Algebra

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Maveric Systems

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Origins of Algebra

- **Mesopotamia & Egypt (c. 2000–1600 BCE)**
  - Early problem-solving (linear/quadratic equations) in word problems
  - No formal symbols, but systematic procedures
- **Greek Era (c. 600 BCE–300 CE)**
  - Geometric methods for solving equations (Euclid, Apollonius)
  - Diophantus introduced proto-symbolic notation
- **Islamic Golden Age (8th–12th Century)**
  - Al-Khwarizmi's work *Al-jabr* $\rightarrow$ term "Algebra"
  - Systematic solutions for linear and quadratic equations
- **Transmission to Europe (12th–17th Century)**
  - Latin translations influenced Fibonacci, others
  - Viète, Descartes established modern symbolic notation & analytic geometry
- **Modern Algebra (19th–20th Century)**
  - Emergence of abstract algebra (groups, rings, fields)
  - Galois, Abel, and others formalized algebraic structures

What is Algebra?

Algebra is a branch of mathematics that deals with numbers, variables, and their relationships. Key concepts include:

- **Variables**: Symbols (like x, y) representing unknown or changing values.
- **Expressions**: Combinations of variables, numbers, and operations. E.g., $2x + 3$.
- **Equations**: Mathematical statements that express equality, e.g., $2x + 3 = 7$.
- **Solving Equations**: Finding values for variables that make an equation true.
- **Polynomials**: Expressions like $3x^2 + 2x - 5$ involving variables raised to powers.
- **Functions**: Describes a relationship between variables, e.g., $y = 2x + 1$.

Integers

- The set of integers is denoted by $\mathbb{Z}$.
- Integers include:

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

- Formally, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- Common properties:
  - $\mathbb{Z}$ is infinite and unbounded in both the negative and positive directions.
  - Closed under addition, subtraction, and multiplication:

$$\forall a, b \in \mathbb{Z}, \quad a \pm b \in \mathbb{Z}, \quad a \cdot b \in \mathbb{Z}.$$

- The quotient of any two integers is not necessarily an integer. So we need to extend arithmetic to **rational numbers**

Rational Numbers

- The set of rational numbers is denoted by $\mathbb{Q}$.
- Definition:

$$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p \in \mathbb{Z},\ q \in \mathbb{Z},\ q \neq 0 \right\}.$$

- Every integer is also a rational number (e.g., $5 = \frac{5}{1}$).
- Examples:

$$\frac{1}{2}, \quad -\frac{3}{4}, \quad 0, \quad 7, \quad \frac{11}{5}, \ldots$$

- Properties:
  - Closed under addition, subtraction, multiplication, and division (except division by zero).
  - Densely packed on the number line: between any two rationals, there is another rational.

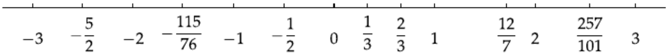
Interesting Facts

- Why division by zero is prohibited ?
  - Division is inverse of multiplication in the sense

$$\frac{m}{n} \cdot n = m$$

  - if $n = 0$ and $m = 1$, we get $\frac{1}{0} \cdot 0 = 1$ which is nonsensical as any number multiplied by zero is zero
- Rational numbers suffice for all actual physical measurements like weight, height and length
- But Geometry, Algebra and Calculus force us to consider **real numbers**

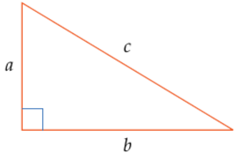
A Real Number Line



Some rational numbers on the real line.

- if n is a positive integer then $\frac{1}{n}$ is to the right of 0 by the length obtained by dividing the segment from $1 to 0$ in to n segments of equal length

Is every Real Number a Rational



- $c^2 = a^2 + b^2$. If $a = 1, b = 1$ then $c^2 = 2$. Then what rational number is c
- By trial and error, $c = \left(\frac{99}{70}\right)^2 = \frac{9801}{4900}$ where the numerator just misses twice the denominator by 1. But this is not 2 but close to 2. Another number is $\left(\frac{9369319}{6625109}\right)^2 = 1.999999999999977$, but not 2
- Greeks proved that it is impossible to find any rational number whose square is 2

Proof: No rational number has a square equal to 2

Let m and n are two integers

$$\left(\frac{m}{n}\right)^2 = 2$$

By canceling any common factors, m and n are reduces to its lowest terms

$$m^2 = 2n^2$$

this makes m^2 even, hence m is an even. (The square of even is even and odd is odd). So $m = 2k$ for some integer k
Substituting $m = 2k$ in the equation gives, $4k^2 = 2n^2$, which results in

$$2k^2 = n^2$$

which means n^2 is even and therefore n is even
$\frac{m}{n}$ has common factors which contradicts the earlier assumption

Irrational Number

Irrational Number

A real number that is not rational is **irrational number**

- $\sqrt(2)$
- $3 + \sqrt(2)$
- $8\sqrt(2)$

Properties of Real Numbers

- **Commutative Properties**
  - Addition: $a + b = b + a$
  - Multiplication: $a \cdot b = b \cdot a$
- **Associative Properties**
  - Addition: $(a + b) + c = a + (b + c)$
  - Multiplication: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- **Distributive Property**
  - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
- **Identity Elements**
  - Additive Identity: $a + 0 = a$
  - Multiplicative Identity: $a \cdot 1 = a$
- **Inverse Elements**
  - Additive Inverse: $a + (-a) = 0$
  - Multiplicative Inverse (if $a \neq 0$): $a \cdot \frac{1}{a} = 1$

Properties of Real Numbers

- **Closure Property**
  - Real numbers are closed under addition, subtraction, multiplication, and division (except division by zero).

Inequalities

Transitivity

- If $a < b$ and $b < c$, then $a < c$

Multiplication

Suppose $a < b$

- If $c > 0$, then $ac < bc$
- If $c < 0$, then $ac > bc$

Exercise

Find all number x such that

$$\frac{x-8}{x-4} < 3$$

Our first step is to multiply by $x-4$ Here there are two conditions:

1. $x-4>0$

$$x-8 < 3(x-4) \implies x-8 < 3x-12 \implies 2x > 4 \implies x > 2$$

But our initial assumption is $x-4>0 \implies x>4$. As $4>2$, original inequality holds if $x>4$

Exercise Conti.

2. $x - 4 < 0$

$$x - 8 > 3(x - 4) \implies x < 2$$

Initial assumption is $x < 4$. As $2 < 4$, inequality holds for $x < 2$

The original inequality holds true for

$$x < 2, x > 4$$

or

$$(-\infty, 2) \cup (4, \infty)$$

Additive Inverse

If $a < b$ then $-a > -b$ Direction of inequalities has to be reversed when taking additive inverses on both sides

Multiplicative Inverse

If $a < b$

- If $a > 0, b > 0$, then $\frac{1}{a} > \frac{1}{b}$
- If $a < 0 < b$, then $\frac{1}{a} < \frac{1}{b}$

What is a Set?

Definition

A **set** is a well-defined collection of distinct objects, called **elements** or **members** of the set.

Representation of a Set:

- **Roster Form:** List elements inside curly braces:

$$A = \{1, 2, 3, 4\}$$

- **Set-Builder Notation:** Describe properties of elements:

$$A = \{x \mid x \text{ is a positive integer less than 5}\}$$

Membership

- If x belongs to A, write $x \in A$.
- If x does not belong to A, write $x \notin A$.

Types of Sets

- **Finite Set:** A set with a countable number of elements.
  Example: $A = \{1, 2, 3, 4\}$
- **Infinite Set:** A set with an uncountable or infinite number of elements.
  Example: $\mathbb{N} = \{1, 2, 3, \dots\}$
- **Empty/Null Set:** A set with no elements, denoted as $\emptyset$ or $\{\}$.
- **Subset:** $A \subseteq B$ if every element of A is in B.
- **Universal Set:** A set containing all objects under consideration, usually denoted by U.
- **Power Set:** The set of all subsets of A, denoted as $P(A)$.
  Example: If $A = \{1, 2\}$, then $P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Set Operations

Union ($\cup$)

Combines elements of two sets:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Intersection ($\cap$)

Elements common to both sets:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Difference ($A - B$)

Elements in A but not in B:

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

Set Operations

Complement (A^c)

Elements not in the set A:

$$A^c = \{x \mid x \notin A\}$$

Examples

- The set of natural numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$.
- The set of integers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.
- The set of even numbers: $\{2, 4, 6, \dots\}$.

What is an Interval?

Definition

An **interval** is a set of real numbers that includes all the numbers between two given endpoints.

Intervals describe ranges of values on the real number line and are widely used in mathematics.

Types of Intervals

- **Closed Interval ($[a, b]$)**: Includes both endpoints a and b.

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

Example: $[2, 5] = \{x \mid 2 \leq x \leq 5\}$.

- **Open Interval ((a, b))**: Excludes both endpoints a and b.

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

Example: $(2, 5) = \{x \mid 2 < x < 5\}$.

Half-Open or Half-Closed Intervals

- **Left-Closed, Right-Open ($[a, b)$)**:

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

  Example: $[2, 5) = \{x \mid 2 \leq x < 5\}$.

- **Left-Open, Right-Closed ($(a, b]$)**:

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

  Example: $(2, 5] = \{x \mid 2 < x \leq 5\}$.

Infinite Intervals

- (a, ∞): All numbers greater than a.

$$(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$$

  Example: $(3, \infty)$ includes all numbers greater than 3.

- $(-\infty, b)$: All numbers less than b.

$$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$$

  Example: $(-\infty, 4)$ includes all numbers less than 4.

- $(-\infty, \infty)$: The entire real number line.

$$(-\infty, \infty) = \mathbb{R}$$

Summary of Interval Types

Type	Interval Notation	Description
Closed	$[a, b]$	Includes both endpoints a, b
Open	(a, b)	Excludes both endpoints a, b
Half-Open Left	$[a, b)$	Includes a, excludes b
Half-Open Right	$(a, b]$	Excludes a, includes b
Infinite Left	$(-\infty, b)$	All $x < b$
Infinite Right	(a, ∞)	All $x > a$
Entire Line	$(-\infty, \infty)$	All real numbers

What is Absolute Value?

Definition

The **absolute value** of a number is its distance from zero on the number line, regardless of direction. It is always non-negative.

For a real number x, the absolute value, denoted as $|x|$, is defined as:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Breaking the absolute value:

- $|f(x)| \leq c \quad \Longrightarrow \quad -c \leq f(x) \leq c$
- $|f(x)| \geq c \quad \Longrightarrow \quad f(x) \leq -c \quad \text{or} \quad f(x) \geq c$

Examples of Absolute Value

- $|3| = 3 \quad$ (because $3 \geq 0$)
- $|-5| = -(-5) = 5 \quad$ (because $-5 < 0$)
- $|0| = 0 \quad$ (because 0 is neither positive nor negative)

Properties of Absolute Value

- **Non-Negativity:** $|x| \geq 0$ for all x.
- **Identity Property:** $|x| = 0 \quad$ if and only if $x = 0$.
- **Multiplicative Property:** $|x \cdot y| = |x| \cdot |y|$.
- **Triangle Inequality:** $|x + y| \leq |x| + |y|$.
- **Distance Interpretation:** $|x - y|$ represents the distance between x and y.

Exercises

Ball bearings need to have extremely accurate sizes to work correctly. The ideal diameter of a particular ball bearing is 0.8 cm, but a ball bearing is declared acceptable if the error in the diameter size is less than 0.001 cm. Write the inequality for acceptance criteria Solution:

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acceptable if diameter d is

$$|d - 0.8| \leq 0.001$$

Exercises

Find all numbers t such that $|3t - 4| = 10$

Solution :

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Find all numbers t such that $|3t - 4| = 10$

Solution :

$$3t - 4 = 10 \; or \; 3t - 4 = -10 \implies t = \frac{14}{3}, t = -2$$

Exercise

Find all numbers x such that

$$\left|\frac{3x-5}{x-1}\right| < 2$$

Solution :

Exercise

Find all numbers x such that

$$\left|\frac{3x-5}{x-1}\right| < 2$$

Solution :

$$|3x-5| < 2|x-1|$$

1. $x - 1 > 0$

Breaking the absolute value:

$$\implies -2(x-1) < 3x-5 < 2(x-1) = -2x+7 < 3x < 2x+3 \quad (1)$$
$$\implies 3x > -2x+7 \; \& \; 3x < 2x+3 \quad (2)$$

Exercise

Solving for $3x < 2x + 3$

$$\implies x < 3 \tag{3}$$

$$\tag{4}$$

Solving for $3x > -2x + 7$

$$\implies 3x > -2x + 7 \implies 5x > 7 \implies x > 7/5 \tag{5}$$

$$\implies x \in (7/5, 3) \tag{6}$$

Exercise

2. $x - 1 < 0 \implies x < 1$

$$\implies |3x-5| < 2|x-1| == |3x-5| < -2(x-1) \quad (7)$$

$$\implies 3x-5 < -2(x-1) \;\; and \;\; -(3x-5) < -2(x-1) \quad (8)$$

$$\implies 3x-5 < -2(x-1) \; and \; 3x-5 > 2(x-1) \quad (9)$$

$$3x-5 < -2(x-1) \implies 3x < -2x+7 \implies 5x < 7 \implies x < 7/5 \quad (10)$$

$$\implies 3x-5 > 2(x-1) \implies 3x > 2x+3 \implies x > 3 \quad (11)$$

Here $x > 3$ is inconsistent with our assumption $x < 1$. So for $x < 1$ there are no values of x satisfying the inequality

What is a Function ?

What is a Function?

A function associates every number in some set of real numbers, called the domain of the function, with exactly one real number

Domain

If a function is defined by a formula, with no domain specified, then the domain is assumed to be the set of all real numbers for which the formula makes sense and produces a real number

Domain

Example 3

Find the domain of the function f defined by

$$f(x) = (3x - 1)^2$$

Example 4

Find the domain of the function f defined by

$$h(t) = \frac{t^2 + 3t + 7}{t - 4}$$

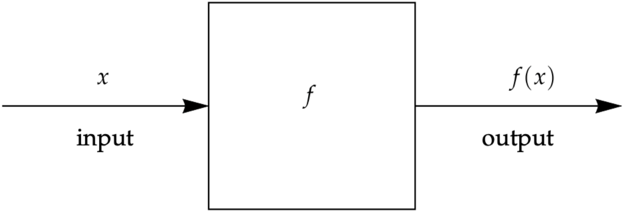
Example 6

Find the domain of the function g defined by

$$g(x) = \sqrt{|x| - 5}$$

Range

The range of a function f is the set of all numbers y such that $f(x) = y$ for at least one x in the domain of f



The set of inputs acceptable by this machine is the domain of f.
The set of outputs is the range of f.

Example 4

The domain of f is the interval $[2,5]$, with f defined on this interval by the equation $f(x) = 3x + 1$

Solution :

Functions

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$$y = f(x) = 3x + 1$$

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Solution :

$$y = f(x) = 3x + 1$$

$$2 \leq \frac{y-1}{3} \leq 5.$$

Functions

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The domain of f is the interval $[2, 5]$, with f defined on this interval by the equation $f(x) = 3x + 1$

Solution :

$$y = f(x) = 3x + 1$$

$$2 \leq \frac{y-1}{3} \leq 5.$$

$$7 \leq y \leq 16.$$

Example 4

The domain of f is the interval $[2, 5]$, with f defined on this interval by the equation $f(x) = 3x + 1$

Solution :

$$y = f(x) = 3x + 1$$

$$2 \leq \frac{y-1}{3} \leq 5.$$

$$7 \leq y \leq 16.$$

Function

Example 5

The domain of g is the interval $[1, 20]$, with g defined on this interval by the equation

$$g(x) = |x - 5|.$$

Is 2 in the range of g?

Solution :

Example 5

The domain of g is the interval $[1, 20]$, with g defined on this interval by the equation

$$g(x) = |x - 5|.$$

Is 2 in the range of g?

Solution :

$$y = |x - 5|$$

for $x - 5 > 0$,$y = x - 5 \implies x = y + 5$

$$5 < y + 5 \leq 20 \implies 0 < y \leq 15$$

for $x - 5 < 0$, $y = -(x - 5) \implies y = -x + 5 \implies 5 - y = x$

$$1 \leq 5 - y \leq 5 \implies -4 \leq -y \leq 0 \implies 4 \geq y \geq 0$$

Equality of Functions

Two functions are equal if and only if they have the same domain and the same value at every number in that domain

Example

Suppose f is the function whose domain is the set of real numbers, with f defined on this domain by

$$f(x) = x^2$$

Suppose g is the function whose domain is the set of positive numbers, with g defined on this domain by

$$g(x) = x^2$$

Are f and g equal functions ?

Equality of functions

Example 2

Suppose f and g are functions whose domain is the set consisting of the two numbers $\{1, 2\}$ with f and g defined on this domain by the formulas

$$f(x) = x^2$$

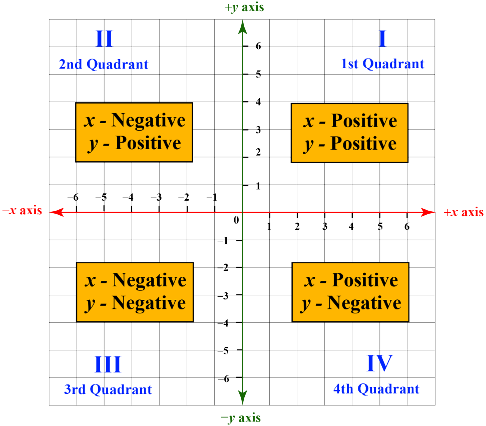
and

$$g(x) = 3x - 2$$

.Are f and g equal functions?

What is Analytic Geometry?

- **Analytic Geometry** (also called *coordinate geometry* or *Cartesian geometry*) bridges algebra and geometry.
- It uses a coordinate system to study geometric shapes and properties.
- Geometric objects are represented as algebraic equations.

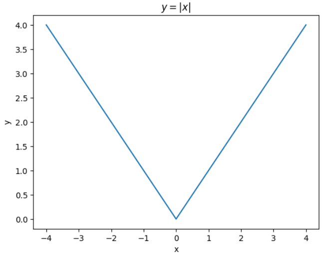


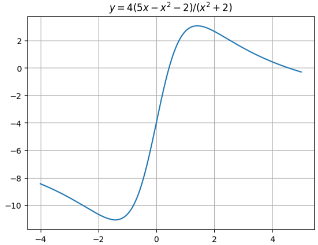
The plane with this system of labeling is often called the **Cartesian plane** in honor of the French mathematician Rene Descartes(1596-1650), who described this technique in his 1637 book Discourse on Method

Graph Functions

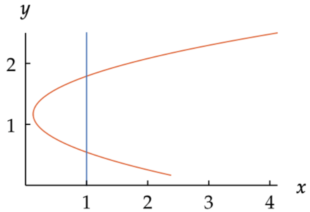
The graph of a function f is the set of points of the form $x, f(x)$ as x varies over the domain of f

Graph of a Function





Checking for a function: Vertical line test



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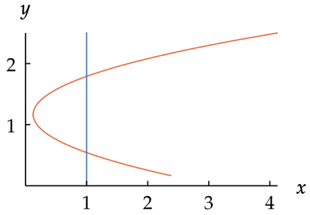


The line $x = 1$ intersects the curve at two points. That is that for each x value there are multiple y values which is contradicting to definition of a function

Vertical Line Test

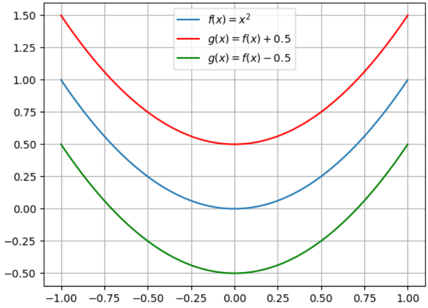
A set of points in the coordinate plane is the graph of some function if and only if every vertical line intersects the set in at most one point

Vertical Transformation

Shifting a graph up or down

Suppose f is a function and $a > 0$. Upshift g and Downshift h by

$$g(x) = f(x) + a \quad h(x) = f(x) - a$$

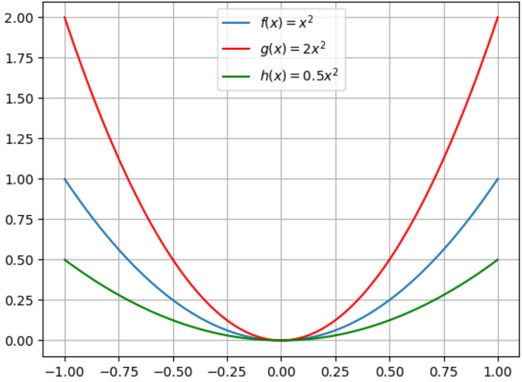


Vertical Transformation

Vertical Stretch

Suppose f is a function and $c > 0$. Define a function g by

$$g(x) = cf(x)$$

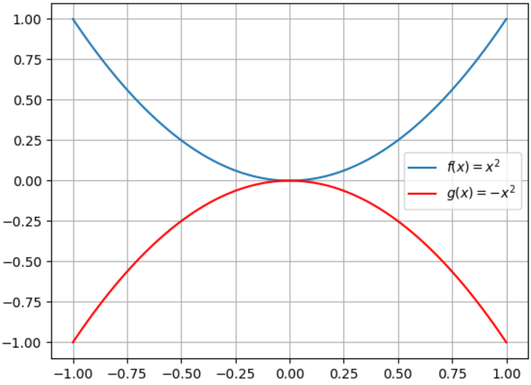


Vertical Transformation

Flipping along the Vertical Axis

Vertical fliiping of $f(x)$ is

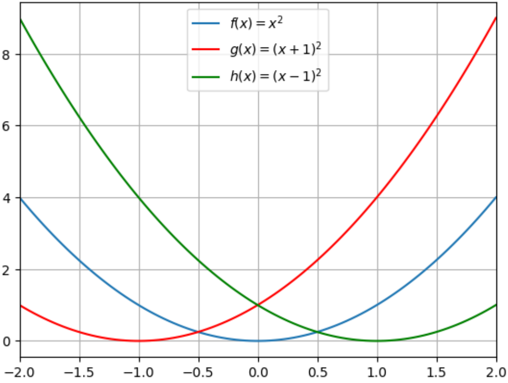
$$g(x) = -f(x)$$



Horizontal Transformation

Horizontal Shift

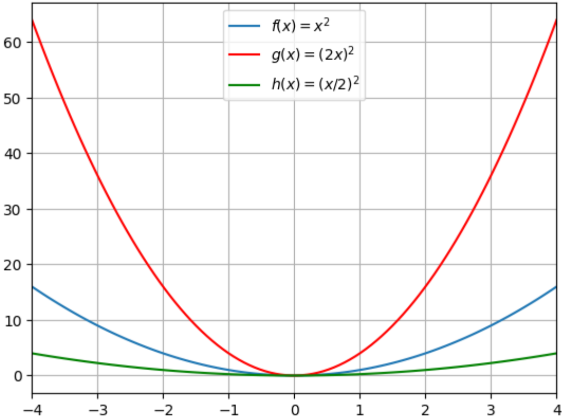
$$g(x) = f(x + a), h(x) = f(x - a)$$



Horizontal Transformation

Horizontal Stretching

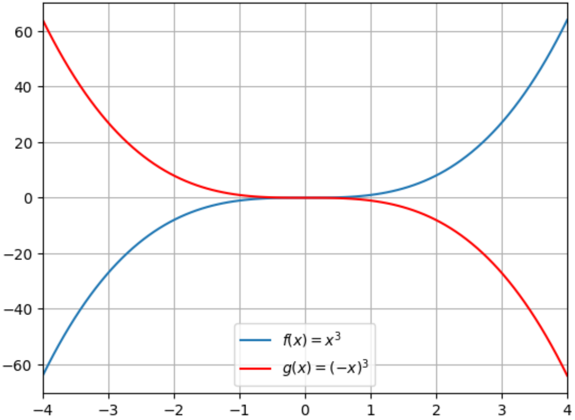
$$g(x) = f(cx)$$



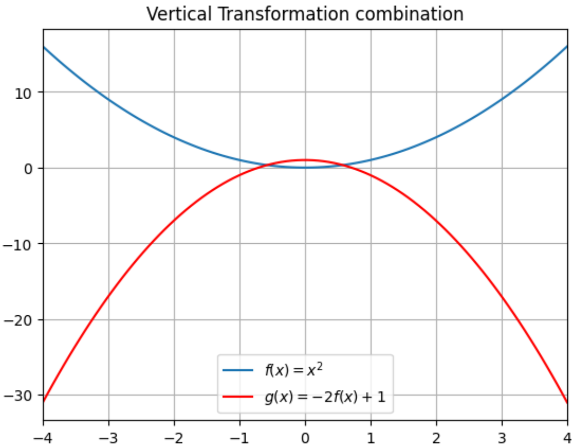
Flipping ac the Vertical Axis

Horizontal Stretching

$$g(x) = f(-x)$$



Combinations of vertical Transformation



Even Functions

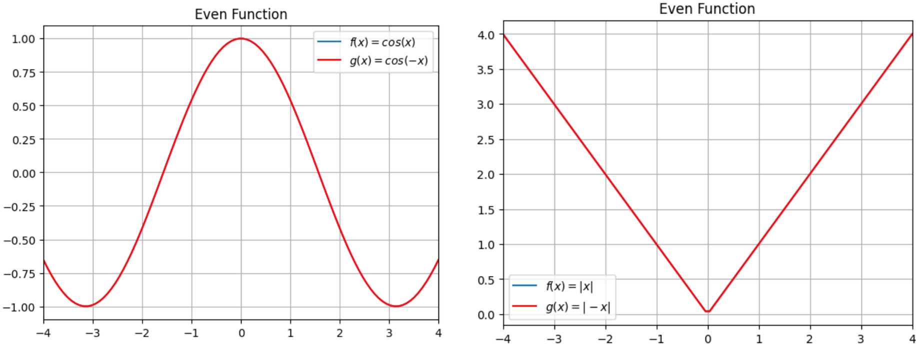
Even

$$f(-x) = f(x) \quad \text{for all } x \text{ in the domain}$$

Example: $f(x) = x^2, \quad f(x) = \cos x$

The graph of an even function is symmetric across the vertical axis

Even Functions



Odd Function

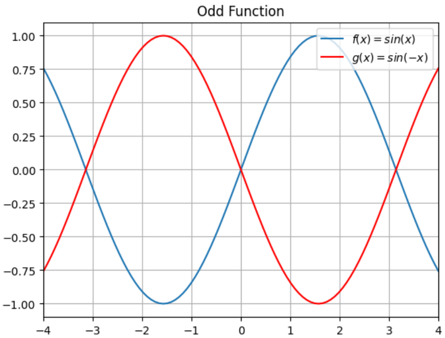
Odd

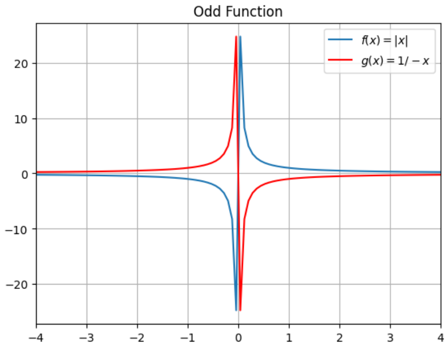
$$f(-x) = -f(x) \quad \text{for all } x \text{ in the domain}$$

Example: $f(x) = x^3, \quad f(x) = \sin x$

The graph of an even function is symmetric if flipped or rotated 18 across the origin

Odd Function





Algebra of Functions

Suppose f and g are functions. We can define new functions from f and g as follows:

Algebra of Functions

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Sum:

$$(f+g)(x) = f(x) + g(x)$$

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Suppose f and g are functions. We can define new functions from f and g as follows:

Sum:

$$(f + g)(x) = f(x) + g(x)$$

Difference:

$$(f - g)(x) = f(x) - g(x)$$

Algebra of Functions

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$$(f \cdot g)(x) = f(x) \cdot g(x)$$

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Product:

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

Quotient:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{provided } g(x) \neq 0.$$

Note: If f and g have domains D_f and D_g, then these operations are defined on the intersection $D_f \cap D_g$. In the case of the quotient, it is defined on

$$\{x \in D_f \cap D_g : g(x) \neq 0\}.$$

Exercise

$f(x) = \sqrt{x-3}$ and $g(x) = \sqrt{8-x}$
Evaluate

a. $(f+g)(x)$
b. $(fg)(x)$
c. Find the domain of above

Exercise

$f(x) = \sqrt{x-3}$ and $g(x) = \sqrt{8-x}$
Evaluate

a. $(f+g)(x)$
b. $(fg)(x)$
c. Find the domain of above

Sol:

a. $\sqrt{x-3} + \sqrt{8-x}$
b. $\sqrt{(x-3)(8-x)}$
c. Domian of
    a. $x \geq 3$
    b. $x \leq 8$
    c. $3 \leq x \leq 8$

Function Composition

Definition:

If $f(x)$ and $g(x)$ are functions, then the composition of f and g, denoted by $f \circ g$, is defined by

$$(f \circ g)(x) = f(g(x)).$$

Example: Consider the function

$$h(x) = \sqrt{x+3}.$$

We can express $h(x)$ as a composition of two functions f and g where:

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = x + 3.$$

Then,

$$h(x) = f(g(x)) = f(x+3) = \sqrt{x+3}.$$

Exercise

$f(x) = \frac{1}{x-4}$ and $g(x) = x^2$

1. $f \circ g$
2. $g \circ f$
3. domian of $f \circ g$
4. domain of $g \circ f$

Sol:

Exercise

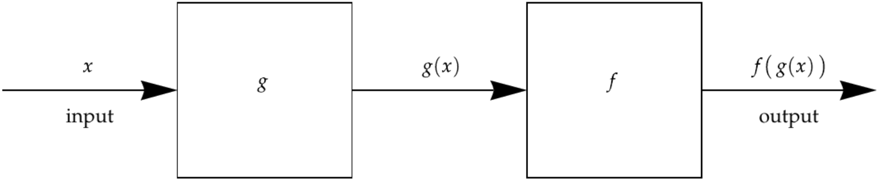
$f(x) = \frac{1}{x-4}$ and $g(x) = x^2$

1. $f \circ g$
2. $g \circ f$
3. domian of $f \circ g$
4. domain of $g \circ f$

Sol:

1. $f(g(x)) = \frac{1}{x^2-4}$
2. $g(f(x)) = (\frac{1}{x-4})^2$
3. $R - \{-2, 2\}$
4. $R - \{4\}$

Composition Machine



The composition $f \circ g$ as the combination of two machines.

Excersise

Problem: Suppose your cell phone company charges \$0.05 per minute plus \$0.47 for each call to China.

(a) Find a function p that gives the amount charged by your cell phone company for a call to China as a function of the number of minutes m.

(b) Suppose the tax on cell phone bills is 6% plus \$0.01 for each call. Find a function t that gives your total cost, including tax, for a call to China as a function of the amount charged by your cell phone company.

(c) Explain why the composition $t \circ p$ gives your total cost, including tax, of making a cell phone call to China as a function of the number of minutes.

(d) Compute a formula for $t \circ p$.

(e) What is your total cost for a ten-minute call to China?

Solution

(a) The company charges \$0.05 per minute plus a fixed charge of \$0.47 per call. Hence, the pre-tax charge function is

$$p(m) = 0.05m + 0.47.$$

(b) The tax on the cell phone bill is 6% of the pre-tax amount plus an additional \$0.01 per call. Thus, if the pre-tax charge is x, the total cost function (including tax) is

$$t(x) = 1.06x + 0.01.$$

(c) The composition $t \circ p$ means we first compute the pre-tax charge $p(m)$ for a call of m minutes, and then we apply the tax function t to this amount. In other words, $t(p(m))$ gives the total cost, including tax, as a function of the number of minutes.

(d) To compute the composition, substitute $p(m)$ into t:

$$(t \circ p)(m) = t(p(m)) = 1.06\big(0.05m + 0.47\big) + 0.01.$$

Solution

Distribute 1.06:

$$1.06(0.05m) = 0.053m \quad \text{and} \quad 1.06(0.47) = 0.4982.$$

Thus,

$$(t \circ p)(m) = 0.053m + 0.4982 + 0.01 = 0.053m + 0.5082.$$

(e) For a ten-minute call ($m = 10$):

$$(t \circ p)(10) = 0.053(10) + 0.5082 = 0.53 + 0.5082 = 1.0382.$$

Rounded to the nearest cent, the total cost is approximately \$1.04.

Identity Function

The identity function is defined by

$$I(x) = x \quad \text{for every number } x.$$

The function I is the identity for composition

If f is any function, then

$$f \circ I = I \circ f = f.$$

Decomposing the Functions

Function Decomposition

$$T(y) = \frac{|y^2 - 3|}{|y^2 - 7|}$$

Sol:

Decomposing the Functions

Function Decomposition

$$T(y) = \frac{|y^2 - 3|}{|y^2 - 7|}$$

Sol:

$$f(y) = |y|, g(y) = \frac{y^2 - 3}{y^2 - 7}$$

$$f(y) = \frac{|y - 3|}{|y - 7|}, g(y) = y^2$$

Composition is associative

if f, g, h are functions then

$$(f \circ g) \circ h = f \circ (g \circ h)$$

Example

Composition of three functions

$$T(x) = \left|\frac{x^2 - 3}{x^2 - 7}\right|$$

Sol:

Example

Composition of three functions

$$T(x) = \left| \frac{x^2 - 3}{x^2 - 7} \right|$$

Sol:

$$f(x) = |x|, g(x) = \frac{x - 3}{x - 7}, h(x) = x^2$$

Linear Functions

Linear Function

A linear function is a function h of the form

$$h(x) = mx + b$$

where m and b are numbers

Linear Functions as Composition

Vertical Transformations as Compositions

A funtion $g(x)$ is defined by

$$g(x) = -2f(x) + 1$$

Write $g(x)$ as a the composition of a linear function with $f(x)$

$$h(x) = -2x + 1$$

$$\implies g(x) = h(f(x)) \implies g = h \circ f$$

Linear Function as Composition

Horizontal Transformations as Compositions

A funtion $g(x)$ is defined by

$$g(x) = f(2x) + 1$$

Write $g(x)$ as a the composition of a linear function, $f(x)$ and other linear function

$$h(x) = x + 1, p(x) = 2x$$

$$\implies g(x) = h(f(p(x))) \implies g = h \circ f \circ p$$

Inverse Function: Example

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$y = \frac{9}{5}x + 32,$$

which converts a temperature x in Celsius to Fahrenheit y. The inverse function f^{-1} converts Fahrenheit back to Celsius:

$$f^{-1}(y) = \frac{5}{9}(y - 32).$$

Verifying that these functions are inverses:

$$f^{-1}(f(x)) = \frac{5}{9}\left(\frac{9}{5}x + 32 - 32\right) = x,$$

$$f(f^{-1}(f)) = \frac{9}{5}\left(\frac{5}{9}(y - 32)\right) + 32 = y.$$

One-to-One Function

One-to-One Function

A function f is called one-to-one if for each number y in the range of f there is exactly one number x in the domain of f such that $f(x) = y$

Inverse Function

Definition

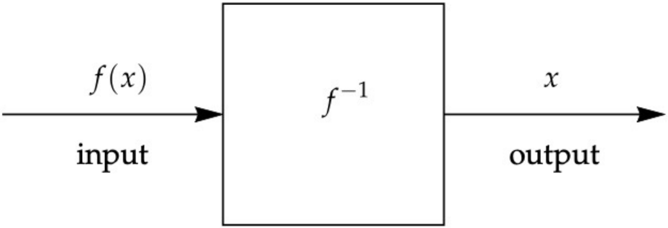
Suppose f is a one-to-one function.

- If y is in the range of f, then $f^{-1}(y)$ is defined to be the number x such that $f(x) = y$.
- The function f^{-1} is called the *inverse function* of f.

Short version:

- $f^{-1}(y) = x$ means $f(x) = y$.

Inverse Function



Domain and Range of an Inverse Function

Properties

If f is a one-to-one function, then:

- The domain of f^{-1} equals the range of f.
- The range of f^{-1} equals the domain of f.

Increasing and Decreasing Function

Increasing

A function f is called increasing if $f(a) < f(b)$ whenever $a < b$ and a, b are in the domain of f

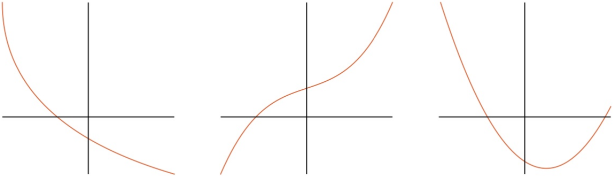
Decreasing

A function f is called decreasing if $f(a) > f(b)$ whenever $a < b$ and a, b are in the domain of f

Increasing and decreasing functions are one-to-one

- Every increasing function is one-to-one
- Every decreasing function is one-to-one.

Exercise



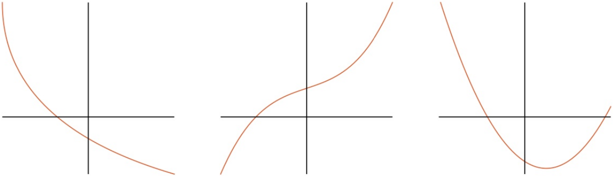
The graph of f. *The graph of g.* *The graph of h.*

(a) Is f increasing, decreasing, or neither?

(b) Is g increasing, decreasing, or neither?

(c) Is h increasing, decreasing, or neither?

Exercise



The graph of f. *The graph of g.* *The graph of h.*

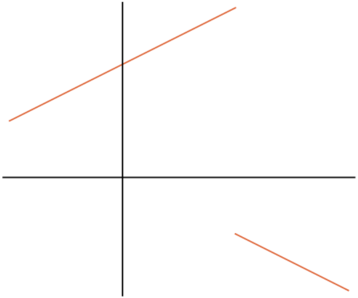
(a) Is f increasing, decreasing, or neither?

(b) Is g increasing, decreasing, or neither?

(c) Is h increasing, decreasing, or neither?

a. Decreasing
b. Increasing
c. Neither

Do all one-to-one maps are increasing or decreasing ?



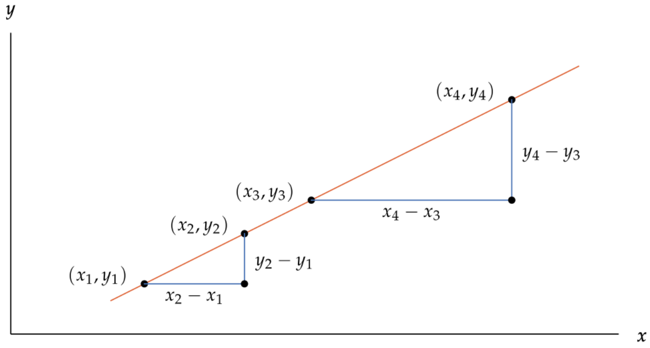
The graph of a one-to-one function that is neither increasing nor decreasing.

Increasing and Decreasing Functions

Inverses of increasing and decreasing functions

- The inverse of an increasing function is increasing.
- The inverse of a decreasing function is decreasing.

Slope



$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}$$

Slope

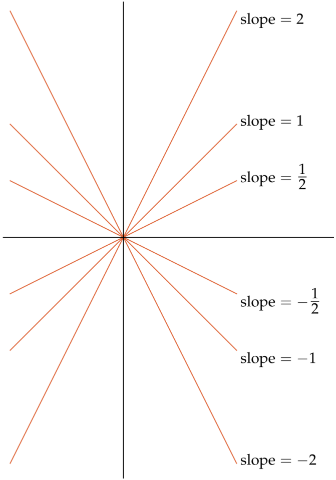
Definition

If x_1, y_1 and x_2, y_2 are any two points on a line with $x_1 \neq x_2$, then the **slope** of the line is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Key Points:

- Positive slope slands up from left to right
- Negative slope slands down from left to right
- Horizontal line has slope $= 0$
- Vertical line has no slope

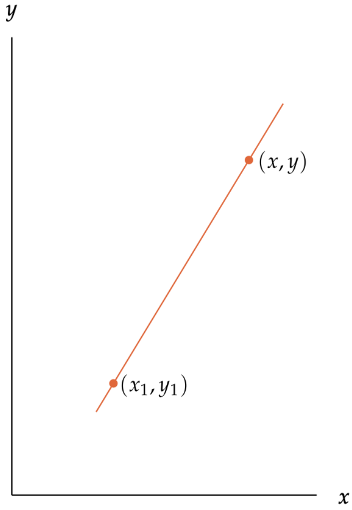


Line Equation

slope and one point on it

The line in the xy-plane that has slope m and contains the point $(x1, y1)$ is given by the equation

$$y - y_1 = m(x - x_1)$$

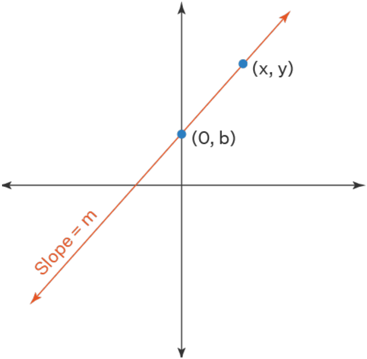


Line Equation

slope and y intercept

The line in the xy-plane with slope m that intersects the y axis at $0, b$ is given by the equation by the equation

$$y = mx + b$$



Line Equation

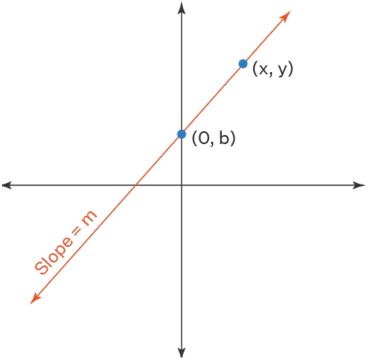
slope and y intercept

The line in the xy-plane that contains the points x_1, y_1 and x_2, y_2 where $x_1 \neq x_2$, is

$$y = mx + b$$

, is given by the equation

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$



Linear Function

Definition

A **linear function** is a function f of the form

$$f(x) = mx + b$$

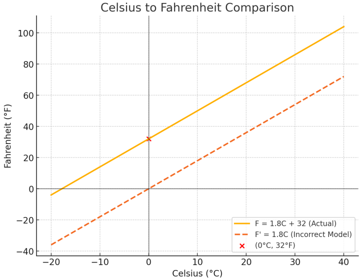
where m and b are numbers

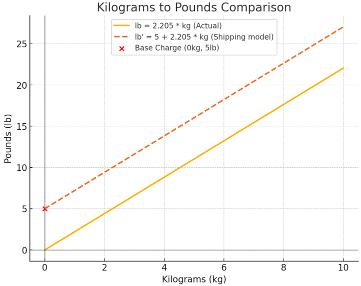
Example 1: Temperature Conversion

- Correct formula: $F = 1.8C + 32$ (Starts at 32°F)
- Incorrect direct proportion: $F' = 1.8C$ (Wrong assumption)

Example 2: Weight Conversion

- True conversion: $lb = 2.205 \times kg$ (Passes through origin)
- Shipping charge model: $lb' = 5 + 2.205 \times kg$ (Has minimum billable weight or fixed cost markup)

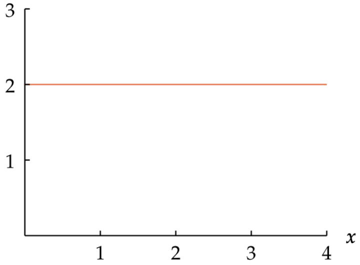




Constant Function

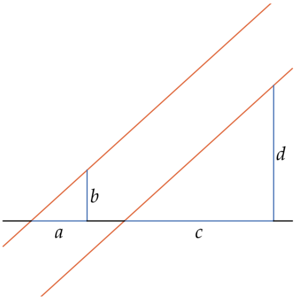
Definition

A constant function is a function f of the form $f(x) = b$, where b is a number



The orange horizontal line is the graph of the constant function f defined by $f(x) = 2$ on the interval $[0, 4]$.

Parallel Lines

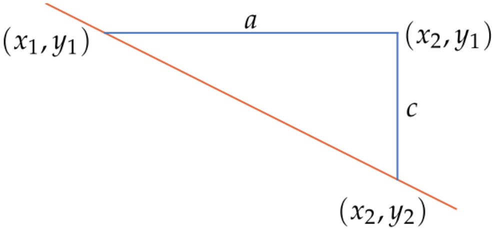


As two lines are parallel, the corresponding angles are concurent and so two triangles are similar so

$$\frac{a}{c} = \frac{b}{d} \implies \frac{b}{a} = \frac{d}{c}$$

it has same slope

Negative Slope



As lengths are positive $a = x_2 - x_1$ and $c = y_1 - y_2$

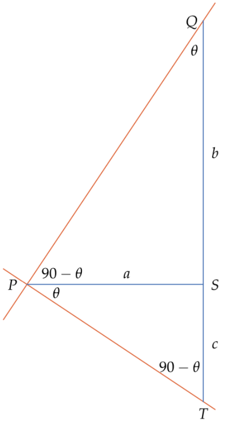
Slope $= \frac{y_2 - y_1}{x_2 - x_1} = -\frac{c}{a}$

Perpendicular Lines

$\triangle PSQ$ and $\triangle TSP$ are similar

$\frac{QS}{SP} = \frac{PS}{ST} \implies \frac{b}{a} = \frac{a}{c}$

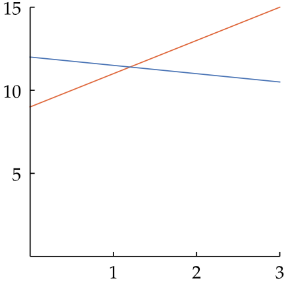
Multiplying by
$-\frac{c}{a} \implies \frac{b}{a} \cdot \left(-\frac{c}{a}\right) = -1$



Unequal Scales

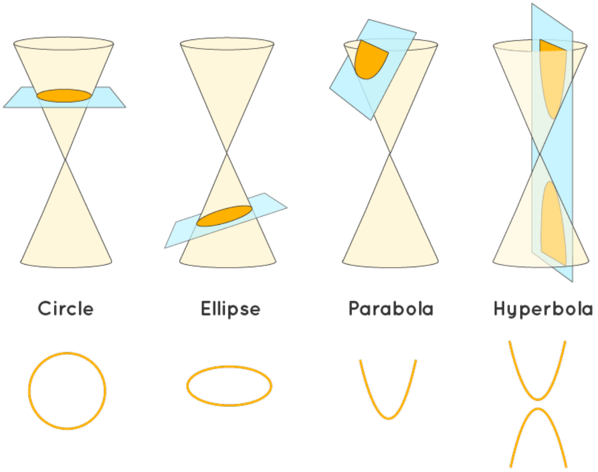
Angles are distorted by unequal scales on coordinate axes

In graphs with unequal scales on the two coordinate axes, angles are not accurately represented



The perpendicular lines
$y = 2x + 9$ *(orange) and*
$y = -\frac{1}{2}x + 12$ *(blue).*

Conics



Quadratic Function

Definition

The function of the form

$$ax^2 + bx + c = 0$$

where a, b, c are real numbers with $a \neq 0$

- if $b^2 - 4ac < 0$, then equation have no real solutions
- if $b^2 - 4ac = 0$, then equation has one solution, $x = -\frac{b}{2a}$
- if $b^2 - 4ac > 0$, then equation has two solutions $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Parabola

A **parabola** is the graph of a quadratic function. The **vertex** of the parabola is the where the line of symmetry of the parabola, intersects the parabola.

Suppose f is a quadratic function. Complete the square to write f in the form

$$f(x) = a(x-h)^2 + k$$

- If $a > 0$ then $f(x)$ attains its minimum value k when $x = h$ and the graph of f is a parabola that opens upward.
- If $a < 0$ then $f(x)$ its maximum value k when $x = h$ and the graph of f is a parabola that opens downward
- The vertex of the graph is h, k

Example

$$f(x) = -3x^2 + 12x - 8$$

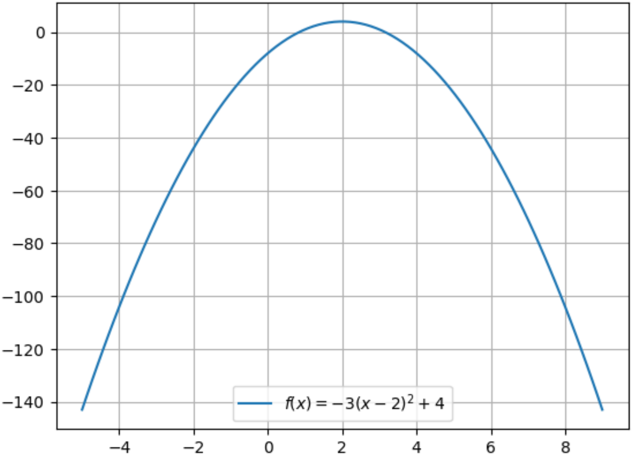
1. For what value of x does $f(x)$ attain its maximum value?
2. What is the maximum value of $f(x)$?
3. Find the vertex

Sol:

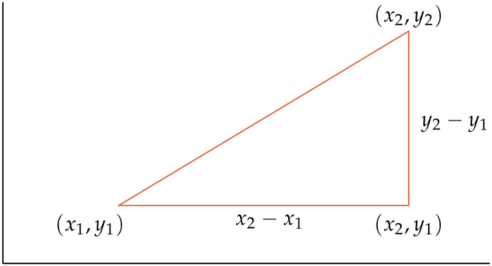
$$f(x) = -3x^2 + 12x - 8 \implies -3(x^2 - 4x + 4) + 4 \implies -3(x-2)^2 + 4$$

1. $x = 2$
2. $f(x = 2) = 4$
3. $(2, 4)$

Parabola



Distance Between Points



Distance Between Points

The distance between points x_1, y_1 and x_2, y_2 is given by

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Circle

Equation of a Circle

The circle with center h, k and radius r is the set of the points x, y that satisfy the equation

$$(x-h)^2 + (y-k)^2 = r^2$$

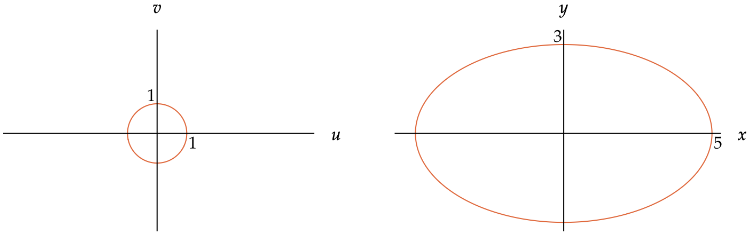
Ellispe

The German mathematician Johannes Kepler, who in 1609 published his discovery that orbits of the planets are ellipses, not circles or combinations of circles as previously thought.

Ellipses

Ellipse

Stretching the circle horizontally and/or vertically produces a curve called an **ellipse**



Stretching horizontally by a factor of 5 and vertically by a factor of 3 transforms the circle on the left into the ellipse on the right.

Ellipse

Equation of the circle is given by $u^2 + v^2 = 1$
By stretching $x = 3u, y = 5v$,
Substituting for u, v

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$$

Ellipses

Ellipse Equation

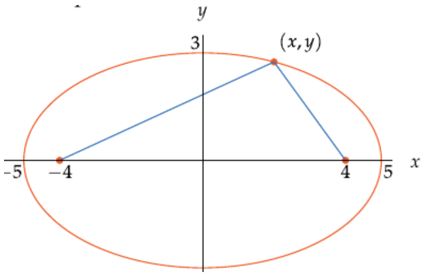
$$\frac{x2}{a^2} + \frac{y^2}{b^2} = 1$$

Foci

The **foci** of an ellispe are two points with the property that the sum of the distances from the **foci** to any point on the ellipse is a constant independent of the point on the ellipse

Ellispe

Isaac Newton showed that the equations of gravity imply that a planet's orbit around a star is an ellipse with the star at one of the foci. For example, if units are chosen so that the orbit of a planet is the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, *then the star must be located at either* $(4, 0)$ *or* $(-4, 0)$.



For every point (x, y) *on the ellipse* $\frac{x^2}{25} + \frac{y^2}{9} = 1$, *the sum of the lengths of the two blue line segments equals* 10.

Eccentricity of an Ellipse

Definition

The **eccentricity** (e) of an ellipse is a measure of how much the ellipse deviates from being a circle. It is defined as

$$e = \frac{c}{a}.$$

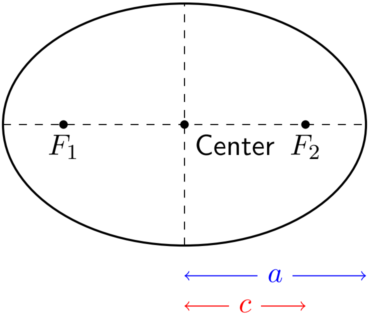
$$c^2 = a^2 - b^2,$$

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

where:

- c is the distance from the center to a focus.
- a is the length of the semi-major axis.

Eccentricity



Additionally, the semi-minor axis b is related to a and c by:

Key Points:

- If $e = 0$, the ellipse is a circle.
- If $0 < e < 1$, the ellipse is elongated, with greater elongation as e increases.

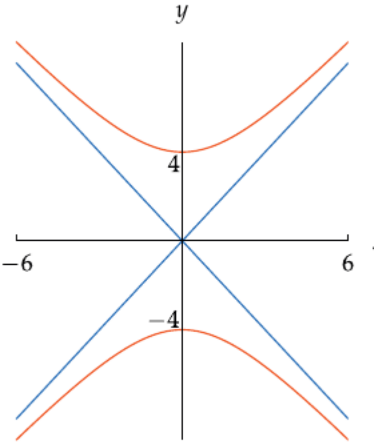
Hyperbola

Definition

The graph of the equation of the form

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

where a, b are non-zero numbers

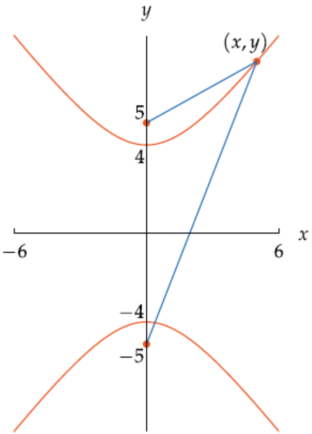


The hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$ *in orange for* $-6 \leq x \leq 6$*, along with the lines* $y = \frac{4}{3}x$ *and* $y = -\frac{4}{3}x$ *in blue.*

Hyperbola

Foci

The foci of a hyperbola are two points with the property that the difference of the distances from the foci to a point on the hyperbola is a constant independent of the point on the hyperbola



For every point (x, y) on the hyperbola $\frac{y^2}{16} - \frac{x^2}{9} = 1$, the difference of the lengths of the two blue line segments equals 8.

A comet whose orbit lies on a hyperbola will come near Earth at most once. A comet whose orbit is an ellipse will return periodically.

Positive Integer Exponent

Positive Integer Exponent

If x is a real number and m is a positive integer, then x^m is defined to be the product with x appearing m times

$$x^m = \underbrace{x \cdot x \cdots x}_{x \;\; appears \;\; m \;\; times}$$

Positive Integer Exponenets

Properties

Suppose x and y are numbers and m and n are positive integers. Then

$$x^m x^n = x^{m+n}$$

$$(x^m)^n = x^{mn}$$

$$x^m y^m = (xy)^m$$

x^0

What is x^0

If $x^m x^n = x^{m+n}$ then we can write

$$x^0 x^n = x^{0+n} = x^n \implies x^0 = 1 \ for \ \ x \neq 0$$

What is 0^0

- The rule $x^0 = 1$ (for $x \neq 0$) suggests that 0^0 should be 1.
- The rule $0^m = 0$ (for $m > 0$) suggests that 0^0 should be 0
- Since these two rules contradict each other, 0^0 is left undefined in general mathematics.
- However, in combinatorics and programming, 0^0 is often defined as 1 for convenience.

Negative Integer Exponents

If $x^m x^n = x^{m+n}$, if we take $m = -n$, then

$$x^m x^{-m} = x^0 = 1 \implies x^m x^{-m} = 1$$

We have to define x^{-m} to equal the multiplicative inverse of x^m

Negative Interger Exponent

If $x \neq 0$ and m is a positive integer, then x^{-m} is defined to multiplicative inverse of x^m

$$x^{-m} = \frac{1}{x^m}$$

Exponents: Some Graphs

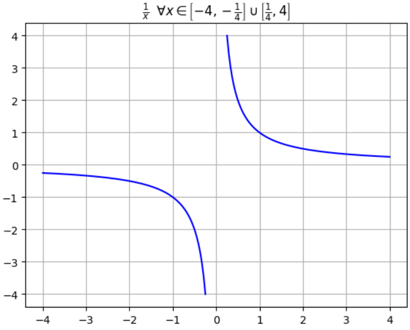


Figure: graph of $\frac{1}{x}$

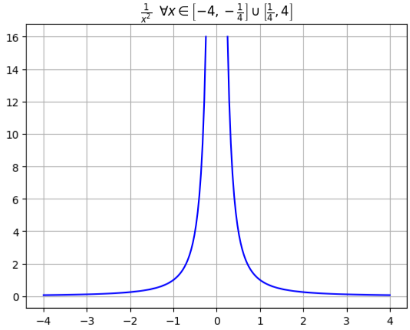


Figure: graph of $\frac{1}{x^2}$

Negative Integer Exponenets

Graph of Negative Integer Exponents

if m is a positive integer then

- $\frac{1}{x^m}$ behaves like $\frac{1}{x}$ if m is odd
- $\frac{1}{x^m}$ behaves like $\frac{1}{x^2}$ if m is even
- Larger values of m correspond to functions whose graphs get closer to the x-axis more rapidly for large values of x and closer to the vertical axis more rapidly for values of x near 0

Roots

m^{th} root

If m is a positive integer and x is a real number, then $x^{1/m}$ is defined to be the real number satisfying the equation

$$\left(x^{1/m}\right)^m = x$$

subject to the following conditions:

- If $x < 0$ and m is an even integer, then $x^{1/m}$is undefined
- If $x > 0$ and m is an even integer, then $x^{1/m}$ is chosen to be the *positive number* satisfying the equation above

The number $x^{1/m}$ is called the m^{th} root of x.

Roots

Example

- $8^{1/3}$ and $-8^{1/3}$
- $9^{1/2}$ and $-9^{1/2}$

Solution

1. $\left(8^{1/3}\right)^3 = 8 \implies 2$
2. $\left(-8^{1/3}\right)^3 = -8 \implies -2.$ *There is no other number other than* -2
3. $\left(9^{1/2}\right)^2 = -3$ *or* $3.$ *But as per the rule, we have to choose positive possibility, that is* 3
4. $\left(-9^{1/2}\right)^2.$ *No number real number exists so no solution*

Rational Exponents

Definition

Suppose $\frac{n}{m}$ is a fraction in reduced form, where n and m are integers and $m > 0$. Then, whenever it makes sense,

$$x^{\frac{n}{m}} = \left(x^{\frac{1}{m}}\right)^n.$$

Note: For the expression $x^{\frac{1}{m}}$ to be defined, additional conditions on x may be required (for example, if m is even, then typically $x \geq 0$).

Algebra of Exponents

Exponent Rules

Let p, q be rational numbers and x, y be positive numbers. Then the following rules hold:

- $x^p \cdot x^q = x^{p+q}$
- $x^p \cdot y^p = (xy)^p$
- $(x^p)^q = x^{pq}$
- $x^0 = 1$
- $x^{-p} = \dfrac{1}{x^p}$
- $\dfrac{x^p}{x^q} = x^{p-q}$
- $\left(\dfrac{x}{y}\right)^p = \dfrac{x^p}{y^p}$

Polynomial Definition

Definition of a Polynomial

A polynomial is a function p such that

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where n is a nonnegative integer and $a_0, a_1, a_2, \ldots, a_n$ are numbers.

Degree of a Polynomial

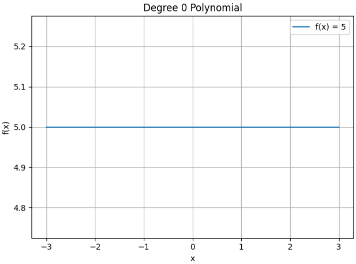
Definition

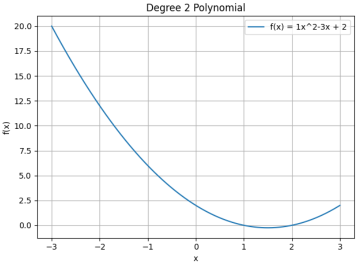
Suppose p is a polynomial defined by

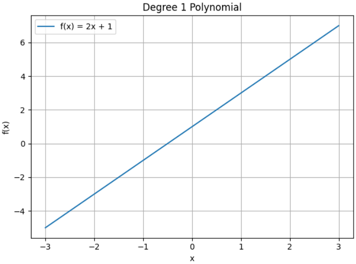
$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

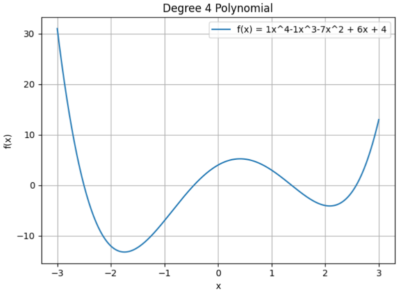
If $a_n \neq 0$, then we say that p has degree n. The degree of p is denoted by $\deg p$.

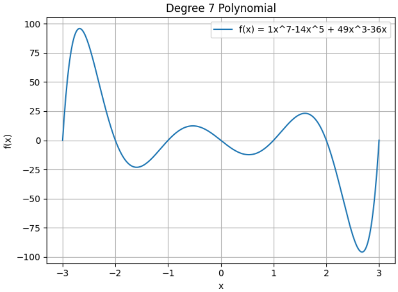
Polynomial Graphs








Degree 4 Polynomial
f(x) = 1x^4-1x^3-7x^2 + 6x + 4
30
20
10
0
−10
f(x)
−3
−2
−1
0
1
2
3
x


Degree 7 Polynomial
f(x) = 1x^7-14x^5 + 49x^3-36x
100
75
50
25
0
−25
−50
−75
−100
f(x)
−3
−2
−1
0
1
2
3
x

The Algebra of Polynomials

Two functions can be added, subtracted, or multiplied, producing another function. Specifically, if p and q are functions, then the functions

$$p+q, \quad p-q, \quad \text{and} \quad pq$$

are defined by

$$(p+q)(x) = p(x) + q(x),$$
$$(p-q)(x) = p(x) - q(x),$$
$$(pq)(x) = p(x)\,q(x).$$

Degree of the Sum and Difference of Two Polynomials

Important Result

If p and q are nonzero polynomials, then

$$\deg(p+q) \leq \max\{\deg p,\, \deg q\},$$

and

$$\deg(p-q) \leq \max\{\deg p,\, \deg q\}.$$

Degree of the Product of Two Polynomials

If p and q are nonzero polynomials, then

$$\deg(pq) = \deg p + \deg q.$$

Example: Polynomials p and q

Problem

Suppose p and q are polynomials defined by

$$p(x) = 2 - 3x^2 \quad \text{and} \quad q(x) = 4x + 7x^5.$$

Answer the following:

1. What is $\deg p$?
2. What is $\deg q$?
3. Find a formula for pq.
4. What is $\deg(pq)$?

Solution

1. Since $p(x) = 2 - 3x^2$ has the highest power x^2, we have $\deg p = 2$.
2. For $q(x) = 4x + 7x^5$, the highest power is x^5, so $\deg q = 5$.
3. The product pq is computed as follows:

$$pq = (2 - 3x^2)(4x + 7x^5) = 2 \cdot 4x + 2 \cdot 7x^5 - 3x^2 \cdot 4x - 3x^2 \cdot 7x^5,$$

which simplifies to:

$$pq = 8x - 12x^3 + 14x^5 - 21x^7.$$

4. The highest power in pq is x^7, so $\deg(pq) = 7$.