

# 03: Boolean Algebra and Circuit Design

IT1206 - Computer Systems

Level I - Semester 1





# **Boolean Algebra**





#### **Boolean Postulates**

$$0.0 = 0$$

$$1 + 1 = 1$$

$$0+0=0$$

$$1.1 = 1$$

$$1.0 = 0.1 = 0$$

$$1 + 0 = 0 + 1 = 1$$





### **Basic Identities of Boolean Algebra**

Identity Name	AND Form	OR Form
Identity Law	1x = x	0+x=x
Null (or Dominance) Law	0x = 0	1+x=1
Idempotent Law	XX = X	X+X=X
Inverse Law	$x\overline{x} = 0$	$x+\overline{x}=1$
Commutative Law	xy = yx	X+y=y+X
Associative Law	(xy)z = x(yz)	(x+y)+z=x+(y+z)
Distributive Law	X+yZ = (X+y)(X+Z)	X(y+z) = Xy+Xz
Absorption Law	X(X+Y)=X	X+XY=X
DeMorgan's Law	$(\overline{xy}) = \overline{x} + \overline{y}$	$(\overline{x+y}) = \overline{x}\overline{y}$
Double Complement Law	$\overline{\overline{x}} = x$	

Source - Linda Null and Julia Lobur, Essentials of Computer Organization and Architecture, 5th Edition



## Laws of Boolean Algebra

• 
$$A.B + A.\overline{B} = A$$
  
 $(A+B)(\overline{A}+\overline{B}) = A$ 

$$A + 0 = A$$

$$A \cdot 0 = 0$$





### Laws of Boolean Algebra

$$A + \overline{A}.B = A + B$$

$$A(\overline{A} + B) = AB$$





### **Boolean Expressions**

- There can be multiple boolean expressions for a single operational behavior (single truth table).
- Thus, there can be multiple circuit structures with different logic gate arrangements for the same behavior.
- Simple and minimal circuit structure is always preferred.
  - Efficiency
  - Cost





### **Simple Expression**

Boolean algebra helps to simplify expressions.

$$-F1 = \bar{x}\bar{y}z + \bar{x}yz + x\bar{y}$$

$$-F2 = x\bar{y} + \bar{x}z$$

 You will get two circuits if you draw circuits for F1 and F2.

$\boldsymbol{x}$	y	Z	<i>F</i> 2	<i>F</i> 1
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	0	0





### **Expression to Truth Table**

- When a boolean expression is given, we can formulate the truth table for the expression.
- Identify the variables in the expression
  - If there are n variables, then there will be  $2^n$  combinations of values
  - 2<sup>n</sup> rows in the turth table
- Derive the resulting value for each value combination in the truth table rows.





## **Expression to Truth Table. (Cont.)**

Ex.

$$F(x, y, z) = x.y.\bar{z} + \bar{y}.z$$

- Three variables in F(x, y, z)
- There will be 2<sup>3</sup> combinations
- There will be 2<sup>3</sup> rows in the truth table





# **Expression to Truth Table. (Cont.)**

• Ex.

$$F(x, y, z) = x.y.\bar{z} + \bar{y}.z$$

х	y	Z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0





### **Truth Table to Expression**

- When a truth table is given, we can formulate the boolean algebric expression to represent it.
- There are two forms (*Canonical* forms) in presenting the expression.
  - Sum of *Minterms* (Standard Products)
  - Product of Maxterms (Standard Sums)





#### **Minterm**

- In a three variable boolean expression, we can combine 3 variables with an AND operator.
- There are 8 possible such arrangements.
  - $-\bar{x}.\bar{y}.\bar{z}$
  - $-\bar{x}.\bar{y}.z$
  - .........
  - -x.y.z
- Each of these AND terms is called *Minterm* (or Standard Product).
- (x. y) is **not** a *Minterm* in this context.



#### **Maxterm**

- In a three variable boolean expression, we can combine 3 variables with an OR operator.
- There are 8 possible such arrangements.

$$-\bar{x} + \bar{y} + \bar{z}$$

$$-\bar{x} + \bar{y} + z$$

$$-x+y+z$$

- Each of these OR terms is called *Maxterm* (or Standard Sum)
- $(\bar{y} + z)$  is not a *Maxterm* in this context.



### [Sum – Product] in Mathematics



A **product** is the result of multiplying, or an expression that identifies factors to be multiplied.



A **sum** is the aggregate of two or more numbers or particulars as determined by or as if by the mathematical process of addition.





#### **Sum of Minterms**

- Disjunction of terms where each term is a conjunction of literals.
- The OR operations are performed on the terms that are made by AND operations.
- Ex.

$$F = \bar{x}.y.\bar{z} + x.\bar{y}.\bar{z} + x.\bar{y}.z + x.y.\bar{z} + x.y.z$$

 Each term in the expression is referred as *Minterm* or Standard Product.





#### **Product of Maxterms**

- Conjunction of terms where each term is a disjunction of literals.
- The AND operations are performed on the terms that are made by OR operations.
- Ex.

$$F = (x + y + z).(x + y + \bar{z}).(x + \bar{y} + \bar{z})$$

 Each term in the expresssion is referred as *Maxterm* or Standard Sum.





#### **Standard Forms**

- In *Cannonical* Form, each term must contain all the variables in the truth table.
- But Standard Form is an alternative way to express Boolean function.
- A term in an expression that is expressed in a standard form doesn't need to contain all the variables in the truth table.
- There are two standard forms.
  - Sum of Products
  - Product of Sums





#### Standard Form – Sum of Products

- OR operation is performed over several terms where each term is made up by performing AND operation over several literals.
- Ex.

$$F = \bar{x}.y + \bar{y}.\bar{z} + x.\bar{y}.z + y$$



$$F = \bar{x}.y + \bar{y}.(x + \bar{z}) + x.\bar{y}.z + y$$







#### **Standard Form - Product of Sums**

- AND operation is performed over several terms where each term is made up by performing OR operation over several literals.
- Ex.

$$F = y.(x + z).(\bar{x} + \bar{y} + \bar{z})$$



$$F = y.(x + z.\bar{y}).(\bar{x} + \bar{y} + \bar{z})$$







# **Logic Equations to Truth Tables**

$$X = A.\overline{B} + \overline{A}.B + AB$$

А	В	X
0	0	
0	1	
1	0	
1	1	





### **Sum of Minterms**

- The OR operation performed on the products of the AND operation
- Fill the corresponding cells with 1 for each product, the other cells with 0

$$X = (A.B) + (\overline{A}.\overline{B}) + (\overline{A}.B)$$

$$A = 1, \overline{A} = 0$$

$$B = 1, \overline{B} = 0$$

Α	В	Χ
0	0	1
0	1	1
1	0	0
1	1	1





### **Product of Maxterms**

- The AND operation performed on the sums of the OR operation
- Fill the corresponding cells with 0 for each sum, the other cells with 1

$$Y = (\overline{A} + \overline{B}).(\overline{A} + B)$$

$$A = 0, \overline{A} = 1$$

$$B = 0, \overline{B} = 1$$

Α	В	Χ
0	0	1
0	1	1
1	0	0
1	1	0



## **Truth Tables to Logic Equations**

Α	В	X
0	0	1
0	1	1
1	0	0
1	1	0

- Sum of Minterms consider 1s
  - Consider A=1,B=1

$$X = (\overline{A}.\overline{B}) + (\overline{A}.B)$$

- Product of Maxterms consider 0s
  - Consider A=0,B=0

$$X = (\overline{A} + B).(\overline{A} + \overline{B})$$



#### **Your Turn: Exercise1**

 Convert the following equation which is in the form of Product-of-sums into the form of Sum-of-products

$$f(ABCD) = (A + \overline{B} + C)(\overline{A} + B + \overline{C} + \overline{D})(\overline{A} + \overline{B} + D)(A + \overline{C})$$





#### **Answer: Exercise1**

$$f(ABCD) = (A + \overline{B} + C)(\overline{A} + B + \overline{C} + \overline{D})(\overline{A} + \overline{B} + D)(A + \overline{C})$$

$$f(ABCD) = (\overline{A} + \overline{B} + C)(\overline{A} + B + \overline{C} + \overline{D})(\overline{A} + \overline{B} + D)(\overline{A} + \overline{C})$$

$$f(ABCD) = (\overline{A} + \overline{B} + C) + (\overline{A} + B + \overline{C} + \overline{D}) + (\overline{A} + \overline{B} + D) + (\overline{A} + \overline{C})$$

$$f(ABCD) = (\overline{A}.\overline{B}.\overline{C}) + (\overline{A}.\overline{B}.\overline{C}.\overline{D}) + (\overline{A}.\overline{B}.\overline{D}) + (\overline{A}.\overline{C})$$

$$f(ABCD) = (\overline{A}.B.\overline{C}) + (\overline{A}.\overline{B}.\overline{C}.D) + (\overline{A}.B.\overline{D}) + (\overline{A}.\overline{C})$$





## Implementation of Boolean Functions

- A Boolean function can be realised in either SOP or POS form
- At this point, it would seem that the choice would depend on whether the truth table contains more 1s and 0s for the output function
- The SOP has one term for each 1, and the POS has one term for each 0





### Implementation of Boolean Functions

- However, there are other considerations:
  - It is generally possible to derive a simpler Boolean expression from truth table than either SOP or POS
  - It may be preferable to implement the function with a single gate type (NAND or NOR)





### Implementation of Boolean Functions

- The significance of this is that, with a simpler Boolean expression, fewer gates will be needed to implement the function
- Methods that can be used to achieve simplification are:
  - Algebraic Simplification
  - Karnaugh Maps





# Your Turn: Algebraic Simplification

 Simplify the following equation using Boolean algebra laws

$$f(ABC) = (A + \overline{B} + \overline{C})(A + \overline{B}C)$$





## **Answer: Algebraic Simplification**

$$f(ABC) = (A + \overline{B} + \overline{C})(A + \overline{B}C)$$

$$f(ABC) = AA + A\overline{B}C + A\overline{B} + \overline{B}\overline{B}C + A\overline{C} + \overline{B}C\overline{C}$$

$$f(ABC) = A(1 + \overline{B}C + \overline{B} + \overline{C}) + \overline{B}C + \overline{B}C\overline{C}$$

$$f(ABC) = A + \overline{B}C$$





### **Karnaugh Maps**

- For purposes of simplification, the Karnaugh map is a convenient way of representing a Boolean function of a small number (up to 4 to 6) of variables
- The map is an array of 2<sup>n</sup> squares, representing the possible combinations of values of **n** binary variables





### **Karnaugh Maps**

- The map can be used to represent any Boolean function in the following way:
  - Each square corresponds to a unique product in the sum-of-products form.
  - With a 1 value corresponding to the variable and a 0 value corresponding to the NOT of that variable





### Karnaugh Maps: 2 Values

ВА	0	1
0	0	1
1	1	1

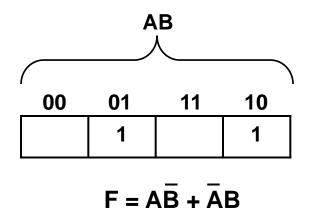
$$X = A.\overline{B} + \overline{A}.B + AB$$





### Karnaugh Maps: 2 Values

- The AB corresponds to the fourth square in the Figure
- For each such production in the function, 1 is placed in the corresponding square







## Karnaugh Maps: 3 Values

C AB	00	01	11	10
0	1	1	0	0
1	0	0	1	1

$$X = A.\overline{B}.C + \overline{A}.B.\overline{C} + A.B.C + \overline{A}.\overline{B}.\overline{C}$$





#### **Karnaugh Maps: 4 Values**

CD AB	00	01	11	10
00	1	0	1	1
01	0	1	1	0
11	0	1	1	0
10	0	1	0	1

$$X = A.\overline{B}.\overline{C}.\overline{D} + A.\overline{B}.\overline{C}.\overline{D} + A.B.\overline{C}.\overline{D} + \overline{A}.\overline{B}.\overline{C}.\overline{D} + \overline{A}.\overline{B}.\overline{C}.D + A.B.\overline{C}.D + A.B.\overline{C}.D + \overline{A}.B.C.D + A.B.C.D$$





#### Karnaugh Maps: Exercise 1

 Simplify the following Karnaugh Map using Boolean equations (Write your answers in both SOP and POS)

C AB	00	01	11	10
0	0	1	0	0
1	1	1	0	1





#### Karnaugh Maps: Answer

$$(\overline{A}.B.\overline{C}) + (\overline{A}.\overline{B}.C) + (\overline{A}.B.C) + (A.\overline{B}.C)$$



$$\overline{A}B + \overline{B}C$$

$$(A+B+C).(\overline{A}+\overline{B}+C).(\overline{A}+B+C).(\overline{A}+\overline{B}+\overline{C})$$



$$(B+C).(\overline{A}+\overline{B})$$

САВ	00	01	11	10	
0	0	1	0	0	
1	1	1	0	1	



#### **Karnaugh Maps: Exercise 2**

 Simplify the following Karnaugh Map using Boolean equations (Write your answers in both SOP and POS)

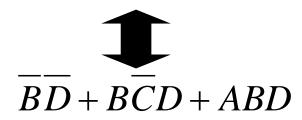
CDAB	00	01	11	10
00	1	0	0	1
01	0	1	1	0
11	0	0	1	0
10	1	0	0	1





#### Karnaugh Maps: Answer

$$(\overline{ABCD}) + (\overline{ABCD}) + (\overline{ABCD}) + (\overline{ABCD}) + (\overline{ABCD}) + (\overline{ABCD}) + (\overline{ABCD}) + (\overline{ABCD})$$



CD	00	01	11	10
00	1	0	0	1
01	0	1	1	0
11 _	0	0	1	0
10	1	0	0	1



#### Karnaugh Maps: Answer

$$(A+\overline{B}+C+D).(\overline{A}+\overline{B}+C+D).(A+\overline{B}+\overline{C}+D).(\overline{A}+\overline{B}+\overline{C}+D).$$

$$(A+B+C+\overline{D}).(A+B+\overline{C}+\overline{D}).(\overline{A}+B+C+\overline{D}).(\overline{A}+B+\overline{C}+\overline{D}).$$

$$(A + \overline{B} + \overline{C} + \overline{D})$$



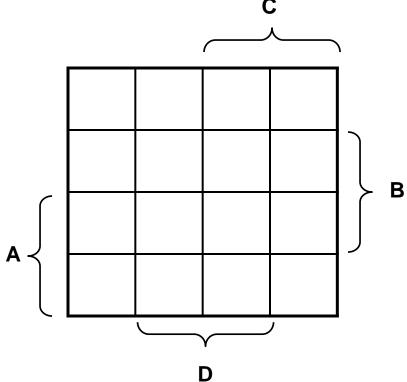
$$(\overline{B}+D).(B+\overline{D}).(A+\overline{C}+\overline{D})$$

CD AB	00	01	11	10
00	1	0	0	1
01	0	1	1	0
11 _	0	0	1	0
10	1	0	0	1



 The labeling used in figure emphasizes the relationship between variables and the rows and columns of the map

The two rows embraced by the symbol A are those in which the variable A has the value 1; the rows not embraced by the symbol A are those in which A is 0







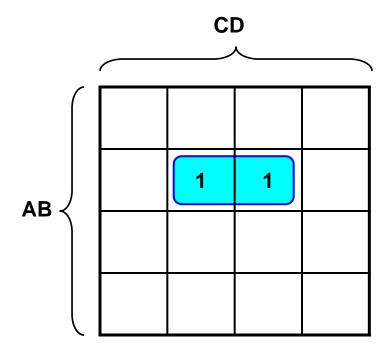
- Once the map of a function is created, we can often write a simple algebraic expression for it by noting the arrangement of the 1s on the map
- The principle is as follows:
  - Any two squares that are adjacent differ in only one of the variables
  - If two adjacent squares both have an entry of 1, then the corresponding product terms differ in only one variable
  - In such a case, the two terms can be merged by eliminating that variable





 For example, in following FIGURE, the two adjacent squares correspond to the two terms ABCD and ABCD

• The function expressed is  $\overline{ABCD} + \overline{ABCD} = \overline{ABD}$ 



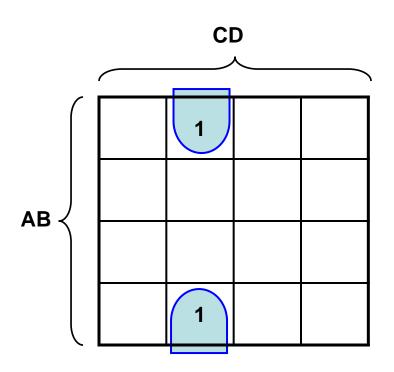


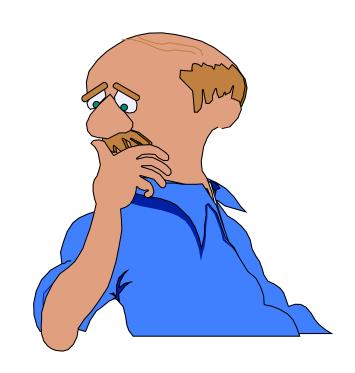
- This process can be extended in several ways:
  - First, the concept of adjacent can be extended to include wrapping around the edge of the map
  - Thus, the top square of a column is adjacent to the bottom square, and the leftmost square of a row is adjacent to the rightmost square
  - Second, we can group not just 2 squares but 2<sup>n</sup> adjacent squares, that is, 4, 8, etc





#### Your turn: Karnaugh Maps

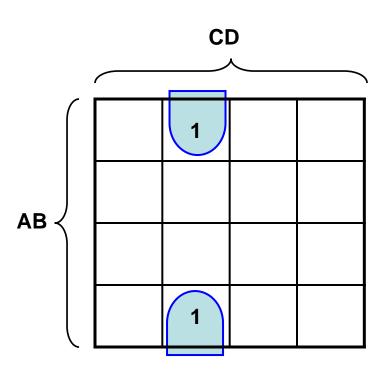








#### **Answer: Karnaugh Maps**

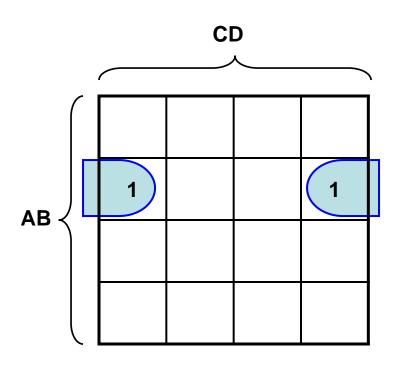


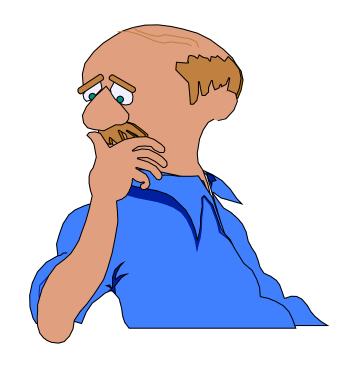






#### Your turn: Karnaugh Maps

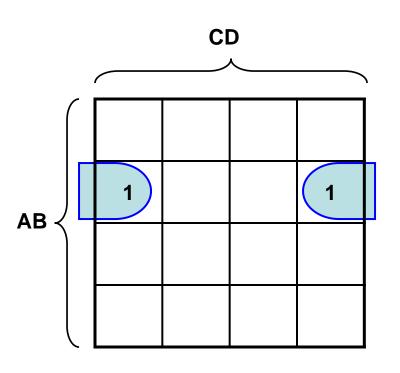








#### **Answer: Karnaugh Maps**

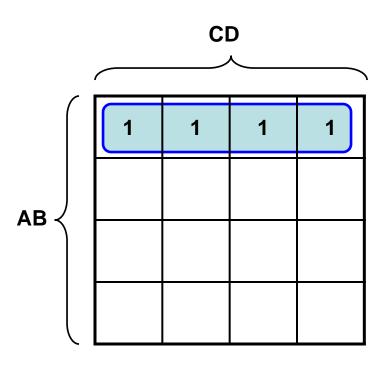


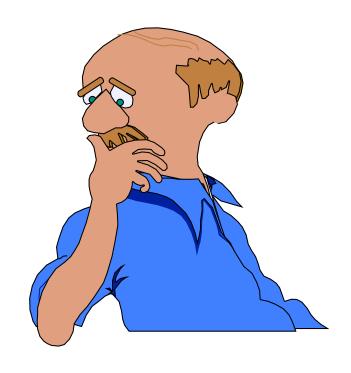






#### Your turn: Karnaugh Maps

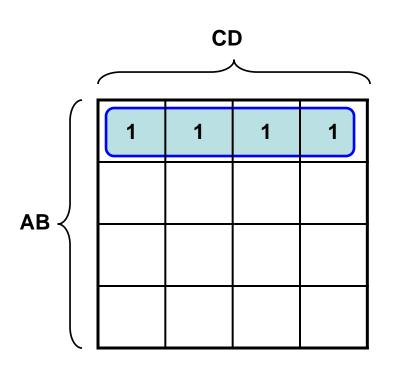








#### **Answer: Karnaugh Maps**

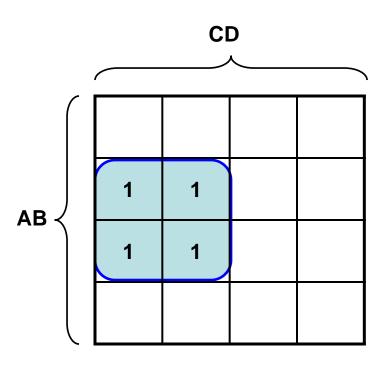


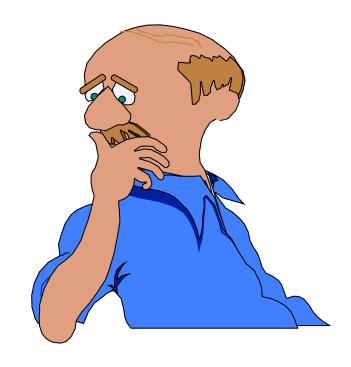






#### Your turn: Karnaugh Maps

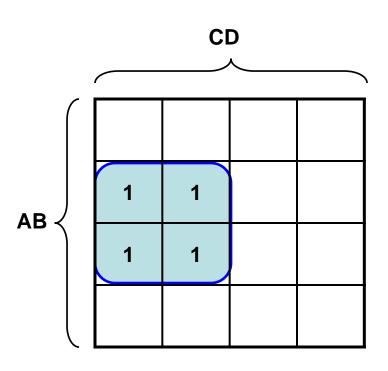








#### **Answer: Karnaugh Maps**

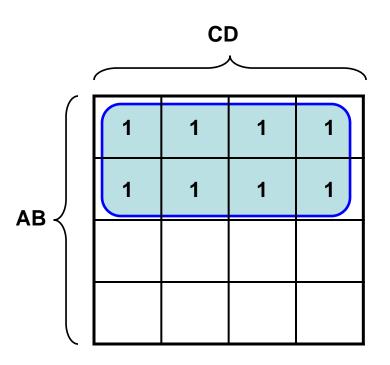


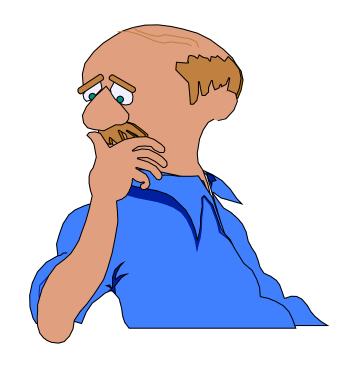
 $B\overline{C}$ 





#### Your turn: Karnaugh Maps

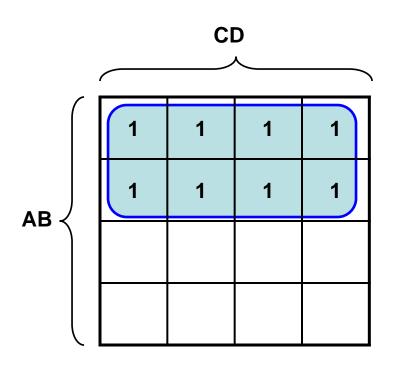








#### **Answer: Karnaugh Maps**

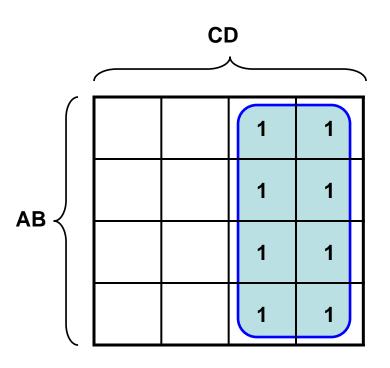


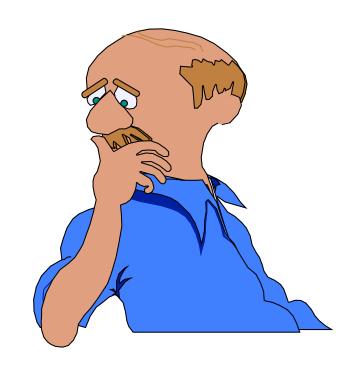






#### Your turn: Karnaugh Maps

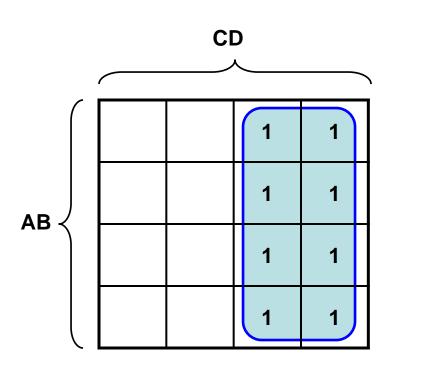








#### **Answer: Karnaugh Maps**



C



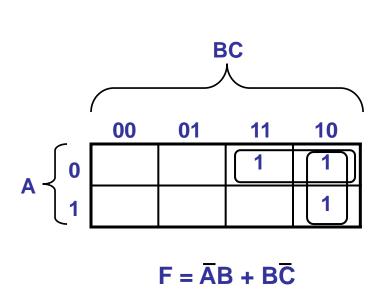


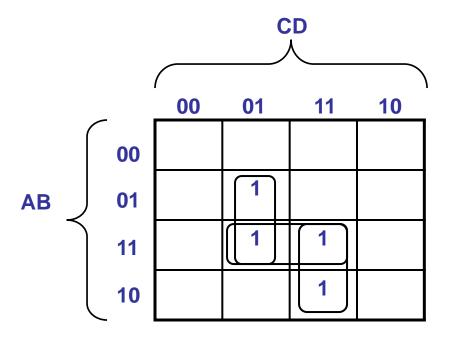
- In attempting to simplify, first look for the largest grouping possible:
  - When you are circling groups, you are allowed to use the same 1 more than once
  - If any isolated 1s remain after the groupings, then each of these is circled as a group of 1s
  - Finally, before going from the map to a simplified Boolean expression, any group of 1s that is completely overlapped by other groups can be eliminated





## Karnaugh Maps: Overlapping Groups



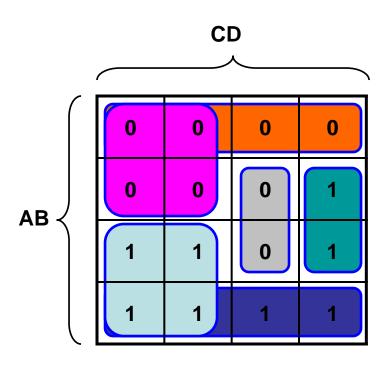


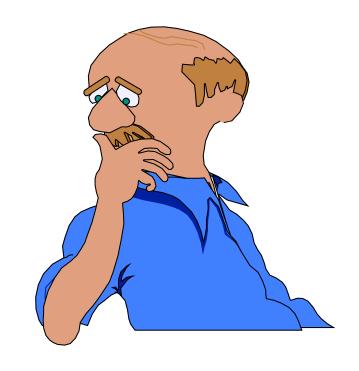
$$F = B\overline{C}D + ACD$$





### Your turn: Karnaugh Maps

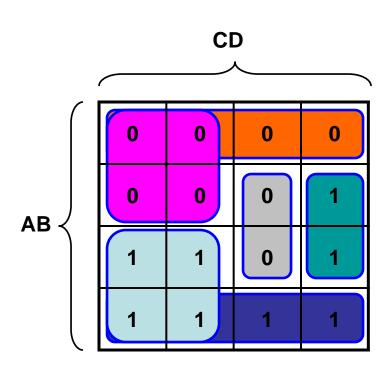








#### **Answer: Karnaugh Maps**



$$F = A\overline{C} + A\overline{B} + BC\overline{D}$$

$$F = (A+C).(A+B).(B+C+D)$$



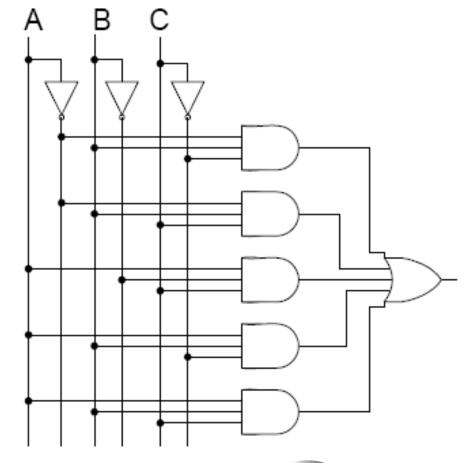


# **Drawing a Circuit**

#### Sum-of-Products Expression

$$\overline{A} B \overline{C} + \overline{A} B C + A \overline{B} C$$
  
+  $A B \overline{C} + A B C$ 

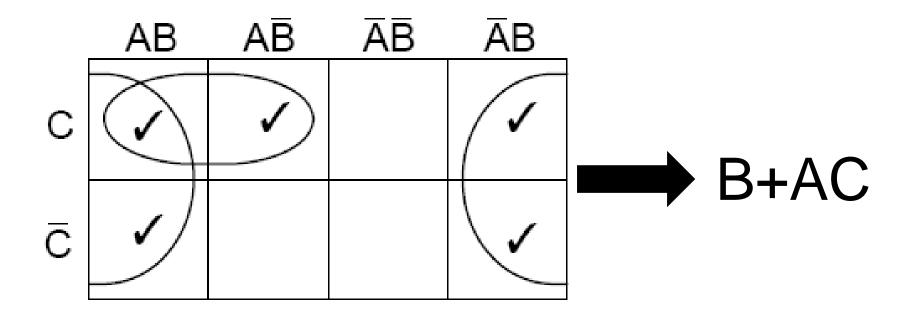
#### Digital Logic Circuit







# **Drawing a Circuit**

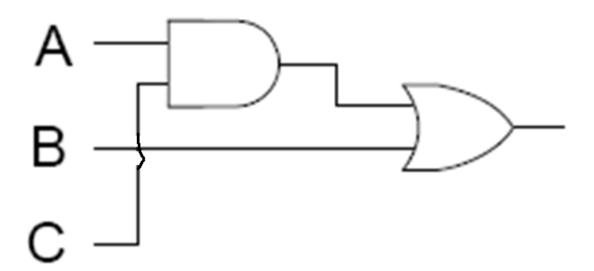






# **Drawing a Circuit**









# **Logic Operators**





#### **Logic Operations**

- Basic logic operators and logic gates
- Boolean algebra
- Combinational Circuits
- Basic circuit design





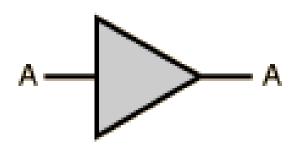
#### **Basic Logic Operators and Logic Gates**

- AND
- OR
- NOT
- XOR (Exclusive OR)
- NOR
- NAND
- XNOR





#### **Buffer**



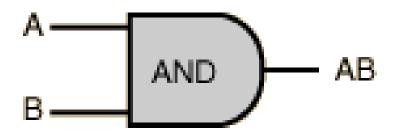
Α	В
0	0
1	1





#### **AND Operation**

- Operator
- ^ Operator
- A.B = A ^ B



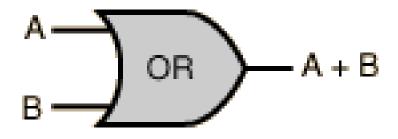
Α	В	A.B
0	0	0
0	1	0
1	0	0
1	1	1





#### **OR Operation**

- + Operator
- v Operator
- $A + B = A \vee B$



Α	В	A + B
0	0	0
0	1	1
1	0	1
1	1	1



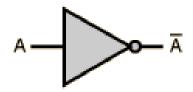


#### **NOT Operation**

- ~ Operator
- ¬ Operator

$$\overline{A} = \neg A = \sim A = A'$$

Α	A'
0	1
1	0



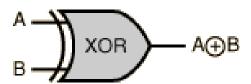




# **XOR Operation**

■ Operator

$$A \oplus B$$



Α	В	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0





# **NAND Operation**

$$(\overline{A.B}) = (A.B)'$$



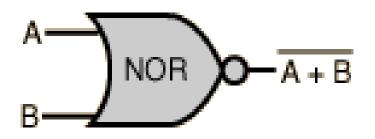
Α	В	$(\overline{A.B})$
0	0	1
0	1	1
1	0	1
1	1	0





# **NOR Operation**

$$(\overline{A+B}) = (A+B)'$$



Α	В	$(\overline{A+B})$
0	0	1
0	1	0
1	0	0
1	1	0





### **XNOR Operation**

 $(\overline{A \oplus B})$ 



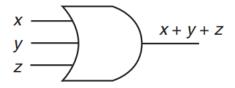
Α	В	$(\overline{A \oplus B})$
0	0	1
0	1	0
1	0	0
1	1	1

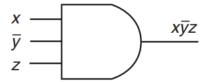




### **Multiple Input Gates**

- Gates are not limited to two inputs.
- There can be logic gates with more than two inputs.









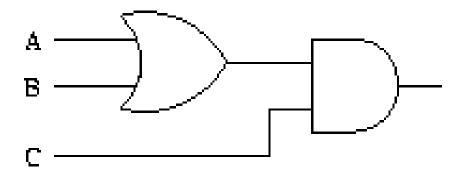
 In addition to the basic gates, gates with 3,4, or more inputs can be used

E.g. x + y + z can be implemented with a single **OR** gate with 3 inputs





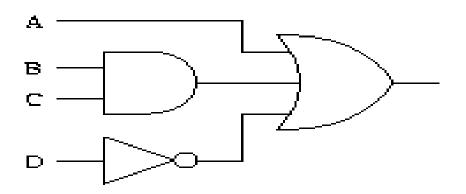
$$X = (A+B)C$$







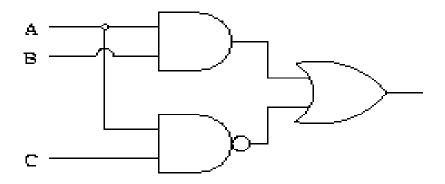
$$X = A + (B.C) + \overline{D}$$







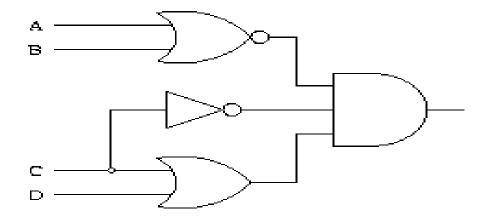
$$X = (A.B) + (\overline{A.C})$$







$$X = (\overline{A+B}).(C+D).\overline{C}$$

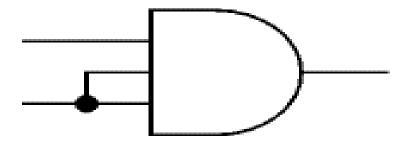






#### **Reducing Logic Gates**

- Reducing the number of inputs
  - The number of inputs to a gate can be reduced by connecting two (or more) inputs together
  - The diagram shows a 3-input AND gate operating as a 2-input AND gate

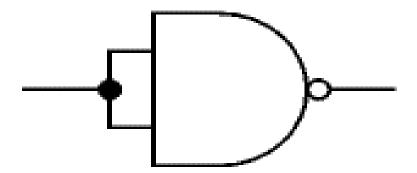






#### **Reducing Logic Gates**

- Reducing the number of inputs
  - Reducing a NOT gate from a NAND or NOR gate
  - ➤ The diagram shows this for a 2-input **NAND** gate







- Typically, not all gate types are used in implementation
  - Design and fabrication are simpler if only one or two types of gates are used
  - Therefore, it is important to identify functionally complete sets of gates
  - This means that any **Boolean function** can be implemented using only the gates in the set





- The following are functionally complete sets:
  - > AND, OR, NOT
  - > AND, NOT
  - > OR, NOT
  - > NAND
  - > NOR





- AND, OR, and NOT gates constitute a functionally complete set, since they represent the 3 operations of Boolean algebra
- For the AND and NOT gates to form a functionally complete set, there must be a way to synthesize the OR operation from the AND and NOT operations

$$A + B = \overline{A \cdot B}$$
  
 $A \circ B = NOT((NOT A) AND(NOT B))$ 





 Similarly, the OR and NOT operations are functionally complete because they can be synthesize the AND operation

A.B = 
$$\overline{A + B}$$
  
A AND B = NOT((NOT A) OR (NOT B))





#### **Universal Gates**

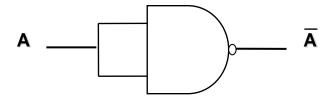
 The AND, OR and NOT functions can be implemented solely with NAND gates, and the same thing for NOR gates.

- For this reason, digital circuits can be, and frequently are, implemented solely with NAND gates or solely with NOR gates
- Therefore, NAND and NOR gates are referred to as a Universal Gate





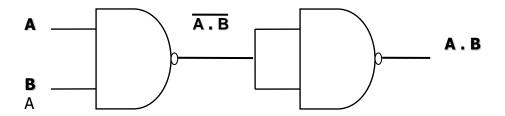
 The diagram shows how the NOT function can be implemented solely with NAND gate







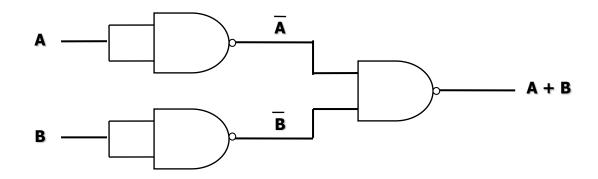
 The diagram shows how the AND function can be implemented solely with NAND gate





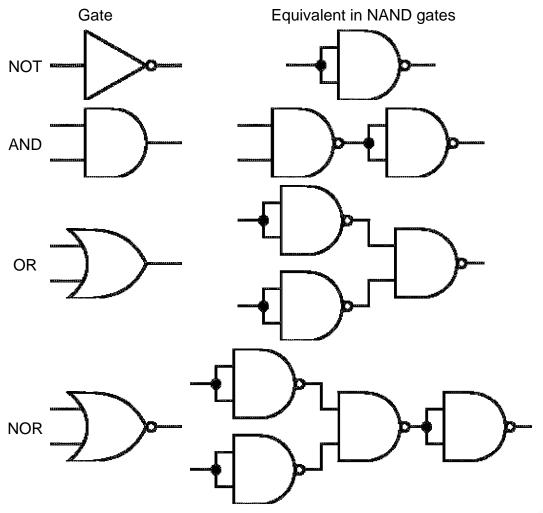


 The diagram shows how the OR function can be implemented solely with NAND gate





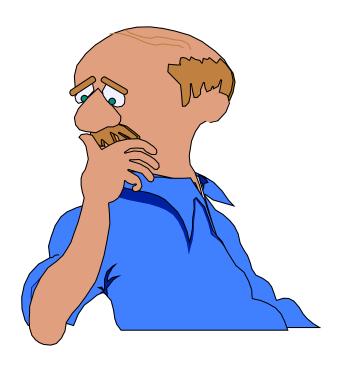






#### Your turn

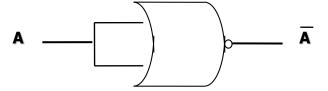
 Draw a diagram that shows how the NOT function can be implemented solely with NOR gate







 The diagram shows how the NOT function can be implemented solely with NOR gate

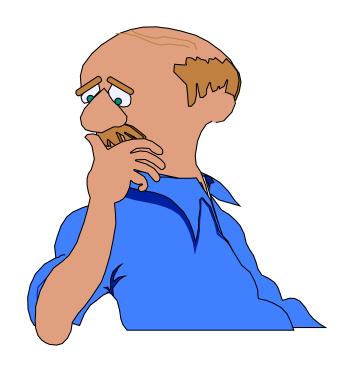






#### Your turn

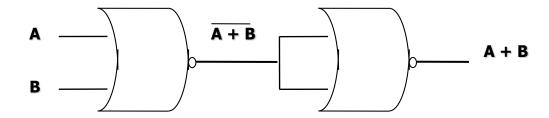
 Draw a diagram that shows how the OR function can be implemented solely with NOR gate







 The diagram shows how the OR function can be implemented solely with NOR gate

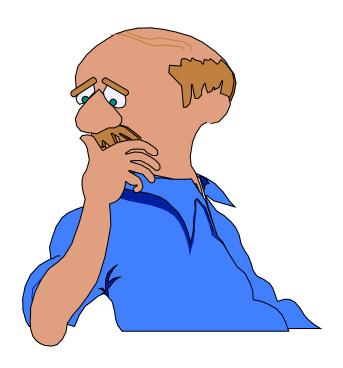






#### Your turn

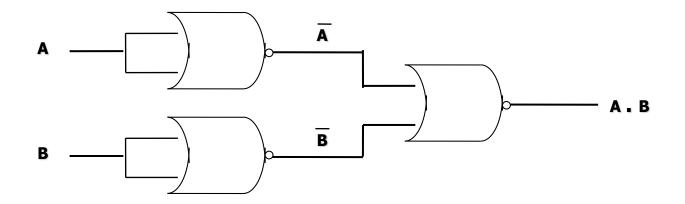
 Draw a diagram that shows how the AND function can be implemented solely with NOR gate







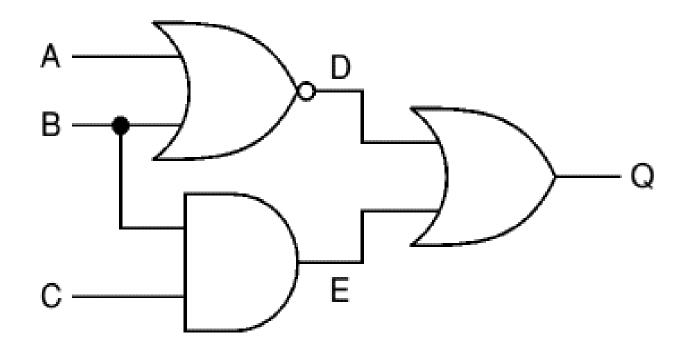
 The diagram shows how the AND function can be implemented solely with NOR gate







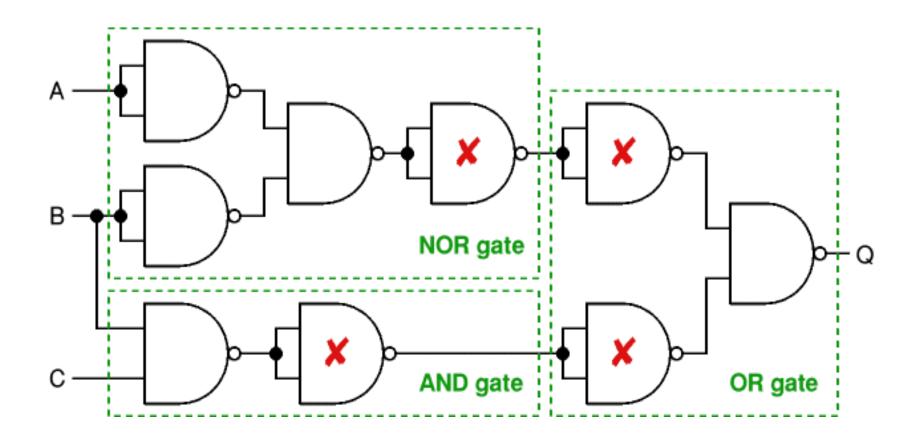
# **Substituting Gates in a Logic System**







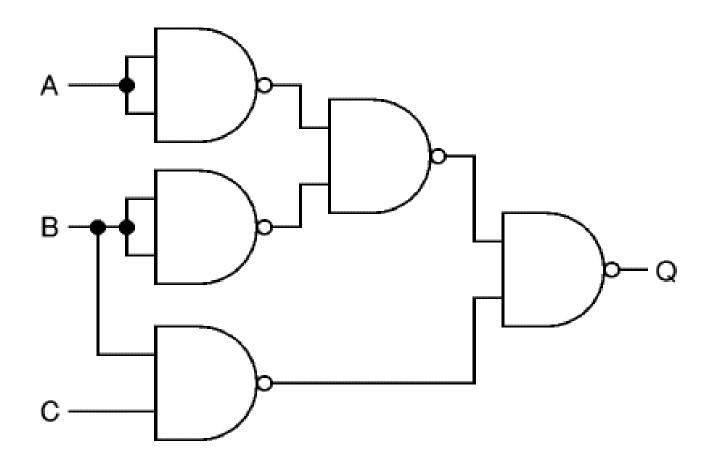
# Substituting Gates in a Logic System







# **Substituting Gates in a Logic System**







# Digital Circuits and Boolean Algebra

- There are digital circuit elements to represent primary boolean algbeabric operations.
  - Boolean OR operation = OR Gate
  - Boolean AND operation = AND Gate
  - Boolean NOT operation = NOT Gate
- Complex boolean functions can be represented as combinations of logic gates.





### **Integrated Circuits**

- Typically gates are not available as individual circuit units.
- Instead, multiple gates which are built into chips are available.
- A chip is typically mounted in a ceramic or plastic container with enternal pins to make connections.
- Not only these chips, but also complex circuits built into such a chip is generally regarded as an Integrated Circuit (IC).





# **Thank You**



