

# Maths Bootcamp

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Overleaf

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# The School of Athens by Raphael



# What is Geometry?

- Branch of mathematics studying shapes, sizes, and spatial relationships.
- Derived from Greek: "*geo*" (earth) + "*metron*" (measure).
- Originally focused on measuring the earth, now much broader.

# Key Concepts in Geometry

- **Points, Lines, and Angles:** Basic building blocks.
- **Shapes and Figures:** Circles, triangles, polygons.
- **Solids:** 3D objects like cubes, spheres, pyramids.
- **Theorems and Proofs:** Logical reasoning based on axioms and postulates.

# Definition of Percentage

- A **percentage** is a way of expressing a number as a fraction of 100.
- The formula to calculate a percentage is:

$$\text{Percentage} = \left( \frac{\text{Part}}{\text{Whole}} \right) \times 100$$

- Example: If you score 45 out of 60 on a test, the percentage is:

$$\left( \frac{45}{60} \right) \times 100 = 75\%$$

# Ratio vs Rate

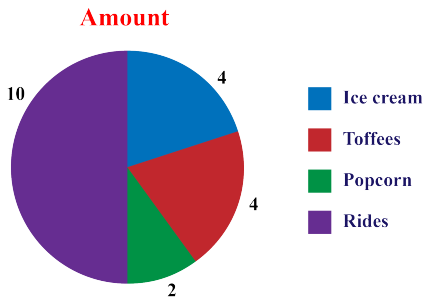


Figure: ratio



Figure: rate

# Key Distinction: Ratio vs Rate

## Ratio

- **Definition:** Comparison of two similar units.
- **Units:** Unitless (dimensionless).
- **Form:** Written as  $a : b$ ,  $\frac{a}{b}$ , or "a to b".
- **Example:** Boys to girls in a classroom is 2 : 3.
- **Key Feature:** Static relationships.

## Rate

- **Definition:** Comparison of two different units.
- **Units:** Includes units (e.g., miles per hour).
- **Form:** Written as  $\frac{a \text{ unit}_1}{b \text{ unit}_2}$  or "a per b".
- **Example:** A car travels 60 mph.
- **Key Feature:** Dynamic relationships (e.g., over time or space).

# Proportional Relationship

## Definition

A proportional relationship is a relationship between two quantities where the ratio between them remains constant. If two variables are proportional, it means they can be expressed in the form:

$$y = kx$$

where  $k$  is the constant of proportionality and it can be an integer or a fraction or an irrational number.



# Proportionality Problem: Mixing Chemicals

## Problem

A person mixes 15mL of bleach with 3.75L of water for sanitizing solution for a daycare. What are the possible combinations

- **A.** 12 mL bleach and 3L water
- **B.** 6 mL bleach and 1.5L water
- **C.** 3 mL leach and 0.75L water
- **D.** 20 mL bleach and 5.5L water

## Problem

Is the area of square is propotional to side length ?

# Proportionality vs. Linearity

- A **proportional relationship** always passes through the origin  $(0, 0)$ .
- The general form of a proportional relationship is:

$$y = kx$$

where  $k$  is the constant of proportionality.

- A **linear relationship** can pass through any point, not necessarily the origin.
- The general form of a linear relationship is:

$$y = mx + b$$

where  $m$  is the slope and  $b$  is the y-intercept.

- Key Difference:
  - In a proportional relationship,  $b = 0$ , so the line always passes through  $(0, 0)$ .
  - In a linear relationship,  $b$  can be any value, so the line does not need to pass through the origin.





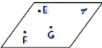
# Main Types of Geometry

- **Euclidean Geometry:** Deals with flat, 2D spaces.
- **Non-Euclidean Geometry:** Studies curved spaces (spherical, hyperbolic).
- **Analytic Geometry:** Combines algebra and geometry using coordinates.
- **Differential Geometry:** Uses calculus to study curves and surfaces.

# Applications of Geometry

- Essential in fields like architecture, physics, engineering.
- Used in navigation, astronomy, and computer graphics.
- Provides tools to understand spatial relationships in various disciplines.

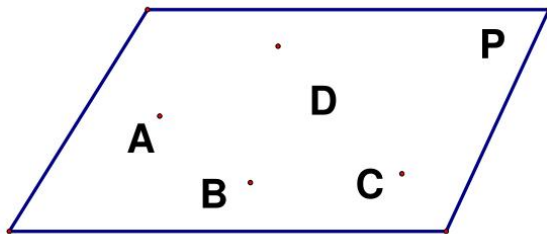
# Geometry Terminology

	Description	Figure	Symbol
Point	A geometric element that has zero dimensions.		P or Point P
Line	A line is a collection of points along a straight path with no end points.		$\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$
Line segment	A line segment is a part of a line that contains every point on the line between its end points.		$\overline{XY}$ or $\overline{YX}$
Ray	A ray is a line with a single end point that goes on and on in one direction.		$\overrightarrow{PQ}$
Plane	A plane is a flat surface that extends to infinity.		Plane EFG or Plane $\tau$

# Geometry Terminology: Coplanar & Collinear

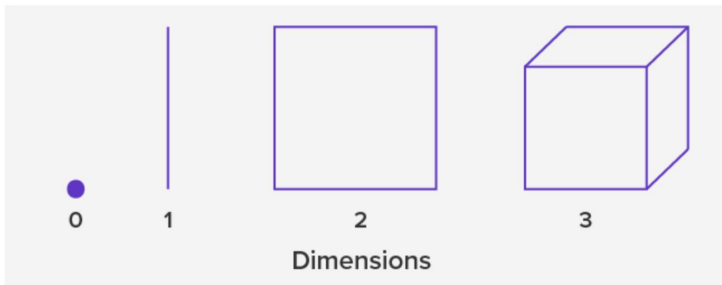


Collinear Points are points on the same line.



Coplanar Points are points that lie in the same plane.

# Geometry Terminology: Dimensions



# Parallel & Perpendicular Lines

## Parallel

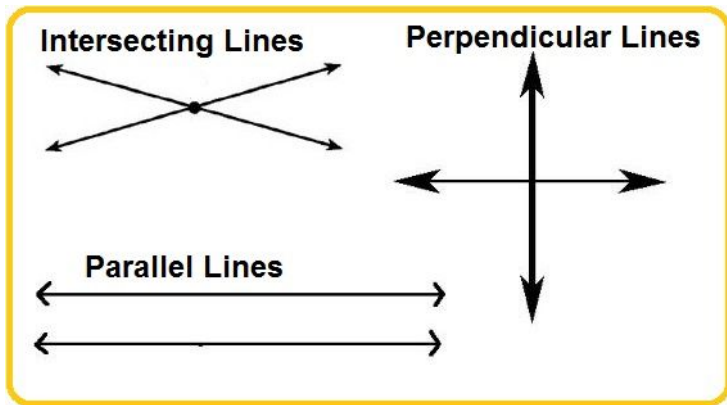
Two lines are said to be **parallel** if they never intersect, no matter how far they are extended, and remain the same distance apart at all points.

## Perpendicular

Two lines are said to be **perpendicular** if they intersect at a right angle (90 degrees).



# Parallel & Perpendicular Lines



# Angles

- **Angle:** Formed by two rays with a common endpoint.
- **Acute Angle:** Less than  $90^\circ$ .
- **Right Angle:** Exactly  $90^\circ$ .
- **Obtuse Angle:** Greater than  $90^\circ$  but less than  $180^\circ$ .
- **Straight Angle:** Exactly  $180^\circ$ .

# Shapes and Figures

- **Polygon:** A closed figure formed by line segments.
- **Triangle:** A polygon with three sides.
  - **Equilateral Triangle:** All sides and angles are equal.
  - **Isosceles Triangle:** Two sides and angles are equal.
  - **Scalene Triangle:** All sides and angles are different.
- **Quadrilateral:** A polygon with four sides (e.g., square, rectangle).
- **Circle:** A set of points equidistant from the center.

# Properties of Shapes

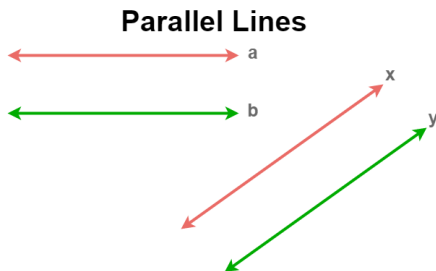
- **Perimeter:** Total distance around a shape.
- **Area:** The measure of space inside a two-dimensional shape.
- **Volume:** The measure of space inside a three-dimensional object.

# Transformations

- **Translation:** Moving a shape without rotating or flipping it.
- **Rotation:** Turning a shape around a fixed point.
- **Reflection:** Flipping a shape over a line to create a mirror image.
- **Dilation:** Resizing a shape while maintaining its proportions.

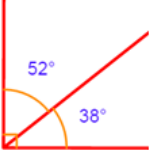
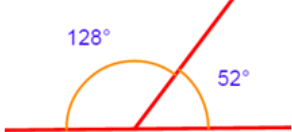
# Parallel Lines

- **Definition:** Two or more lines that are always the same distance apart and never meet, no matter how far they are extended.
- They run in the same direction and have the same slope.
- Example: Think of train tracks running side by side—they never cross each other.



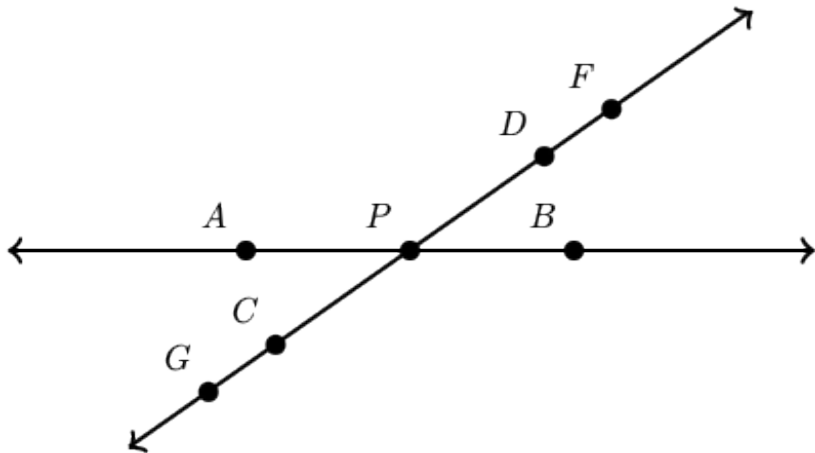
# Angles

# Supplementary and Complementary Angles

Type of Angles	Description	Example
Complementary Angles	Angles that add up to $90^\circ$	 A diagram showing a right angle (90 degrees) divided into two adjacent angles by a red ray. The top angle is labeled 52 degrees and the bottom angle is labeled 38 degrees. A small square at the vertex indicates the total angle is 90 degrees.
Supplementary Angles	Angles that add up to $180^\circ$	 A diagram showing a straight line (180 degrees) divided into two adjacent angles by a red ray. The left angle is labeled 128 degrees and the right angle is labeled 52 degrees. A semi-circular arc at the vertex indicates the total angle is 180 degrees.

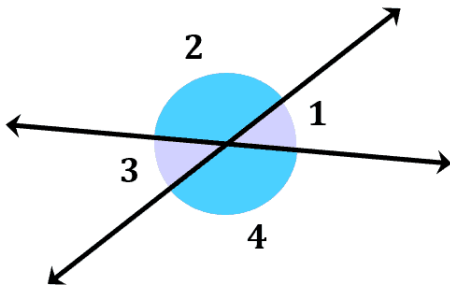


# Problem 1



# Vertical Angles

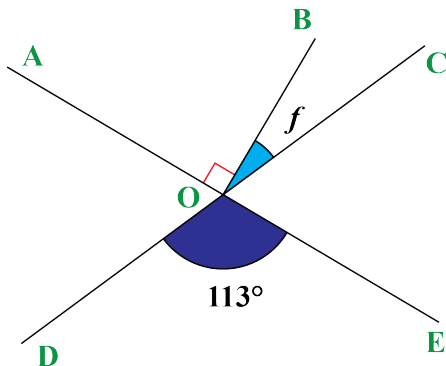
Vertical angles, also known as opposite angles, are the angles formed when two lines intersect



$$\angle 1 \cong \angle 3$$

$$\angle 2 \cong \angle 4$$

## Problem 2: Vertical Angles



# Transversal & Parallel Lines

A **transversal line** is a line that crosses or intersects two or more other lines at different points

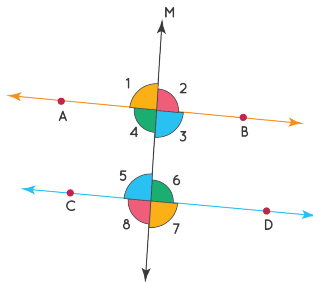
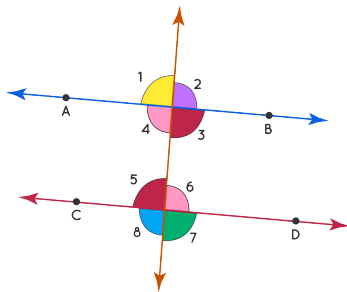


Figure: alternate interior angles      Figure: alternate exterior angles

# Altitudes, Medians and Centroid

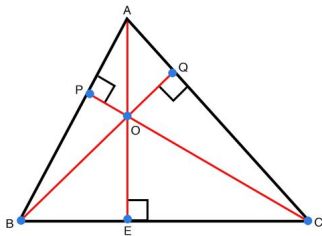


Figure: altitudes

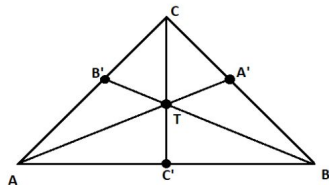
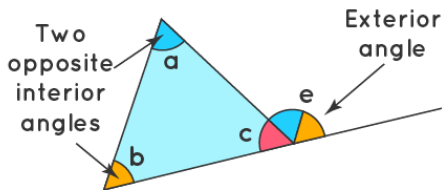


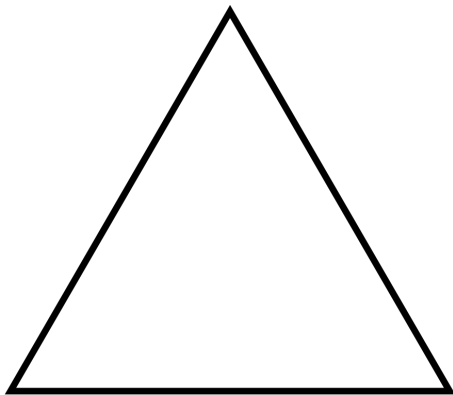
Figure: medians

# Exterior Angle Property



$$\angle e = \angle a + \angle b$$

# Angles of a Triangle Measure to $180^\circ$



# Equilateral and Isosceles Triangle

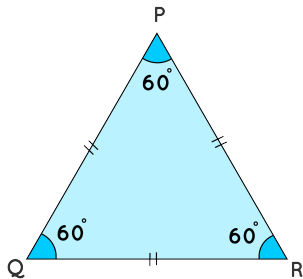


Figure: Equilateral

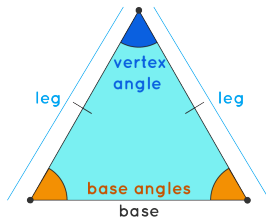


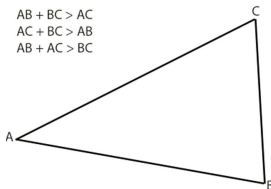
Figure: Isosceles



# Triangle Inequality

For any triangle with sides  $a$ ,  $b$ , and  $c$ , the following conditions must hold true:

- $a + b > c$
- $a + c > b$
- $b + c > a$



# What are Rigid Transformations?

- Rigid transformations (or isometries) preserve the shape and size of geometric objects.
- They do not alter:
  - Distances between points.
  - Angles between lines or curves.
- The object remains congruent to itself after the transformation.

# Types of Rigid Transformations

- 1 **Translation:** Moves every point by the same distance in a given direction.
- 2 **Rotation:** Rotates an object around a fixed point by a certain angle.
- 3 **Reflection:** Flips an object over a specified line (the "mirror line").
- 4 **Glide Reflection:** Combines a reflection and a translation along the direction of the reflection line.

## Definition:

- Moves every point of an object by the same distance in a specific direction.

## Mathematical Representation:

$$(x', y') = (x + a, y + b)$$

## Properties:

- Preserves distances and angles.
- Does not change the orientation of the object.

# Rotation

## Definition:

- Rotates an object around a fixed point by a certain angle.

**Mathematical Representation:** For a rotation by angle  $\theta$  around the origin:

$$x' = x \cos \theta - y \sin \theta$$

$$y' = x \sin \theta + y \cos \theta$$

## Properties:

- Preserves distances and angles.
- Changes the orientation depending on the direction of rotation.

# Reflection

## Definition:

- Flips an object over a line (the "mirror line").

## Examples:

- Reflection across the  $x$ -axis,  $y$ -axis, or any line  $y = mx + c$ .

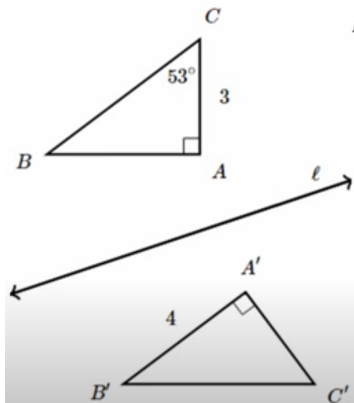
## Properties:

- Preserves distances and angles.
- Changes the orientation of the object.

# Properties of Rigid Transformations

- **Distance Preservation:** The distance between any two points remains unchanged.
- **Angle Preservation:** Angles between lines or curves are preserved.
- **Parallelism:** Parallel lines remain parallel.
- **Co-ordinates** Co-ordinates are not preserved
- **Congruence:** The original and transformed shapes are congruent.

# Exercise: Rigid Transformations



$\triangle ABC$  is reflected across line  $\ell$  to get  $\triangle A'B'C'$ .

What's  $A'C'$ ?

What's the measure of angle  $B'$ ?

What's the area of  $\triangle ABC$ ?

What's the perimeter of  $\triangle A'B'C'$ ?



## Definition:

- Dilation involves scaling distances from a point (the center of dilation) by a constant factor  $k$ . It changes the size of a figure but not its shape.
- A non rigid transformation where lengths are not preserved
- Dilation will preserve angles

# What is Congruence

## Definition

Congruence means that two figures or shapes are identical in shape and size. They can be transformed into each other using rigid transformations such as translation, rotation, or reflection

- Vertical angles are congruent
- Alternate interior angles are congruent
- Alternate exterior angles are congruent
- Corresponding angles are congruent

# Congruence in Triangles

## SSS

if three sides of one triangle are congruent to the three sides of another triangle

## SAS

if two sides and an included angle of one triangle is congruent to another

## ASA

if two angles and included side of a triangle is congruent to another

## AAS

if two angles and non included side of a triangle is congruent to another

# Similarity in Triangles

AAA

if three angles of one triangle are congruent to another

SSS

if three sides of a triangle are proportional to another

SAS

if two sides of a triangle are proportional and included angle is congruent

# What is a Unit Circle

## The Unit circle

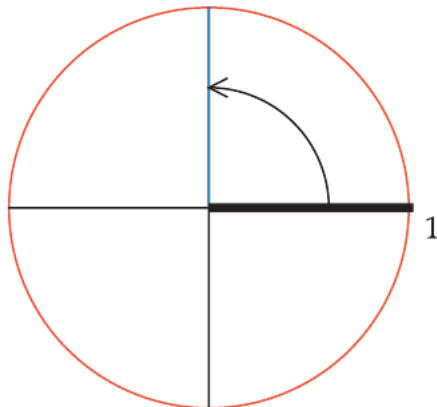
The unit circle is the circle with radius 1 centered at the origin

## Equation of unit Circle

The unit circle in the  $xy$ -plane is the set of points  $(x,y)$  such that

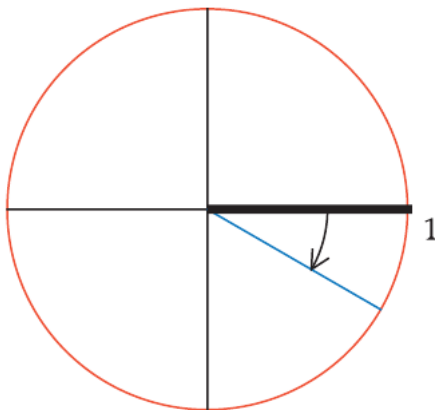
$$x^2 + y^2 = 1$$

# Radius corresponding to a positive angle



$90^\circ$

# Radius corresponding to a negative angle

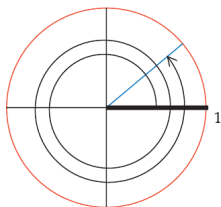


$-30^\circ$

## Positive and Negative Angles

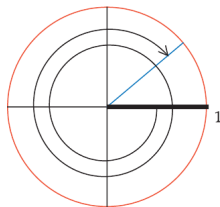
- Angle measurements for a radius on the unit circle are made from the positive horizontal axis.
- Positive angles correspond to moving counterclockwise from the positive horizontal axis.
- Negative angles correspond to moving clockwise from the positive horizontal axis.





$760^\circ$

Figure: +ve angle



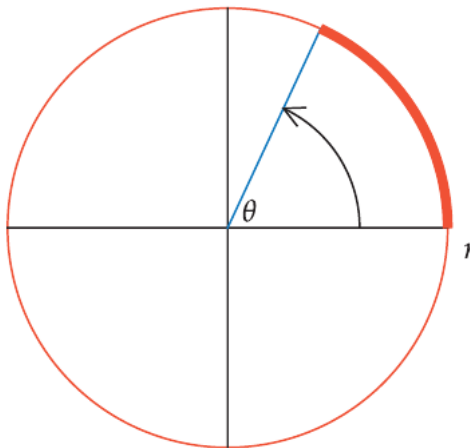
$-680^\circ$

Figure: -ve angle

## cyclic behaviour of angles

A radius of the unit circle corresponding to  $\theta$  degrees also corresponds to  $\theta + 360n$  degrees for every integer  $n$ .

# Length of a Circular Arc



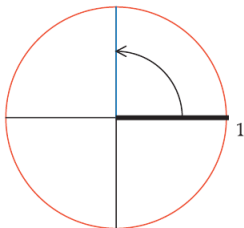
*This circular arc has length  $\frac{\theta\pi r}{180}$ .*

# Radians

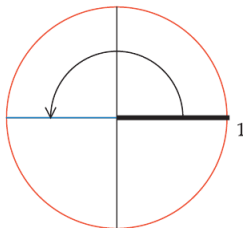
## Radians

Radians are a unit of measurement for angles such that  $2\pi$  radians correspond to a rotation through an entire circle.

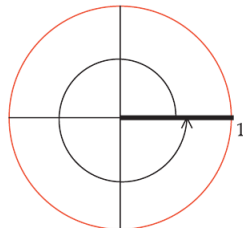
# Radians



$\frac{\pi}{2}$  radians



$\pi$  radians



$2\pi$  radians

## Degree to Radians

$$360^\circ = 2\pi \text{ radians}$$

$$1^\circ = \frac{2\pi}{360} \text{ radians}$$

# Arc Length

length of a circular arc

If  $0 < \theta \leq 2\pi$ , then a circular arc on the unit circle corresponding to  $\theta$  radians has length  $\theta$

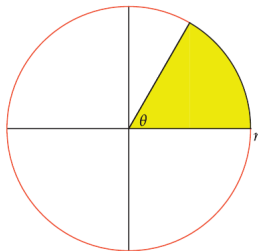


Figure: Area of slice

## Area of slice

A slice with angle  $\theta$  radians inside a circle with radius  $r$  has area  $\frac{1}{2}\theta r^2$ .

# Cosine and Sine

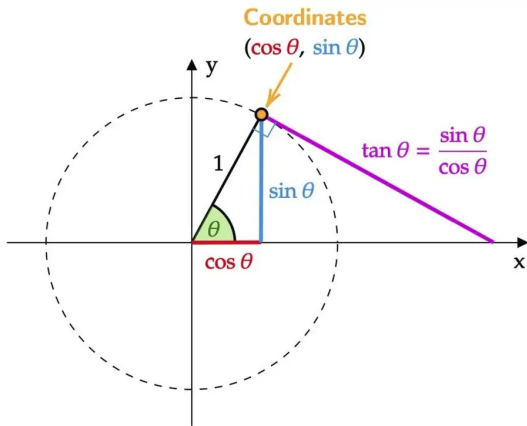


Figure: sin and cos



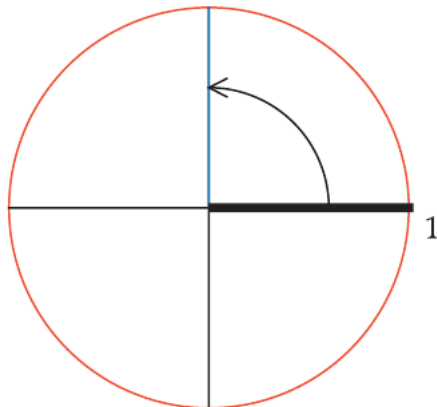
## Variability

Variability refers to how data points differ from one another within a data set. In real-world data, there is almost always some variation because no two measurements, observations, or events are exactly the same.

# Variability Problems

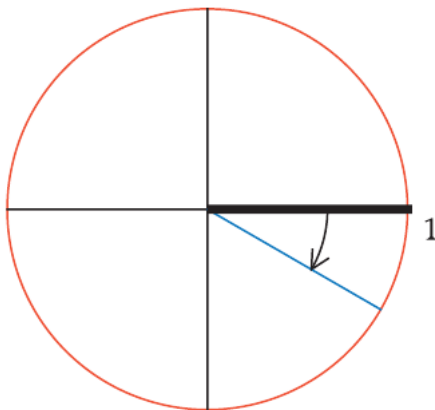
- How much does my pet weight ?
- What is the average number of cars in a parking lot on Monday mornings ?
- Am i hungry?
- How often am I hungry after lunch ?
- How much time do you spend on facebook every month?

# Radius corresponding to a positive angle



$90^\circ$

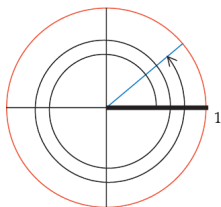
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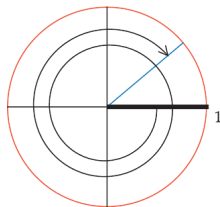
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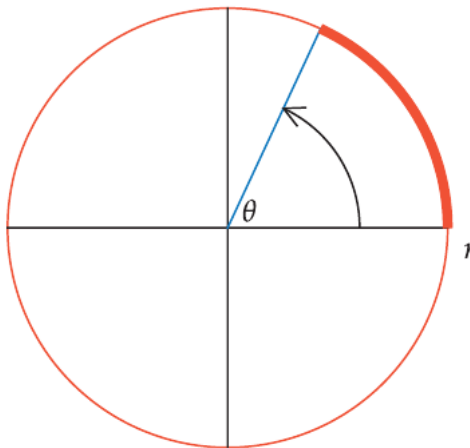
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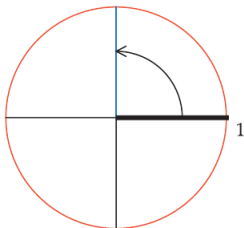
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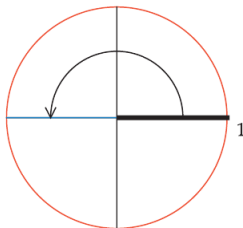
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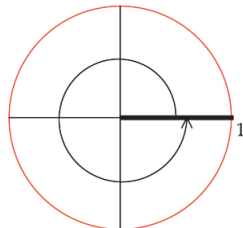
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$\frac{\pi}{2}$  radians



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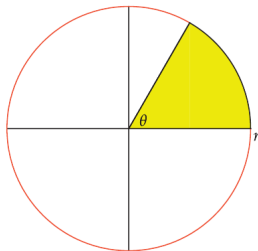
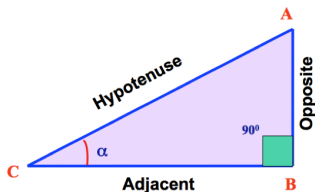


Figure: Area of slice

## Area of slice

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# Sine, Cosine and Tangent



$$\sin \alpha = \frac{AB}{CA} = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos \alpha = \frac{CB}{CA} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan \alpha = \frac{AB}{CB} = \frac{\text{opposite}}{\text{adjacent}}$$

# Unit Circle Co-ordinates

