DEPARTMENT OF MATHEMATICS
(2014/2015)SEMESTER 1

MATH 223–CALCULUS II

Exercise 2

1. Given that $f(x) = b^x$ for $b > 0$. Determine the values of b that will make the function f an increasing or a decreasing function.

2. Given that $f(x) = \log_b x$ for $b > 0$. Determine the values of b that will make the function f an increasing or a decreasing function.

3. Explain with graphs why the coordinate (a, b) is on the graph of f whenever the coordinate (b, a) is o the graph of f^{-1}.

4. Explain the meaning of $\log_b x$.

5. Sketch the graphs of each of the following functions on the same set of axes.
   $y = 2^x$, $y = 2^{-x}$, $y = 2^{x-1}$, $y = 2^x + 1$, $y = 2^{2x}$.

6. Do as in question 5. for the following functions
   $y = \log_2 x$, $y = \log_2(x-1)$, $y = \log_2 x^2$.

7. Find the derivatives of the following.
   a) $y = \log_8(|\tan x|)$ b) $y = \log_2(\log_2 x)$
   c) $f(x) = (\sin x)^{\tan x}$, $0 < x < \pi$, $x \neq \frac{\pi}{2}$ c) $f(x) = \frac{e^{2x} x (\sin x)^{\cos x}}{x^{\cot x}}$, $x \neq \frac{\pi}{2}$.

8. Suppose that f and g are differentiable functions of x. Show, by logarithimic differentiation, that

$$\frac{d}{dx}\left(f^g\right) = g\left(f^{g-1}\right)\frac{df}{dx} + \left(f^g \ln g\right)\frac{dg}{dx}.$$

   Interpret the two terms on the right in the special cases where f is constant, and where g is constant.

9. Use the definition of the derivative to evaluate the following limits.
   a) $\lim_{x\to e} \frac{\ln x - 1}{x - e}$ b) $\lim_{h\to 0} \frac{\ln\left(e^8 + h\right) - 8}{h}$ c) $\lim_{h\to 0} \frac{(3+h)^{3+h} - 27}{h}$ d) $\lim_{x\to 0} \frac{2^x - 1}{3^x - 1}$
   e) $\lim_{x\to 2} \frac{5^x - 25}{x - 2}$. f) $\lim_{x\to\infty} x\left(e^{1/x} - 1\right)$

10. By Choosing a suitable function, derive the limit definition of the following using the defniition of the derivative
    a) $\ln(\sin b)$, $\sin b > 0$ b) $\ln(a^a)$, $a > 0$

11. By using the definition of the natural logarithmic function, prove that $2 < e < 3$.

12. By using the definition of the natural logarithmic function, prove the following

    (a) $\frac{x}{1+x} < \ln(1+x) < x$ $(x > 0)$,
    (b) $x < -\ln(1-x) < \frac{x}{1-x}$ $(0 < x < 1)$.
    (c) $x - \frac{1}{2}x^2 < \ln(1+x)$ $(x > 0)$,
    (d) $\frac{x-1}{x} < \log x < x - 1$ $(x > 1)$.

13. In each of the following, determine which function (f, g) has a higher order of magnitude. Find another function h such that its order of magnitude is between that of f and g (i.e say $f(x) < h(x) < g(x)$ for large values of x).
    a) $g(x) = \sqrt{x}$ and $f(x) = \sqrt[3]{x}$ b) $f(x) = \sqrt{\ln x}$ and $g(x) = \sqrt[3]{\ln x}$
    c) $f(x) = \sqrt{x}$ and $g(x) = \ln^3 x$.

14. Arrange the functions

$$\frac{x}{\sqrt{\ln x}}, \frac{x\sqrt{\ln x}}{\ln\ln x}, \frac{x\ln\ln x}{\sqrt{\ln x}}, \frac{x\ln\ln\ln x}{\sqrt{\ln\ln x}}$$

according to their order of magnitude.

15. Use limit methods to determine which of the two given functions grows faster, or state that they have comparable growth rates.

    (a) $x^{10}; e^{0.01x}$
    (b) $x^2 \ln x; \ln^2 x$
    (c) $e^{x^2}; x^{x/10}$
    (d) $e^x; x^x$

16. Evaluate the following limits:

    (a) $\lim_{x\to 0^+} x\ln x$
    (b) $\lim_{x\to\infty} x^{-3}e^x$
    (c) $\lim_{x\to 0^+} \frac{\ln x}{x}$
    (d) $\lim_{x\to\infty} x\ln x$

17. Use l'hopital's rule to evaluate these limits

    (a) $\lim_{h\to 0}(1+2h)^{1/h}$
    (b) $\lim_{x\to 0} \frac{\ln(1+x)}{x}$
    (c) $\lim_{u\to\pi/4} \frac{\tan u - \cot u}{u-\pi/4i}$
    (d) $\lim_{x\to 0^+}(\sin x)\sqrt{\frac{1-x}{x}}$
    (e) $\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right)$
    (f) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\ln x}$

18. **Miscellaneous limits by any means:** Use analytical methods to evaluate the following limits.

    (a) $\lim_{x\to 6} \frac{\sqrt[5]{5x+2}-2}{1/x-1/6}$
    (b) $\lim_{x\to\infty}(\log_2 x - \log_3 x)$
    (c) $\lim_{n\to\infty} \frac{1+2+\ldots+n}{n^2}$
    (d) $\lim_{x\to 1^+}\left(\frac{1}{x-1} - \frac{1}{\sqrt{x-1}}\right)$

19. Consider the limit

$$\lim_{x\to\infty} \frac{\sqrt{ax+b}}{\sqrt{cx+d}},$$

where a, b, c, d are positive real numbers. Show that L'hopital's rule fails for this limit. Find the limit using another method.