It Can Be Shown

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Contents

Part I

Probability Distributions

The Bernoulli Distribution

A random variable is said to have a Bernoulli Distribution with parameter p if its probability mass function is:

$$p(x) = \begin{cases} p^x(1-p)^{1-x}, & x = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Where p is the probability of a sucess.

1 Cumulative Distribution Function

$$P(x) = \begin{cases} 0 & x < 0 \\ 1-p & x = 0 \\ 1 & 1 \leq x \end{cases}$$

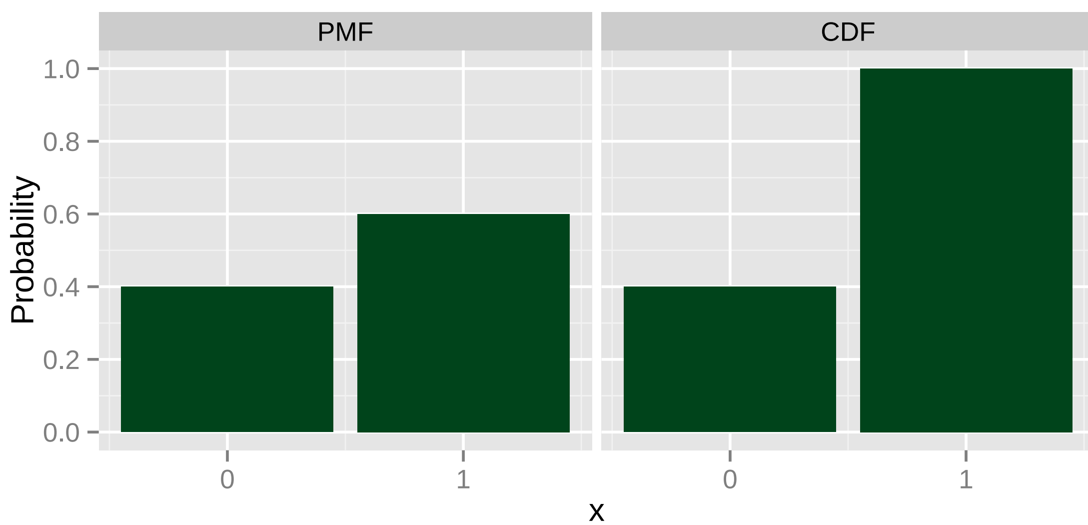

Figure .1: The graphs on the left and right show a Binomial Probability Distribution and Cumulative Distribution Function, respectively, with $p = .4$. Note that this is identical to a Binomial Distribution with parameters $n = 1$ and $p = .4$.

2 Expected Values

$$\begin{aligned}
E(X) &= \sum_{i=0}^{1} x \cdot p(x) \\
&= \sum_{i=0}^{1} x \cdot p^x (1-p)^{1-x} \\
&= 0 \cdot p^0 (1-p)^{1-0} + 1 \cdot p^1 (1-p)^{1-1} \\
&= 0 + p(1-p)^0 \\
&= p
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= \sum_{i=0}^{1} x^2 \cdot p(x) \\
&= \sum_{i=0}^{1} x^2 \cdot p^x (1-p)^{1-x} \\
&= \sum_{i=0}^{1} 0^2 \cdot p^0 (1-p)^{1-0} + 1^2 \cdot p^1 (1-p)^{1-1} \\
&= 0 \cdot 1 \cdot 1 + 1 \cdot p \cdot 1 \\
&= 0 + p \\
&= p
\end{aligned}$$

$\mu = E(X) = p$

$\sigma^2 = E(X^2) - E(X)^2 = p - p^2 = p(1-p)$

---

3 Moment Generating Function

$$\begin{aligned}
M_X(t) = E(e^{tX}) &= \sum_{i=0}^{1} e^{tx} p(x) = \sum_{i=0}^{1} e^{tx} p^x (1-p)^{1-x} \\
&= e^{t0} p^0 (1-p)^{1-0} + e^t p^t (1-p)^{1-1} = (1-p) + e^t p = pe^t + (1-p)
\end{aligned}$$

$$\begin{aligned}
M_X^{(1)}(t) &= pe^t \\
M_X^{(2)}(t) &= pe^t
\end{aligned}$$

$$E(X) = M_X^{(1)}(0) = pe^0 = p \cdot 1 = p$$

$$E(X^2) = M_X^{(2)}(0) = pe^0 = p$$

$$\mu = E(X) = p$$

$$\sigma^2 = E(X^2) - E(X)^2 = p - p^2 = p(1-p)$$

4 Theorems for the Bernoulli Distribution

4.1 Validity of the Distribution

$$\sum_{x=0}^{1} p^x (1-p)^{1-x} = 1$$

Proof:

$$\sum_{x=0}^{1} p^x (1-p)^{1-x} = p^0 (1-p)^1 + p^1 (1-p)^0 = (1-p) + p = 1$$

■

4.2 Sum of Bernoulli Random Variables

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables from a Bernoulli distribution with parameter p. Let $Y = \sum_{i=1}^{n} X_i$.
Then $Y \sim \text{Binomial}(n, p)$

Proof:

$$\begin{aligned} M_Y(t) = E(e^{tY}) = E(e^{tX_1} e^{tX_2} \cdots e^{tX_n}) = E(e^{tX_1}) E(e^{tX_2}) \cdots E(e^{tX_n}) \\ = (pe^t + (1-p))(pe^t + (1-p)) \cdots (pe^t + (1-p)) = (\end{aligned}$$

Which is the mgf of a Binomial random variable with parameters n and p.
Thus, $Y \sim \text{Binomial}(n, p)$. ■

The Binomial Distribution

A random variable is said to follow a Binomial distribution with parameters n and p if its probability mass function is:\

$$p(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, 2, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

Where n is the number of trials performed and p is the probability of a success on each individual trial.

5 Cumulative Distribution Function

$$P(x) = \begin{cases} 0 & x < 0 \\ \sum_{i=0}^{x} \binom{n}{i} p^i (1-p)^{n-i} & 0 \le x = 0, 1, 2, \dots, n \\ 1 & n \le x \end{cases}$$

A recursive form of the cdf can be derived and has some usefulness in computer applications. With it, one need only initiate the first value and additional cumulative probabilities can be calculated. It is derived as follows:

$$\begin{aligned} F(x+1) &= \binom{n}{x+1} p^{x+1} (1-p)^{n-(x+1)} \\ &= \frac{n!}{(x+1)!(n-(x+1))!} p^{x+1} (1-p)^{n-(x+1)} \\ &= \frac{n!}{(x+1)!(n-x-1)!} p^{x+1} (1-p)^{n-x-1} \\ &= \frac{(n-x)n!}{(x+1)x!(n-x)(n-x-1)!} p \cdot p^x \frac{(1-p)^{n-x}}{(1-p)} \\ &= \frac{(n-x)n!}{(x+1)x!(n-x)!} \cdot \frac{p}{1-p} p^x (1-p)^{n-x} \\ &= \frac{p}{1-p} \cdot \frac{n-x}{x+1} \cdot \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\ &= \frac{p}{1-p} \cdot \frac{n-x}{x+1} \cdot \binom{n}{x} p^x (1-p)^{n-x} \\ &= \frac{p}{1-p} \cdot \frac{n-x}{x+1} \cdot F(x) \end{aligned}$$

6 Expected Values

Let X be a binomial random variable with parameters n and p. The expected value of X is:

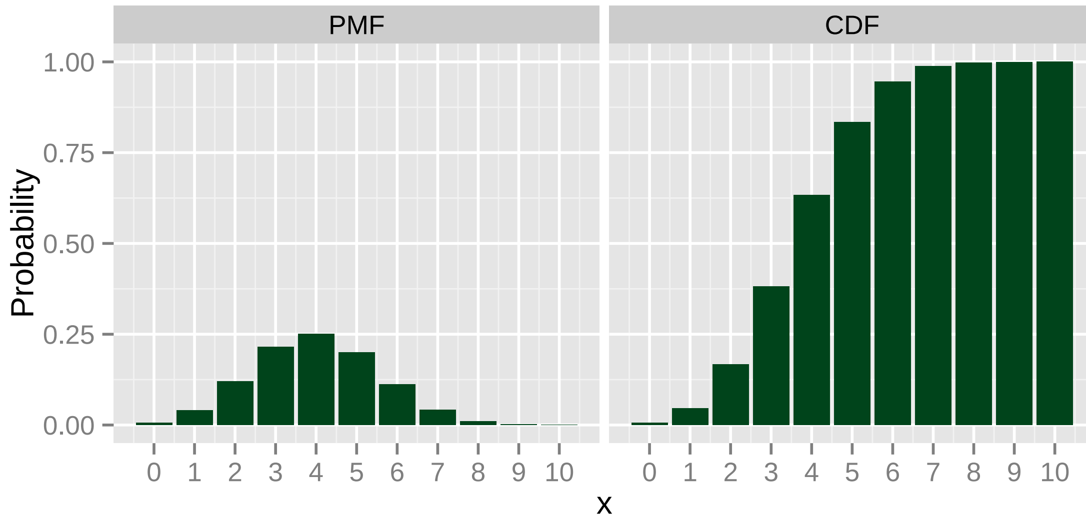

Figure .1: plot of chunk unnamed-chunk-8

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{n} x \cdot p(x) \\
&= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}
\end{aligned}
$$

For convenience, let $q = (1-p)$

$$
\begin{aligned}
&= \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} \\
&= \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} \\
&= 0 \cdot \binom{n}{0} p^0 q^n + 1 \cdot \binom{n}{1} p^1 q^{n-1} + \cdots + n \binom{n}{n} p^n q^{n-n} \\
&= 0 + 1 \binom{n}{1} p^1 q^{n-1} + 2 \binom{n}{2} p^2 q^{n-2} + \cdots + n \binom{n}{n} p^n q^{n-n} \\
&= np^1 q^{n-1} + n(n-1) p^2 q^{n-2} + \cdots + n(n-1) p^{n-1} q^{n-(n-1)} + np^n \\
&= np[q^{n-1} + (n-1) p q^{n-2} + \cdots + p^{n-1}] \\
&= np\Big[\binom{n-1}{0} p^0 q^{n-1} + \binom{n-1}{1} p^1 q^{(n-1)-1} + \cdots + \binom{n-1}{n-1} p^{n-1} q^{(n-} \\
&= np(\sum_{x=0}^{n-1} \binom{n-1}{x} p^x q^{(n-1)-x})
\end{aligned}
$$

By the Binomial Theorem, $\sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x} = (a+b)^n$

$$
= np(p+q)^{n-1}
$$

Resubstituting $(1-p)$ for q gives us

$$
\begin{aligned}
&= np(p + (1-p))^{n-1} \\
&= np(p + 1 - p)^{n-1} \\
&= np(1)^{n-1} \\
&= np(1) \\
&= np
\end{aligned}
$$

The Expected Value of X^2 is:

$$\begin{aligned}
E(X^2) &= \sum_{x=0}^{n} x^2 p(x) \\
&= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}
\end{aligned}$$

For convenience, let $q = (1-p)$

$$\begin{aligned}
&= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x} \\
&= 0^2 \frac{n!}{0!(n-0)!} p^0 q^n + 1^2 \frac{n!}{1!(n-1)!} p^1 q^{n-1} + \cdots + n^2 \frac{n!}{n!(n-n)!} p^n q^{n-n} \\
&= 0 + 1 \frac{n!}{(n-1)!} pq^{n-1} + 2 \frac{n!}{1 \cdot (n-2)!} p^2 q^{n-2} + \cdots + n \frac{n!}{(n-1)!(n-n)!} p^n \\
&= np\Big[1 \frac{(n-1)!}{(n-1)!} p^0 q^{n-1} + 2 \frac{(n-1)!}{1(n-2)!} p^2 q^{n-2} + \cdots + n \frac{(n-1)!}{(n-1)!(n-n)!} p^{n-1}\Big] \\
&= np\Big[1 \frac{(n-1)!}{(1-1)!((n-1)-(-1-1))!} p^{1-1} q^{n-1} + \cdots + n \frac{(n-1)!}{(n-1)!((n-1)-(n-1))!} p^{n-1} q \\
&= np \sum_{x=1}^{n} x \binom{n-1}{x-1} p^{x-1} 1^{(n-1)-(x-1)}
\end{aligned}$$

Let $y = x - 1$ and $n = m + 1$

$\Rightarrow x = y + 1$ and $m = n - 1$

$$\begin{aligned}
&= \sum_{y=0}^{m} (y+1) \binom{m}{y} p^y q^{m-y} \\
&= np\Big[\sum_{y=0}^{m} y \binom{m}{y} p^y q^{m-y} + \binom{m}{y} p^y q^{m-y}\Big] \\
&= np\Big[\sum_{y=0}^{m} y \binom{m}{y} p^y q^{m-y} + \sum_{y=0}^{m} \binom{m}{y} p^y q^{m-y}\Big]
\end{aligned}$$

$\sum_{y=0}^{m} y \binom{m}{y} p^y q^{m-y}$ is of the form of the expected value of Y, and $E(Y) = mp = (n-1)p$

$\sum_{y=0}^{m} \binom{m}{y} p^y q^{m-y}$ is the sum of all probabilities over the domain of Y, which is 1.

$$\begin{aligned}
&= np(mp+1) \\
&= np[(n-1)p+1] \\
&= np(np-p+1) \\
&= n^2p^2 - np^2 + np
\end{aligned}$$

The mean of X can be calculated as

$$\mu = E(X) = np$$

And the variance of X can be calculated by

$$\begin{aligned}
\sigma^2 &= E(X^2) - E(X)^2 \\
&= n^2p^2 - np^2 + np - n^2p^2 \\
&= -np^2 + np \\
&= np(-p-1) \\
&= np(1-p)
\end{aligned}$$

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7 Moment Generating Function

$$\begin{aligned}
M_X(t) = E(e^{tX}) &= \sum_{x=0}^{n} e^{tx} p(x) \\
&= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} \binom{n}{x} e^{tx} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} \binom{n}{x} (pe^{tx})^x (1-p)^{n-x}
\end{aligned}$$

By Binomial Theorem REF

$$\sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x} = (a+b)^n$$

$$= [(1-p) + pe^t]^n$$

$$M_X^{(1)}(t) = n[(1-p) + pe^t]^{n-1} pe^t$$

$$\begin{aligned}
M_X^{(2)}(t) &= n[(1-p) + pe^t]^{n-1} pe^t + n(n-1)[(1-p) + pe^t]^{n-2} (pe^t)^2 \\
&= npe^t[(1-p) + pe^t]^{n-1} + n(n-1)pe^{2t}[(1-p) + pe^t]^{n-2}
\end{aligned}$$

$$\begin{aligned}
E(X) &= M_X^{(1)}(0) \\
&= n[(1-p)+pe^0]^{n-1}pe^0 \\
&= n[1-p+p^{n-1}p \\
&= n(1)^{n-1}p = np
\end{aligned}$$

$$\begin{aligned}
E(X^2) &= M_X^{(2)}(0) = npe^0[(1-p)+pe^0]^{n-1} + n(n-2)pe^{2\cdot 0}[(1-p)+pe^0]^{n-2} \\
&= np(1-p+p)^{n-2} + n(n-1)p^2(1-p+p^{n-2} \\
&= np(1)^{n-1} + n(n-1)p^2(1)^{n-2} = np + n(n-1)p^2 = np + (n^2-n)p^2 \\
&= np + n^2 + n^2p^2 - np^2
\end{aligned}$$

$$\mu = E(X) = np$$

$$\begin{aligned}
\sigma^2 &= E(X^2) - E(X)^2 \\
&= np + n^2p^2 - np^2 - n^2p^2 \\
&= np - np^2 \\
&= np(1-p)
\end{aligned}$$

8 Maximum Likelihood Estimator

Since n is fixed in each Binomial experiment, and must therefore be given, it is unnecessary to develop an estimator for n. The mean and variance can both be estimated from the single parameter p.

Let X be a Binomial random variable with parameter p and n outcomes $(x_1, x_2, \ldots, x_n)$. Let $x_i = 0$ for a failure and $x_i = 1$ for a success. In other words, X is the sum of n Bernoulli trials with equal probability of success and $X = \sum_{i=1}^{n} x_i$.

8.1 Likelihood Function

$$\begin{aligned}
L(\theta) &= L(x_1, x_2, \ldots, x_n|\theta) \\
&= P(x_1|\theta)P(x_2|\theta)\cdots P(x_n|\theta) \\
&= [\theta^{x_1}(1-\theta)^{1-x_1}][\theta^{x_2}(1-\theta)^{1-x_2}]\cdots[\theta^{x_n}(1-\theta)^{1-x_n}] \\
&= \exp_\theta\left\{\sum_{i=1}^{n} x_i\right\}\exp_{(1-\theta)}\left\{n - \sum_{i=1}^{n} x_i\right\} \\
&= \theta^X(1-\theta)^{n-X}
\end{aligned}$$

8.2 Log-likelihood Function

$$\begin{aligned}\ell(\theta) &= \ln L(\theta)\\ &= \ln\left(\theta^X(1-\theta)^{n-X}\right)\\ &= X\ln(\theta) + (n-X)\ln(1-\theta)\end{aligned}$$

8.3 MLE for p

$$\begin{aligned}\frac{d\ell(p)}{dp} &= \frac{X}{p} - \frac{n-X}{1-p}\\ 0 &= \frac{X}{p} - \frac{n-X}{1-p}\\ \frac{X}{p} &= \frac{n-X}{1-p}\\ (1-p)X &= p(n-X)\\ X - pX &= np - pX\\ X &= np\\ \frac{X}{n} &= p\end{aligned}$$

So $\hat{p} = \frac{X}{n} = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$ is the maximum likelihood estimator for p.

9 Theorems for the Binomial Distribution

9.1 Validity of the Distribution

$$\sum_{x=0}^{n}\binom{n}{x}p^x(1-p)^{n-x} = 1$$

Proof:

$$\sum_{x=0}^{n}\binom{n}{x}p^x(1-p)^{n-x} =$$

$$\sum_{x=0}^{n}\binom{n}{x}a^xb^{n-x}(a+b)^n.$$

See Binomial Theorem REF.

$$\begin{aligned}&= big(p+(1-p)\big)^n\\ &= (1)^n\\ &= 1\end{aligned}$$

■

9.2 Sum of Binomial Random Variables

Let $X_1, X_2, \ldots, X_k$ be independent random variables where X_i comes from a Binomial distribution with parameters n_i and p. That is $X_i \sim (n_i, p)$.\ Let $Y = \sum\limits_{i=1} kX_i$. Then $Y \sim$Binomial$(\sum\limits_{i=1}^{k} n_i, p)$.

Proof:

$$\begin{aligned}
M_Y(t) &= E(e^{tY}) \\
&= E(e^{t(x_1+X_2+\cdots+X_k)} \\
&= E(e^{tX_1}e^{tX_2}\cdots e^{tX_k}) \\
&= E(e^{tX_1})E(e^{tX_2})\cdots E(e^{tX_k}) \\
&= \prod_{i=1}^{k}[(1-p)+pe^t]^{n_i} \\
&= [(1-p)+pe^t]^{\sum\limits_{i=1}^{k} n_i}
\end{aligned}$$

\ \ Which is the mgf of a Binomial random variable with parameters $\sum\limits_{i=1}^{k} n_i$ and p.

Thus $Y \sim$Binomial$(\sum\limits_{i=1}^{k} n_i, p)$.■

9.3 Sum of Bernoulli Random Variables

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables from a Bernoulli distribution with parameter p. Let $Y = \sum\limits_{i=1}^{n} X_i$.

Then $Y \sim$Binomial(n, p)

Proof:

$$\begin{aligned}
M_Y(t) &= E(e^{tY}) \\
&= E(e^{tX_1}e^{tX_2}\cdots e^{tX_n}) \\
&= E(e^{tX_1})E(e^{tX_2})\cdots E(e^{tX_n}) \\
&= (pe^t+(1-p))(pe^t+(1-p))\cdots(pe^t+(1-p)) \\
&= (pe^t+(1-p))^n
\end{aligned}$$

Which is the mgf of a Binomial random variable with parameters n and p. Thus, $Y \sim$ Binomial(n, p). ■

The Chi-Square Distribution

A random variable X is said to have a Chi-Square Distribution with parameter ν if its probability distribution function is

$$f(x) = \begin{cases} \frac{x^{\frac{\nu}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} & 0 < x,\ 0 < \nu \\ 0 & otherwise \end{cases}$$

ν is commonly referred to as the *degrees of freedom*.

10 Cumulative Distribution Function

The cumulative distribution function for the Chi-Square Distribution cannot be written in closed form. It's integral form is expressed as

$$F(x) = \begin{cases} \displaystyle\int_0^x \frac{t^{\frac{\nu}{2}-1}e^{-\frac{t}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}dt & 0 < x,\ 0 < \nu \\ \\ 0 & otherwise \end{cases}$$

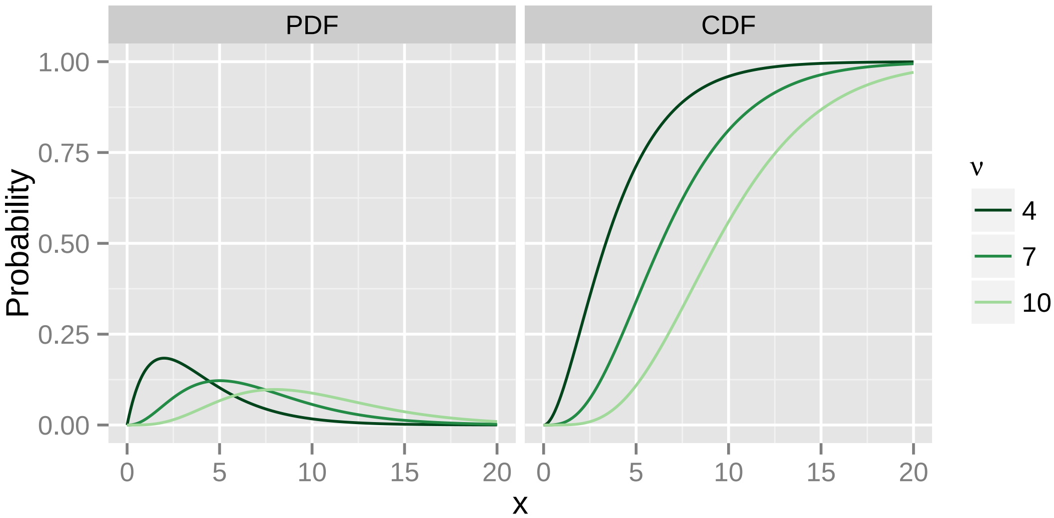

Figure .1: The graphs on the left and right depict the Chi-Square probability distribution and cumulative distribution functions, respectively, for $\nu = 4, 7, 10$. As ν gets larger, the distribution becomes flatter with thicker tails.

11 Expected Values

$$
\begin{aligned}
E(X) &= \int_0^\infty x \frac{x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} dx \\
&= \frac{1}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} \int_0^\infty x \cdot x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}} dx \\
&= \frac{1}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} \int_0^\infty x^{\frac{\nu}{2}} e^{-\frac{x}{2}} dx
\end{aligned}
$$

$$\int_0^\infty x^{\alpha-1} e^{-\frac{x}{\beta}} dx = \beta^\alpha \Gamma(\alpha)$$

$$
\begin{aligned}
&= \frac{1}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} \left[\Gamma\left(\frac{\nu}{2} + 1\right) 2^{\frac{\nu}{2}+1} \right] \\
&= \frac{\Gamma(\frac{\nu}{2} + 1) 2^{\frac{\nu}{2}+1}}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} \\
&= \frac{\frac{\nu}{2} \Gamma(\frac{\nu}{2}) 2^{\frac{\nu}{2}+1}}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} \\
&= \frac{2\nu}{2} \\
&= \nu
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \int_0^\infty x^2 \frac{x^{\frac{\nu}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}dx \\
&= \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}\int_0^\infty x^2 \cdot x^{\frac{\nu}{2}-1}e^{-\frac{x}{2}}dx \\
&= \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}\int_0^\infty x^{\frac{\nu}{2}+1}e^{-\frac{x}{2}}dx
\end{aligned}
$$

$$\int_0^\infty x^{\alpha-1}e^{-\frac{x}{\beta}}dx = \beta^\alpha\Gamma(\alpha)$$

$$
\begin{aligned}
&= \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}\left[\Gamma(\frac{\nu}{2}+2)2^{\frac{\nu}{2}+2}\right] \\
&= \frac{\Gamma\left(\frac{\nu}{2}+2\right)2^{\frac{\nu}{2}+2}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \\
&= \frac{(\frac{\nu}{2}+1)\Gamma(\frac{\nu}{2}+1)2^{\frac{\nu}{2}+2}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \\
&= \frac{\left(\frac{\nu}{2}+1\right)\frac{\nu}{2}\Gamma(\frac{\nu}{2})2^{\frac{\nu}{2}+2}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \\
&= \left(\frac{\nu}{2}+1\right)\frac{\nu}{2}\cdot 2^2 = 2\left(\frac{\nu}{2}+1\right)\nu \\
&= (\nu+2)\nu = \nu^2+2\nu
\end{aligned}
$$

$$\mu = E(X) = \nu$$

$$\sigma^2 = E(X^2) - E(X)^2 = \nu^2 + 2\nu - \nu^2 = 2\nu$$

12 Moment Generating Function

$$
\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \int_0^\infty e^{tx} \frac{x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} dx \\
&= \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \int_0^\infty e^{tx} \cdot x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}} dx \\
&= \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \int_0^\infty x^{\frac{\nu}{2}-1} e^{tx} e^{-\frac{x}{2}} dx \\
&= \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \int_0^\infty x^{\frac{\nu}{2}-1} e^{tx-\frac{x}{2}} dx \\
&= \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \int_0^\infty x^{\frac{\nu}{2}-1} e^{\frac{2tx}{2}-\frac{x}{2}} dx \\
&= \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \int_0^\infty x^{\frac{\nu}{2}-1} e^{-\frac{2tx-x}{2}} dx \\
&= \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \int_0^\infty x^{\frac{\nu}{2}-1} e^{-x\frac{-2t+1}{2}} dx \\
&= \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \int_0^\infty x^{\frac{\nu}{2}-1} e^{-x\frac{1-2t}{2}} dx \\
&= \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \int_0^\infty x^{\frac{\nu}{2}-1} e^{\frac{\frac{-x}{2}}{1-2t}} dx
\end{aligned}
$$

$$\int_0^\infty x^{\alpha-1} e^{-\frac{x}{\beta}} dx = \beta^\alpha \Gamma(\alpha)$$

$$
\begin{aligned}
&= \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \left[\left(\frac{2}{1-2t}\right)^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})\right] \\
&= \frac{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})(1-2t)^{\frac{\nu}{2}}} \\
&= \frac{1}{(1-2t)^{\frac{\nu}{2}}} \\
&= (1-2t)^{-\frac{\nu}{2}}
\end{aligned}
$$

$$
\begin{aligned}
M_X^{(1)}(t) &= -\frac{\nu}{2}(1-2t)^{-\frac{\nu}{2}-1}(-2) \\
&= \frac{2\nu}{2}(1-2t)^{-\frac{\nu}{2}-1} \\
&= \nu(1-2t)^{-\frac{\nu}{2}-1}
\end{aligned}
$$

$$\begin{aligned} M_X^{(2)}(t) &= (-\frac{\nu}{2}-1)\nu(1-2t)^{-\frac{\nu}{2}-2}(-2) \\ &= (\frac{2\nu}{2}+2)\nu(1-2t)^{-\frac{\nu}{2}-2} \\ &= (\nu+2)\nu)(1-2t)^{-\frac{\nu}{2}-2} \\ &= (\nu^2+2\nu)(1-2t)^{-\frac{\nu}{2}-2} \end{aligned}$$

$$\begin{aligned} M_X^{(1)}(0) &= \nu(1-2\cdot 0)^{-\frac{\nu}{2}-1} \\ &= \nu(1-0)^{-\frac{\nu}{2}-1} \\ &= \nu(1)^{-\frac{\nu}{2}-1} \\ &= \nu \end{aligned}$$

$$\begin{aligned} M_X^{(2)}(0) &= (\nu^2+2\nu)(1-2\cdot 0)^{-\frac{\nu}{2}-2} \\ &= (\nu^2+2\nu)(1-0)^{-\frac{\nu}{2}-2} \\ &= (\nu^2+2\nu)(1)^{-\frac{\nu}{2}-2} \\ &= (\nu^2+2\nu) \end{aligned}$$

$$E(X) = M_X^{(1)}(0) = \nu$$

$$E(X^2) = M_X^{(2)}(0) = (\nu^2+2\nu)$$

$$\mu = E(X) = \nu$$

$$\sigma^2 = E(X^2) - E(X)^2 = \nu^2 + 2\nu - \nu^2 = 2\nu$$

13 Maximum Likelihood Function

Let $x_1, x_2, \ldots, x_n$ be a random sample from a Chi-square distribution with parameter ν.

13.1 Likelihood Function

$$
\begin{aligned}
L(\theta) &= f(x_1|\theta)f(x_2|\theta)\cdots f(x_n|\theta) \\
&= \frac{x_1^{\nu/2-1}e^{-x_1/2}}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{x_2^{\nu/2-1}e^{-x_2/2}}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)} \cdots \frac{x_n^{\nu/2-1}e^{-x_n/2}}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)} \\
&= \prod_{i=1}^{n} \frac{x_i^{\nu/2-1}e^{-x_i/2}}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)} \\
&= \left(2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)\right) \prod_{i=1}^{n} x_i^{\nu/2-1}e^{-x_i/2} \\
&= \left(2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)\right) \cdot \exp\left\{\sum_{i=1}^{n}\frac{x_i}{2}\right\} \cdot \prod_{i=1}^{n} x_i^{\nu/2-1} \\
&= \left(2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)\right) \cdot \exp\left\{\frac{1}{2}\sum_{i=1}^{n}x_i\right\} \cdot \prod_{i=1}^{n} x_i^{\nu/2-1}
\end{aligned}
$$

13.2 Log-likelihood Function

$$
\begin{aligned}
\ell(\theta) &= \ln\big(L(\theta)\big) \\
&= \ln\left[\left(2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)\right) \cdot \exp\left\{\frac{1}{2}\sum_{i=1}^{n}x_i\right\} \cdot \prod_{i=1}^{n} x_i^{\nu/2-1}\right] \\
&= \ln\left[\left(2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)\right)\right] + \ln\left(\exp\left\{\frac{1}{2}\sum_{i=1}^{n}x_i\right\}\right) + \ln\left(\prod_{i=1}^{n} x_i^{\nu/2-1}\right) \\
&= -n\ln\left(2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)\right) + \frac{1}{2}\sum_{i=1}^{n}x_i + \left(\frac{\nu}{2}-1\right)\ln\left(\prod_{i=1}^{n}x_i\right) \\
&= -n\left(\ln(2^{\nu/2}) + \Gamma\left(\frac{\nu}{2}\right)\right) + \frac{1}{2}\sum_{i=1}^{n}x_i + \left(\frac{\nu}{2}-1\right)\sum_{i=1}^{n}\ln x_i \\
&= -n\left(\frac{\nu}{2}\ln 2 + \ln\Gamma\left(\frac{\nu}{2}\right)\right) + \frac{1}{2}\sum_{i=1}^{n}x_i + \left(\frac{\nu}{2}-1\right)\sum_{i=1}^{n}\ln x_i \\
&= -\frac{n\nu}{2}\ln 2 - n\ln\Gamma\left(\frac{\nu}{2}\right) + \frac{1}{2}\sum_{i=1}^{n}x_i + \left(\frac{\nu}{2}-1\right)\sum_{i=1}^{n}\ln x_i
\end{aligned}
$$

13.3 MLE for ν

$$
\begin{aligned}
\frac{d\ell}{d\nu} &= -\frac{n}{2}\ln 2 - \frac{n}{\Gamma\left(\frac{\nu}{2}\right)}\Gamma'\left(\frac{\nu}{2}\right)\cdot\frac{1}{2} + 0 + \frac{1}{2}\sum_{i=1}^{n}\ln x_i \\
&= -\frac{n}{2}\ln 2 - \frac{n}{2\Gamma\left(\frac{\nu}{2}\right)}\Gamma'\left(\frac{\nu}{2}\right) + \frac{1}{2}\sum_{i=1}^{n}\ln x_i
\end{aligned}
$$

$$
\begin{aligned}
0 &= -\frac{n}{2}\ln 2 - \frac{n}{2\Gamma\left(\frac{\nu}{2}\right)}\Gamma'\left(\frac{\nu}{2}\right) + \frac{1}{2}\sum_{i=1}^{n}\ln x_i \\
\frac{n}{2}\ln 2 - \frac{1}{2}\sum_{i=1}^{n}\ln x_i &= -\frac{n}{2\Gamma\left(\frac{\nu}{2}\right)}\Gamma'\left(\frac{\nu}{2}\right) \\
n\ln 2 - \sum_{i=1}^{n}\ln x_i &= -\frac{n}{\Gamma\left(\frac{\nu}{2}\right)}\Gamma'\left(\frac{\nu}{2}\right) \\
\frac{\sum\limits_{i=1}^{n}\ln x_i - n\ln 2}{n} &= \frac{\Gamma'\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}
\end{aligned}
$$

Due to the complexity of the Gamma function in this equation, no solution can be developed for ν in closed form. Thus, we have to rely on numerical methods to obtain a solution to the equation and find the maximum likelihood estimator.

14 Theorems for the Chi-Square Distribution

14.1 Validity of the Distribution

$$\int_0^\infty \frac{x^{\frac{\nu}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}dx = 1$$

Proof:

$$\int_0^\infty \frac{x^{\frac{\nu}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}dx = \frac{1}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}\int_0^\infty x^{\frac{\nu}{2}-1}e^{-\frac{x}{2}}dx$$

$$\int_0^\infty x^{\alpha-1}e^{-\frac{x}{\beta}}dx = \beta^\alpha\Gamma(\alpha)$$

$$
\begin{aligned}
&= \frac{1}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}\left[2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)\right] \\
&= \frac{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} \\
&= 1
\end{aligned}
$$

■

14.2 Sum of Chi-Square Random Variables

Let $X_1, X_2, \ldots, X_n$ be independent Chi-Square random variables with parameter ν_i, that is $X_i \sim \chi^2(\nu_i),\ i = 1, 2, \ldots, n$.

Suppose $Y = \sum\limits_{i=1}^{n} X_i$.

Then $Y \sim \chi^2(\sum\limits_{i=1}^{n} \nu_i)$.

_Proof:

$$
\begin{aligned}
M_Y(t) &= E(e^{tY} = E(e^{t(X_1+X_2+\cdots+X_n}) \\
&= E(e^{tX_1}e^{tX_2}\cdots e^{tX_n}) \\
&= E(e^{tX_1})E(e^{tX_2})\cdots E(e^{tX_n}) \\
&= (1-2t)^{-\frac{\nu_1}{2}}(1-2t)^{-\frac{\nu_2}{2}}\cdots(1-2t)^{-\frac{\nu_n}{2}} \\
&= (1-2t)^{\sum\limits_{i=1}^{n}\nu_i}
\end{aligned}
$$

Which is the mgf of a Chi-Square random variable with parameter $\sum\limits_{i=1}^{n}\nu_i$.

Thus $Y \sim \chi^2\left(\sum\limits_{i=1}^{n}\nu_i\right)$. ■

14.3 Square of a Standard Normal Random Variable

If $Z \sim N(0,1)$, then $Z^2 \sim \chi^2(1)$.

Proof:

$$\begin{aligned}
M_{Z^2}(t) &= E(e^{tZ^2}) \\
&= \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz^2} e^{-\frac{z^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}(-2t+1)} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}(1-2t)} dz
\end{aligned}$$

$$\int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx$$

when $f(x)$ is an even function (**??**)

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{z^2}{2}(1-2t)} dz$$

$$\text{Let } u = \frac{z^2}{2}(1-2t)$$

$$\Rightarrow z = \frac{\sqrt{2}u^{\frac{1}{2}}}{(1-2t)^{\frac{1}{2}}}$$

$$\text{So } dz = \frac{\sqrt{2}u^{-\frac{1}{2}}}{2(1-2t)^{\frac{1}{2}}}$$

$$\begin{aligned}
&= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-u} \frac{\sqrt{2}u^{-\frac{1}{2}}}{2(1-2t)^{\frac{1}{2}}} du \\
&= \frac{2\sqrt{2}}{2\sqrt{2\pi}(1-2t)^{\frac{1}{2}}} \int_{0}^{\infty} e^{-u} u^{-\frac{1}{2}} du \\
&= \frac{2\sqrt{2}}{2\sqrt{2\pi}(1-2t)^{\frac{1}{2}}} \int_{0}^{\infty} u^{\frac{1}{2}-1} e^{-u} du
\end{aligned}$$

$$\int_{0}^{\infty} x^{\alpha-1} e^{-\frac{x}{\beta}} dx = \beta^{\alpha}\Gamma(\alpha)$$

$$\begin{aligned}
&= \frac{1}{\sqrt{\pi}(1-2t)^{\frac{1}{2}}} \Gamma(\frac{1}{2}) \\
&= \frac{\sqrt{\pi}}{\sqrt{\pi}(1-2t)^{\frac{1}{2}}} \\
&= \frac{1}{(1-2t)^{\frac{1}{2}}} = (1-2t)^{-\frac{1}{2}}
\end{aligned}$$

Which is the mgf of a Chi-Square random variable with 1 degree of freedom. Thus $Z^2 \sim \chi^2(1)$. ∎

The Exponential Distribution

The Gamma Distribution

The Geometric Distribution

The Hypergeometric Distribution

The Multinomial Distribution

The Normal Distribution

The Poisson Distribution

The Skew-Normal Distribution

The Uniform Distribution

The Weibull Distribution

Part II

Frequentist Hypothesis Testing

Mantel-Haenszel Test

Part III

Supplemental Subjects

Chebychev's Theorem

Combinations

The Correlation Coefficient

Covariance

Experimental Designs

Moments and Moment Generating Functions

Summation

The Method of Transformations

Variance Paramter

Part IV

Non-Statistical Proofs

The Binomial Theorem

Functions

The Geometric Series

Integraion: Techniques and Theorems

Logarithmic and Exponential Functions

The Real Number System