Algebra

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Origins of Algebra

- **Mesopotamia & Egypt (c. 2000–1600 BCE)**
  - Early problem-solving (linear/quadratic equations) in word problems
  - No formal symbols, but systematic procedures
- **Greek Era (c. 600 BCE–300 CE)**
  - Geometric methods for solving equations (Euclid, Apollonius)
  - Diophantus introduced proto-symbolic notation
- **Islamic Golden Age (8th–12th Century)**
  - Al-Khwarizmi's work *Al-jabr* $\rightarrow$ term "Algebra"
  - Systematic solutions for linear and quadratic equations
- **Transmission to Europe (12th–17th Century)**
  - Latin translations influenced Fibonacci, others
  - Viète, Descartes established modern symbolic notation & analytic geometry
- **Modern Algebra (19th–20th Century)**
  - Emergence of abstract algebra (groups, rings, fields)
  - Galois, Abel, and others formalized algebraic structures

What is Algebra?

Algebra is a branch of mathematics that deals with numbers, variables, and their relationships. Key concepts include:

- **Variables**: Symbols (like x, y) representing unknown or changing values.
- **Expressions**: Combinations of variables, numbers, and operations. E.g., $2x + 3$.
- **Equations**: Mathematical statements that express equality, e.g., $2x + 3 = 7$.
- **Solving Equations**: Finding values for variables that make an equation true.
- **Polynomials**: Expressions like $3x^2 + 2x - 5$ involving variables raised to powers.
- **Functions**: Describes a relationship between variables, e.g., $y = 2x + 1$.

Integers

- The set of integers is denoted by $\mathbb{Z}$.
- Integers include:

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

- Formally, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- Common properties:
  - $\mathbb{Z}$ is infinite and unbounded in both the negative and positive directions.
  - Closed under addition, subtraction, and multiplication:

$$\forall a, b \in \mathbb{Z}, \quad a \pm b \in \mathbb{Z}, \quad a \cdot b \in \mathbb{Z}.$$

- The quotient of any two integers is not necessarily an integer. So we need to extend arithmetic to **rational numbers**

Rational Numbers

- The set of rational numbers is denoted by $\mathbb{Q}$.
- Definition:

$$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p \in \mathbb{Z},\ q \in \mathbb{Z},\ q \neq 0 \right\}.$$

- Every integer is also a rational number (e.g., $5 = \frac{5}{1}$).
- Examples:

$$\frac{1}{2}, \quad -\frac{3}{4}, \quad 0, \quad 7, \quad \frac{11}{5}, \ldots$$

- Properties:
  - Closed under addition, subtraction, multiplication, and division (except division by zero).
  - Densely packed on the number line: between any two rationals, there is another rational.

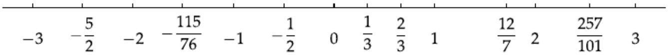
Interesting Facts

- Why division by zero is prohibited ?
  - Division is inverse of multiplication in the sense

$$\frac{m}{n} \cdot n = m$$

  - if $n = 0$ and $m = 1$, we get $\frac{1}{0} \cdot 0 = 1$ which is nonsensical as any number multiplied by zero is zero
- Rational numbers suffice for all actual physical measurements like weight, height and length
- But Geometry, Algebra and Calculus force us to consider **real numbers**

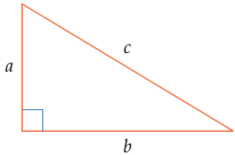
A Real Number Line



Some rational numbers on the real line.

- if n is a positive integer then $\frac{1}{n}$ is to the right of 0 by the length obtained by dividing the segment from $1 to 0$ in to n segments of equal length

Is every Real Number a Rational



- $c^2 = a^2 + b^2$. If $a = 1, b = 1$ then $c^2 = 2$. Then what rational number is c
- By trial and error, $c = \left(\frac{99}{70}\right)^2 = \frac{9801}{4900}$ where the numerator just misses twice the denominator by 1. But this is not 2 but close to 2. Another number is $\left(\frac{9369319}{6625109}\right)^2 = 1.999999999999977$, but not 2
- Greeks proved that it is impossible to find any rational number whose square is 2

Proof: No rational number has a square equal to 2

Let m and n are two integers

$$\left(\frac{m}{n}\right)^2 = 2$$

By canceling any common factors, m and n are reduces to its lowest terms

$$m^2 = 2n^2$$

this makes m^2 even, hence m is an even. (The square of even is even and odd is odd). So $m = 2k$ for some integer k
Substituting $m = 2k$ in the equation gives, $4k^2 = 2n^2$, which results in

$$2k^2 = n^2$$

which means n^2 is even and therefore n is even
$\frac{m}{n}$ has common factors which contradicts the earlier assumption

Irrational Number

Irrational Number

A real number that is not rational is **irrational number**

- $\sqrt(2)$
- $3 + \sqrt(2)$
- $8\sqrt(2)$

Properties of Real Numbers

- **Commutative Properties**
  - Addition: $a + b = b + a$
  - Multiplication: $a \cdot b = b \cdot a$
- **Associative Properties**
  - Addition: $(a + b) + c = a + (b + c)$
  - Multiplication: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- **Distributive Property**
  - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
- **Identity Elements**
  - Additive Identity: $a + 0 = a$
  - Multiplicative Identity: $a \cdot 1 = a$
- **Inverse Elements**
  - Additive Inverse: $a + (-a) = 0$
  - Multiplicative Inverse (if $a \neq 0$): $a \cdot \frac{1}{a} = 1$

Properties of Real Numbers

- **Closure Property**
  - Real numbers are closed under addition, subtraction, multiplication, and division (except division by zero).

Inequalities

- **Transitivity**
    - If $a < b$ and $b < c$, then $a < c$