Algebra

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Maveric Systems

June 3, 2025

1. Exponential Functions, Logarithms and e

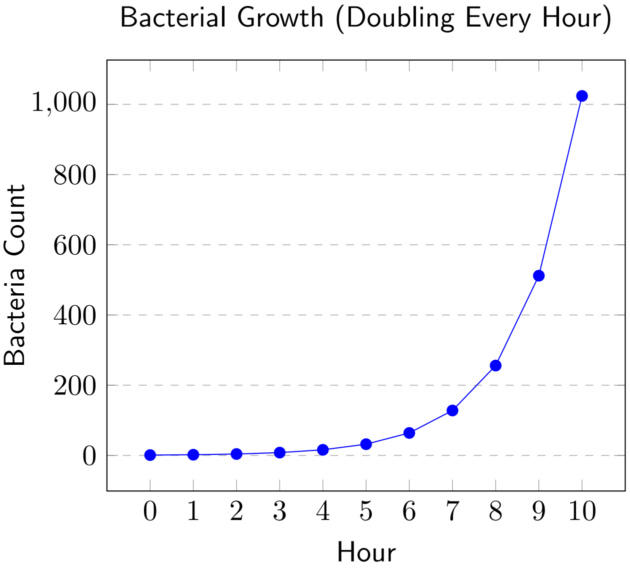
Bacterial Growth on the Human Body

- Our skin (and other areas like the mouth, nose, and intestines) hosts hundreds of thousands of microscopic organisms.
- In fact, bacterial cells in our body outnumber our own cells.
- While some bacteria can cause illness, many are essential for our health.
- Bacteria reproduce through binary fission—each cell splits into two.
- Under ideal conditions, a single bacterium doubling every hour can lead to over 1,000 cells in 10 hours and more than 16 million in 24 hours.

Bacterial Growth Over Time

Hour	0	1	2	3	4	5	6	7	8	9	10
Bacteria	1	2	4	8	16	32	64	128	256	512	1024

Table: Bacterial cell count doubling every hour.


Bacterial Growth (Doubling Every Hour)
1,000
800
600
400
200
0
Bacteria Count
0
1
2
3
4
5
6
7
8
9
10
Hour

Population Growth in India

- India is the second most populous country, with about 1.39 billion people in 2021.
- Its population grows at an annual rate of roughly 1.2%.
- If this trend continues, India's population is projected to exceed China's by 2027.
- While rapid population increases are often described as "exponential," in mathematics the term has a very precise meaning.

Defining Exponential Growth

Key Concepts

- **Percentage Change:**
  - refers to a change based on a percent of the original amount
- **Exponential Growth:**
  - refers to an increase based on a constant multiplicative rate of change over equal increments of time, that is, a percent increase of the original amount over time.
  - For example, if a quantity doubles each period, that is a 100% increase per period.
- **Linear Growth:**
  - The original value increases by a fixed **amount** (additive rate) over equal time intervals.
- **Exponent Decay:**
  - refers to a decrease based on a constant multiplicative rate of change over equal increments of time, that is, a percent decrease of the original amount over time.

Exponential Function and Its Behavior

Definition

Suppose $b > 0$ with $b \neq 1$. Then the *exponential function* with base b is defined by

$$f(x) = b^x.$$

For example, if $b = 2$, then $f(x) = 2^x$. (Note that 2^x is different from x^2.)

Behavior (for $b > 1$)

- **Domain:** All real numbers, $\mathbb{R}$.
- **Range:** All positive numbers, $(0, \infty)$.
- $f(x) = b^x$ is an *increasing* function.
- b^x becomes very large as x increases.
- b^x approaches 0 as x becomes very negative.

Comparing Exponential and Linear Growth

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8

Table: Exponential vs. Linear Growth.

- Linear growth (e.g., $g(x) = 2x$) increases by a constant amount (2) for each increase in x, that is, it is adding or subtracting a constant value. A constant amount $\rightarrow$ linear growth or additive growth.
- Exponential growth (e.g., $f(x) = 2^x$) increases by a constant factor (2) for each increase in x, that is, it is multiplying or dividing by a constant value. A constant factor $\rightarrow$ exponential growth or multiplicative growth.

Example: The Function $f(x) = 2^x$

Exponential Growth Illustrated (Table 2)

x	-3	-2	-1	0	1	2	3
2^x	$2^{-3} = \frac{1}{8}$	$2^{-2} = \frac{1}{4}$	$2^{-1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$

Table: Exponential values of 2^x for $x = -3, \ldots, 3$.

Observation: As x increases by 1, the output of 2^x doubles, clearly illustrating exponential growth.

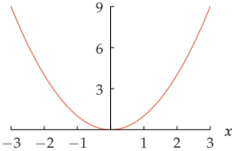
Algebraic Properties of Exponents

Properties

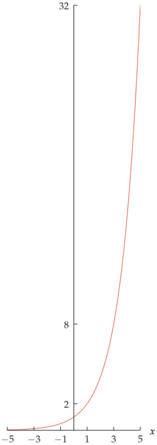
Let $a, b > 0$ and $x, y \in \mathbb{R}$. Then:

- $b^x \cdot b^y = b^{x+y}$
- $(b^x)^y = b^{xy}$
- $a^x \cdot b^x = (ab)^x$
- $b^0 = 1$
- $b^{-x} = \dfrac{1}{b^x}$
- $\dfrac{b^x}{b^y} = b^{x-y}$
- $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

Exponent Graph



The graph of x^2 on the interval $[-3, 3]$. Unlike the graph of 2^x, the graph of x^2 is symmetric about the vertical axis.



The graph of the exponential function 2^x on the interval $[-5, 5]$. Here the same scale is used on both axes to emphasize the rapid growth of this function.

Logarithm

Definition

Suppose b and y are positive numbers with $b \neq 1$.

- The logarithm base b of y, denoted $\log_b y$, is defined as the unique number x such that
$$b^x = y.$$
- Short Version
$$\log_b y = x \quad \text{means} \quad b^x = y.$$

Logarithm of 1 and the Base

Key Properties

Let $b > 0$ with $b \neq 1$. Then:

- $\log_b 1 = 0$ because $b^0 = 1$,
- $\log_b b = 1$ because $b^1 = b$.

Logarithm as an Inverse Function

Definition

Suppose b is a positive number with $b \neq 1$ and the exponential function f is defined by

$$f(x) = b^x.$$

Then the inverse function f^{-1} is given by

$$f^{-1}(y) = \log_b y.$$

Inverse Properties of Logarithms - Summary

- **Inverse Relationship:**
  - $\log_b x$ is the inverse of b^x.
  - Flipping the graph of b^x across the line $y = x$ yields the graph of $\log_b x$.
- **Monotonicity:**
  - For $b > 1$, b^x is increasing, so $\log_b x$ is also increasing.
- **Key Equations:**
  - $b^{\log_b y} = y$
  - $\log_b(b^x) = x$
- **Function-Inverse Properties:**
  - These can be written as $(f \circ f^{-1})(y) = y$ and $(f^{-1} \circ f)(x) = x$.
- **Understanding:**
  - These properties are fundamental to the relationship between exponential and logarithmic functions.

Logarithm of a Power

Property

If b and y are positive numbers, with $b \neq 1$, and t is a real number, then

$$\log_b \left(y^t\right) = t \log_b y.$$

Radioactive Decay

If a radioactive isotope has half-life h, then the function modeling the number of atoms in a sample of this isotope is

$$a(t) = a_0 2^{-t/h}$$

where a_0 is the number of atoms of the isotope in the sample at time 0

Exponential Growth

A story

A mathematician in ancient India invented the game of chess. Filled with gratitude for the remarkable entertainment of this game, the king offered themathematician anything he wanted. The king expected the mathematician to ask for rare jewels or a majestic palace. But the mathematician asked only that he be given one grain of rice for the first square on a chessboard, plus two grains of rice for the next square, plus four grains for the next square, and so on, doubling the amount for each square, until the 64th square on an 8-by-8 chessboard had been reached. The king was pleasantly surprised that the mathematician had asked for such a modest reward. A bag of rice was opened, and first 1 grain was set aside, then 2, then 4, then 8, and so on. As the eighth square was reached, 128 grains of rice were counted out. The king was secretly delighted to be paying such a small reward and also wondering at the foolishness of the mathematician.

Story Cont..

As the 16th square was reached, 32,768 grains of rice were counted out—this was a small part of a bag of rice. But the 21st square required a full bag of rice, and the 24th square required eight bags of rice. This was more than the king had expected. However, it was a trivial amount because the royal granary contained about 200,000 bags of rice to feed the kingdom during the coming winter. As the 31st square was reached, over a thousand bags of rice were required and were delivered from the royal granary. Now the king was worried. By the 37th square, the royal granary was two-thirds empty. The 38th square would have required more bags of rice than were left. The king then stopped the process and ordered that the mathematician's head be chopped off as a warning about the greed induced by exponential growth

Mathematical analysis

- 64^{th} square requires 2^{63} grains $\approx$ $2^3 * (2^{10})^6 = 8 * (10^3)^6 = 8 * 10^{18} \approx 10^{19}$
- if one large bag $= 10^6$ grains of rice , then total bags $= 10^{19}/10^6$
- In 2025 India's population is $\approx 1 * 10^9$

Exponential Growth

Definition

A function f is said to have **exponential growth** if $f(x) = cb^{kx}$ where c and k are positive numbers and $b > 1$

- $f(x) > p(x)$ where f is exponenetial and p is polynomial for sufficiently large x
- $2^x > x^{1000} \ \forall \ \ x > 13747$
- A function f has exponential growth if and only if the graph of $\log f(x)$ is a line with a positive slope

Population Growth

Exponential Growth

$$p(t) = p_0\, e^{rt}$$

- p_0: initial population
- r: constant per-capita growth rate
- Assumes unlimited resources $\rightarrow$ population grows without bound
- Populations of various organisms, ranging from bacteria to humans, often exhibit exponential growth

Population Growth: Example

Suppose a colony of bacteria in a petri dish has 700 cells at 1 pm. These bacteria reproduce at a rate that leads to doubling every three hours. How many bacteria cells will be in the petri dish at 9 pm on the same day?

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$$p(t) = p_0 2^{t/3} \implies 700 \cdot 2^{8/3}$$

Population Growth

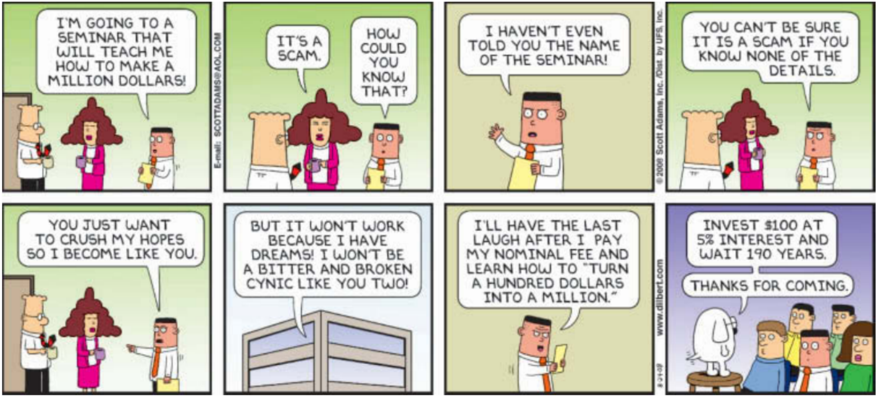
Exponential growth and doubling

If a population doubles every d time units, then the function p modeling this population growth is given by the formula

$$p(t) = p_0 \cdot 2^{(t-t_0)/d}$$

where p_0 is the population at time t_0

Compound Interest



Example

Suppose you deposit 8000 in a bank account that pays 5% annual interest. Assume the bank pays interest once per year at the end of the year, and that each year you place the interest in a cookie jar for safekeeping.

1. How much will you have (original amount plus interest) at the end of two years?
2. How much will you have (original amount plus interest) at the end of t years?

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Interest per year $= 8000 * 0.05 = 400$. For 2 years $= 400 * 2 = 800$
After t years $= 8000 + 8000 * 0.05 * t = 8000(1 + 0.05t)$

Simple Interest

If interest is paid once per year at the annual rate of r, with no interest paid on the interest, then after t years an initial amount P grows to

$$P(t) = P_0(1 + rt)$$

Example

Suppose you deposit 8000 in a bank account that pays 5% annual interest. Assume the bank pays interest once per year at the end of the year, and that each year the interest is deposited in the bank account

1. How much will you have at the end of one year?
2. How much will you have at the end of two years?
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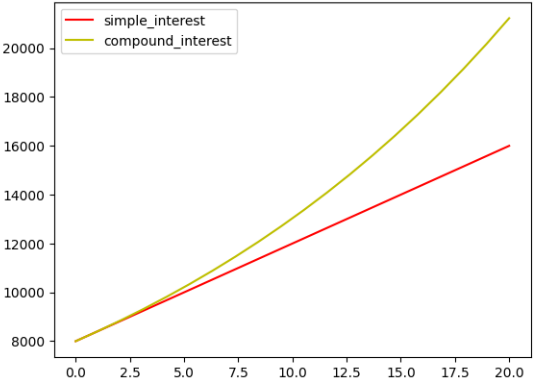
1. How much will you have at the end of one year?
2. How much will you have at the end of two years?
3. How much will you have at the end of t years?

1. At the end of an year
$= 8000 + 8000 * 0.05 = 8400 \implies 8000(1 + 0.05)$
2. At the end of 2 year
$= 8400 + 8400 * 0.05 \implies 8400(1 + 0.05) = 8000(1.05)^2$
3. At the end of t years $= 8000(1.05)^t$

SI vs CI


simple_interest
compound_interest
20000
18000
16000
14000
12000
10000
8000
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0

Example

- Interest is often compounded more than once per year
- In the above example,if the interest is compunded twice an year means instead of 5% being paid every year the interest comes as two payments of 2.5% each year with each payment made at the end of every 6 months

Suppose you deposit 8000 in a bank account that pays 5% annual interest, com- pounded twice per year. How much will you have at the end of one year?

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Suppose you deposit 8000 in a bank account that pays 5% annual interest, com- pounded twice per year. How much will you have at the end of one year?

- At the end of 6 months $= 8000(1 + .025)$
- At the end of 1 year $= (8000 * 1.025)(1.025) = (8000 * 1.05/2)^2$
- At the end of t years $= 8000 * (1 + \frac{0.05}{2})^{2*t}$

Compound Interest

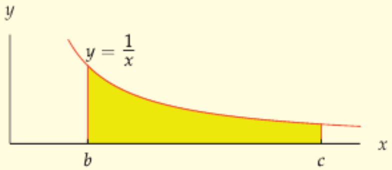
n times per year

If the interest is compounded n times per year at annual interest rate r then after t years an initial amount of P_0 grows to

$$P(t) = P_0(1 + \frac{r}{n})^{nt}$$

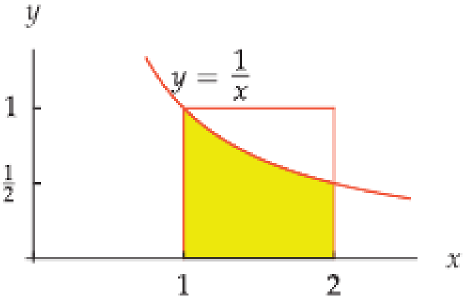
area$\left(\frac{1}{x}, b, c\right)$

For positive numbers b and c with $b < c$, let area$\left(\frac{1}{x}, b, c\right)$ denote the area of the yellow region below:



In other words, area$\left(\frac{1}{x}, b, c\right)$ is the area of the region under the curve $y = \frac{1}{x}$, above the x-axis, and between the lines $x = b$ and $x = c$.

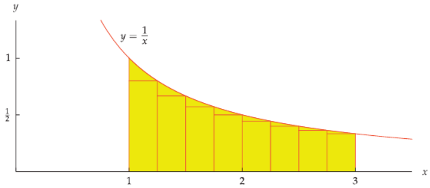
area$(\frac{1}{x}, 1, 2) < 1$



- The area of the rectangle between $x = 1$ and $x = 2$ is 1
- The yellow region lies inside the rectangle and the area of the yellow region is less than 1

area$(\frac{1}{x}, 1, 3) > 1$

- Interval $[1, 3]$ divided into 8 equal parts; each has width $\frac{1}{4}$.
- Heights are calculated using $f(x) = \frac{1}{x}$ at left endpoints of subintervals.
- First three rectangles:
  - 1st: Height $= \frac{1}{\frac{5}{4}} = \frac{4}{5}$, Area $= \frac{1}{4} \cdot \frac{4}{5} = \frac{1}{5}$
  - 2nd: Height $= \frac{1}{\frac{7}{4}} = \frac{4}{7}$, Area $= \frac{1}{4} \cdot \frac{4}{7} = \frac{1}{7}$
  - 3rd: Height $= \frac{1}{\frac{9}{4}} = \frac{4}{9}$, Area $= \frac{1}{4} \cdot \frac{4}{9} = \frac{1}{9}$
- Guess for all areas: $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots, \frac{1}{12}$
- Total area: $\sum_{k=5}^{12} \frac{1}{k} = \frac{28271}{27720} > 1$



Defining e

- Consider the area under $y = \frac{1}{x}$ from 1 to c.
- area$(\frac{1}{x}, 1, 2^2) = 2 *$ area$(\frac{1}{x}, 1, 2)$
- area$(\frac{1}{x}, 1, 3^2) = 2 *$ area$(\frac{1}{x}, 1, 3)$
- area$(\frac{1}{x}, 1, 2^3) = 3 *$ area$(\frac{1}{x}, 1, 2)$
- In general, area$(\frac{1}{x}, 1, c^t) = t *$ area$(\frac{1}{x}, 1, c)$ for every $t > 0$ and $c > 1$.

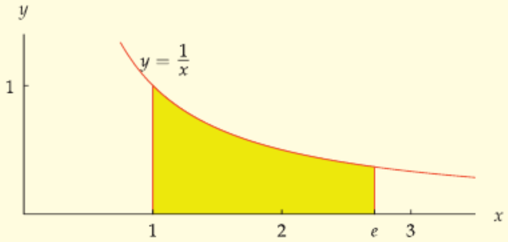
c	Area $(\frac{1}{x}, 1, c)$
2	0.693147
3	1.098612
4	1.386294
5	1.609438
6	1.791759
7	1.945910
8	2.079442
9	2.197225

e

e is the number such that

$$\text{area}\left(\frac{1}{x}, 1, e\right) = 1.$$

In other words, e is the number such that the yellow region below has area 1.



Irrationality of e

- The number e is irrational.
- Here is a 40-digit approximation of e:
- $e \approx 2.718281828459045235360287471352662497757$

Defining the Natural Logarithm

Area under $y = \frac{1}{x}$

$\text{area}(\frac{1}{x}, 1, c^t) = t * \text{area}(\frac{1}{x}, 1, c)$

- The formula resembles the bhaviour of logarithms.
- Thus, area uner the curve $y = \frac{1}{x}$ is connected with a logarithm

$$\text{area}(\frac{1}{x}, 1, e) = 1$$

$$\text{area}(\frac{1}{x}, 1, e^t) = t$$

Assume $t = \log_e c$ (the natural logarithm of c). Then we have

$$\text{area}(\frac{1}{x}, 1, c) = \text{area}(\frac{1}{x}, 1, e^{\log_e c}) = \log_e c$$

Natural Logarithm

Natural Logarithm

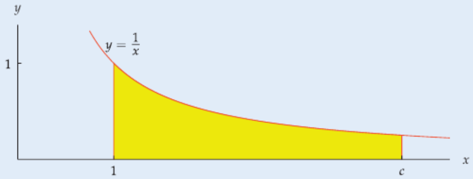
The natural logarithm, denoted $\ln$, is defined as follows:

$$\ln c = \log_e c$$

for $c > 1$.

Natural logarithms as areas

For $c > 1$, the natural logarithm of c is the area of the yellow region below:



In other words,

$$\ln c = \text{area}\left(\tfrac{1}{x}, 1, c\right).$$

The exponenetial function

The **exponential function** is the function f defined by

$$f(x) = e^x$$

where e is the base of the natural logarithm.

- The exponential function with base b is defined as b^x.
- If no base is mentioned, assume the base is e.
- The graph of e^x resembles 2^x, 3^x, etc., for $b > 1$.
- The function b^x is defined as $b^x = e^{\ln b^x}$