Maths Bootcamp

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Overleaf

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The School of Athens by Raphel

What is Geometry?

- Branch of mathematics studying shapes, sizes, and spatial relationships.
- Derived from Greek: *"geo"* (earth) + *"metron"* (measure).
- Originally focused on measuring the earth, now much broader.

Key Concepts in Geometry

- **Points, Lines, and Angles**: Basic building blocks.
- **Shapes and Figures**: Circles, triangles, polygons.
- **Solids**: 3D objects like cubes, spheres, pyramids.
- **Theorems and Proofs**: Logical reasoning based on axioms and postulates.

Why Learning Geometry is Important in ML

- **Understanding Data:** ML operates in high-dimensional spaces, where geometry helps analyze structure and relationships.
- **Feature Engineering:** Transformations (rotations, scaling, projections) improve model performance.
- **Distance Metrics:** Algorithms rely on distances (Euclidean, cosine, Manhattan) for clustering and classification.
- **Optimization:** Gradient descent follows geometric paths to minimize loss functions.
- **Manifold Learning:** Real-world data often lies on curved manifolds, requiring non-Euclidean methods (e.g., t-SNE, UMAP).
- **Deep Learning:** CNNs use geometric transformations; GNNs handle graph structures.
- **Model Interpretability:** Decision boundaries (e.g., SVMs) are geometric constructs that explain model behavior.

Definition of Percentage

- A **percentage** is a way of expressing a number as a fraction of 100.
- The formula to calculate a percentage is:

$$\text{Percentage} = \left(\frac{\text{Part}}{\text{Whole}}\right) \times 100$$

- Example: If you score 45 out of 60 on a test, the percentage is:

$$\left(\frac{45}{60}\right) \times 100 = 75\%$$

Ratio vs Rate

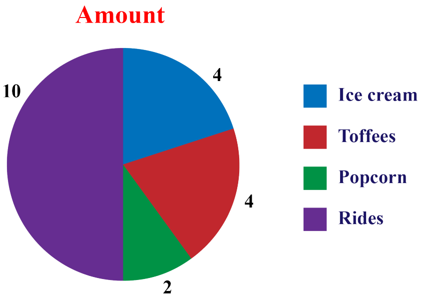


Figure: ratio

Figure: rate

Key Distinction: Ratio vs Rate

Ratio

- **Definition:** Comparison of two similar units.
- **Units:** Unitless (dimensionless).
- **Form:** Written as $a : b$, $\frac{a}{b}$, or "a to b".
- **Example:** Boys to girls in a classroom is $2 : 3$.
- **Key Feature:** Static relationships.

Rate

- **Definition:** Comparison of two different units.
- **Units:** Includes units (e.g., miles per hour).
- **Form:** Written as $\frac{a\,\text{unit}_1}{b\,\text{unit}_2}$ or "a per b".
- **Example:** A car travels 60 mph.
- **Key Feature:** Dynamic relationships (e.g., over time or space).

Proportional Relationship

Definition

A proportional relationship is a relationship between two quantities where the ratio between them remains constant. If two variables are proportional, it means they can be expressed in the form:

$$y = kx$$

where k is the constant of proportionality and it can be an integer or a fraction or an irrational number.

Proportionality Problem: Mixing Chemicals

Problem

A person mixes 15ml of bleach with 3.75L of water for sanitizing solution for a daycare. What are the possible combinations

- **A.** 12 mL bleach and 3L water
- **B.** 6 mL bleach and 1.5L water
- **C.** 3 mL leach and 0.75L water
- **D.** 20 mL bleach and 5.5L water

Problem

Is the area of square is propotional to side length ?

Proportionality vs. Linearity

- A **proportional relationship** always passes through the origin $(0, 0)$.
- The general form of a proportional relationship is:

  $$y = kx$$

  where k is the constant of proportionality.
- A **linear relationship** can pass through any point, not necessarily the origin.
- The general form of a linear relationship is:

  $$y = mx + b$$

  where m is the slope and b is the y-intercept.
- Key Difference:
  - In a proportional relationship, $b = 0$, so the line always passes through $(0, 0)$.
  - In a linear relationship, b can be any value, so the line does not need to pass through the origin.

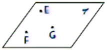
Main Types of Geometry

- **Euclidean Geometry**: Deals with flat, 2D spaces.
- **Non-Euclidean Geometry**: Studies curved spaces (spherical, hyperbolic).
- **Analytic Geometry**: Combines algebra and geometry using coordinates.
- **Differential Geometry**: Uses calculus to study curves and surfaces.

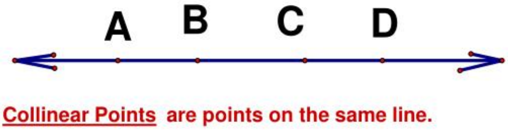
Applications of Geometry

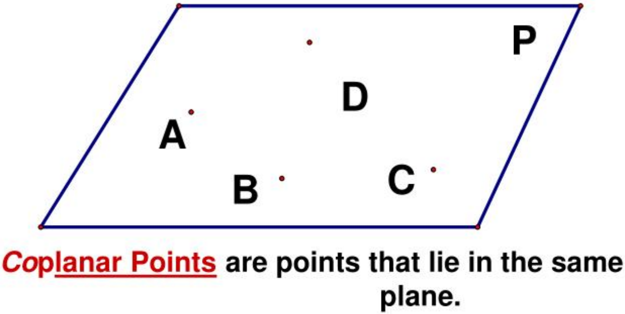
- Essential in fields like architecture, physics, engineering.
- Used in navigation, astronomy, and computer graphics.
- Provides tools to understand spatial relationships in various disciplines.

Geometry Terminology

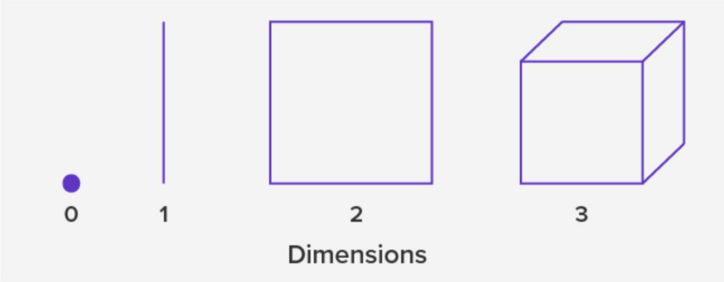
	Description	Figure	Symbol
Point	A geometric element that has zero dimensions.		P or Point P
Line	A line is a collection of points along a straight path with no end points.		$\overline{AB}$ or $\overline{BA}$
Line segment	A line segment is a part of a line that contains every point on the line between its end points.		$\overline{XY}$ or $\overline{YX}$
Ray	A ray is a line with a single end point that goes on and on in one direction.		$\overline{PQ}$
Plane	A plane is a flat surface that extends to infinity.		Plane EFG or Plane τ

Geometry Terminology: Coplanar & Colinear





Geometry Terminology: Dimensions



Parallel & Perpendicular Lines

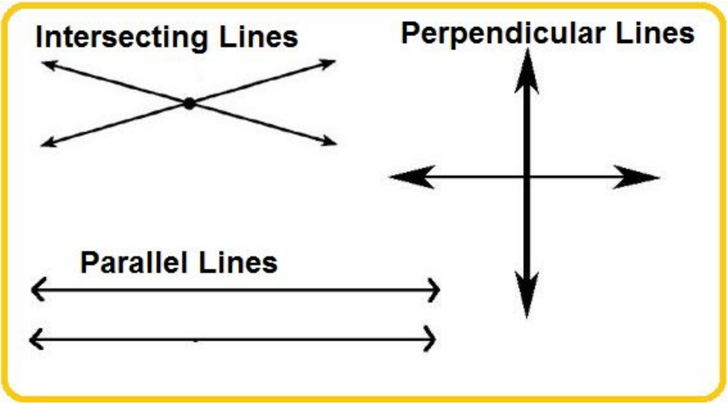
Parallel

Two lines are said to be **parallel** if they never intersect, no matter how far they are extended, and remain the same distance apart at all points.

Perpendicular

Two lines are said to be **perpendicular** if they intersect at a right angle (90 degrees).

Parallel & Perpendicular Lines



Angles

- **Angle**: Formed by two rays with a common endpoint.
- **Acute Angle**: Less than 90°.
- **Right Angle**: Exactly 90°.
- **Obtuse Angle**: Greater than 90° but less than 180°.
- **Straight Angle**: Exactly 180°.

Shapes and Figures

- **Polygon**: A closed figure formed by line segments.
- **Triangle**: A polygon with three sides.
  - **Equilateral Triangle**: All sides and angles are equal.
  - **Isosceles Triangle**: Two sides and angles are equal.
  - **Scalene Triangle**: All sides and angles are different.
- **Quadrilateral**: A polygon with four sides (e.g., square, rectangle).
- **Circle**: A set of points equidistant from the center.

Properties of Shapes

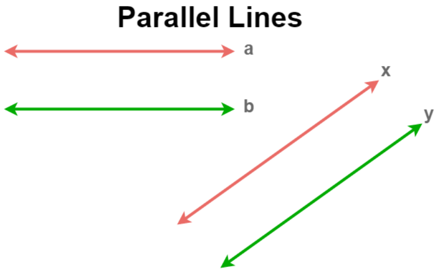
- **Perimeter**: Total distance around a shape.
- **Area**: The measure of space inside a two-dimensional shape.
- **Volume**: The measure of space inside a three-dimensional object.

Transformations

- **Translation**: Moving a shape without rotating or flipping it.
- **Rotation**: Turning a shape around a fixed point.
- **Reflection**: Flipping a shape over a line to create a mirror image.
- **Dilation**: Resizing a shape while maintaining its proportions.

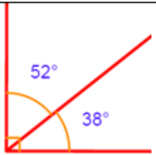
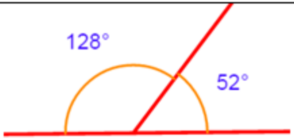
Parallel Lines

- **Definition:** Two or more lines that are always the same distance apart and never meet, no matter how far they are extended.
- They run in the same direction and have the same slope.
- Example: Think of train tracks running side by side—they never cross each other.

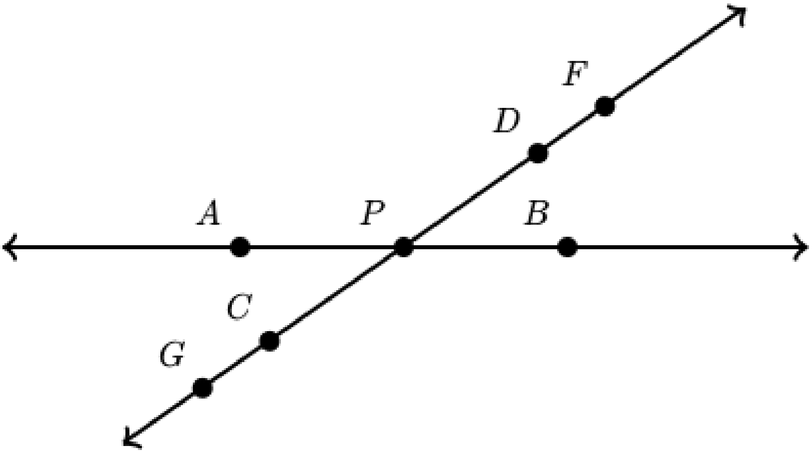


Angles

Supplementary and Complementary Angles

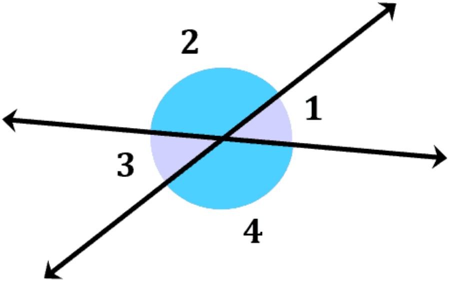
Type of Angles	Description	Example
Complementary Angles	Angles that add up to 90°	
Supplementary Angles	Angles that add up to 180°	

Problem 1



Vertical Angles

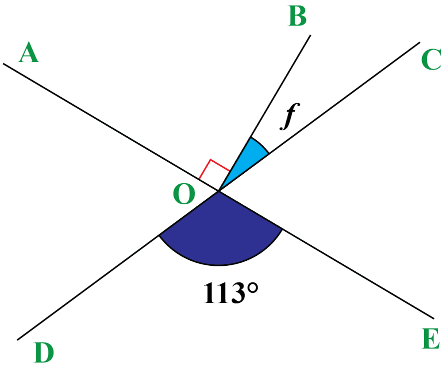
Vertical angles, also known as opposite angles, are the angles formed when two lines intersect



$$\angle 1 \cong \angle 3$$

$$\angle 2 \cong \angle 4$$

Problem 2: Vertical Angles



Transversal & Parallel Lines

A **transversal line** is a line that crosses or intersects two or more other lines at different points

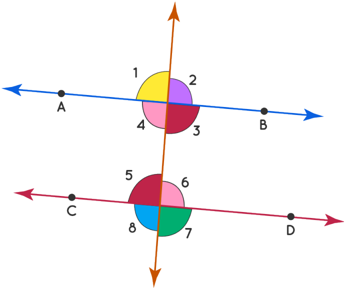


Figure: alternate interior angles

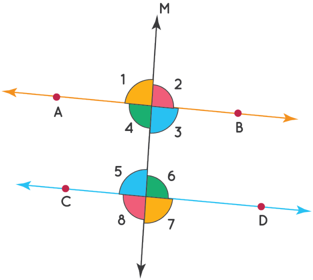


Figure: alternate exterior angles

Altitudes, Medians and Centroid

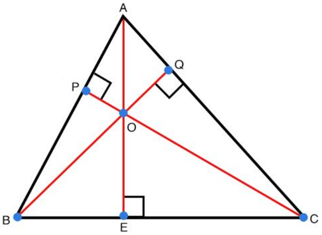


Figure: altitudes

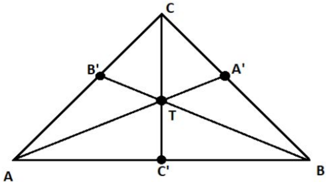
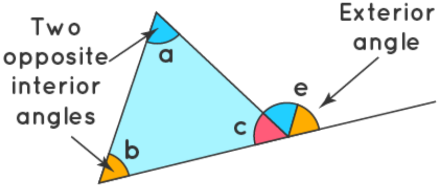


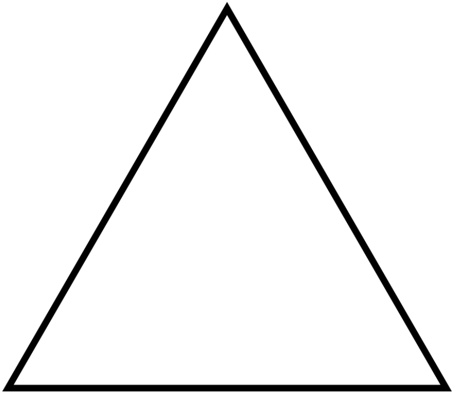
Figure: medians

Exterior Angle Property



$\angle e = \angle a + \angle b$

Angles of a Triangle Measure to 180°



Equilateral and Isosceles Triangle

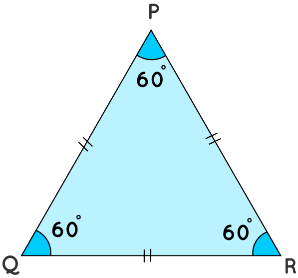


Figure: Equilateral

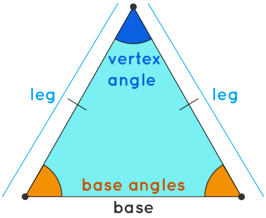
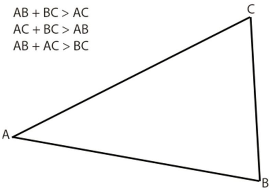


Figure: Isosceles

Triangle Inequality

For any triangle with sides a, b, and c, the following conditions must hold true:

- $a + b > c$
- $a + c > b$
- $b + c > a$



What are Rigid Transformations?

- Rigid transformations (or isometries) preserve the shape and size of geometric objects.
- They do not alter:
  - Distances between points.
  - Angles between lines or curves.
- The object remains congruent to itself after the transformation.

Types of Rigid Transformations

1. **Translation:** Moves every point by the same distance in a given direction.
2. **Rotation:** Rotates an object around a fixed point by a certain angle.
3. **Reflection:** Flips an object over a specified line (the "mirror line").
4. **Glide Reflection:** Combines a reflection and a translation along the direction of the reflection line.

Translation

Definition:

- Moves every point of an object by the same distance in a specific direction.

Mathematical Representation:

$$(x', y') = (x + a, y + b)$$

Properties:

- Preserves distances and angles.
- Does not change the orientation of the object.

Rotation

Definition:

- Rotates an object around a fixed point by a certain angle.

Mathematical Representation: For a rotation by angle θ around the origin:

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

Properties:

- Preserves distances and angles.
- Changes the orientation depending on the direction of rotation.

Reflection

Definition:

- Flips an object over a line (the "mirror line").

Examples:

- Reflection across the x-axis, y-axis, or any line $y = mx + c$.

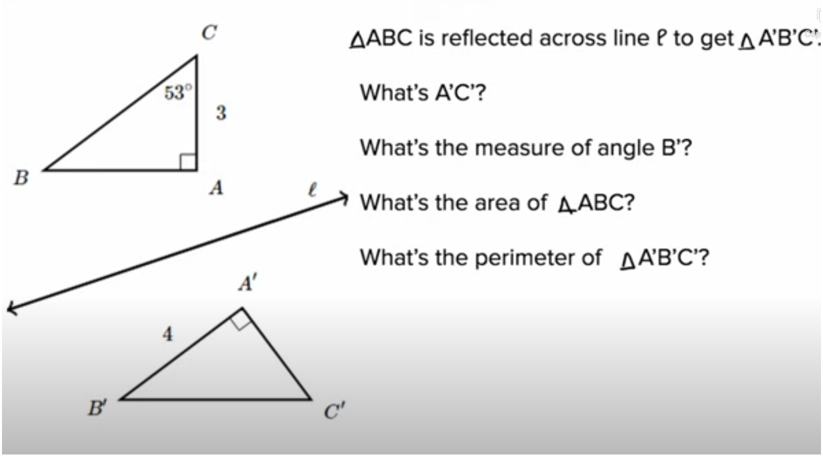
Properties:

- Preserves distances and angles.
- Changes the orientation of the object.

Properties of Rigid Transformations

- **Distance Preservation:** The distance between any two points remains unchanged.
- **Angle Preservation:** Angles between lines or curves are preserved.
- **Parallelism:** Parallel lines remain parallel.
- **Co-ordinates** Co-ordinates ae not preserved
- **Congruence:** The original and transformed shapes are congruent.

Exercise: Rigid Transformations



Dilations

Definition:

- Dilation involves scaling distances from a point (the center of dilation) by a constant factor k. It changes the size of a figure but not its shape.
- A non rigid transformation where lengths are not preserved
- Dilation will preserve angles

What is Congruence

Definition

Congruence means that two figures or shapes are identical in shape and size. They can be transformed into each other using rigid transformations such as translation, rotation, or reflection

- Vertical angles are congruent
- Alternate interior angles are congruent
- Alternate exterior angles are congruent
- Corresponding angles are congruent

Congruence in Triangles

SSS
if three sides of one triangle are congruent to the three sides of another triangle

SAS
if two sides and an included angle of one triangle is congruent to another

ASA
if two angles and included side of a triangle is congruent to another

AAS
if two angles and non included side of a triangle is congruent to another

Similarity in Triangles

AAA

if three angles of one triangle are congruent to another

SSS

if three sides of a triangle are proportional to another

SAS

if two sides of a triangle are proportional and included angle is congruent

What is a Unit Circle

The Unit circle

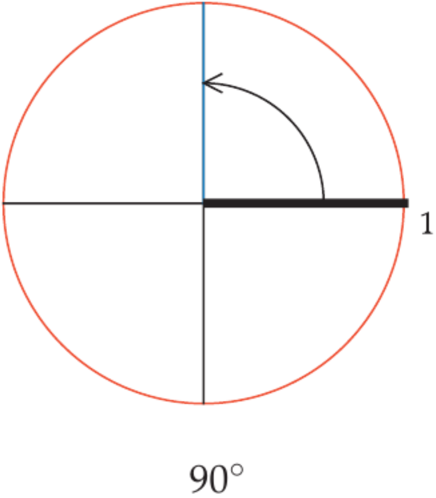
The unit circle is the circle with radius 1 centered at the origin

Equation of unit Circle

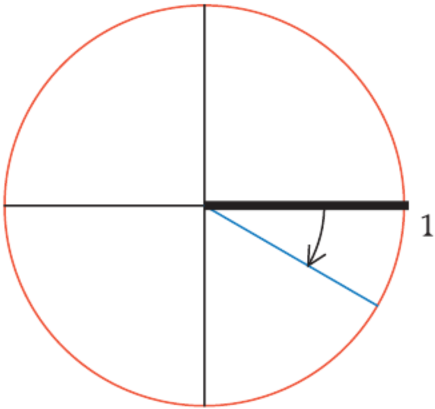
The unit circle in the xy-plane is the set of points (x,y) such that

$$x^2 + y^2 = 1$$

Radius corresponding to a positive angle



Radius corresponding to a negative angle



$-30°$

Positive and Negative Angles

- Angle measurements for a radius on the unit circle are made from the positive horizontal axis.
- Positive angles correspond to moving counterclockwise from the positive horizontal axis.
- Negative angles correspond to moving clockwise from the positive hori- zontal axis.

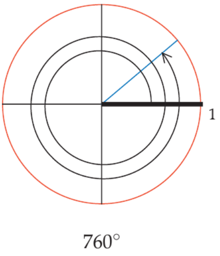


Figure: +ve angle

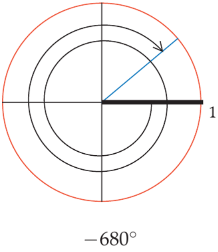
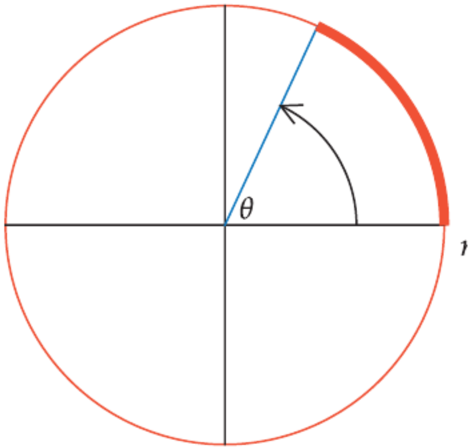


Figure: -ve angle

cyclic hehaviour of angles

A radius of the unit circle corresponding to θ degrees also corresponds to $\theta + 360n$ degrees for every integer n.

Length of a Circular Arc



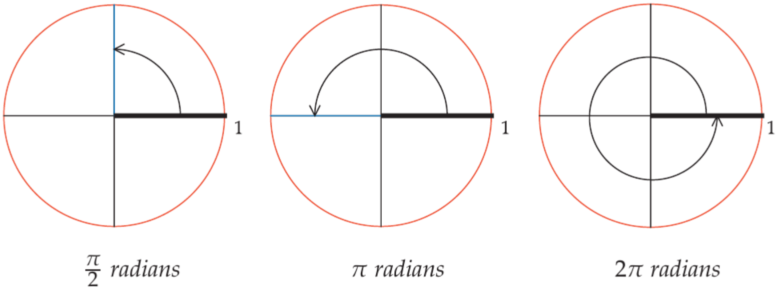
This circular arc has length $\dfrac{\theta\pi r}{180}$.

Radians

Radians

Radians are a unit of measurement for angles such that 2π radians correspond to a rotation through an entire circle.

Radians



Radians

Degree to Radians

$$360^\circ = 2\pi\, radians$$
$$1^\circ = \frac{2\pi}{360} radians$$

Arc Length

length of a circular arc

If $0 < \theta \leq 2\pi$, then a circular arc on the unit circle corresponding to θ radians has length θ

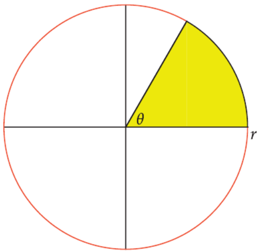


Figure: Area of slice

Area of slice

A slice with angle θ radians inside a circle with radius r has area $\frac{1}{2}\theta r^2$.

Cosine and Sine

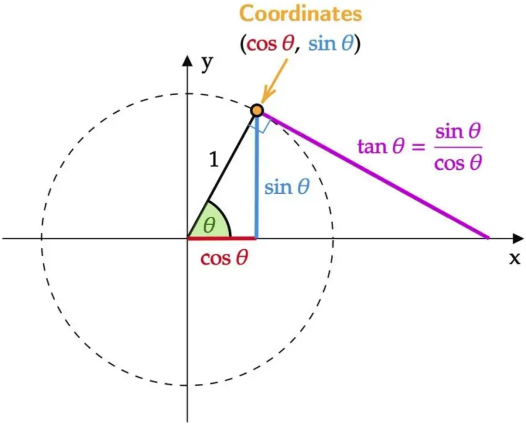


Figure: sin and cos

Variability

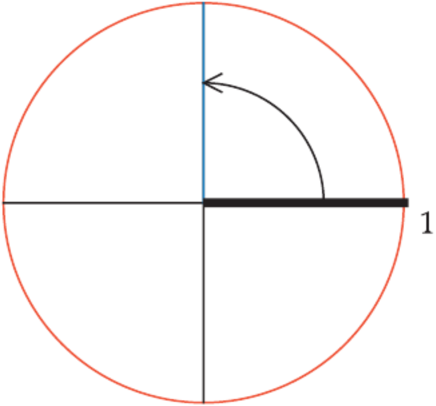
Variability

Variability refers to how data points differ from one another within a data set. In real-world data, there is almost always some variation because no two measurements, observations, or events are exactly the same.

Variability Problems

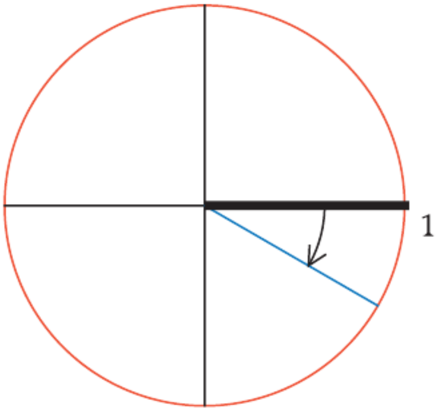
- How much does my pet weight ?
- What is the average number of cars in a parking lot on Monday mornings ?
- Am i hungry?
- How often am I hungry after lunch ?
- How much time do you spend on facebook every month?

Radius corresponding to a positive angle



90°

Radius corresponding to a negative angle



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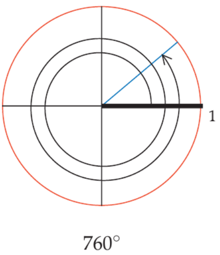

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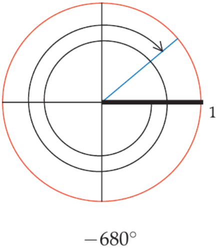

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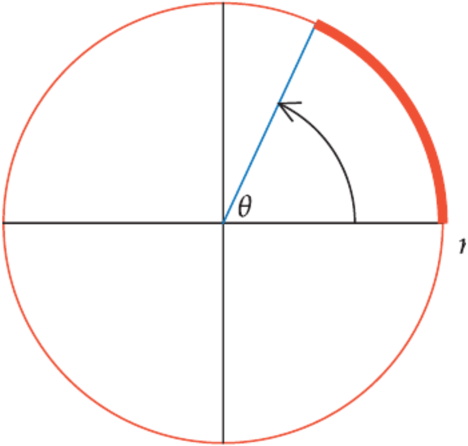

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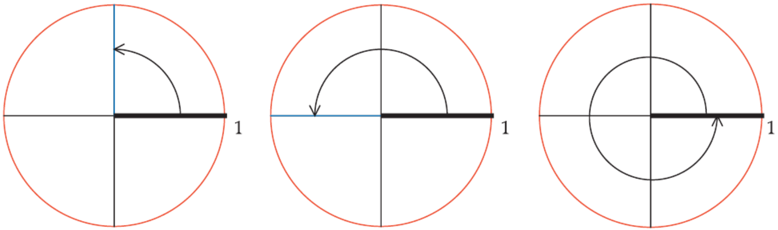

$\frac{\pi}{2}$ *radians* $\qquad$ π *radians* $\qquad$ 2π *radians*

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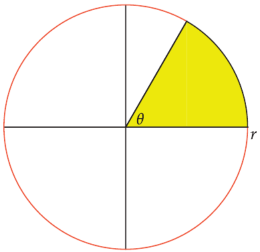

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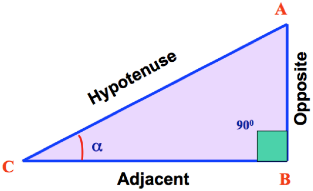
Sine, Cosine and Tangent



$$\sin\alpha = \frac{AB}{CA} = \frac{opposite}{hypotenuse}$$

$$\cos\alpha = \frac{CB}{CA} = \frac{adjacent}{hypotenuse}$$

$$tan\,\alpha = \frac{AB}{CB} = \frac{opposite}{adjacent}$$

Unit Circle Co-ordinates