Algebra

Nithin

Maveric Systems

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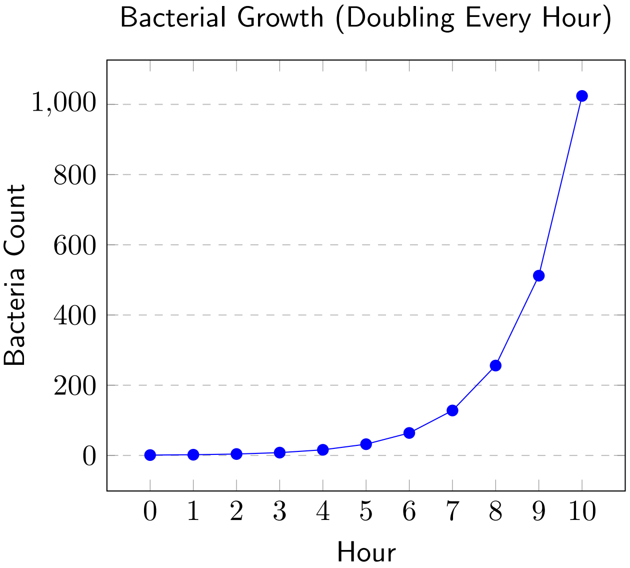
Bacterial Growth on the Human Body

- Our skin (and other areas like the mouth, nose, and intestines) hosts hundreds of thousands of microscopic organisms.
- In fact, bacterial cells in our body outnumber our own cells.
- While some bacteria can cause illness, many are essential for our health.
- Bacteria reproduce through binary fission—each cell splits into two.
- Under ideal conditions, a single bacterium doubling every hour can lead to over 1,000 cells in 10 hours and more than 16 million in 24 hours.

Bacterial Growth Over Time

Hour	0	1	2	3	4	5	6	7	8	9	10
Bacteria	1	2	4	8	16	32	64	128	256	512	1024

Table: Bacterial cell count doubling every hour.


Bacterial Growth (Doubling Every Hour)
Bacteria Count
1,000
800
600
400
200
0
0
1
2
3
4
5
6
7
8
9
10
Hour

Population Growth in India

- India is the second most populous country, with about 1.39 billion people in 2021.
- Its population grows at an annual rate of roughly 1.2%.
- If this trend continues, India's population is projected to exceed China's by 2027.
- While rapid population increases are often described as "exponential," in mathematics the term has a very precise meaning.

Defining Exponential Growth

Key Concepts

- **Percentage Change:**
  - refers to a change based on a percent of the original amount
- **Exponential Growth:**
  - refers to an increase based on a constant multiplicative rate of change over equal increments of time, that is, a percent increase of the original amount over time.
  - For example, if a quantity doubles each period, that is a 100% increase per period.
- **Linear Growth:**
  - The original value increases by a fixed **amount** (additive rate) over equal time intervals.
- **Exponent Decay:**
  - refers to a decrease based on a constant multiplicative rate of change over equal increments of time, that is, a percent decrease of the original amount over time.

Exponential Function and Its Behavior

Definition

Suppose $b > 0$ with $b \neq 1$. Then the *exponential function* with base b is defined by

$$f(x) = b^x.$$

For example, if $b = 2$, then $f(x) = 2^x$. (Note that 2^x is different from x^2.)

Behavior (for $b > 1$)

- **Domain:** All real numbers, $\mathbb{R}$.
- **Range:** All positive numbers, $(0, \infty)$.
- $f(x) = b^x$ is an *increasing* function.
- b^x becomes very large as x increases.
- b^x approaches 0 as x becomes very negative.

Comparing Exponential and Linear Growth

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

Table: Exponential vs. Linear Growth.

Example: The Function $f(x) = 2^x$

Exponential Growth Illustrated (Table 2)

x	-3	-2	-1	0	1	2	3
2^x	$2^{-3} = \frac{1}{8}$	$2^{-2} = \frac{1}{4}$	$2^{-1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$

Table: Exponential values of 2^x for $x = -3, \ldots, 3$.

Observation: As x increases by 1, the output of 2^x doubles, clearly illustrating exponential growth.

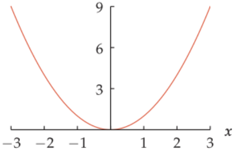
Algebraic Properties of Exponents

Properties

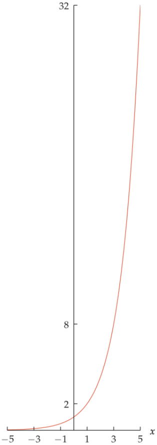
Let $a, b > 0$ and $x, y \in \mathbb{R}$. Then:

- $b^x \cdot b^y = b^{x+y}$
- $(b^x)^y = b^{xy}$
- $a^x \cdot b^x = (ab)^x$
- $b^0 = 1$
- $b^{-x} = \dfrac{1}{b^x}$
- $\dfrac{b^x}{b^y} = b^{x-y}$
- $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

Exponent Graph



The graph of x^2 on the interval $[-3, 3]$. Unlike the graph of 2^x, the graph of x^2 is symmetric about the vertical axis.



The graph of the exponential function 2^x on the interval $[-5, 5]$. Here the same scale is used on both axes to emphasize the rapid growth of this function.

Logarithm

Definition

Suppose b and y are positive numbers with $b \neq 1$.

- The logarithm base b of y, denoted $\log_b y$, is defined as the unique number x such that
$$b^x = y.$$
- Short Version
$$\log_b y = x \quad \text{means} \quad b^x = y.$$

Logarithm of 1 and the Base

Key Properties

Let $b > 0$ with $b \neq 1$. Then:

- $\log_b 1 = 0$ because $b^0 = 1$,
- $\log_b b = 1$ because $b^1 = b$.

Logarithm as an Inverse Function

Definition

Suppose b is a positive number with $b \neq 1$ and the exponential function f is defined by

$$f(x) = b^x.$$

Then the inverse function f^{-1} is given by

$$f^{-1}(y) = \log_b y.$$

Inverse Properties of Logarithms - Summary

- **Inverse Relationship:**
  - $\log_b x$ is the inverse of b^x.
  - Flipping the graph of b^x across the line $y = x$ yields the graph of $\log_b x$.
- **Monotonicity:**
  - For $b > 1$, b^x is increasing, so $\log_b x$ is also increasing.
- **Key Equations:**
  - $b^{\log_b y} = y$
  - $\log_b(b^x) = x$
- **Function-Inverse Properties:**
  - These can be written as $(f \circ f^{-1})(y) = y$ and $(f^{-1} \circ f)(x) = x$.
- **Understanding:**
  - These properties are fundamental to the relationship between exponential and logarithmic functions.

Logarithm of a Power

Property

If b and y are positive numbers, with $b \neq 1$, and t is a real number, then

$$\log_b \left(y^t\right) = t \log_b y.$$

Radioactive Decay) If a radioactive isotope has half-life , then the function modeling the number of atoms in a sample of this isotope is

$$a(t) = a_0 2^{-t/h}$$

where a_0 is the number of atoms of the isotope in the sample at time 0