

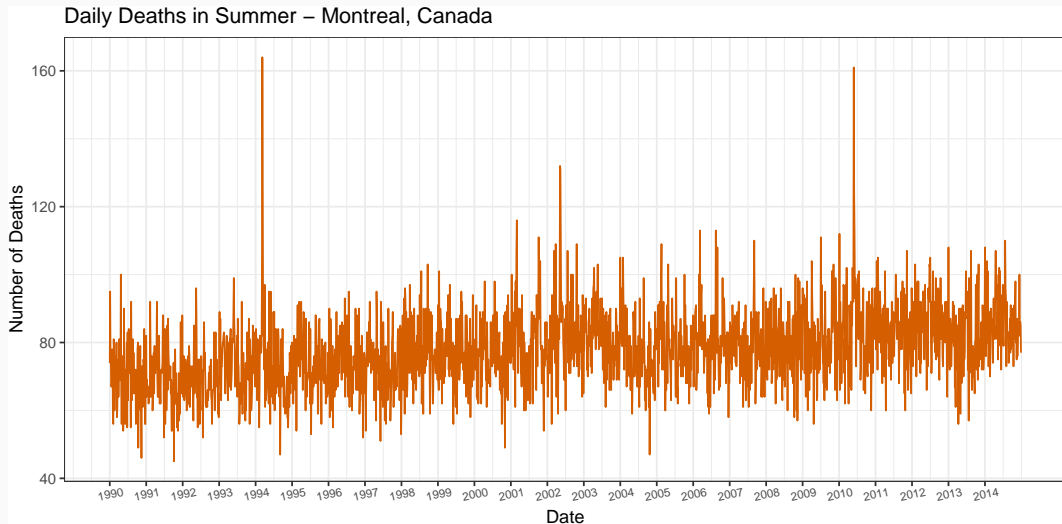
# **Sparse Multiple Index (SMI) Models for High-dimensional Nonparametric Forecasting**

Nuwani Palihawadana

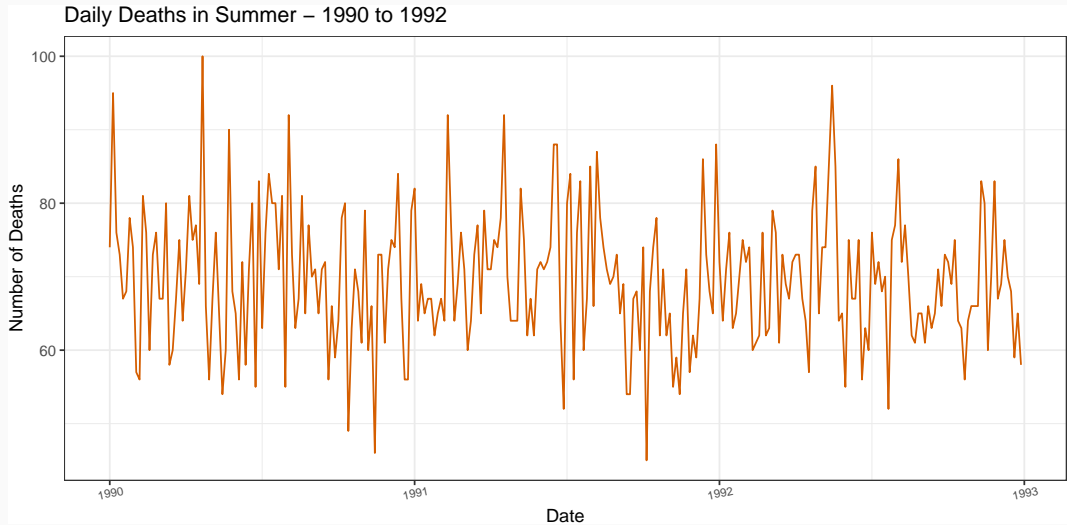
**Joint work with :** Rob Hyndman, Xiaoqian Wang

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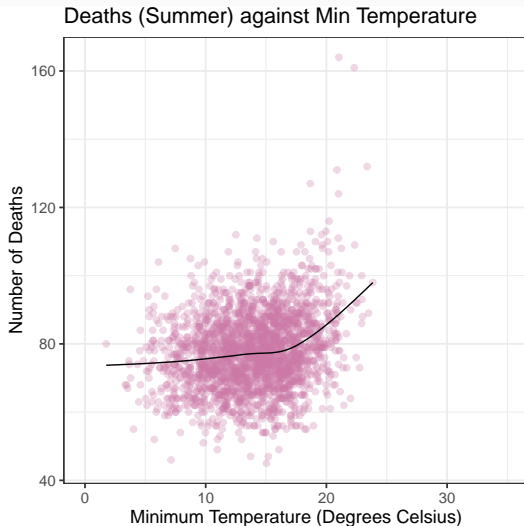
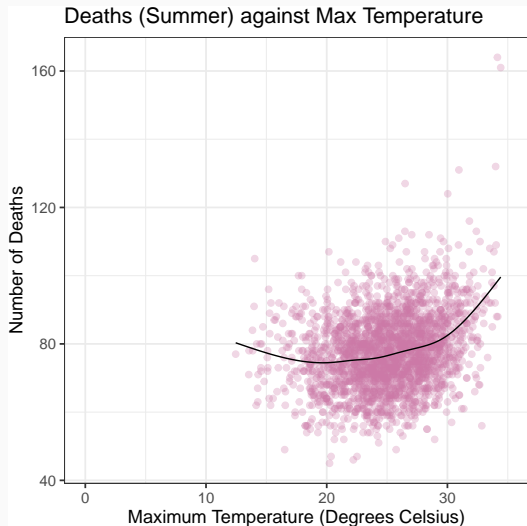
# Heat Exposure Related Daily Mortality



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## ■ *Nonlinear "Transfer Function" model*

$$y_t = f(\mathbf{x}_t, \mathbf{x}_{t-1}, \dots, \mathbf{x}_{t-p}, y_1, \dots, y_{t-k}) + \varepsilon_t$$

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## i Index Models:

- **Mitigate difficulty of estimating a nonparametric component for each predictor**
- **Improve flexibility**

$$y_i = g(\alpha^T x_i) + \varepsilon_i$$

## Semi-parametric model

$$y_i = \beta_0 + \sum_{j=1}^p g_j(\boldsymbol{\alpha}_j^T \mathbf{x}_{ij}) + \sum_{k=1}^d f_k(w_{ik}) + \boldsymbol{\theta}^T \mathbf{u}_i + \varepsilon_i, \quad i = 1, \dots, n,$$

- $y_i$  – univariate response
- $\mathbf{x}_{ij} \in \mathbb{R}^{\ell_j}$ ,  $j = 1, \dots, p$  –  $p$  subsets of predictors entering indices
- $\boldsymbol{\alpha}_j$  –  $\ell_j$ -dimensional vectors of index coefficients
- $g_j, f_k$  – smooth nonlinear functions
- Additional predictors :
  - ▶  $w_{ik}$  – nonlinear
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Allow elements equal to zero in  $\boldsymbol{\alpha}_j$  – "Sparse"

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Both "p" and the predictor grouping among indices are unknown.

Overlapping of predictors among indices is not allowed.

# Optimisation Problem

Let  $q$  be the *total number of predictors* entering indices.

$$\begin{aligned} \min_{\beta_0, p, \alpha, g, f, \theta} \quad & \sum_{i=1}^n \left[ y_i - \beta_0 - \sum_{j=1}^p g_j(\alpha_j^T \mathbf{x}_i) - \sum_{k=1}^d f_k(w_{ik}) - \theta^T \mathbf{u}_i \right]^2 \\ & + \lambda_0 \sum_{j=1}^p \sum_{m=1}^q \mathbb{1}(\alpha_{jm} \neq 0) + \lambda_2 \sum_{j=1}^p \|\alpha_j\|_2^2 \\ \text{s.t.} \quad & \sum_{j=1}^p \mathbb{1}(\alpha_{jm} \neq 0) \in \{0, 1\} \quad \forall m \end{aligned}$$

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- $\lambda_0 > 0$  – controls the number of selected predictors
- $\lambda_2 \geq 0$  – controls the strength of the additional shrinkage



# MIQP Formulation

$$\min_{\beta_0, p, \alpha, g, f, \theta, z} \sum_{i=1}^n \left[ y_i - \beta_0 - \sum_{j=1}^p g_j(\alpha_j^T \mathbf{x}_i) - \sum_{k=1}^d f_k(w_{ik}) - \theta^T \mathbf{u}_i \right]^2$$
$$+ \lambda_0 \sum_{j=1}^p \sum_{m=1}^q z_{jm} + \lambda_2 \sum_{j=1}^p \sum_{m=1}^q \alpha_{jm}^2$$

$$\text{s.t. } |\alpha_{jm}| \leq M z_{jm} \quad \forall j, \forall m,$$

$$\sum_{j=1}^p z_{jm} \leq 1 \quad \forall m,$$

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- $M < \infty$ : If  $\alpha^*$  is an optimal solution, then  $\max(\{|\alpha_{jm}^*|\}_{j \in [p], m \in [q]}) \leq M$

# Estimation Algorithm

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**Step 6:** Increase  $p$  by 1 in each iteration of step 5 – until:

- no. of indices reaches  $q$
- loss increases after the increment model OR
- solution maintains same no. of indices as previous iteration, and  $\text{abs}(\text{difference of index coefficients between two successive iterations}) \leq \text{tolerance}$

# Forecasting Heat Exposure Related Daily Mortality

## Variables

- **Response: Daily deaths in Summer**
  - 1990 to 2014 – Montreal, Canada
- **Index Variables:**
  - ▶ Death lags
  - ▶ Max temperature lags
  - ▶ Min temperature lags
  - ▶ Vapor pressure lags
- **Nonlinear:** DOS (day of the season), Year

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$$\text{Deaths} = \beta_0 + \sum_{j=1}^p g_j(X\alpha_j) + f_1(\text{DOS}) + f_2(\text{Year}) + \epsilon,$$

# Results

Model	Predictors	Indices	Test Set 1		Test Set 2	
			MSE	MAE	MSE	MAE
SMI Model (5, 12) - PPR	61	7	<b>85.233</b>	<b>7.140</b>	<b>97.353</b>	<b>7.772</b>
SMI Model (1, 0) - Additive	61	59	96.398	7.481	112.199	8.156
SMI Model (6, 11) - Linear	61	2	100.231	7.719	120.542	8.598
Backward Elimination	40	NA	136.204	9.319	140.867	9.385
GAIM	61	4	90.763	7.247	106.251	7.928
PPR	61	4	90.698	7.343	110.497	8.057

**SMI Model (a, b)**  $\rightarrow \lambda_0 = a, \lambda_2 = b$

- **Test Set 1:** Three months (June, July and August 2014)
- **Test Set 2:** One month (June 2014)

## **i** Key features:

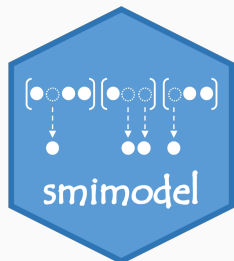
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## Things to improve:

- Initialisation: we encourage trial-and-error
- Applicability: more applications are needed
- Computational time: increases with number of predictors and indices



■ **R package :**

[github.com/nuwani-palihawadana/smimodel](https://github.com/nuwani-palihawadana/smimodel)

■ **Paper :**

[github.com/nuwani-palihawadana/smimodel\\_paper](https://github.com/nuwani-palihawadana/smimodel_paper)

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