Sparse Multiple Index (SMI) Models for High-dimensional Nonparametric Forecasting

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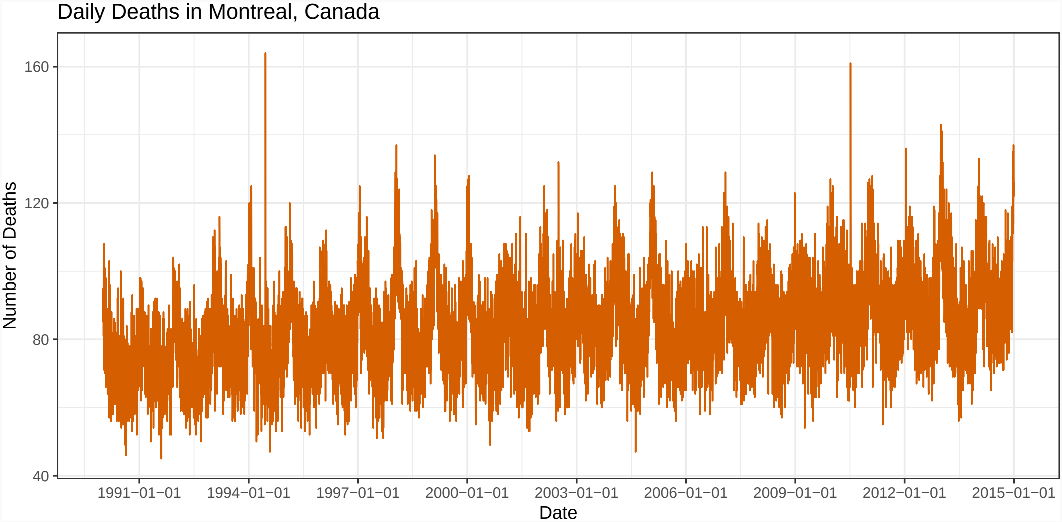
Daily Mortality Data



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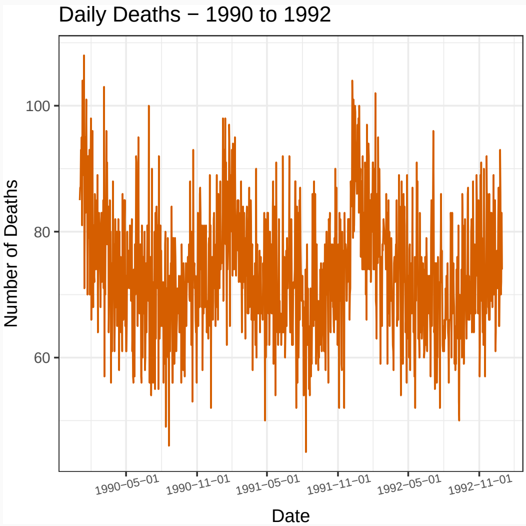

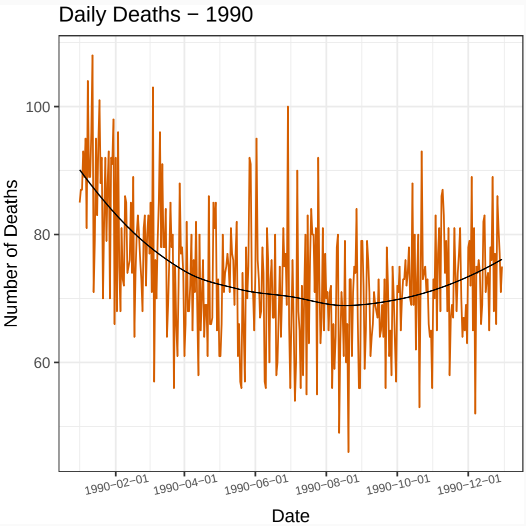

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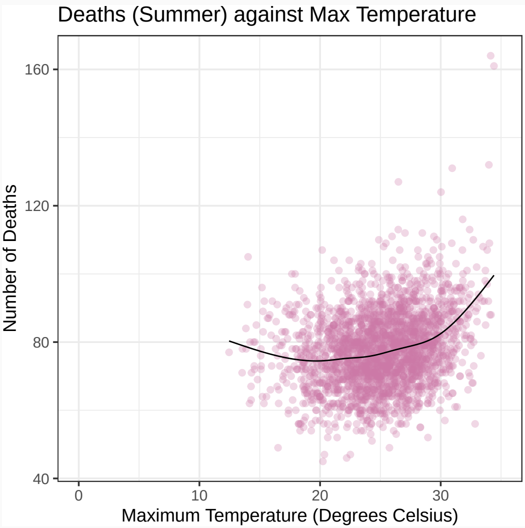

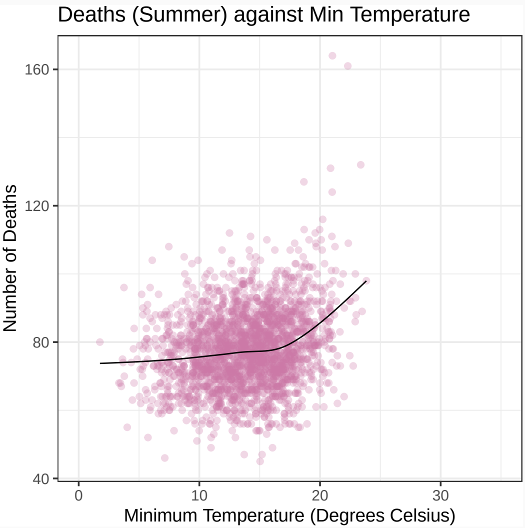

Background

- ***Nonlinear "Transfer Function" model***

$$y_t = f(\boldsymbol{x}_t, \boldsymbol{x}_{t-1}, \dots, \boldsymbol{x}_{t-p}, y_1, \dots, y_{t-k}) + \varepsilon_t$$

y_t – variable to forecast
$\boldsymbol{x}_t$ – a vector of predictors
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Index Models:

- **Mitigate difficulty of estimating a nonparametric component for each predictor**
- **Improve flexibility**

SMI Model

Semi-parametric model

$$y_i = \beta_0 + \sum_{j=1}^{p} g_j(\boldsymbol{\alpha}_j^T \boldsymbol{x}_{ij}) + \sum_{k=1}^{d} f_k(w_{ik}) + \boldsymbol{\theta}^T \boldsymbol{u}_i + \varepsilon_i, \quad i = 1, \dots, n,$$

- y_i – univariate response
- $\boldsymbol{x}_{ij} \in \mathbb{R}^{\ell_j}$, $j = 1, \dots, p$ – p subsets of predictors entering indices
- $\boldsymbol{\alpha}_j$ – ℓ_j-dimensional vectors of index coefficients
- g_j, f_k – smooth nonlinear functions
- Additional predictors :
  - w_{ik} – nonlinear
  - $\boldsymbol{u}_i$ – linear

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Allow elements equal to zero in $\boldsymbol{\alpha}_j$ – "Sparse"

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Both "p" and the predictor grouping among indices are unknown.

Overlapping of predictors among indices is not allowed.

Optimisation Problem

Let q be the *total number of predictors* entering indices.

$$\min_{\beta_0, p, \boldsymbol{\alpha}, \boldsymbol{g}, \boldsymbol{f}, \boldsymbol{\theta}} \quad \sum_{i=1}^{n} \left[y_i - \beta_0 - \sum_{j=1}^{p} g_j(\boldsymbol{\alpha}_j^T \boldsymbol{x}_i) - \sum_{k=1}^{d} f_k(w_{ik}) - \boldsymbol{\theta}^T \boldsymbol{u}_i \right]^2$$

$$+ \lambda_0 \sum_{j=1}^{p} \sum_{m=1}^{q} \mathbb{1}(\alpha_{jm} \neq 0) + \lambda_2 \sum_{j=1}^{p} \|\boldsymbol{\alpha}_j\|_2^2$$

$$\text{s.t.} \quad \sum_{j=1}^{p} \mathbb{1}(\alpha_{jm} \neq 0) \in \{0, 1\} \quad \forall m$$

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- $\lambda_0 > 0$ – controls the number of selected predictors
- $\lambda_2 \geq 0$ – controls the strength of the additional shrinkage

MIQP Formulation

$$
\begin{aligned}
\min_{\beta_0, p, \boldsymbol{\alpha}, \boldsymbol{g}, \boldsymbol{f}, \boldsymbol{\theta}, \boldsymbol{z}} \quad & \sum_{i=1}^{n} \left[y_i - \beta_0 - \sum_{j=1}^{p} g_j(\boldsymbol{\alpha}_j^T \boldsymbol{x}_i) - \sum_{k=1}^{d} f_k(w_{ik}) - \boldsymbol{\theta}^T \boldsymbol{u}_i \right]^2 \\
& + \lambda_0 \sum_{j=1}^{p} \sum_{m=1}^{q} z_{jm} + \lambda_2 \sum_{j=1}^{p} \sum_{m=1}^{q} \alpha_{jm}^2 \\
\text{s.t.} \quad & |\alpha_{jm}| \leq M z_{jm} \quad \forall j, \forall m, \\
& \sum_{j=1}^{p} z_{jm} \leq 1 \quad \forall m, \\
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- $M < \infty$: If $\boldsymbol{\alpha}^*$ is an optimal solution, then $\max\left(\{|\alpha_{jm}^*|\}_{j\in[p], m\in[q]}\right) \leq M$

Estimation Algorithm

Step 1 : Initialise index structure and index coefficients

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Step 6 : Increase p by 1 in each iteration of step 5 – until:

- no.of indices reaches q
- loss increases after the increment model OR
- solution maintains same no.of indices as previous iteration, and abs(difference of index coefficients between two successive iterations) $<=$ tolerance

Forecasting Heat Exposure Related Mortality

Variables

- **Response:** **Daily deaths in Summer** – 1990 to 2014 – Montreal, Canada
- **Index Variables:**
  - Death lags
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$$\textbf{Deaths} = \beta_0 + \sum_{j=1}^{p} g_j(\boldsymbol{X}\boldsymbol{\alpha}_j) + f_1(\textbf{DOS}) + f_2(\textbf{Year}) + \varepsilon,$$

Results

Model	Predictors	Indices	Test Set 1		Test Set 2	
			MSE	MAE	MSE	MAE
SMI Model (5, 12) - PPR	61	7	**85.233**	**7.140**	**97.353**	**7.772**
SMI Model (1, 0) - Additive	61	59	96.398	7.481	112.199	8.156
SMI Model (6, 11) - Linear	61	2	100.231	7.719	120.542	8.598
Backward Elimination	40	—	136.204	9.319	140.867	9.385
Group-wise Additive Index Model	61	4	90.763	7.247	106.251	7.928
Projection Pursuit Regression	61	4	90.698	7.343	110.497	8.057

SMI Model (a, b) $\rightarrow$ $\boldsymbol{\lambda}_0 = \boldsymbol{a}, \boldsymbol{\lambda}_2 = \boldsymbol{b}$

- **Test Set 1:** Three months (June, July and August 2014)
- **Test Set 2:** One month (June 2014)

Conclusion

Key features:

- Automatic selection of number of indices and predictor grouping
- Automatic predictor selection
- A wide spectrum: from single index models to additive models
- Flexibility to include separate nonlinear and linear predictors

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Things to improve:

- Initialisation: we encourage trial-and-error
- Applicability: more applications are needed
- Computational time: increases with number of predictors and indices

- **R package :**
  github.com/nuwani-palihawadana/smimodel

- **Paper :**
  github.com/nuwani-palihawadana/smimodel_paper

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