Further Transcendental Functions

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1 Introduction

What is the transcendental function?

These are the transcendental functions :
$\sin x$, $\cos x$, $\tan x$, $\sec x$, $\operatorname{cosec} x$, $\cot x$,(Trigonometric)
$\log x$ or $\ln x$, e^x,(Logarithmic and Exponential)
$\sinh x$, $\cosh x$, $\tanh x$, $\operatorname{sech} x$, $\operatorname{cosech} x$,(Hyperbolic)

AND the inverse function of the above trigonometric and hyperbolic functions :
$\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$, $\operatorname{cosec}^{-1} x$, $\cot^{-1} x$,
$\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$, $\operatorname{sech}^{-1} x$, $\operatorname{cosech}^{-1} x$ $\coth^{-1} x$.

We will first learn about the properties of these functions, and then using the properties we will solve the equations involving the transcendental functions, and also prove some identities.

Later on in the next topic, we will learn how to find the derivative of these transcendental functions.

Most of the calculation in this topic will involve transforming the hyperbolic function, into exponential function,(e^x) or logarithmic function ($\ln x$), and then evaluate the resulting equation.

Hence, it is useful to familiarise with the properties of logarithmic and exponential function.

Some useful formula :

$$
\begin{aligned}
\ln x &= \log_e x \qquad \text{log with base } e\text{, with } e \approx 2.718\text{, also known as natural logarithm} \\
e^{\ln a} &= a \\
e^{-\ln a} &= e^{\ln \frac{1}{a}} = \frac{1}{a} \\
(e^x)^2 &= e^x \times e^x = e^{x+x} = e^{2x} \\
e^{-x} &= \frac{1}{e^x} \\
e^x &= a \Rightarrow \ln a = x.
\end{aligned}
$$

Another formula that will come in handy is the roots of quadratic formula

$$Ax^2 + Bx + C = 0, \Rightarrow x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

or learn how to use scientific calculator on how to find the factors/roots of the quadratic equation.

2 Hyperbolic Function

Hyperbolic function can be written in terms of exponential function

$$
\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},
\end{aligned}
$$

in which we can also find the $\operatorname{sech} x$, $\operatorname{cosech} x$, and coth by using the definition.

We can now prove the following identities

$$\cosh^2 x - \sinh^2 x = 1 \tag{2.1}$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x \tag{2.2}$$

$$\coth^2 x - 1 = \operatorname{cosech}^2 x \tag{2.3}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \tag{2.4}$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \tag{2.5}$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \tag{2.6}$$

The following are the graphs for the hyperbolic functions.

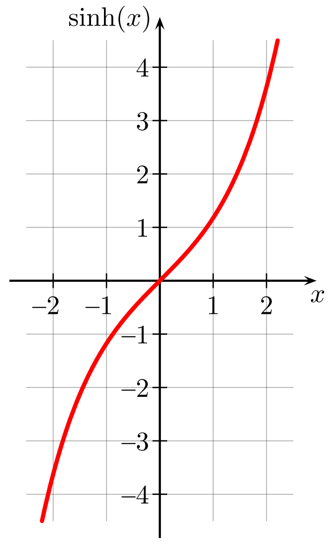


Figure 1: Graph of hyperbolic sine. $D_f = \mathbb{R} \quad \text{and} \quad R_f = \mathbb{R}$.

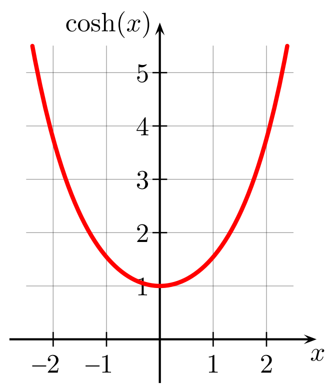


Figure 2: Graph of hyperbolic cosine. $D_f = \mathbb{R} \quad \text{and} \quad R_f = \{y \,|\, y > 1\}$.

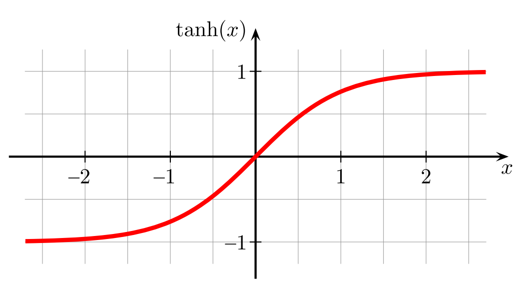


Figure 3: Graph of hyperbolic tangent. $D_f = \mathbb{R} \quad \text{and} \quad R_f = \{y \,|\, -1 < y < 1\}$.

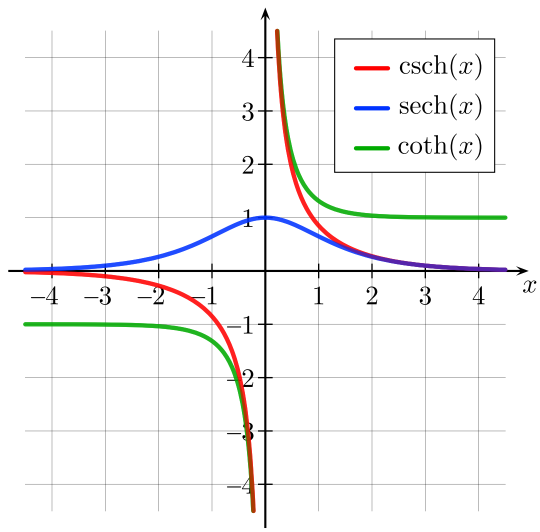


Figure 4: Graph of hyperbolic secant cosecant and cotangent. For $\operatorname{sech} x$, $D_f = \mathbb{R}$ and $R_f = \{y \,|\, 0 < y \leq 1\}$. For $\operatorname{cosech} x$, $D_f = \{x \,|\, x \neq 0, x = \mathbb{R}\}$ and $R_f = \{y \,|\, y \neq 0, y = \mathbb{R}\}$. For $\coth x$, $D_f = \{x \,|\, x \neq 0, x = \mathbb{R}\}$ and $R_f = \{y \,|\, -\infty < y < \infty,\ \text{or } 1 < y < \infty\}$.

Now using the definition of hyperbolic functions, we can use it to estimate these transcendental function.

Example 1 : Estimate each of the following to four decimal places

a) $\sinh 4$

b) $\cosh(\ln 3)$

c) $\tanh(-1)$

d) $\operatorname{sech}(2)$

For this chapter, we are also required to prove the equations involving hyperbolic functions.

In mathematics, to prove and equation, the method can be direct, but it is still need to be generalize, meaning that it must be true for all result. Note that in order to prove an equation, we can do it from left hand side to obtain the right hand side, or the other way around. But it is better to go as shown in the question accordingly, or start from the side that has more terms and arithmetic operations. If you find it really hard to prove either way, you can show that the resulting equation of the left, is equal to the resulting equation on the right after you applied the definition on both sides, ie show that by using definition on both sides that they are the same. Depends on the question, you can use definition to prove the equation. Or if it is not stated particularly, you can use the known identities from the formula.

Example 2 : Prove that

a) $\cosh x + \sinh x = e^x$

b) $\sinh 2x = 2 \sinh x \cosh x$

c) $1 + 2\sinh^2 x = \cosh 2x$

d) $\cosh^2 x - \sinh^2 x = 1$

e) Prove $\cosh 3x = 4\cosh^3 x - 3\cosh x$ using identities ONLY.

f) Prove $\cosh 3x = 4\cosh^3 x - 3\cosh x$ using definition.

We are usually given an equation involving transcendental function with x as the independent variable, and we are required to solve it. In Algebraic terms, "solving equations" means we are looking for the value of x that satisfies the equation. It is also known as finding the roots since we set the right hand side of the equations to be zero.

Example 3: Solve the following equation (Find the roots of the following equation) giving your answer in terms of natural logarithm

1. $\cosh x = \dfrac{17}{8}$
2. $\sinh x = \dfrac{3}{4}$
3. $\sinh x = \cosh x - 4$
4. $2\cosh x = \sinh x + 4$
5. $3\tanh 2x = 2$
6. $4\cosh x - \sinh x = 8$
7. $11\cosh x - 5\sinh x = 10$
8. $2\cosh 2x - \sinh 2x = 2$
9. $\cosh x = 5 - \sinh x$

10. $2\cosh 2x + 10\sinh 2x = 5$

11. $4\cosh x + \sinh x = 4$

12. $3\sinh x - \cosh x = 1$

13. Find the possible values of $\sinh x$ for which

$$12\cosh^2 x + 7\sinh x = 24.$$

Hint : use identity $\cosh^2 x - \sinh^2 x = 1$. Hence find the possible values of x in natural logarithms.

3 Inverse Function

If $f : X \Longrightarrow Y$ is a one-to-one and onto functions with the domain X and the range Y, then there exists an inverse function

$$f^{-1} : Y \longrightarrow X,$$

where the domain is Y and the range is X such that

$$y = f(x) \Longleftrightarrow x = f^{-1}(y).$$

Also, this implies that, $f^{-1}(f(x)) = x$ for all values of x in the domain f.

Now we want to find the inverse function for trigonometry $\sin x$, $\cos x$, $\tan x$ and the hyperbolic $\sinh x$,$\cosh x$, and $\tanh x$. However, these functions are not one-to-one. But we can restrict the domain so that it will be one-to-one.

As it turns out, the graph of the inverse function f^{-1} is a reflection of the graph f at $y = x$.

The term $\sin^{-1} x$ is sometimes written as $a\sin x$ or $\arcsin x$. This applies to all other trigonometrics and hyperbolic functions.

It is important to note that

$$\frac{1}{\sin^{-1} x} \neq \mathrm{cosec}^{-1}\, x, \quad \text{or} \quad \sin^{-1} x \neq (\sin x)^{-1}$$

and for other trigonometric functions.

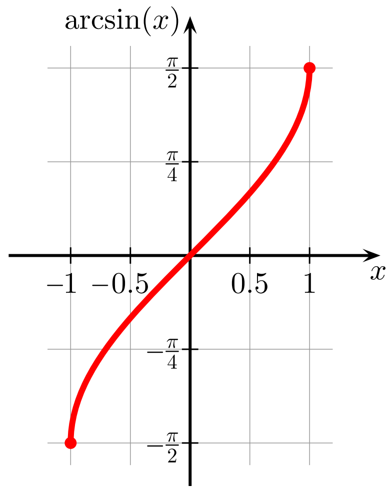
The above statement also applies for inverse hyperbolic functions.

4 Inverse Sine Function

The inverse sine function is defined as

$$y = \sin^{-1} x \iff \sin y = x$$

where the domain is $-1 \leq x \leq 1$ and the range is $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$.

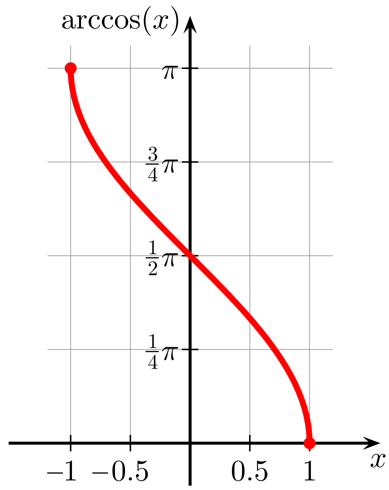


5 Inverse Cosine Function

The inverse cosine function is defined as

$$y = \cos^{-1} x \iff \cos y = x$$

where the domain is $-1 \leq x \leq 1$ and the range is $0 \leq y \leq \pi$.

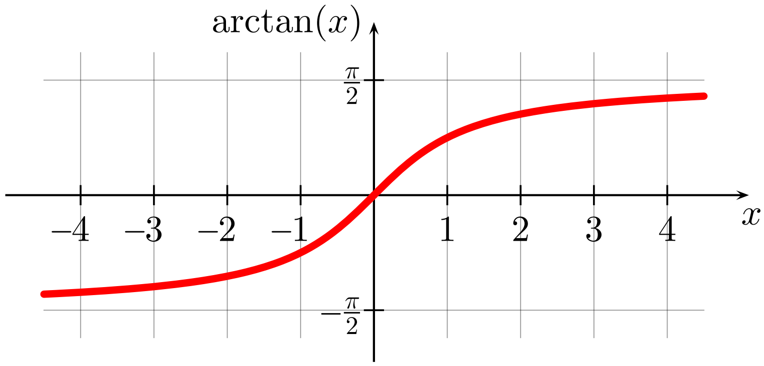


6 Inverse Tangent Function

The inverse tangent function is defined as

$$y = \tan^{\text{-1}} x \iff \tan y = x$$

where the domain is $-\infty < x < \infty$ and the range is $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$.



7 Inverse Secant Function

The inverse secant function is defined as

$$y = \sec^{\text{-1}} x \iff \sec y = x.$$

For the domain and range, we can use either $1 \leq x < \infty$, $0 \leq y < \frac{\pi}{2}$, OR $-\infty < x \leq -1$, $\frac{3\pi}{2} < y \leq \pi$.

8 Inverse Cosecant Function

The inverse secant function is defined as

$$y = \operatorname{cosec} x \iff \operatorname{cosec} y = x.$$

For the domain and range, we can use either $1 \leq x < \infty$, $0 \leq y < \frac{\pi}{2}$, OR $-\infty < x \leq -1$, $-\pi < y \leq -\dfrac{\pi}{2}$.

9 Inverse Cotangent Function

The inverse secant function is defined as

$$y = \cot^{\text{-1}} x \iff \cot y = x.$$

The domain is $-\infty < x < \infty$ and range is $0 < y < \pi$.

10 Identities Involving Trigonometric Functions

From the inverse function, we have $f^{-1}(f(x)) = x$, hence we have

$$\sin(\sin^{-1} x) = x$$
$$\cos(\cos^{-1} x) = x$$
$$\tan(\tan^{-1} x) = x$$

Other important identities

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$$
$$\sec^{-1} x + \mathrm{cosec}^{-1} = \frac{\pi}{2}$$
$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

From the formula of compound angles,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$
$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$$

Other three indentities which can be obtained form the definition

$$\sin^{-1}(-x) = -\sin^{-1} x$$
$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$
$$\tan^{-1}(-x) = -\tan^{-1} x$$

Example 4
Prove the following identities

1. $\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$.

2. $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$.

3. $\sec^{-1} x + \mathrm{cosec}^{-1} x = \frac{\pi}{2}$.

Find x (Solve the equation or find the roots)

4. $\sin^{-1} 2x + \cos^{-1} x = \frac{\pi}{6}$.

5. $\sin^{-1} 2x + \sin^{-1} x = \dfrac{\pi}{2}$.

6. $\cos^{-1} x + \tan^{-1} x = \dfrac{\pi}{2}$.

7. $\sin^{-1} x + \tan^{-1} x = 0$.

Inverse Hyperbolic Function

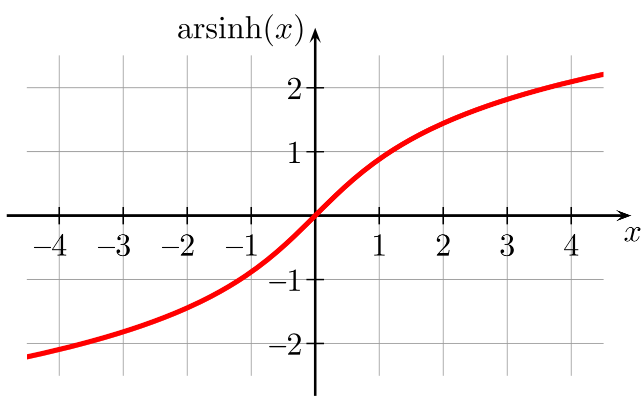
The inverse of hyperbolic function is defined in the same way as the trigonometric function. Additionally, the inverse of hyperbolic function can also be expressed in terms of logartihmic functions by using the definition of hyperbolic functions.

11 Inverse Hyperbolic Sine Function

The inverse hyperbolic sine function is defined as

$$y = \sinh^{-1} x \Longleftrightarrow \sinh y = x$$

where the domain is $\infty < x < \infty$ and the range is $\infty < y < \infty$.

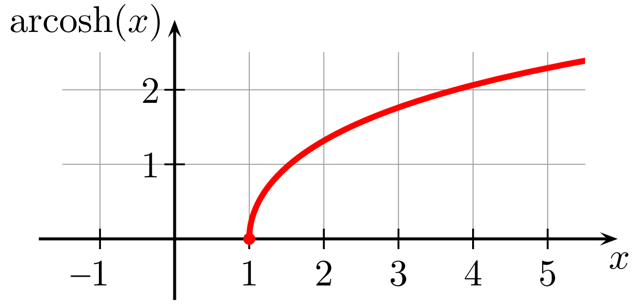


12 Inverse Hyperbolic Cosine Function

The inverse hyperbolic cosine function is defined as

$$y = \cosh^{-1} x \Longleftrightarrow \cosh y = x$$

where the domain is $x \geq 1$ and the range is $y \geq 0$.

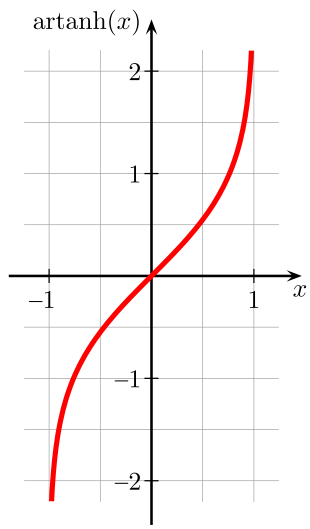


13 Inverse Hyperbolic Tangent Function

The inverse hyperbolic tangent function is defined as

$$y = \cosh^{-1} x \iff \cosh y = x$$

where the domain is $-1 < x < 1$ and the range is $\infty < y < \infty$.



Similarly, the other three inverse hyperbolic functions can be defined as

a) $\operatorname{sech}^{-1} x = \cosh^{-1} \dfrac{1}{x}$.

b) $\operatorname{cosech}^{-1} x = \sinh^{-1} \dfrac{1}{x}$.

b) $\coth^{-1} x = \tanh^{-1} \dfrac{1}{x}$.

Example 5

a) Prove $\operatorname{sech}^{-1} x = \cosh^{-1} \dfrac{1}{x}$.

b) Prove $\operatorname{cosech}^{-1} x = \sinh^{-1} \dfrac{1}{x}$.

b) Prove $\coth^{-1} x = \tanh^{-1} \dfrac{1}{x}$.

14 Inverse Hyperbolic Function in Logarithmic Form

By using the definition of hyperbolic function, we can transform the inverse hyperbolic into logarithmic form such that

a) $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$.

b) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$.

c) $\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)$.

d) $\coth^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{x+1}{x-1}\right)$.

e) $\operatorname{sech}^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$.

f) $\operatorname{cosech}^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{|x|}\right)$.

Example 6 : Prove the following identities

a) $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$.

b) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$.

c) $\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)$.

Example 7: Find the value of (using logarithmic formula)

a) $\sinh^{-1}(1.5)$.

b) $\cosh^{-1}(1.5)$.

Example 8: Solve the following equations

a) $\sinh^{-1} x = 2\cosh^{-1} x$.

b) $\tanh^{-1} x = \ln(2x+1)$.

c) $\cosh^{-1} 5x = \sinh^{-1} 4x$.

d) $2\tanh^{-1} x = \tanh^{-1}\left(\frac{6x}{5}\right)$.

e) $\sinh^{-1} x = \ln(2-x)$.

15 Exercise

1. Solve the following equations

   a) $\sinh x + 2 = \cosh x$.

   b) $\tanh x = e^{-x}$.

2. Express the following in logarithmic form

   a) $\sinh^{-1}(-2x)$.

   b) $\operatorname{cosech}^{-1}(-2x)$.

3. Given that $2\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{2}$. Show that

$$x = \frac{1}{2}(\sqrt{3} - 1).$$

4. Show that

   a) $\cosh 3x = 4\cosh^3 x - 3\cosh x$.

   b) $\dfrac{1+\tanh^2 x}{1-\tanh^2 x} = \cosh 2x$.

5. Solve $3\sinh x - \cosh x = 1$. and give the answer in logarithmic form.

6. Solve $\cosh^{-1} 5x = \sinh^{-1} 4x$.

7. Find the positive root of the equation

$$\sin^{-1} x + \cos^{-1} 4x = \frac{\pi}{6}.$$

16 Conclusion

In this chapter, you should be able to :

1. Prove the identities of hyperbolic function using definiton and/or formulas.
2. Solve the equations involving hyperbolic function.
3. Prove the identities of inverse trigonometric functions.
4. Solve the equations involving inverse trigonometric functions.
5. Prove the identities of the inverse hyperbolic functions using definition.
6. Solve the equations involving inverse hyperbolic functions.