DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GHANA
MATH 223-CALCULUS II

1. a) Evaluate the following integrals

   1- $\int x \sec x \tan x dx$.

   2- $\int x^2 e^{-x} dx$.

   3- $\int \frac{\ln^2 x}{x} dx$.

   4- $\int_1^e \frac{dx}{x\sqrt{4-(\ln x)^2}}$.

   b) Prove that if f' is continuous on the interval $[a, b] \subset \mathbb{R}$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt.$$

   Hence show that

$$\int_0^{\frac{\pi}{4}} \ln(\tan x)) \sec^2(x)dx = -1$$

   and

$$\int_1^e \frac{\ln(\ln x)}{x} dx = -1.$$

   c) Let $a, b, c \in \mathbb{R}^+$ be pairwise distinct. Show that the equation

$$a \cosh x + b \sinh x = c$$

   has two distinct real solutions only if $b^2 < a^2 < b^2 + c^2$.

   N. B. There is no need to find the solutions.

2. a) Assume that f is continuous on the ranges of the differentiable functions u and v. Using the FTC, prove that

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = v'(x)f(v(x)) - u'(x)f(u(x)).$$

Deduce

$$\frac{d}{dx}\int_{x}^{\tan x} \tan^{-1} t\, dt.$$

b) Let

$$I_n := \int_0^{\frac{\pi}{2}} \cos^n x dx, \quad n = 0, 1, 2, \ldots.$$

(i) Evaluate I_0, I_1, I_2.

(ii) Prove that $I_n = \frac{n-1}{n} I_{n-2}$.

(iii) Deduce the value of I_3, I_4, I_5.

(iv) Prove that $I_{2n} = \frac{(2n)!}{2^{2n}n!}\frac{\pi}{2}$ and $I_{2n+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}\frac{\pi}{2}$.

c) Evaluate the following limits

$$\lim_{x\to 0} x^{\tanh x}; \quad \lim_{x\to 1^+} (x-1)^{\ln x}.$$

d) Prove that $\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$ for all $x \in \mathbb{R}$.

3. a) Evaluate the following limits

i) $\lim\limits_{x\to -\infty} \frac{x\ln|x| + e^{x-1} + 5}{x^2+1}$

ii) $\lim\limits_{x\to 0} \frac{\sinh x^5}{x \tan x^4}$

Prove that

$$\lim_{x\to 0^+} \int_x^{2x} \sin t\, dt = 0.$$

You are not allowed to compute the integral.

b) Prove that

(i) for any $n \in \mathbb{N}$, if $I_n = \int_1^e x \ln^n x dx$, then $I_n = \frac{1}{2}e^2 - \frac{1}{2}nI_{n-1}$.

(ii) if $f_n(x) = \frac{1}{n!}\int_0^x t^n e^{-t} dt$, then $f_{n+1}(x) - f_n(x) = \frac{1}{n!}f_n'(x)$.

c) Given that

$$y = x + \ln\left(\frac{(x-3)^5}{(y-1)^2}\right),$$

show that

$$(xy - 3y + x - 3)\frac{dy}{dx} = xy + 2y - x - 2.$$

d) (i) Determine the surface area of the solid obtained by rotation about the x-axis for $x = \cos^3\theta \;\; y = \sin^3\theta \;\; 0 \le \theta \le \frac{\pi}{2}$.

(ii) Determine the length of $r = \theta$, $0 \le \theta \le 1$.

(iii) Determine the surface area obtained by rotation the curve $x^2 + y^2 = 16$, $-1 \le x \le 1$ about the x-axis.

(iv) Find the volume of the solid formed by revolving the following region about the x-axis: $y = e^{-x}$, $y = 0$, $x = 0, x = 1$.

4. a) Find the Taylor series about $a = 0$ of $\ln(1+x)$. What is its domain of convergence?
Deduce the value of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

b) By first finding the Taylor series of the derivative, deduce the Taylor series for each of the following functions for $-1 < x < 1$:

$$\ln(1-x);\ \arctan x;\ \arctan^{-1} x.$$

c) Evaluate

(i) $\int e^t \cos t \, dt$.

(ii) $\int \sin^5 x \, dx$.

(iii) $\int \sin^6 x \cos^3 x \, dx$.

(iv)

d) By using the proposed substitution, evaluate the following.

(i) $\int \frac{1}{x^4\sqrt{9-x^2}} dx$; $x = 3\sec\theta$.

(ii) $\int e^{4x}\sqrt{1+e^{2x}} dx$; $e^x = \tan\theta$.

(iii) $\int \frac{x-1}{\sqrt[3]{x+2}} dx$; $u = \sqrt[3]{x+2}$

5. a) Show that the following series is not convergent

$$\sum_{n \ge 1} \ln\left(1 + \frac{1}{n}\right).$$

b) Using definition of the Riemann integral to evaluate the following limit

(i)

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\ln\left(1+\frac{k}{n}\right).$$

(ii)

$$\lim_{n\to\infty}\sum_{p=1}^{n}\sin\left(\frac{p\pi}{n}\right).$$

(iii)

$$\lim_{n\to\infty}\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots}{n\sqrt{n}}.$$

(iv)

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\ln 2}{n}\cosh\left(\frac{k\ln 2}{n}\right)$$

c) Evaluate using the Riemann sum the following

$$\int_{-1}^{2}(1-2x^2+x^3)\,dx.$$

d) (i) Determine the length of $x=\frac{1}{2}y^2$ for $0\le x\le\frac{1}{2}$ where y is assumed to be positive.

(ii) Find the surface area of the solid obtained by rotation $y=\sqrt{4-x^2}$, about the x-axis, $-1\le x\le 1$.

(iii) Find the surface area of the solid obtained by rotation $y=\sqrt[3]{x}$, about the y-axis, $1\le y\le 2$.

(iv) Find the area under the parametric curve given by the following parametric equations,

$$x=6(\theta-\sin\theta)\quad y=6(1-\cos\theta),\quad 0\le\theta\le 2\pi.$$

Arc Length and Surface Area: L and S Formula:

$$L=\int ds;$$

$$S=\int 2\pi y ds\quad\text{rotation about x-axis;}$$

$$S = \int 2\pi x ds \text{ rotation about y-axis.}$$

: Arc differential element:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \text{ If } y = f(x), a \leq x \leq b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \text{ If } x = g(y), c \leq y \leq d$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \text{ If } x = f(t), y = g(t), \alpha \leq t \leq \beta$$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \text{ If } r = f(\theta), \alpha \leq \theta \leq \beta.$$

Volume of revolution:

- If $y = f(x)$, then the volume of the solid obtained by rotating the portion of the curve between $x = a$ and $x = b$ about the x-axis is given by
$$V = \int_a^b \pi y^2 dx.$$
- If $x = g(y)$, then the volume of the solid obtained by rotating the portion of the curve between $y = c$ and $y = d$ about the y-axis is given by
$$V = \int_c^d \pi x^2 dy.$$