UNIVERSITY OF GHANA

DEPARTMENT OF MATHEMATICS

MATH 223: CALCULUS II (3 credits)
CHAPTER 5: Hyperbolic Functions

At the end of the chapter, learners are expected to be able to determine:

1. Define hyperbolic functions and their inverses
2. Apply hyperbolic identities to solve problems
3. Find derivatives of hyperbolic functions and their inverses

LESSON HIGHLIGHTS

Definition

Let f be a function defined on an interval I. If $\forall x \in I$,

- $f(-x) = f(x)$, then f is an even function.
- $f(-x) = -f(x)$, then f is an odd function.

Theorem

Any given function can be written as the sum of an even and an odd function.

Basic Propertied of Even and Odd Functions

1. The sum of two even (odd) functions is an even (odd) function.
2. The difference of two even (odd) functions is an even (odd) function.
3. The constant multiple of an even (odd) function is an even (odd) function.
4. The product of two even or two odd functions is an even function.
5. The product of an even and an odd function is an odd function.
6. The quotient of two even or two odd functions is an even function.
7. The quotient of an even and an odd function is an odd function.
8. The derivative of an even (odd) function is an odd (even) function.
9. $\int_{-a}^{a} f_o(x) = 0$, for $a \in \mathbb{R}$, $a < \infty$ and f_o an odd function.

10. $\int_{-a}^{a} f_e(x) = 2\int_0^a f_e(x)$, for $a \in \mathbb{R}$, $a < \infty$ and f_e an even function.

Definitions: Hyperbolic Functions

- $\cosh : \mathbb{R} \to [1, \infty)$ defined by $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$
- $\sinh : \mathbb{R} \to (-\infty, \infty)$ defined by $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$
- $\tanh : \mathbb{R} \to (-1, 1)$ defined by $\tanh x = \frac{\sinh x}{\cosh x}$
- $\coth : \mathbb{R}\backslash 0 \to \mathbb{R}\backslash[-1, 1]$ defined by $\coth x = \frac{\cosh x}{\sinh x}$
- sech $: \mathbb{R} \to (0, 1]$ defined by sech $x = \frac{1}{\cosh x}$
- csch $: \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ defined by $\text{csch}\, x = \frac{1}{\sinh x}$

Hyperbolic Identities

1. $\cosh^2 x - \sinh^2 x = 1$
2. $1 - \tanh^2 x = \text{sech}^2 x$
3. $\coth^2 x - 1 = \text{csch}^2 x$
4. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
5. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
6. $\tanh(x \pm y) = \frac{\tanh x x \pm \tanh y}{1 \pm \tanh x \tanh y}$

Derivatives of Hyperbolic Functions

1. $\frac{d}{dx}[\sinh x] = \cosh x$
2. $\frac{d}{dx}[\cosh x] = \sinh x$
3. $\frac{d}{dx}[\tanh x] = \text{sech}^2 x$
4. $\frac{d}{dx}[\text{sech}\, x] = -\tanh x \,\text{sech}\, x$
5. $\frac{d}{dx}[\coth x] = -\text{csch}^2 x$
6. $\frac{d}{dx}[\text{csch}\, x] = -\coth x \,\text{csch}\, x$

Remark

If u is a function of x, then we apply the chain rule in finding derivatives of hyperbolic functions. For example,

$$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx}$$

Inverse Hyperbolic Functions and Their Domains

	Domain
$y = \sinh^{-1} x \Leftrightarrow x = \sinh y$	$(-\infty, \infty)$
$y = \cosh^{-1} x \Leftrightarrow x = \cosh y$	$[1, \infty)$
$y = \tanh^{-1} x \Leftrightarrow x = \tanh y$	$(-1, 1)$
$y = \operatorname{csch}^{-1} x \Leftrightarrow x = \operatorname{csch} y$	$(-\infty, 0) \cup (0, \infty)$
$y = \operatorname{sech}^{-1} x \Leftrightarrow x = \operatorname{sech} y$	$(0, 1]$
$y = \coth^{-1} x \Leftrightarrow x = \coth y$	$(-\infty, -1) \cup (1, \infty)$

Inverse Hyperbolic Functions as Logarithmic Functions

$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$	defined on $(-\infty, \infty)$
$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$	defined on $[1, \infty)$
$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$	defined on $(-1, 1)$
$\operatorname{csch}^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right)$	defined on $(-\infty, 0) \cup (0, \infty)$
$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	defined on $(0, 1]$
$\coth^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	defined on $(-\infty, -1) \cup (1, \infty)$

Derivatives of Inverse Hyperbolic Functions

1. $\frac{d}{dx}[\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$
2. $\frac{d}{dx}[\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$
3. $\frac{d}{dx}[\tanh^{-1} x = \frac{1}{1-x^2}$
4. $\frac{d}{dx}[\operatorname{sech}^{-1} x] = -\frac{1}{x\sqrt{1-x^2}}$
5. $\frac{d}{dx}[\operatorname{csch}^{-1} x] = -\frac{1}{\lvert x\rvert\sqrt{x^2+1}}$
6. $\frac{d}{dx}[\coth^{-1} = \frac{1}{1-x^2}$

IMPORTANT THINGS TO NOTE

- Spend time trying the exercises on your own. This would give you an idea of what you truly understand and what you need to work on.
- Revise your notes before class and make an effort to read ahead of each class.
- Seek help before it is too late.