply the fundamental theorem of Calculus to solve questions.
2. Evaluate Riemann sums.
3. Use the method of substitution, integration by parts, trigonometric substitutions, partial fractions, and the tables of anti-derivatives to evaluate definite and indefinite integrals.

LESSON HIGHLIGHTS

Definition (The average value of a function)

If f is integrable on $[a, b]$, then the average value of f over $[a, b]$ is the number $f_{\text{av}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$.

The Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then $\exists\ c \in [a, b]$ such that $f(c) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$.

Fundamental Theorem of Calculus

Part 1

If f is continuous on $[a, b[$, then the function F defined by $F(x) = \displaystyle\int_a^x f(t)dt$ for $a \leq x \leq b$ is differentaible on (a, b) and $F'(x) = \dfrac{d}{dx}\displaystyle\int_a^x f(t)dt = f(x)$

Part 2

If f is continuous on $[a, b]$, then $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f; i.e $F' = f$.

Defintion of an antiderivative

A function F is an antiderivative of a function f on an interval I if $f(x) = F'(x)$, for all $x \in I$.

Definition of a Partition

A partition P of an interval $I = [a, b]$ is a finite set $P = \{x_0, x_1, \cdots, x_n\}$ with the property that $a = x_0 < x_1 < \cdots < x_n = b$. A partition divides an interval into n sub-intervals $[x_0, x_1], [x_1, x_2], \cdots, [x_{k-1}, x_k], \cdots, [x_{n-1}, x_n]$. The length of each subinterval in the partition $[x_{k-1}, x_k]$ is given by $\Delta x_k = x_k - x_{k-1}$, for $1 \leq k \leq n$.

If Δx_k is the same $\forall\, 1 \leq k \leq n$, then we have a *regular partition* and $\Delta x = \frac{b-a}{n}$.

Definiton of Riemann Sums

Let f be a function defined on $[a, b]$ with $P\{x_0, x_1, \cdots, x_n\}$, a regular partition of $[a, b]$, i.e. $\Delta x = \frac{b-a}{n}$. If $\bar{x_k}$ is any point in the k^{th} subinterval $[x_{k-1}, x_k]$, for $1 \leq k \leq n$, then the Riemann sum of f on $[a, b]$ is given by

$$\sum_{k=1}^{n} f(\bar{x_k})\Delta x = f(\bar{x_1})\Delta x + f(\bar{x_2})\Delta x + \cdots + f(\bar{x_n})\Delta x$$

If $\bar{x_k}$ is the left end point of the k^{th} subinterval, i.e. $\bar{x_k} = x_{k-1}$, then $\displaystyle\sum_{k=1}^{n} f(\bar{x_k})\Delta x$ is the Left Riemann Sum or the Lower Riemann Sum.

If $\bar{x_k}$ is the right end point of the k^{th} subinterval, i.e. $\bar{x_k} = x_k$, then $\displaystyle\sum_{k=1}^{n} f(\bar{x_k})\Delta x$ is the Right Riemann Sum or the Upper Riemann Sum.

If $\bar{x_k}$ is the midpoint of the k^{th} subinterval, i.e. $\bar{x_k} = x_{k-1} + \frac{x_k - x_{k-1}}{2}$ or $\bar{x_k} = x_k - \frac{x_k - x_{k-1}}{2}$, then $\displaystyle\sum_{k=1}^{n} f(\bar{x_k})\Delta x$ is the Midpoint Riemann Sum.

Note

For f defined on $[a, b]$ with a regular partition $P = \{x_0, x_1, \cdots, x_n\}$, we have the following:

- $\Delta x = \frac{b-a}{n}$
- $x_0 = a$
- $x_1 + \frac{b-a}{n}$
- $x_2 = a + 2\left(\frac{b-a}{n}\right)$
- $x_i = a + i\left(\frac{b-a}{n}\right)$
- $x_n = a + n\left(\frac{b-a}{n}\right) = a + b - a = a$

For every partition on which f is increasing, the Lower Riemann Sum is less than or equal to the Upper Riemann Sum.

For every partition on which f is decreasing, the Upper Riemann Sum is less than or equal to the Lower Riemann Sum.

As n gets infinitely large, the Lower and Upper Riemann sums coincide, and this is the Riemann Integral of the function on the interval.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \text{Upper Riemann Sum}$$

or

$$\int_a^b f(x)dx = \lim n \to \infty \text{Lower Riemann Sum}$$

As $n \to \infty$, $\Delta x \to 0$, so we choose $\bar{x_k} = x$ and the Riemann Integral is defined as

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x)\Delta x.$$

Theorem

Suppose f is a function which is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.

Sumation Formuale

1. $\displaystyle\sum_{i=1}^{n} k = kn$
2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{i=1}^{n} i\right)^2$
5. $\displaystyle\sum_{i=1}^{n} r^{i-1} = \frac{r^n - 1}{r - 1}$ where $r \neq 1$

Properties of the Definite Integral

Let f and g be ontinuous real-valued functions on $[a, b]$. Then we have the following:

1. $\displaystyle\int_a^a f(x)dx = 0.$
2. $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\displaystyle\int_a^b (rf(x) \pm sg(x))\, dx = r\int_a^b f(x)dx \pm s\int_a^b g(x)dx$ for constants r, s.

4. $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for $c \in [a, b]$.

5. If $f(x) = k$ for $x \in [a, b]$, then $\displaystyle\int_a^b f(x) = k(b - a)$

6. $\displaystyle\int_a^b f(x)dx \geq 0$, if $\forall\, x \in [a, b],\ f(x) \geq 0$.

7. $\displaystyle\int_a^b f(x)dx \leq \int_a^b g(x)dx$ if $f(x) \leq g(x)\ \forall x \in [a, b]$.

8. $\displaystyle\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx\ \forall x \in [a, b]$.

9. $\displaystyle\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

10. $\displaystyle\int_0^a f(x)dx = \int_0^a f(a - x)dx$

11. $\displaystyle\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^{2a} f(2a - x)dx$

12. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(2a - x) = f(x)[f \text{ even}] \\ 0 & \text{if } f(2a - x) = -f(x)[\ f \text{ odd} \end{cases}$

13. $\int_{-a}^a f(x)dx = \begin{cases} 2\int_a^a f(x)dx & \text{if } f \text{ is even} \\ 0 & \text{if } f \text{ is odd} \end{cases}$

Theorem

Let f be a piecewise continuous function on $[a, b]$. Let $a = x_0 < x_1 < \cdots < x_n = b$ such that f_i is a continuous on $(x_{i-1}, x_i), \forall\, 1 \leq i \leq n$ and $f(x) = f_i(x)\,\forall\, i$. Then $\displaystyle\int_a^b f(x)dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f_i(x)dx$.

Basic Integratiom Formulas

C is the constant of integration.

1. $\displaystyle\int 0dx = C$

2. $\displaystyle\int dx = x + C$

3. $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1}$ where $n \neq 1$

4. $\int \frac{dx}{x} = \ln x + C$

5. $\int e^x = e^x + C$

6. $\int a^x dx = \frac{a^x}{\ln a} + C$

7. $\int \sin x dx = -\cos x + C$

8. $\int \cos x = \sin x + C$

9. $\int \sec^2 x dx = \tan x + C$

10. $\int \sec x \tan x dx = \sec x + C$

11. $\int \mathrm{csc} x \cot x dx = -\mathrm{csc} x + C$

12. $\int \mathrm{csc}^2 x dx = -\cot x + C$

13. $\int \tan x dx = -\ln |\cos x| + C$

14. $\int \mathrm{csc} x dx = \ln |\mathrm{csc} x - \cot x| + C$

15. $\int \cot x dx = \ln |\sin x| + C$

16. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

17. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

18. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

19. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$

20. $\int \sinh x dx = \cosh x + C$

21. $\int \cosh x dx = \sinh x + C$

22. $\int \tanh x dx = \ln \cosh x + C$

23. $\displaystyle\int \coth x dx = \ln|\sinh x| + C$

24. $\displaystyle\int \text{sech} x = \tan^{-1}|\sinh x| + C$

25. $\displaystyle\int \text{csch} x = \ln\left|\tanh \frac{1}{2}x\right| + C$

26. $\displaystyle\int \text{sech}^2 dx = \tanh x + C$

27. $\displaystyle\int \text{csch}^2 x dx = -\coth x + C$

28. $\displaystyle\int \text{sech} x \tanh x dx = -\text{sech} x + C$

29. $\displaystyle\int \text{csch} x \coth x dx = -\text{csch} x + C$

Method of Substitution

Let $y = f(g(x))$, where f and g are differentiable functions of x. Then by the chain rule, $\dfrac{dy}{dx} = f'(g(x))g'(x)$.

If $u = g(x)$, then $du = g'(x)dx$ and

$$\begin{array}{rcl} \int f'(g(x))g'(x)dx & = & \int f'(u)du \\ & = & f(u) + C \\ & = & f(g(x)) + C \end{array}$$

Integration by parts

Let $y = uv$ where u and v are differentiable functions of x.

$$\begin{array}{rcl} \frac{d}{dx}(uv) & = & udv + vdu \\ \int \frac{d}{dx}(uv) & = & \int udv + \int vdu \\ uv & = & \int udv + \int vdu \end{array}$$

So,

$$\int udv = uv - \int vdu$$

IMPORTANT THINGS TO NOTE

- Spend time trying the exercises on your own. This would give you an idea of what you truly understand and what you need to work on.
- Revise your notes before class and make an effort to read ahead of each class.
- Seek help before it is too late.