UNIVERSITY OF GHANA

DEPARTMENT OF MATHEMATICS

MATH 223: CALCULUS II (3 credits)

EXERCISE 4 (Indeterminate forms & Hyperbolic functions)

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1. Find the following limits

   - $\lim\limits_{x\to -1} \dfrac{2x^2-2}{x+1}$ • $\lim\limits_{x\to\infty} \dfrac{3x^2-1}{2x^2+1}$ • $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{x}$ • $\lim\limits_{x\to\infty} \dfrac{\ln x}{x}$
   - $\lim\limits_{x\to -\infty} \dfrac{x^2}{e^{-x}}$ • $\lim\limits_{x\to 0} \dfrac{x^3}{x-\tan x}$ • $\lim\limits_{x\to\infty} e^{-x}\sqrt{x}$ • $\lim\limits_{x\to 0^+} x\ln x$
   - $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x$ • $\lim\limits_{x\to 0^+} (\sin x)^x$ • $\lim\limits_{x\to 0^+} x^x$ • $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x}\right)^{\sin x}$
   - $\lim\limits_{x\to 1^+} \left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right)$ • $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{e^x-1}\right)$

2. Determine whether or not the following functions are even or odd

   •$f(x) = -3x^2+4$ •$g(x) = 2x^3-4x$ •$h(x) = 2x^3-3x^2-4x+4$

3. Establish the following hyperbolic identities:

   (a) $\cosh^2 x - \sinh^2 x = 1$

   (b) $1-\tanh^2 x = \text{sech}^2 x$

   (c) $\coth^2 x - 1 = \text{csch}^2 x$

   (d) $\sinh(x\pm y) = \sinh x\cosh y \pm \cosh x \sinh y$

   (e) $\cosh(x\pm y) = \cosh x\cosh y \pm \sinh x \sinh y$

   (f) $\tanh(x\pm y) = \frac{\tanh x x \pm \tanh y}{1\pm \tanh x \tanh y}$

4. Establish the following relations:

   (a) $\sinh^{-1} x = \ln(x+\sqrt{x^2+1}) \qquad \forall x \in (-\infty,\infty)$

   (b) $\cosh^{-1} x = \ln(x+\sqrt{x^2-1}) \qquad \forall x \in [1,\infty)$

   (c) $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \forall x \in (-1,1)$

   (d) $\text{csch}^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \qquad \forall x \in (-\infty,0)\cup(0,\infty)$

   (e) $\text{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \forall x \in (0,1]$

   (f) $\coth^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right) \qquad \forall x \in (-\infty,-1)\cup(1,\infty)$

5. Prove the following
   (a) $\frac{d}{dx}[\sinh x] = \cosh x$
   (b) $\frac{d}{dx}[\cosh x] = \sinh x$
   (c) $\frac{d}{dx}[\tanh x] = \text{sech}^2 x$
   (d) $\frac{d}{dx}[\text{sech}\, x] = -\tanh x\, \text{sech}\, x$
   (e) $\frac{d}{dx}[\coth x] = -\text{csch}^2 x$
   (f) $\frac{d}{dx}[\text{csch}\, x] = -\coth x\, \text{csch}\, x$
   (g) $\frac{d}{dx}[\sinh^{-1} x\,] = \frac{1}{\sqrt{x^2+1}}$
   (h) $\frac{d}{dx}[\cosh^{-1} x] = \frac{1}{\sqrt{x^2-1}}$
   (i) $\frac{d}{dx}[\tanh^{-1} x] = \frac{1}{1-x^2}$
   (j) $\frac{d}{dx}[\text{sech}^{-1} x] = -\frac{1}{x\sqrt{1-x^2}}$
   (k) $\frac{d}{dx}[\text{csch}^{-1} x] = -\frac{1}{|x|\sqrt{x^2+1}}$
   (l) $\frac{d}{dx}[\coth^{-1}] = \frac{1}{1-x^2}$
6. Show that $\lim\limits_{x\to\infty} \tanh \ln x = 1$
7. Simplify the expression $\dfrac{\cosh \ln x + \sinh \ln x}{\cosh \ln x - \sinh \ln x}$
8. Find the following derivatives
   (a) $y = \tanh^{-1}(\sinh x)$
   (b) $f(x) = x\sqrt{1+x^2} + \sinh^{-1} x$
   (c) $g(x) = x\sqrt{x^2-a^2} - a^2 \cosh^{-1}\left(\frac{x}{a}\right)$
   (d) $h(x) = x^{\sinh x}$
9. Prove that $\tanh^{-1} x - \tanh^{-1} y = \tanh^{-1}\left(\dfrac{x-y}{1-xy}\right)$
10. Show that if $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $y > 0$ and $\cos x \cosh y = 1$, then $y = \ln(\sec x + \tan x)$, $\dfrac{dy}{dx} = \sec x$ and $\dfrac{dx}{dy} = \text{sechy}$
11. Prove that $16 \sinh^2 x \cosh^3 x = \cosh 5x + \cosh 3x - 2\cosh x$. Hence or otherwise, deduce that $\displaystyle\int_0^1 \sinh^2 x \cosh^3 x dx = \frac{1}{5}\sin 5 + \frac{1}{3}\sinh 3 - 2\sinh 1$
12. Solve the following:
   (a) $7\sinh x + 3\cosh x = 9$
   (b) $\text{csch}^{-1} x + \ln x = 3$