UNIVERSITY OF GHANA

DEPARTMENT OF MATHEMATICS

MATH 223: CALCULUS II (3 credits)
CHAPTER 3: Logarithmic and Exponential Functions

At the end of the chapter, learners are expected to be able to determine:

1. Define the natural logarithmic and exponential functions.
2. Define the general logarithmic and exponential functions.
3. Apply the properties of logarithmic and exponential functions in solving problems.
4. Find derivatives and anti-derivatives of logarithmic and exponential functions.
5. Perform logarithmic differentiation.
6. Compare the order of magnitude of two functions.

LESSON HIGHLIGHTS

Definition

The natural logarithmic function, denoted by ln is defined by

$$\ln x = \int_1^x \frac{1}{t}dt \qquad\qquad \forall x > 0$$

Derivative of $\ln x$

- $\frac{d}{dx}[\ln x] = \frac{1}{x}$ for $x > 0$
- $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}$, where u is a differentiable function of x.

Laws of Logarithms

Let x and y be positive numbers and let r be a rational number. Then we have the following properties

1. $\ln 1 = 0$
2. $\ln xy = \ln x + \ln y$
3. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

4. $\ln x^r = r \ln x$

The Graph of the Natural Logarithmic Function

Let $f(x) = \ln x$, then we have the following;

1. Domain is $(0, \infty)$ by definition
2. f is continuous on $(0, \infty)$ since f if differentiable on $(0, \infty)$
3. f is increasing on $(0, \infty)$ since $f'(x) = \frac{1}{x} > 0$, $\forall x > 0$
4. The graph of f is concave down on $(0, \infty)$ since $f''(x) = \frac{1}{x^2} < 0$, $\forall x > 0$
5. As $x \to \infty$, $\ln x \to \infty$
6. As $x \to 0^+$, $\ln x \to -\infty$

Proceedure for Logarithmic Differentiation

Suppose we have to find $\frac{dy}{dx}$, given $y = f(x)$,

1. Take the logarithm of both sides of the equation and simplify using the laws of logarithms.
2. Differentiate implicitly with respect to x.
3. Solve the equation for $\frac{dy}{dx}$.
4. Substitute for y.

Definition

The number e is the number such that

$$\ln e = \int_1^e \frac{1}{t} dt = 1$$

Definition

If x is any real number, then $e^x = y \Leftrightarrow \ln y = x$.

Note

- $\ln e^x = x \qquad \forall x \in (-\infty, \infty)$
- $e^{\ln x} = x \qquad \forall x \in (0, \infty)$

Graph of the Natural Exponential Function

Let $f(x) = e^x$.

1. The doman of f is $(-\infty, \infty)$.
2. The range of f if $(0, \infty)$.
3. f is continuous amd increasing on $(-\infty, \infty)$.

4. The graph of f is concave upward on $(-\infty, \infty)$.
5. $\lim_{x\to-\infty} e^x = 0$
6. $\lim_{x\to\infty} e^x = \infty$

Differentiation and Integration of the Natural Exponential Function

Let u be a differentiable function of x. then

- $\frac{d}{dx}[e^x] = e^x$
- $\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$
- $\int e^x dx = e^x + c$, where c is a constant

The General Exponential Function

Let a be a positive real number with $a \neq 1$. The exponential function with base a is the function f defined by $f(x) = a^x$, $\forall x \in \mathbb{R}$. Note that $a^x = e^{x \ln a}$.

Laws of Exponents

Let a and b be positive numbers. If x and y are real numbers, then

- $a^x a^y = a^{x+y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$
- $\frac{a^x}{a^y} = a^{x-y}$
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

Derivatives of a^x and a^u

Let a be a positive number with $a \neq 1$. If u is a differentiable function of x,

- $\frac{d}{dx}[a^x] = (\ln a)a^x$
- $\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$

Graph of $y = a^x$

- If $a > 1$, $y = a^x$ is an increasing function $\forall x \in (-\infty, \infty)$.
- If $0 < a < 1$, $y = a^x$ is a decreasing function $\forall x \in (-\infty, \infty)$.

Integrating a^x

$$\int a^x dx = \frac{a^x}{\ln a} + c, \qquad \text{where } a > 0, a \neq 1 \text{ and } c \text{ is a constant}$$

Logarithmic functions with base a

- If $a > 0$ and $a \neq 1$, then $\forall x \in (0, \infty)$, the logarithmic function with base a is defined by $y = \log_a x \Leftrightarrow x = a^y$
- $\log_a x = \frac{\ln x}{\ln a}$, for $a > 0$ and $a \neq 1$
- $\frac{d}{dx} \log_a |x| = \frac{1}{x \ln a}$ where $x \neq 0$
- $\frac{d}{dx} \log_a |u| = \frac{1}{u \ln a} \frac{du}{dx}$, where u is a differentiable function of x and $u \neq 0$

Logarithms and Exponents as Limits

- $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)$
- $\ln a = \lim_{x \to 0} \frac{a^x - 1}{x}$
- $e^x = \lim_{h \to 0} (1 + xh)^{\frac{1}{h}}$

Order of Magnitude

If $f(x)$ is a function of the $k - th$ order of magnitude, then $\lim_{x \to \infty} \frac{f(x)}{x^k} = L$, where $L \neq 0$ and L is a constant.

Relative rates of Growth and Relative rates of Decay

Let $f, g : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$ and $\lim_{x \to a} |f(x)| = \lim_{x \to a} |g(x)| = \infty$

1. If $\lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| = \infty$, then f appproaches ∞ on a higher order of magnitude than g.
2. If $\lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| = 0$, then f appproaches ∞ on a lower order of magnitude than g.
3. If $\lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| = c$, $c \in \mathbb{R}$, with $c \neq 0$, then f and g appproach ∞ on the same order of magnitude.

Let $f, g : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$ and $\lim_{x \to a} |f(x)| = \lim_{x \to a} |g(x)| = 0$

1. If $\lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| = 0$, then f appproaches 0 on a higher order of magnitude than g.
2. If $\lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| = \infty$, then f appproaches 0 on a lower order of magnitude than g.
3. If $\lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| = c$, $c \in \mathbb{R}$, with $c \neq 0$, then f and g appproach 0 on the same order of magnitude.

Theorem

As x apporaches ∞, exponential, power and logarithmic functions approach ∞. The following is the order of magnitude with which they approach ∞;

1. Exponential functions
2. Power functions

3. Logarithmic functions

Added on, we have the following:

1. If $a, b > 0$, $a \neq 1$, $b \neq 1$ and $a > b$, then a^x approaches ∞ faster that b^x as $x \to \infty$
2. If $n, m \in \mathbb{N}$ and $n > m$, then $x^n \to \infty$ faster than x^m as $x \to \infty$
3. If $a, b > 0$, $a \neq 1$, $b \neq 1$, $a > b$, then $\log_a x$ appraoches ∞ at the same rate as $\log_b x$.

Definition

Suppose f can be written as a linear combination of functions $\{f_1, f_2, \cdots, f_n\}$ and $\lim_{x\to a} |f(x)| = \infty$ where $a \in \mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$, the dominant term of f is the function f_i which approaches ∞ on the highest order of magnitude. This is denoted by $\hat{f}$.

Theorem

Let $f(x)$ and $g(x)$ be functions such that $\lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty$, where $a \in \mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$. If $\hat{f}$ and $\hat{g}$ are the dominant terms of f and g respectively, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\hat{f}(x)}{\hat{g}(x)}$$

Remark

If f grow faster than g, then the reciprocal of f would decay faster than the reciprocal of g.

IMPORTANT THINGS TO NOTE

- Spend time trying the exercises on your own. This would give you an idea of what you truly understand and what you need to work on.
- Revise your notes before class and make an effort to read ahead of each class.
- Seek help before it is too late.