UNIVERSITY OF GHANA

FIRST SEMESTER EXAMINATIONS: 2018/2019
EXAM SAMPLE

DEPARTMENT OF MATHEMATICS

BSc/BA

MATH 223: CALCULUS II (3 credits)

INSTRUCTION:

Answer any Four out of the following Six questions

TIME ALLOWED:

Two hours and Thirty minutes
($2\frac{1}{2}$ hours)

1

(a) Using the Riemann sum, evaluate the following integral

$$\int_2^7 (3x^2 - x + 4).$$

(b) If u is differentiable over an interval I, and f is continuous on the domain of u, find a formula for

$$\frac{d}{dx}\left[\int_a^{u(x)} f(t)dt\right]$$

where a is a constant.
N.B.:Clearly specify which theorem you are using.
Hence find

$$\frac{d^2}{dx^2}\left[\int_0^x \left(\int_1^{\sin t} \sqrt{1+u^4}du\right) dt\right].$$

[**From 2016/2017 Exam**]

(c) Evaluate

$$\lim_{x\to 3^+}\left[\frac{\sin(\frac{\pi}{x})}{x-3}\int_3^x \frac{\sin t}{t}dt\right].$$

[**From 2016/2017 Exam**]

2

(a) Let be the function defined by

$$f(x) = \sum_{k=0}^{\infty} \frac{2^{2k}k!}{(2k+1)!}x^{2k+1} = x + \frac{2}{3}x^3 + \frac{4}{3\times 5}x^5 + \frac{8}{3\times 5\times 7}x^7 + \ldots.$$

(i) Find the domain of convergence of f.

(ii) Prove that $f'(x) = 1 + 2xf(x)$.

(iii) Prove that

$$\frac{d}{dx}\left(e^{-x^2}f(x)\right) = e^{-x^2}.$$

[**Hint:** You may use the previous question.]
Hence express $f(x)$ as an integral.

(iv) Evaluate $\lim_{x\to 0}\frac{f(x)}{x}$.

(b) Evaluate the following limits.

$$\lim_{x\to 0}\frac{\tanh 5x^3}{x\sin 3x^2};$$

$$\lim_{x\to 0}(1+3x)^{\tanh 5x}.$$

(c) For each of the following series, find its partial sum. Hence deduce the exact value of the series.

$$\sum_{n=0}^{\infty}\frac{3}{n^2+4n+3};$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{\ln(n+1)} - \frac{1}{\ln(n+3)}\right).$$

3

(a) Evaluate the following integrals

(i) $\int \frac{1}{x^4\sqrt{9-x^2}}dx$

Hint: You may use $x = 3\sin\theta$ as a first substitution.

(ii) $\int e^{4x}\sqrt{1+e^{2x}}dx$

Hint: You may use $e^x = 3\tan\theta$ as a first substitution.

(iii) $\int x^5\sqrt{1+x^3}dx$.

(b) Prove that $\int_0^{\ln 2}\frac{\sinh x - x\cosh x}{\sinh^2 x}dx = \frac{4\ln 2 - 3}{3}$.

(c) Determine the length of $x = \frac{1}{2}y^2$ for $0 \le x \le \frac{1}{2}$, $y > 0$.

(d) Use the Mean Value Theorem to establish that for any $x > e$,

$$e^{-1}(x-e) < \ln x - 1 < x - e.$$

4

(a) Show that if $I_m = \int \sec^m \theta d\theta$, $m = 2, 3, \ldots$, then

$$(m-1)I_m = \tan\theta \sec^{m-2}\theta + (m-2)I_{m-2}.$$

(b) Let $f(x) = 5\cosh x + 3\sinh x - 4\sqrt{2}$.

(i) Find all the solutions to $f(x) = 0$ in the interval $(0, \infty)$.

(ii) Show that f is invertible on $(0, \infty)$.

(iii) Find $(f^{-1})'(0)$.

(c) For $n = 2, 3, \ldots$, put $J_n = \int \tan^n x dx$. Prove that

$$I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}.$$

(d) Let $(u_n)_{n\geq 0}$ be a geometric sequence such that $u_0 = \frac{1}{2}$ and $u_5 \times u_7 = 1024$. What is its common ratio?

5

(a) Let $f(z) = z\exp(z^2)$. Find an antiderivative F of f and deduce

$$\int_0^{\frac{\pi}{4}} f(z)dz.$$

(b) Evaluate $\int \frac{3}{x-3\sqrt{x+10}}dx$.

(c) Determine the surface area of the solid obtained by rotating $y = x^{\frac{3}{2}}$, $1 \leq y \leq 2\sqrt{2}$ about the y-axis.

(d) Prove that

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx,$$

and

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(a+x)dx.$$

6

(a) Find the MacLaurin series of $f(x) = \ln(1-x)$. What is its domain of convergence? Deduce the value of

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}.$$

(b) If $x^y = e^{x-y}$, prove that

$$\frac{dy}{dx} = \frac{\ln x}{(1+\ln x)^2}.$$

[**From 2017/2018 Exam**]

(c) Prove that the function $y = g(x) = \sin(\sin x)$ satisfies

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y \cos^2 x = 0.$$

[**From 2017/2018 Exam**]

d) Show that the MacLaurin series of $\cosh x$ is

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Deduce the value of the sum

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!}.$$