

# Algebra

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February 9, 2025

# Origins of Algebra

- **Mesopotamia & Egypt (c. 2000–1600 BCE)**
  - Early problem-solving (linear/quadratic equations) in word problems
  - No formal symbols, but systematic procedures
- **Greek Era (c. 600 BCE–300 CE)**
  - Geometric methods for solving equations (Euclid, Apollonius)
  - Diophantus introduced proto-symbolic notation
- **Islamic Golden Age (8th–12th Century)**
  - Al-Khwarizmi's work *Al-jabr* → term “Algebra”
  - Systematic solutions for linear and quadratic equations
- **Transmission to Europe (12th–17th Century)**
  - Latin translations influenced Fibonacci, others
  - Viète, Descartes established modern symbolic notation & analytic geometry
- **Modern Algebra (19th–20th Century)**
  - Emergence of abstract algebra (groups, rings, fields)
  - Galois, Abel, and others formalized algebraic structures

# What is Algebra?

Algebra is a branch of mathematics that deals with numbers, variables, and their relationships. Key concepts include:

- **Variables:** Symbols (like  $x$ ,  $y$ ) representing unknown or changing values.
- **Expressions:** Combinations of variables, numbers, and operations. E.g.,  $2x + 3$ .
- **Equations:** Mathematical statements that express equality, e.g.,  $2x + 3 = 7$ .
- **Solving Equations:** Finding values for variables that make an equation true.
- **Polynomials:** Expressions like  $3x^2 + 2x - 5$  involving variables raised to powers.
- **Functions:** Describes a relationship between variables, e.g.,  $y = 2x + 1$ .

# Integers

- The set of integers is denoted by  $\mathbb{Z}$ .
- Integers include:

$$\dots, -3, -2, -1, 0, 1, 2, 3, \dots$$

- Formally,  $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ .
- Common properties:
  - $\mathbb{Z}$  is infinite and unbounded in both the negative and positive directions.
  - Closed under addition, subtraction, and multiplication:

$$\forall a, b \in \mathbb{Z}, \quad a \pm b \in \mathbb{Z}, \quad a \cdot b \in \mathbb{Z}.$$

- The quotient of any two integers is not necessarily an integer. So we need to extend arithmetic to **rational numbers**

# Rational Numbers

- The set of rational numbers is denoted by  $\mathbb{Q}$ .

- Definition:

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0 \right\}.$$

- Every integer is also a rational number (e.g.,  $5 = \frac{5}{1}$ ).

- Examples:

$$\frac{1}{2}, \quad -\frac{3}{4}, \quad 0, \quad 7, \quad \frac{11}{5}, \dots$$

- Properties:

- Closed under addition, subtraction, multiplication, and division (except division by zero).
- Densely packed on the number line: between any two rationals, there is another rational.

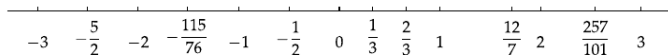
# Interesting Facts

- Why division by zero is prohibited ?
  - Division is inverse of multiplication in the sense

$$\frac{m}{n} \cdot n = m$$

- if  $n = 0$  and  $m = 1$ , we get  $\frac{1}{0} \cdot 0 = 1$  which is nonsensical as any number multiplied by zero is zero
- Rational numbers suffice for all actual physical measurements like weight, height and length
- But Geometry, Algebra and Calculus force us to consider **real numbers**

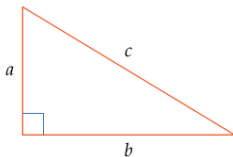
# A Real Number Line



*Some rational numbers on the real line.*

- if  $n$  is a positive integer then  $\frac{1}{n}$  is to the right of 0 by the length obtained by dividing the segment from 1 to 0 in to  $n$  segments of equal length

# Is every Real Number a Rational



- $c^2 = a^2 + b^2$ . If  $a = 1, b = 1$  then  $c^2 = 2$ . Then what rational number is  $c$
- By trial and error,  $c = \left(\frac{99}{70}\right)^2 = \frac{9801}{4900}$  where the numerator just misses twice the denominator by 1. But this is not 2 but close to 2. Another number is  $\left(\frac{9369319}{6625109}\right)^2 = 1.9999999999999977$ , but not 2
- Greeks proved that it is impossible to find any rational number whose square is 2



# Proof: No rational number has a square equal to 2

Let  $m$  and  $n$  are two integers

$$\left(\frac{m}{n}\right)^2 = 2$$

By canceling any common factors,  $m$  and  $n$  are reduces to its lowest terms

$$m^2 = 2n^2$$

this makes  $m^2$  even, hence  $m$  is an even. (The square of even is even and odd is odd). So  $m = 2k$  for some integer  $k$

Substituting  $m = 2k$  in the equation gives,  $4k^2 = 2n^2$ , which results in

$$2k^2 = n^2$$

which means  $n^2$  is even and therefore  $n$  is even

$\frac{m}{n}$  has common factors which contradicts the earlier assumption

# Irrational Number

## Irrational Number

A real number that is not rational is **irrational number**

- $\sqrt{2}$
- $3 + \sqrt{2}$
- $8\sqrt{2}$

# Properties of Real Numbers

## • Commutative Properties

- Addition:  $a + b = b + a$
- Multiplication:  $a \cdot b = b \cdot a$

## • Associative Properties

- Addition:  $(a + b) + c = a + (b + c)$
- Multiplication:  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

## • Distributive Property

- $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

## • Identity Elements

- Additive Identity:  $a + 0 = a$
- Multiplicative Identity:  $a \cdot 1 = a$

## • Inverse Elements

- Additive Inverse:  $a + (-a) = 0$
- Multiplicative Inverse (if  $a \neq 0$ ):  $a \cdot \frac{1}{a} = 1$

- **Closure Property**

- Real numbers are closed under addition, subtraction, multiplication, and division (except division by zero).

## Transitivity

- If  $a < b$  and  $b < c$ , then  $a < c$

## Multiplication

Suppose  $a < b$

- If  $c > 0$ , then  $ac < bc$
- If  $c < 0$ , then  $ac > bc$

# Exercise

Find all number  $x$  such that

$$\frac{x-8}{x-4} < 3$$

Our first step is to multiply by  $x-4$  Here there are two conditions:

①  $x-4 > 0$

$$x-8 < 3(x-4) \implies x-8 < 3x-12 \implies 2x > 4 \implies x > 2$$

But our initial assumption is  $x-4 > 0 \implies x > 4$ . As  $4 > 2$ , original inequality holds if  $x > 4$

## Exercise Conti.

②  $x - 4 < 0$

$$x - 8 > 3(x - 4) \implies x < 2$$

Initial assumption is  $x < 4$ . As  $2 < 4$ , inequality holds for  $x < 2$

The original inequality holds true for

$$x < 2, x > 4$$

or

$$(-\infty, 2) \cup (4, \infty)$$

# Inequalities

## Additive Inverse

If  $a < b$  then  $-a > -b$  Direction of inequalities has to be reversed when taking additive inverses on both sides

## Multiplicative Inverse

If  $a < b$

- If  $a > 0, b > 0$ , then  $\frac{1}{a} > \frac{1}{b}$
- If  $a < 0 < b$ , then  $\frac{1}{a} < \frac{1}{b}$



# What is a Set?

## Definition

A **set** is a well-defined collection of distinct objects, called **elements** or **members** of the set.

## Representation of a Set:

- **Roster Form:** List elements inside curly braces:

$$A = \{1, 2, 3, 4\}$$

- **Set-Builder Notation:** Describe properties of elements:

$$A = \{x \mid x \text{ is a positive integer less than } 5\}$$

## Membership

- If  $x$  belongs to  $A$ , write  $x \in A$ .
- If  $x$  does not belong to  $A$ , write  $x \notin A$ .

## Types of Sets

- **Finite Set:** A set with a countable number of elements.  
Example:  $A = \{1, 2, 3, 4\}$
- **Infinite Set:** A set with an uncountable or infinite number of elements.  
Example:  $\mathbb{N} = \{1, 2, 3, \dots\}$
- **Empty/Null Set:** A set with no elements, denoted as  $\emptyset$  or  $\{\}$ .
- **Subset:**  $A \subseteq B$  if every element of  $A$  is in  $B$ .
- **Universal Set:** A set containing all objects under consideration, usually denoted by  $U$ .
- **Power Set:** The set of all subsets of  $A$ , denoted as  $P(A)$ .  
Example: If  $A = \{1, 2\}$ , then  $P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ .

# Set Operations

## Union ( $\cup$ )

Combines elements of two sets:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

## Intersection ( $\cap$ )

Elements common to both sets:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

## Difference ( $A - B$ )

Elements in  $A$  but not in  $B$ :

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

# Set Operations

## Complement ( $A^c$ )

Elements not in the set  $A$ :

$$A^c = \{x \mid x \notin A\}$$

## Examples

- The set of natural numbers:  $\mathbb{N} = \{1, 2, 3, \dots\}$ .
- The set of integers:  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ .
- The set of even numbers:  $\{2, 4, 6, \dots\}$ .

# What is an Interval?

## Definition

An **interval** is a set of real numbers that includes all the numbers between two given endpoints.

Intervals describe ranges of values on the real number line and are widely used in mathematics.

# Types of Intervals

- **Closed Interval**  $[a, b]$ : Includes both endpoints  $a$  and  $b$ .

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

Example:  $[2, 5] = \{x \mid 2 \leq x \leq 5\}$ .

- **Open Interval**  $(a, b)$ : Excludes both endpoints  $a$  and  $b$ .

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

Example:  $(2, 5) = \{x \mid 2 < x < 5\}$ .

# Half-Open or Half-Closed Intervals

- **Left-Closed, Right-Open**  $([a, b))$ :

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

Example:  $[2, 5) = \{x \mid 2 \leq x < 5\}$ .

- **Left-Open, Right-Closed**  $((a, b])$ :

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

Example:  $(2, 5] = \{x \mid 2 < x \leq 5\}$ .

# Infinite Intervals

- $(a, \infty)$ : All numbers greater than  $a$ .

$$(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$$

Example:  $(3, \infty)$  includes all numbers greater than 3.

- $(-\infty, b)$ : All numbers less than  $b$ .

$$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$$

Example:  $(-\infty, 4)$  includes all numbers less than 4.

- $(-\infty, \infty)$ : The entire real number line.

$$(-\infty, \infty) = \mathbb{R}$$



# Summary of Interval Types

Type	Interval Notation	Description
Closed	$[a, b]$	Includes both endpoints $a, b$
Open	$(a, b)$	Excludes both endpoints $a, b$
Half-Open Left	$[a, b)$	Includes $a$ , excludes $b$
Half-Open Right	$(a, b]$	Excludes $a$ , includes $b$
Infinite Left	$(-\infty, b)$	All $x < b$
Infinite Right	$(a, \infty)$	All $x > a$
Entire Line	$(-\infty, \infty)$	All real numbers

# What is Absolute Value?

## Definition

The **absolute value** of a number is its distance from zero on the number line, regardless of direction. It is always non-negative.

For a real number  $x$ , the absolute value, denoted as  $|x|$ , is defined as:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Breaking the absolute value:

- $|f(x)| \leq c \implies -c \leq f(x) \leq c$
- $|f(x)| \geq c \implies f(x) \leq -c \text{ or } f(x) \geq c$

# Examples of Absolute Value

- $|3| = 3$  (because  $3 \geq 0$ )
- $|-5| = -(-5) = 5$  (because  $-5 < 0$ )
- $|0| = 0$  (because 0 is neither positive nor negative)

# Properties of Absolute Value

- **Non-Negativity:**  $|x| \geq 0$  for all  $x$ .
- **Identity Property:**  $|x| = 0$  if and only if  $x = 0$ .
- **Multiplicative Property:**  $|x \cdot y| = |x| \cdot |y|$ .
- **Triangle Inequality:**  $|x + y| \leq |x| + |y|$ .
- **Distance Interpretation:**  $|x - y|$  represents the distance between  $x$  and  $y$ .

Ball bearings need to have extremely accurate sizes to work correctly. The ideal diameter of a particular ball bearing is 0.8 cm, but a ball bearing is declared acceptable if the error in the diameter size is less than 0.001 cm. Write the inequality for acceptance criteria Solution:

Ball bearings need to have extremely accurate sizes to work correctly. The ideal diameter of a particular ball bearing is 0.8 cm, but a ball bearing is declared acceptable if the error in the diameter size is less than 0.001 cm. Write the inequality for acceptance criteria Solution: The ball bearings are acceptable if diameter  $d$  is

$$|d - 0.8| \leq 0.001$$

Find all numbers  $t$  such that  $|3t - 4| = 10$

Solution :

Find all numbers  $t$  such that  $|3t - 4| = 10$

Solution :

$$3t - 4 = 10 \text{ or } 3t - 4 = -10 \implies t = \frac{14}{3}, t = -2$$



# Exercise

Find all numbers  $x$  such that

$$\left| \frac{3x - 5}{x - 1} \right| < 2$$

Solution :

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Find all numbers  $x$  such that

$$\left| \frac{3x - 5}{x - 1} \right| < 2$$

Solution :

$$|3x - 5| < 2|x - 1|$$

①  $x - 1 > 0$

Breaking the absolute value:

$$\implies -2(x - 1) < 3x - 5 < 2(x - 1) = -2x + 7 < 3x < 2x + 3 \quad (1)$$

$$\implies 3x > -2x + 7 \text{ \& } 3x < 2x + 3 \quad (2)$$

# Exercise

Solving for  $3x < 2x + 3$

$$\implies x < 3 \quad (3)$$

(4)

Solving for  $3x > -2x + 7$

$$\implies 3x > -2x + 7 \implies 5x > 7 \implies x > 7/5 \quad (5)$$

$$\implies x \in (7/5, 3) \quad (6)$$

# Exercise

$$\textcircled{2} \quad x - 1 < 0 \implies x < 1$$

$$\implies |3x - 5| < 2|x - 1| \implies |3x - 5| < -2(x - 1) \quad (7)$$

$$\implies 3x - 5 < -2(x - 1) \text{ and } -(3x - 5) < -2(x - 1) \quad (8)$$

$$\implies 3x - 5 < -2(x - 1) \text{ and } 3x - 5 > 2(x - 1) \quad (9)$$

$$3x - 5 < -2(x - 1) \implies 3x < -2x + 7 \implies 5x < 7 \implies x < 7/5 \quad (10)$$

$$\implies 3x - 5 > 2(x - 1) \implies 3x > 2x + 3 \implies x > 3 \quad (11)$$

Here  $x > 3$  is inconsistent with our assumption  $x < 1$ . So for  $x < 1$  there are no values of  $x$  satisfying the inequality

# What is a Function ?

## What is a Function?

A function associates every number in some set of real numbers, called the domain of the function, with exactly one real number

If a function is defined by a formula, with no domain specified, then the domain is assumed to be the set of all real numbers for which the formula makes sense and produces a real number

## Example 3

Find the domain of the function  $f$  defined by

$$f(x) = (3x - 1)^2$$

## Example 4

Find the domain of the function  $f$  defined by

$$h(t) = \frac{t^2 + 3t + 7}{t - 4}$$

## Example 6

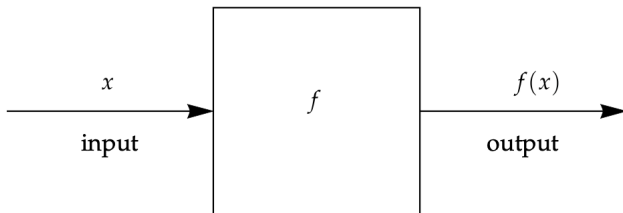
Find the domain of the function  $g$  defined by

$$g(x) = \sqrt{|x| - 5}$$

# Functions

## Range

The range of a function  $f$  is the set of all numbers  $y$  such that  $f(x) = y$  for at least one  $x$  in the domain of  $f$



*The set of inputs acceptable by this machine is the domain of  $f$ .  
The set of outputs is the range of  $f$ .*



## Example 4

The domain of  $f$  is the interval  $[2, 5]$ , with  $f$  defined on this interval by the equation  $f(x) = 3x + 1$

Solution :

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$$y = f(x) = 3x + 1$$

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Solution :

$$y = f(x) = 3x + 1$$

$$2 \leq \frac{y - 1}{3} \leq 5.$$

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Solution :

$$y = f(x) = 3x + 1$$

$$2 \leq \frac{y - 1}{3} \leq 5.$$

$$7 \leq y \leq 16.$$

## Example 4

The domain of  $f$  is the interval  $[2, 5]$ , with  $f$  defined on this interval by the equation  $f(x) = 3x + 1$

Solution :

$$y = f(x) = 3x + 1$$

$$2 \leq \frac{y - 1}{3} \leq 5.$$

$$7 \leq y \leq 16.$$

## Example 5

The domain of  $g$  is the interval  $[1, 20]$ , with  $g$  defined on this interval by the equation

$$g(x) = |x - 5|.$$

Is 2 in the range of  $g$ ?

Solution :

## Example 5

The domain of  $g$  is the interval  $[1, 20]$ , with  $g$  defined on this interval by the equation

$$g(x) = |x - 5|.$$

Is 2 in the range of  $g$ ?

Solution :

$$y = |x - 5|$$

$$\text{for } x - 5 > 0, y = x - 5 \implies x = y + 5$$

$$5 < y + 5 \leq 20 \implies 0 < y \leq 15$$

$$\text{for } x - 5 < 0, y = -(x - 5) \implies y = -x + 5 \implies 5 - y = x$$

$$1 \leq 5 - y \leq 5 \implies -4 \leq -y \leq 0 \implies 4 \geq y \geq 0$$

# Equality of Functions

Two functions are equal if and only if they have the same domain and the same value at every number in that domain

## Example

Suppose  $f$  is the function whose domain is the set of real numbers, with  $f$  defined on this domain by

$$f(x) = x^2$$

Suppose  $g$  is the function whose domain is the set of positive numbers, with  $g$  defined on this domain by

$$g(x) = x^2$$

Are  $f$  and  $g$  equal functions ?



# Equality of functions

## Example 2

Suppose  $f$  and  $g$  are functions whose domain is the set consisting of the two numbers  $\{1, 2\}$  with  $f$  and  $g$  defined on this domain by the formulas

$$f(x) = x^2$$

and

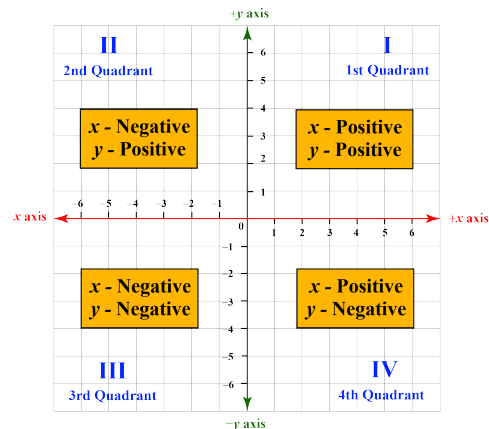
$$g(x) = 3x - 2$$

.Are  $f$  and  $g$  equal functions?

# What is Analytic Geometry?

- **Analytic Geometry** (also called *coordinate geometry* or *Cartesian geometry*) bridges algebra and geometry.
- It uses a coordinate system to study geometric shapes and properties.
- Geometric objects are represented as algebraic equations.

# Co-ordinate Plane

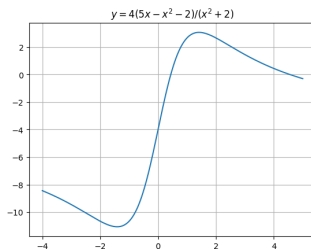
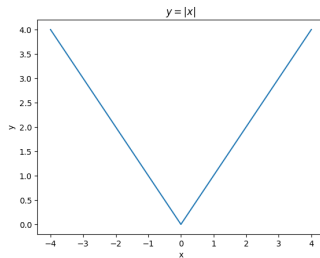


The plane with this system of labeling is often called the **Cartesian plane** in honor of the French mathematician Rene Descartes(1596-1650), who described this technique in his 1637 book Discourse on Method

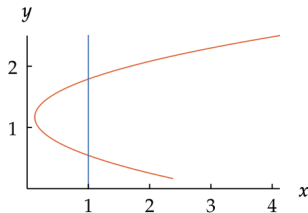
# Graph Functions

The graph of a function  $f$  is the set of points of the form  $x, f(x)$  as  $x$  varies over the domain of  $f$

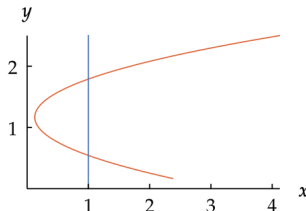
# Graph of a Function



# Checking for a function: Vertical line test



# Checking for a function: Vertical line test



The line  $x = 1$  intersects the curve at two points. That is that for each  $x$  value there are multiple  $y$  values which is contradicting to definition of a function

## Vertical Line Test

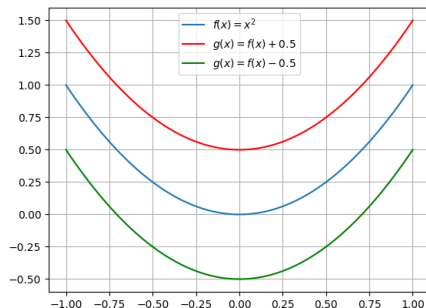
A set of points in the coordinate plane is the graph of some function if and only if every vertical line intersects the set in at most one point

# Vertical Transformation

## Shifting a graph up or down

Suppose  $f$  is a function and  $a > 0$ . Upshift  $g$  and Downshift  $h$  by

$$g(x) = f(x) + a \quad h(x) = f(x) - a$$



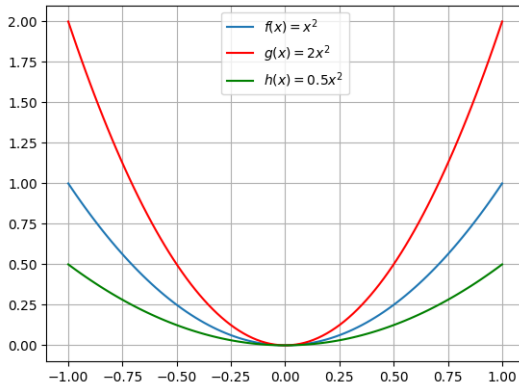


# Vertical Transformation

## Vertical Stretch

Suppose  $f$  is a function and  $c > 0$ . Define a function  $g$  by

$$g(x) = cf(x)$$

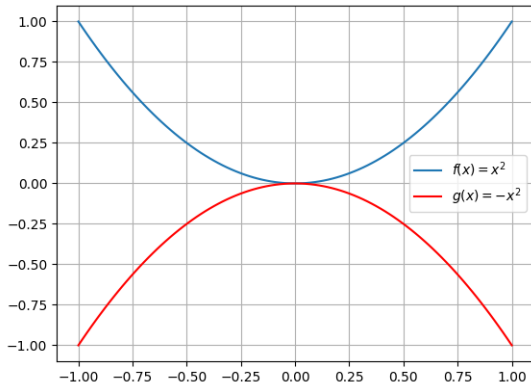


# Vertical Transformation

## Flipping along the Vertical Axis

Vertical flipping of  $f(x)$  is

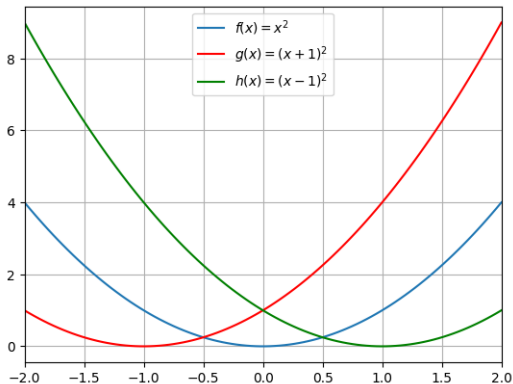
$$g(x) = -f(x)$$



# Horizontal Transformation

## Horizontal Shift

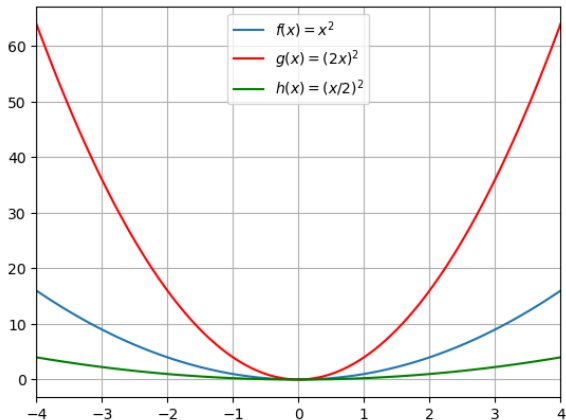
$$g(x) = f(x + a), h(x) = f(x - a)$$



# Horizontal Transformation

## Horizontal Stretching

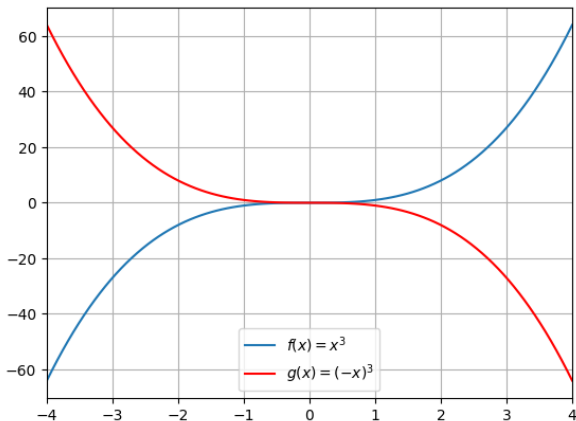
$$g(x) = f(cx)$$



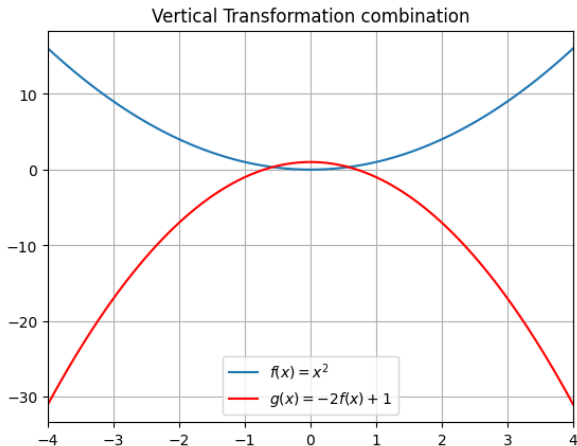
# Flipping ac the Vertical Axis

## Horizontal Stretching

$$g(x) = f(-x)$$



# Combinations of vertical Transformation



# Even Functions

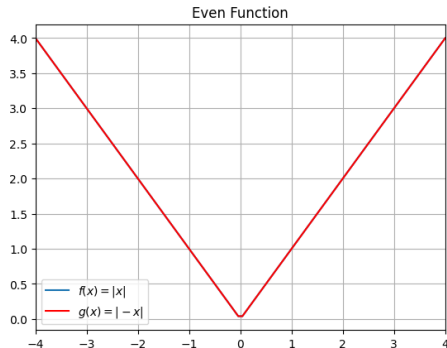
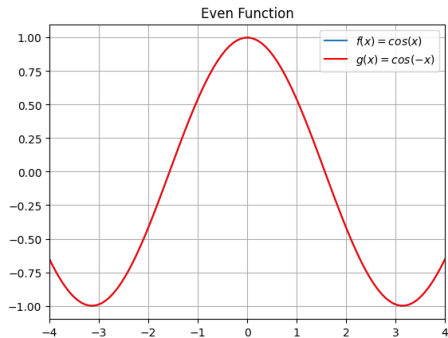
## Even

$$f(-x) = f(x) \quad \text{for all } x \text{ in the domain}$$

Example:  $f(x) = x^2$ ,  $f(x) = \cos x$

The graph of an even function is symmetric across the vertical axis

# Even Functions





# Odd Function

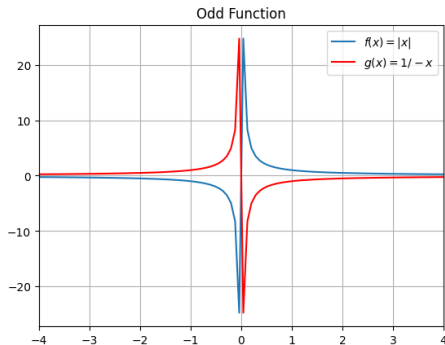
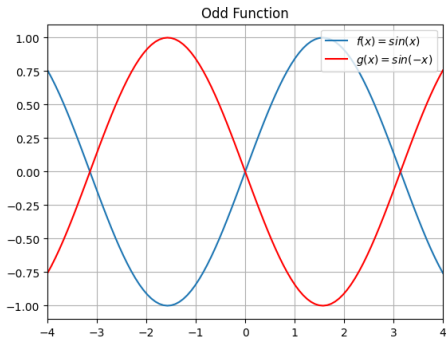
## Odd

$$f(-x) = -f(x) \quad \text{for all } x \text{ in the domain}$$

Example:  $f(x) = x^3$ ,  $f(x) = \sin x$

The graph of an even function is symmetric if flipped or rotated 180 across the origin

# Odd Function



# Algebra of Functions

Suppose  $f$  and  $g$  are functions. We can define new functions from  $f$  and  $g$  as follows:

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**Product:**

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

**Quotient:**

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{provided } g(x) \neq 0.$$

**Note:** If  $f$  and  $g$  have domains  $D_f$  and  $D_g$ , then these operations are defined on the intersection  $D_f \cap D_g$ . In the case of the quotient, it is defined on

$$\{x \in D_f \cap D_g : g(x) \neq 0\}.$$

# Exercise

$$f(x) = \sqrt{x-3} \text{ and } g(x) = \sqrt{8-x}$$

Evaluate

- a.  $(f+g)(x)$
- b.  $(fg)(x)$
- c. Find the domain of above



# Exercise

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- a.  $(f+g)(x)$
- b.  $(fg)(x)$
- c. Find the domain of above

Sol:

- a.  $\sqrt{x-3} + \sqrt{8-x}$
- b.  $\sqrt{(x-3)(8-x)}$
- c. Domain of
  - a.  $x \geq 3$
  - b.  $x \leq 8$
  - c.  $3 \leq x \leq 8$

# Function Composition

## Definition:

If  $f(x)$  and  $g(x)$  are functions, then the composition of  $f$  and  $g$ , denoted by  $f \circ g$ , is defined by

$$(f \circ g)(x) = f(g(x)).$$

**Example:** Consider the function

$$h(x) = \sqrt{x+3}.$$

We can express  $h(x)$  as a composition of two functions  $f$  and  $g$  where:

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = x + 3.$$

Then,

$$h(x) = f(g(x)) = f(x+3) = \sqrt{x+3}.$$

# Exercise

$$f(x) = \frac{1}{x-4} \text{ and } g(x) = x^2$$

①  $f \circ g$

②  $g \circ f$

③ domain of  $f \circ g$

④ domain of  $g \circ f$

Sol:

# Exercise

$$f(x) = \frac{1}{x-4} \text{ and } g(x) = x^2$$

①  $f \circ g$

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Sol:

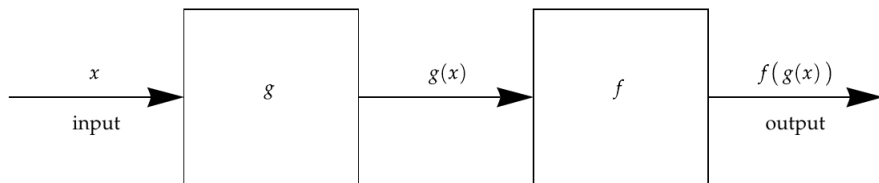
①  $f(g(x)) = \frac{1}{x^2-4}$

②  $g(f(x)) = \left(\frac{1}{x-4}\right)^2$

③  $\mathbb{R} - \{-2, 2\}$

④  $\mathbb{R} - \{4\}$

# Composition Machine



*The composition  $f \circ g$  as the combination of two machines.*

# Excercise

**Problem:** Suppose your cell phone company charges \$0.05 per minute plus \$0.47 for each call to China.

- (a) Find a function  $p$  that gives the amount charged by your cell phone company for a call to China as a function of the number of minutes  $m$ .
- (b) Suppose the tax on cell phone bills is 6% plus \$0.01 for each call. Find a function  $t$  that gives your total cost, including tax, for a call to China as a function of the amount charged by your cell phone company.
- (c) Explain why the composition  $t \circ p$  gives your total cost, including tax, of making a cell phone call to China as a function of the number of minutes.
- (d) Compute a formula for  $t \circ p$ .
- (e) What is your total cost for a ten-minute call to China?

# Solution

**(a)** The company charges \$0.05 per minute plus a fixed charge of \$0.47 per call. Hence, the pre-tax charge function is

$$p(m) = 0.05m + 0.47.$$

**(b)** The tax on the cell phone bill is 6% of the pre-tax amount plus an additional \$0.01 per call. Thus, if the pre-tax charge is  $x$ , the total cost function (including tax) is

$$t(x) = 1.06x + 0.01.$$

**(c)** The composition  $t \circ p$  means we first compute the pre-tax charge  $p(m)$  for a call of  $m$  minutes, and then we apply the tax function  $t$  to this amount. In other words,  $t(p(m))$  gives the total cost, including tax, as a function of the number of minutes.

**(d)** To compute the composition, substitute  $p(m)$  into  $t$ :

$$(t \circ p)(m) = t(p(m)) = 1.06(0.05m + 0.47) + 0.01.$$

# Solution

Distribute 1.06:

$$1.06(0.05m) = 0.053m \quad \text{and} \quad 1.06(0.47) = 0.4982.$$

Thus,

$$(t \circ p)(m) = 0.053m + 0.4982 + 0.01 = 0.053m + 0.5082.$$

**(e)** For a ten-minute call ( $m = 10$ ):

$$(t \circ p)(10) = 0.053(10) + 0.5082 = 0.53 + 0.5082 = 1.0382.$$

Rounded to the nearest cent, the total cost is approximately \$1.04.



## Identity Function

The identity function is defined by

$$I(x) = x \quad \text{for every number } x.$$

The function  $I$  is the identity for composition

If  $f$  is any function, then

$$f \circ I = I \circ f = f.$$

# Decomposing the Functions

## Function Decomposition

$$T(y) = \frac{|y^2 - 3|}{|y^2 - 7|}$$

Sol:

# Decomposing the Functions

## Function Decomposition

$$T(y) = \frac{|y^2 - 3|}{|y^2 - 7|}$$

Sol:

$$f(y) = |y|, g(y) = \frac{y^2 - 3}{y^2 - 7}$$

$$f(y) = \frac{|y - 3|}{|y - 7|}, g(y) = y^2$$

## Composition is associative

if  $f, g, h$  are functions then

$$(f \circ g) \circ h = f \circ (g \circ h)$$

# Example

## Composition of three functions

$$T(x) = \left| \frac{x^2 - 3}{x^2 - 7} \right|$$

Sol:

# Example

## Composition of three functions

$$T(x) = \left| \frac{x^2 - 3}{x^2 - 7} \right|$$

Sol:

$$f(x) = |x|, g(x) = \frac{x - 3}{x - 7}, h(x) = x^2$$

# Linear Functions

## Linear Function

A linear function is a function  $h$  of the form

$$h(x) = mx + b$$

where  $m$  and  $b$  are numbers

# Linear Functions as Composition

## Vertical Transformations as Compositions

A function  $g(x)$  is defined by

$$g(x) = -2f(x) + 1$$

Write  $g(x)$  as a the composition of a linear function with  $f(x)$

$$h(x) = -2x + 1$$

$$\implies g(x) = h(f(x)) \implies g = h \circ f$$

# Linear Function as Composition

## Horizontal Transformations as Compositions

A function  $g(x)$  is defined by

$$g(x) = f(2x) + 1$$

Write  $g(x)$  as a the composition of a linear function,  $f(x)$  and other linear function

$$h(x) = x + 1, p(x) = 2x$$

$$\implies g(x) = h(f(p(x))) \implies g = h \circ f \circ p$$



# Inverse Function: Example

Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$y = \frac{9}{5}x + 32,$$

which converts a temperature  $x$  in Celsius to Fahrenheit  $y$ . The inverse function  $f^{-1}$  converts Fahrenheit back to Celsius:

$$f^{-1}(y) = \frac{5}{9}(y - 32).$$

Verifying that these functions are inverses:

$$f^{-1}(f(x)) = \frac{5}{9}\left(\frac{9}{5}x + 32 - 32\right) = x,$$

$$f(f^{-1}(y)) = \frac{9}{5}\left(\frac{5}{9}(y - 32)\right) + 32 = y.$$

# One-to-One Function

## One-to-One Function

A function  $f$  is called one-to-one if for each number  $y$  in the range of  $f$  there is exactly one number  $x$  in the domain of  $f$  such that  $f(x) = y$

## Definition

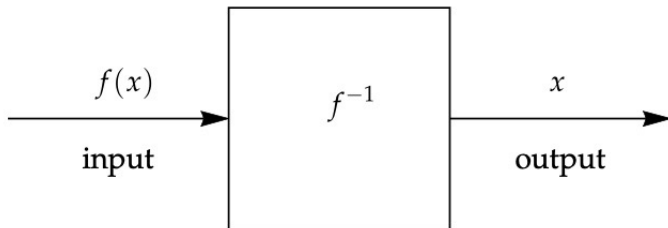
Suppose  $f$  is a one-to-one function.

- If  $y$  is in the range of  $f$ , then  $f^{-1}(y)$  is defined to be the number  $x$  such that  $f(x) = y$ .
- The function  $f^{-1}$  is called the *inverse function* of  $f$ .

## Short version:

- $f^{-1}(y) = x$  means  $f(x) = y$ .

# Inverse Function



# Domain and Range of an Inverse Function

## Properties

If  $f$  is a one-to-one function, then:

- The domain of  $f^{-1}$  equals the range of  $f$ .
- The range of  $f^{-1}$  equals the domain of  $f$ .

# Increasing and Decreasing Function

## Increasing

A function  $f$  is called increasing if  $f(a) < f(b)$  whenever  $a < b$  and  $a, b$  are in the domain of  $f$

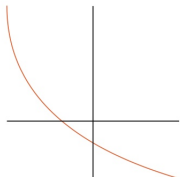
## Decreasing

A function  $f$  is called decreasing if  $f(a) > f(b)$  whenever  $a < b$  and  $a, b$  are in the domain of  $f$

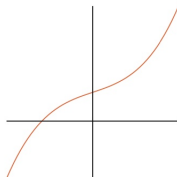
## Increasing and decreasing functions are one-to-one

- Every increasing function is one-to-one
- Every decreasing function is one-to-one.

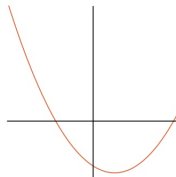
# Exercise



*The graph of  $f$ .*



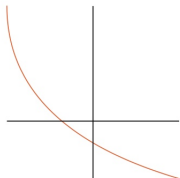
*The graph of  $g$ .*



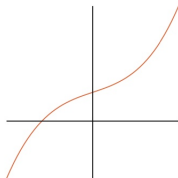
*The graph of  $h$ .*

- (a) Is  $f$  increasing, decreasing, or neither?
- (b) Is  $g$  increasing, decreasing, or neither?
- (c) Is  $h$  increasing, decreasing, or neither?

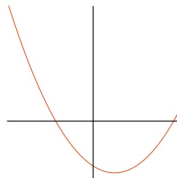
# Exercise



*The graph of  $f$ .*



*The graph of  $g$ .*



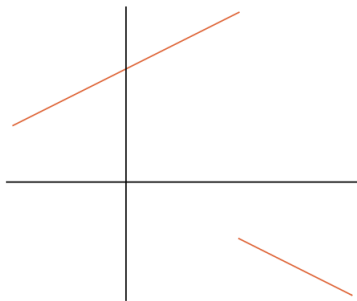
*The graph of  $h$ .*

- (a) Is  $f$  increasing, decreasing, or neither?
- (b) Is  $g$  increasing, decreasing, or neither?
- (c) Is  $h$  increasing, decreasing, or neither?

- a. Decreasing
- b. Increasing
- c. Neither



# Do all one-to-one maps are increasing or decreasing ?



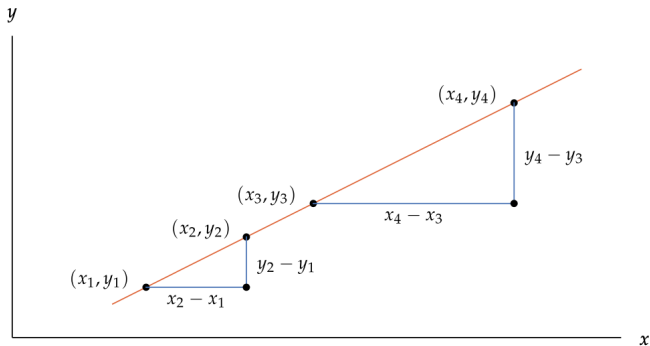
*The graph of a one-to-one function  
that is neither increasing nor  
decreasing.*

# Increasing and Decreasing Functions

## Inverses of increasing and decreasing functions

- The inverse of an increasing function is increasing.
- The inverse of a decreasing function is decreasing.

# Slope



$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}$$

## Definition

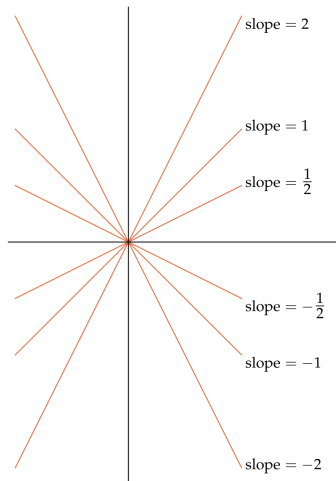
If  $x_1, y_1$  and  $x_2, y_2$  are any two points on a line with  $x_1 \neq x_2$ , then the **slope** of the line is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

# Slope

## Key Points:

- Positive slope slants up from left to right
- Negative slope slands down from left to right
- Horizontal line has slope = 0
- Vertical line has no slope

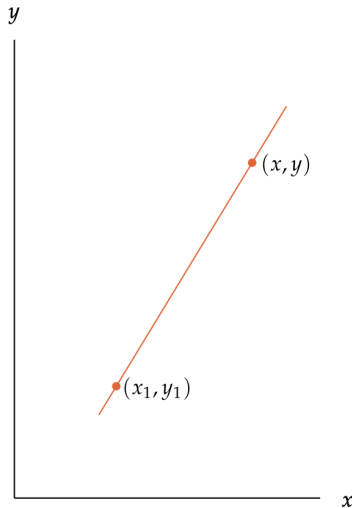


# Line Equation

## slope and one point on it

The line in the  $xy$ -plane that has slope  $m$  and contains the point  $(x_1, y_1)$  is given by the equation

$$y - y_1 = m(x - x_1)$$

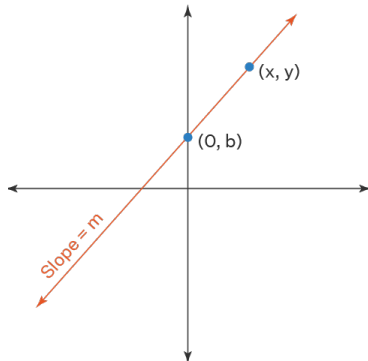


# Line Equation

## slope and $y$ intercept

The line in the  $xy$ -plane with slope  $m$  that intersects the  $y$  axis at  $0, b$  is given by the equation

$$y = mx + b$$



# Line Equation

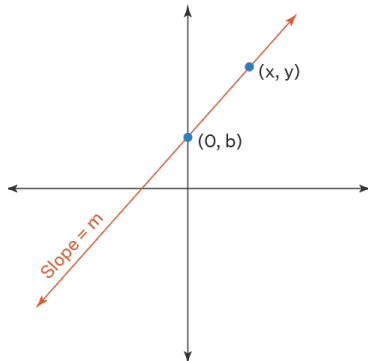
## slope and $y$ intercept

The line in the  $xy$ -plane that contains the points  $x_1, y_1$  and  $x_2, y_2$  where  $x_1 \neq x_2$ , is

$$y = mx + b$$

, is given by the equation

$$y - y_1 = \left( \frac{y_2 - y_1}{x_2 - x_1} \right) (x - x_1)$$





# Linear Function

## Definition

A **linear function** is a function  $f$  of the form

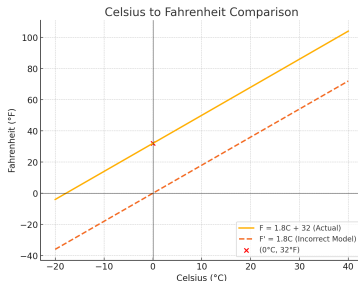
$$f(x) = mx + b$$

where  $m$  and  $b$  are numbers

# Linear Functions: Origin vs Y-Intercept

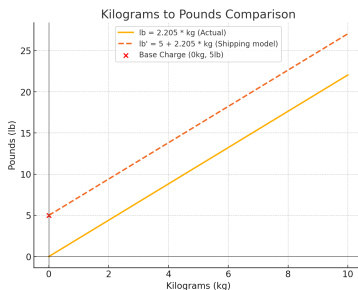
## Example 1: Temperature Conversion

- Correct formula:  $F = 1.8C + 32$   
(Starts at 32°F)
- Incorrect direct proportion:  
 $F' = 1.8C$  (Wrong assumption)



## Example 2: Weight Conversion

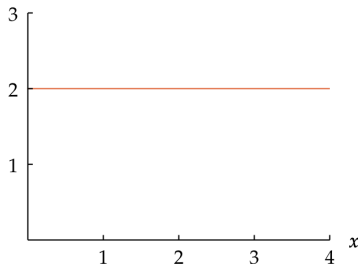
- True conversion:  
 $lb = 2.205 \times kg$  (Passes through origin)
- Shipping charge model:  
 $lb' = 5 + 2.205 \times kg$  (Has minimum billable weight or fixed cost markup)



# Constant Function

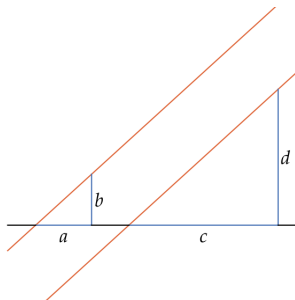
## Definition

A constant function is a function  $f$  of the form  $f(x) = b$ , where  $b$  is a number



*The orange horizontal line is the graph of the constant function  $f$  defined by  $f(x) = 2$  on the interval  $[0, 4]$ .*

# Parallel Lines

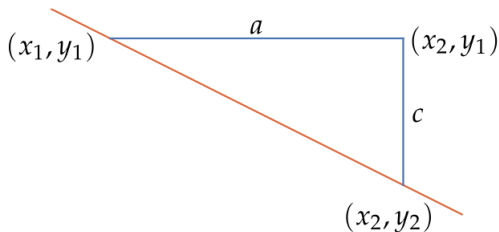


As two lines are parallel, the corresponding angles are congruent and so two triangles are similar so

$$\frac{a}{c} = \frac{b}{d} \implies \frac{b}{a} = \frac{d}{c}$$

it has same slope

# Negative Slope



As lengths are positive  $a = x_2 - x_1$  and  $c = y_1 - y_2$

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = -\frac{c}{a}$$

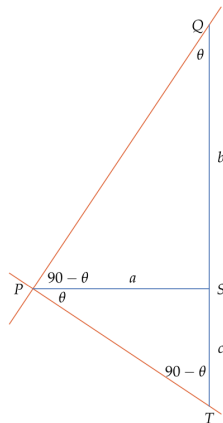
# Perpendicular Lines

$\triangle PSQ$  and  $\triangle TSP$  are similar

$$\frac{QS}{SP} = \frac{PS}{ST} \implies \frac{b}{a} = \frac{a}{c}$$

Multiplying by

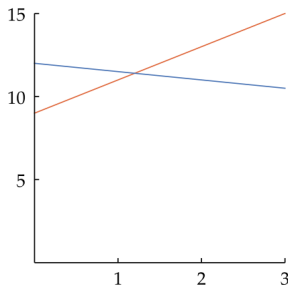
$$-\frac{c}{a} \implies \frac{b}{a} \cdot \left(-\frac{c}{a}\right) = -1$$



# Unequal Scales

Angles are distorted by unequal scales on coordinate axes

In graphs with unequal scales on the two coordinate axes, angles are not accurately represented



*The perpendicular lines*  
 $y = 2x + 9$  (orange) and  
 $y = -\frac{1}{2}x + 12$  (blue).

# Quadratic Function

## Definition

The function of the form

$$ax^2 + bx + c = 0$$

where  $a, b, c$  are real numbers with  $a \neq 0$

- if  $b^2 - 4ac < 0$ , then equation have no real solutions
- if  $b^2 - 4ac = 0$ , then equation has one solution,  $x = -\frac{b}{2a}$
- if  $b^2 - 4ac > 0$ , then equation has two solutions  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$



## Parabola

A **parabola** is the graph of a quadratic function. The **vertex** of the parabola is the where the line of symmetry of the parabola, intersects the parabola.

Suppose  $f$  is a quadratic function. Complete the square to write  $f$  in the form

$$f(x) = a(x - h)^2 + k$$

- If  $a > 0$  then  $f(x)$  attains its minimum value  $k$  when  $x = h$  and the graph of  $f$  is a parabola that opens upward.
- If  $a < 0$  then  $f(x)$  its maximum value  $k$  when  $x = h$  and the graph of  $f$  is a parabola that opens downward
- The vertex of the graph is  $h, k$

## Example

$$f(x) = -3x^2 + 12x - 8$$

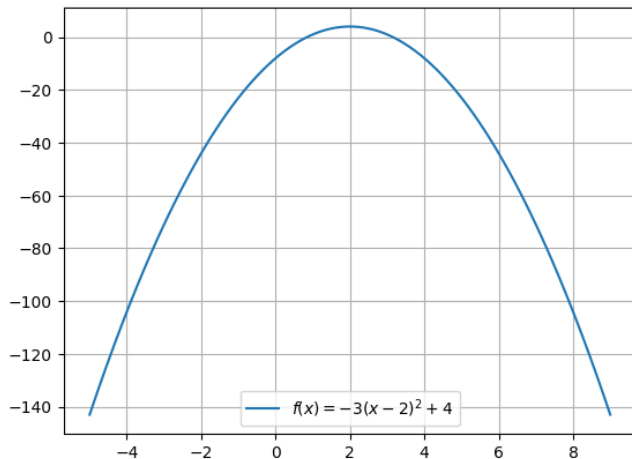
- 1 For what value of  $x$  does  $f(x)$  attain its maximum value?
- 2 What is the maximum value of  $f(x)$ ?
- 3 Find the vertex

Sol:

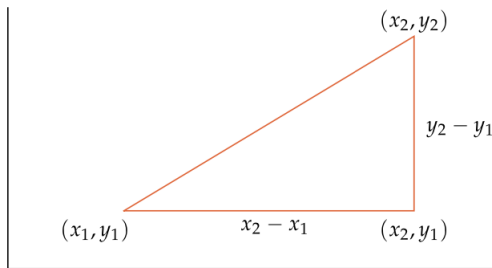
$$f(x) = -3x^2 + 12x - 8 \implies -3(x^2 - 4x + 4) + 4 \implies -3(x - 2)^2 + 4$$

- 1  $x = 2$
- 2  $f(x = 2) = 4$
- 3  $(2, 4)$

# Parabola



# Distance Between Points



## Distance Between Points

The distance between points  $x_1, y_1$  and  $x_2, y_2$  is given by

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

## Equation of a Circle

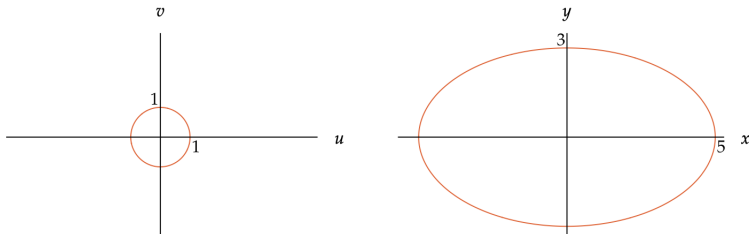
The circle with center  $h, k$  and radius  $r$  is the set of the points  $x, y$  that satisfy the equation

$$(x - h)^2 + (y - k)^2 = r^2$$

# Ellipses

## Ellipse

Stretching the circle horizontally and/or vertically produces a curve called an **ellipse**



*Stretching horizontally by a factor of 5 and vertically by a factor of 3 transforms the circle on the left into the ellipse on the right.*

## Ellipse

Equation of the circle is given by  $u^2 + v^2 = 1$

By stretching  $x = 3u, y = 5v$ ,

Substituting for  $u, v$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$$