Algebra

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Maveric Systems

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Origins of Algebra

- **Mesopotamia & Egypt (c. 2000–1600 BCE)**
  - Early problem-solving (linear/quadratic equations) in word problems
  - No formal symbols, but systematic procedures
- **Greek Era (c. 600 BCE–300 CE)**
  - Geometric methods for solving equations (Euclid, Apollonius)
  - Diophantus introduced proto-symbolic notation
- **Islamic Golden Age (8th–12th Century)**
  - Al-Khwarizmi's work *Al-jabr* $\rightarrow$ term "Algebra"
  - Systematic solutions for linear and quadratic equations
- **Transmission to Europe (12th–17th Century)**
  - Latin translations influenced Fibonacci, others
  - Viète, Descartes established modern symbolic notation & analytic geometry
- **Modern Algebra (19th–20th Century)**
  - Emergence of abstract algebra (groups, rings, fields)
  - Galois, Abel, and others formalized algebraic structures

What is Algebra?

Algebra is a branch of mathematics that deals with numbers, variables, and their relationships. Key concepts include:

- **Variables**: Symbols (like x, y) representing unknown or changing values.
- **Expressions**: Combinations of variables, numbers, and operations. E.g., $2x + 3$.
- **Equations**: Mathematical statements that express equality, e.g., $2x + 3 = 7$.
- **Solving Equations**: Finding values for variables that make an equation true.
- **Polynomials**: Expressions like $3x^2 + 2x - 5$ involving variables raised to powers.
- **Functions**: Describes a relationship between variables, e.g., $y = 2x + 1$.

Integers

- The set of integers is denoted by $\mathbb{Z}$.
- Integers include:

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

- Formally, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- Common properties:
  - $\mathbb{Z}$ is infinite and unbounded in both the negative and positive directions.
  - Closed under addition, subtraction, and multiplication:

$$\forall a, b \in \mathbb{Z}, \quad a \pm b \in \mathbb{Z}, \quad a \cdot b \in \mathbb{Z}.$$

- The quotient of any two integers is not necessarily an integer. So we need to extend arithmetic to **rational numbers**

Rational Numbers

- The set of rational numbers is denoted by $\mathbb{Q}$.
- Definition:

$$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p \in \mathbb{Z},\ q \in \mathbb{Z},\ q \neq 0 \right\}.$$

- Every integer is also a rational number (e.g., $5 = \frac{5}{1}$).
- Examples:

$$\frac{1}{2}, \quad -\frac{3}{4}, \quad 0, \quad 7, \quad \frac{11}{5}, \ldots$$

- Properties:
  - Closed under addition, subtraction, multiplication, and division (except division by zero).
  - Densely packed on the number line: between any two rationals, there is another rational.

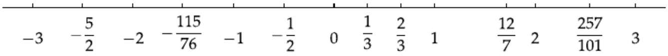
Interesting Facts

- Why division by zero is prohibited ?
  - Division is inverse of multiplication in the sense

$$\frac{m}{n} \cdot n = m$$

  - if $n = 0$ and $m = 1$, we get $\frac{1}{0} \cdot 0 = 1$ which is nonsensical as any number multiplied by zero is zero
- Rational numbers suffice for all actual physical measurements like weight, height and length
- But Geometry, Algebra and Calculus force us to consider **real numbers**

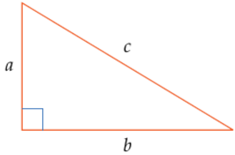
A Real Number Line



Some rational numbers on the real line.

- if n is a positive integer then $\frac{1}{n}$ is to the right of 0 by the length obtained by dividing the segment from $1 to 0$ in to n segments of equal length

Is every Real Number a Rational



- $c^2 = a^2 + b^2$. If $a = 1, b = 1$ then $c^2 = 2$. Then what rational number is c
- By trial and error, $c = \left(\frac{99}{70}\right)^2 = \frac{9801}{4900}$ where the numerator just misses twice the denominator by 1. But this is not 2 but close to 2. Another number is $\left(\frac{9369319}{6625109}\right)^2 = 1.999999999999977$, but not 2
- Greeks proved that it is impossible to find any rational number whose square is 2

Proof: No rational number has a square equal to 2

Let m and n are two integers

$$\left(\frac{m}{n}\right)^2 = 2$$

By canceling any common factors, m and n are reduces to its lowest terms

$$m^2 = 2n^2$$

this makes m^2 even, hence m is an even. (The square of even is even and odd is odd). So $m = 2k$ for some integer k
Substituting $m = 2k$ in the equation gives, $4k^2 = 2n^2$, which results in

$$2k^2 = n^2$$

which means n^2 is even and therefore n is even
$\frac{m}{n}$ has common factors which contradicts the earlier assumption

Irrational Number

Irrational Number

A real number that is not rational is **irrational number**

- $\sqrt(2)$
- $3 + \sqrt(2)$
- $8\sqrt(2)$

Properties of Real Numbers

- **Commutative Properties**
  - Addition: $a + b = b + a$
  - Multiplication: $a \cdot b = b \cdot a$
- **Associative Properties**
  - Addition: $(a + b) + c = a + (b + c)$
  - Multiplication: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- **Distributive Property**
  - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
- **Identity Elements**
  - Additive Identity: $a + 0 = a$
  - Multiplicative Identity: $a \cdot 1 = a$
- **Inverse Elements**
  - Additive Inverse: $a + (-a) = 0$
  - Multiplicative Inverse (if $a \neq 0$): $a \cdot \frac{1}{a} = 1$

Properties of Real Numbers

- **Closure Property**
  - Real numbers are closed under addition, subtraction, multiplication, and division (except division by zero).

Inequalities

Transitivity

- If $a < b$ and $b < c$, then $a < c$

Multiplication

Suppose $a < b$

- If $c > 0$, then $ac < bc$
- If $c < 0$, then $ac > bc$

Exercise

Find all number x such that

$$\frac{x-8}{x-4} < 3$$

Our first step is to multiply by $x-4$ Here there are two conditions:

1. $x-4>0$

$$x-8 < 3(x-4) \implies x-8 < 3x-12 \implies 2x > 4 \implies x > 2$$

But our initial assumption is $x-4>0 \implies x>4$. As $4>2$, original inequality holds if $x>4$

Exercise Conti.

2. $x - 4 < 0$

$$x - 8 > 3(x - 4) \implies x < 2$$

Initial assumption is $x < 4$. As $2 < 4$, inequality holds for $x < 2$

The original inequality holds true for

$$x < 2, x > 4$$

or

$$(-\infty, 2) \cup (4, \infty)$$

Inequalities

Additive Inverse

If $a < b$ then $-a > -b$ Direction of inequalities has to be reversed when taking additive inverses on both sides

Multiplicative Inverse

If $a < b$

- If $a > 0, b > 0$, then $\frac{1}{a} > \frac{1}{b}$
- If $a < 0 < b$, then $\frac{1}{a} < \frac{1}{b}$

What is a Set?

Definition

A **set** is a well-defined collection of distinct objects, called **elements** or **members** of the set.

Representation of a Set:

- **Roster Form:** List elements inside curly braces:

$$A = \{1, 2, 3, 4\}$$

- **Set-Builder Notation:** Describe properties of elements:

$$A = \{x \mid x \text{ is a positive integer less than 5}\}$$

Membership

- If x belongs to A, write $x \in A$.
- If x does not belong to A, write $x \notin A$.

Types of Sets

Types of Sets

- **Finite Set:** A set with a countable number of elements.
  Example: $A = \{1, 2, 3, 4\}$
- **Infinite Set:** A set with an uncountable or infinite number of elements.
  Example: $\mathbb{N} = \{1, 2, 3, \dots\}$
- **Empty/Null Set:** A set with no elements, denoted as $\emptyset$ or $\{\}$.
- **Subset:** $A \subseteq B$ if every element of A is in B.
- **Universal Set:** A set containing all objects under consideration, usually denoted by U.
- **Power Set:** The set of all subsets of A, denoted as $P(A)$.
  Example: If $A = \{1, 2\}$, then $P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Set Operations

Union ($\cup$)

Combines elements of two sets:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Intersection ($\cap$)

Elements common to both sets:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Difference ($A - B$)

Elements in A but not in B:

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

Complement (A^c)

Elements not in the set A:

$$A^c = \{x \mid x \notin A\}$$

Examples

- The set of natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- The set of integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- The set of even numbers: $\{2, 4, 6, \ldots\}$.

What is an Interval?

Definition

An **interval** is a set of real numbers that includes all the numbers between two given endpoints.

Intervals describe ranges of values on the real number line and are widely used in mathematics.

Types of Intervals

- **Closed Interval ($[a, b]$):** Includes both endpoints a and b.

  $$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

  Example: $[2, 5] = \{x \mid 2 \leq x \leq 5\}$.

- **Open Interval ((a, b)):** Excludes both endpoints a and b.

  $$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

  Example: $(2, 5) = \{x \mid 2 < x < 5\}$.

Half-Open or Half-Closed Intervals

- **Left-Closed, Right-Open ($[a, b)$)**:

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

  Example: $[2, 5) = \{x \mid 2 \leq x < 5\}$.

- **Left-Open, Right-Closed ($(a, b]$)**:

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

  Example: $(2, 5] = \{x \mid 2 < x \leq 5\}$.

Infinite Intervals

- (a, ∞): All numbers greater than a.

$$(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$$

  Example: $(3, \infty)$ includes all numbers greater than 3.

- $(-\infty, b)$: All numbers less than b.

$$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$$

  Example: $(-\infty, 4)$ includes all numbers less than 4.

- $(-\infty, \infty)$: The entire real number line.

$$(-\infty, \infty) = \mathbb{R}$$

Summary of Interval Types

Type	Interval Notation	Description
Closed	$[a, b]$	Includes both endpoints a, b
Open	(a, b)	Excludes both endpoints a, b
Half-Open Left	$[a, b)$	Includes a, excludes b
Half-Open Right	$(a, b]$	Excludes a, includes b
Infinite Left	$(-\infty, b)$	All $x < b$
Infinite Right	(a, ∞)	All $x > a$
Entire Line	$(-\infty, \infty)$	All real numbers

What is Absolute Value?

Definition

The **absolute value** of a number is its distance from zero on the number line, regardless of direction. It is always non-negative.

For a real number x, the absolute value, denoted as $|x|$, is defined as:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Breaking the absolute value:

- $|f(x)| \leq c \quad \implies \quad -c \leq f(x) \leq c$
- $|f(x)| \geq c \quad \implies \quad f(x) \leq -c \quad \text{or} \quad f(x) \geq c$

Examples of Absolute Value

- $|3| = 3 \quad$ (because $3 \geq 0$)
- $|-5| = -(-5) = 5 \quad$ (because $-5 < 0$)
- $|0| = 0 \quad$ (because 0 is neither positive nor negative)

Properties of Absolute Value

- **Non-Negativity:** $|x| \geq 0$ for all x.
- **Identity Property:** $|x| = 0 \quad$ if and only if $x = 0$.
- **Multiplicative Property:** $|x \cdot y| = |x| \cdot |y|$.
- **Triangle Inequality:** $|x + y| \leq |x| + |y|$.
- **Distance Interpretation:** $|x - y|$ represents the distance between x and y.

Ball bearings need to have extremely accurate sizes to work correctly. The ideal diameter of a particular ball bearing is 0.8 cm, but a ball bearing is declared acceptable if the error in the diameter size is less than 0.001 cm. Write the inequality for acceptance criteria

Solution:

Exercises

Ball bearings need to have extremely accurate sizes to work correctly. The ideal diameter of a particular ball bearing is 0.8 cm, but a ball bearing is declared acceptable if the error in the diameter size is less than 0.001 cm. Write the inequality for acceptance criteria

Solution: The ball bearings are acceptable if diameter d is

$$|d - 0.8| \leq 0.001$$

Exercises

Find all numbers t such that $|3t - 4| = 10$

Solution :

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Solution :

$$3t - 4 = 10 \; or \; 3t - 4 = -10 \implies t = \frac{14}{3}, t = -2$$

Exercise

Find all numbers x such that

$$\left|\frac{3x-5}{x-1}\right| < 2$$

Solution :

Exercise

Find all numbers x such that

$$\left|\frac{3x-5}{x-1}\right| < 2$$

Solution :

$$|3x-5| < 2|x-1|$$

1. $x - 1 > 0$

Breaking the absolute value:

$$\implies -2(x-1) < 3x-5 < 2(x-1) = -2x+7 < 3x < 2x+3 \quad (1)$$
$$\implies 3x > -2x+7 \;\&\; 3x < 2x+3 \quad (2)$$

Exercise

Solving for $3x < 2x + 3$

$$\implies x < 3 \tag{3}$$

$$\tag{4}$$

Solving for $3x > -2x + 7$

$$\implies 3x > -2x + 7 \implies 5x > 7 \implies x > 7/5 \tag{5}$$

$$\implies x \in (7/5, 3) \tag{6}$$

Exercise

2. $x - 1 < 0 \implies x < 1$

$$\implies |3x-5| < 2|x-1| == |3x-5| < -2(x-1) \quad (7)$$
$$\implies 3x-5 < -2(x-1) \;\; and \;\; -(3x-5) < -2(x-1) \quad (8)$$
$$\implies 3x-5 < -2(x-1) \; and \; 3x-5 > 2(x-1) \quad (9)$$

$$3x-5 < -2(x-1) \implies 3x < -2x+7 \implies 5x < 7 \implies x < 7/5 \quad (10)$$
$$\implies 3x-5 > 2(x-1) \implies 3x > 2x+3 \implies x > 3 \quad (11)$$

Here $x > 3$ is inconsistent with our assumption $x < 1$. So for $x < 1$ there are no values of x satisfying the inequality

What is a Function?

A function associates every number in some set of real numbers, called the domain of the function, with exactly one real number

If f is a function and x is a number in the domain of f , then the number that f associates with x is denoted by $f(x)$ and is called the value of f at x

Equality of Functions

Two functions are equal if and only if they have the same domain and the same value at every number in that domain

Example

Suppose f is the function whose domain is the set of real numbers, with f defined on this domain by

$$f(x) = x^2$$

Suppose g is the function whose domain is the set of positive numbers, with g defined on this domain by

$$g(x) = x^2$$

Are f and g equal functions ?

Equality of functions

Example 2

Suppose f and g are functions whose domain is the set consisting of the two numbers $\{1, 2\}$ with f and g defined on this domain by the formulas

$$f(x) = x^2$$

and

$$g(x) = 3x - 2$$

.Are f and g equal functions?

Domain

If a function is defined by a formula, with no domain specified, then the domain is assumed to be the set of all real numbers for which the formula makes sense and produces a real number

Domain

Example 3

Find the domain of the function f defined by

$$f(x) = (3x - 1)^2$$

Example 4

Find the domain of the function f defined by

$$h(t) = \frac{t^2 + 3t + 7}{t - 4}$$

Example 6

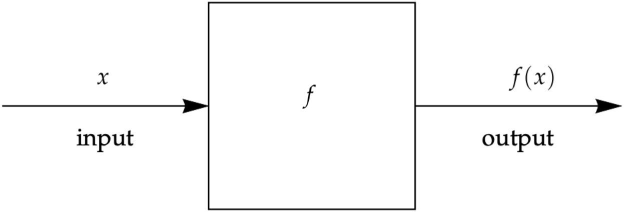
Find the domain of the function g defined by

$$g(x) = \sqrt{|x| - 5}$$

Functions

Range

The range of a function f is the set of all numbers y such that $f(x) = y$ for at least one x in the domain of f



The set of inputs acceptable by this machine is the domain of f.
The set of outputs is the range of f.

Example 4

The domain of f is the interval $[2, 5]$, with f defined on this interval by the equation $f(x) = 3x + 1$

Solution :

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Solution :

$$y = f(x) = 3x + 1$$

Functions

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Solution :

$$y = f(x) = 3x + 1$$

$$2 \leq \frac{y-1}{3} \leq 5.$$

Example 4

The domain of f is the interval $[2, 5]$, with f defined on this interval by the equation $f(x) = 3x + 1$

Solution :

$$y = f(x) = 3x + 1$$

$$2 \leq \frac{y-1}{3} \leq 5.$$

$$7 \leq y \leq 16.$$

Example 4

The domain of f is the interval $[2,5]$, with f defined on this interval by the equation $f(x) = 3x + 1$

Solution :

$$y = f(x) = 3x + 1$$

$$2 \leq \frac{y-1}{3} \leq 5.$$

$$7 \leq y \leq 16.$$

Example 5

The domain of g is the interval $[1, 20]$, with g defined on this interval by the equation

$$g(x) = |x - 5|.$$

Is 2 in the range of g?

Solution :

Example 5

The domain of g is the interval $[1, 20]$, with g defined on this interval by the equation

$$g(x) = |x - 5|.$$

Is 2 in the range of g?

Solution :

$$y = |x - 5|$$

for $x - 5 > 0$,$y = x - 5 \implies x = y + 5$

$$5 < y + 5 \leq 20 \implies 0 < y \leq 15$$

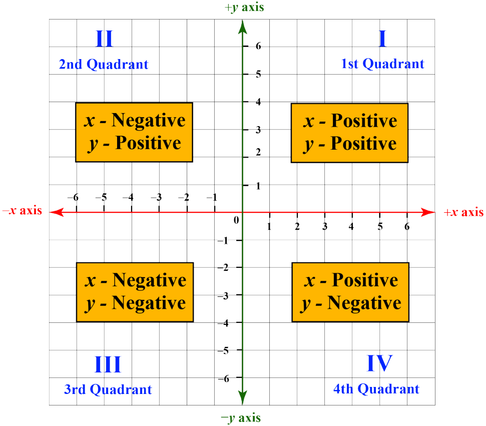
for $x - 5 < 0$, $y = -(x - 5) \implies y = -x + 5 \implies 5 - y = x$

$$1 \leq 5 - y \leq 5 \implies -4 \leq -y \leq 0 \implies 4 \geq y \geq 0$$

What is Analytic Geometry?

- **Analytic Geometry** (also called *coordinate geometry* or *Cartesian geometry*) bridges algebra and geometry.
- It uses a coordinate system to study geometric shapes and properties.
- Geometric objects are represented as algebraic equations.

Co-ordinate Plane

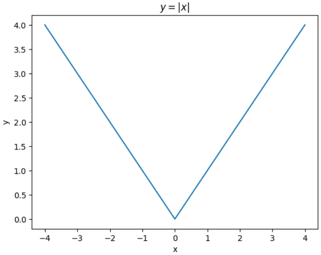


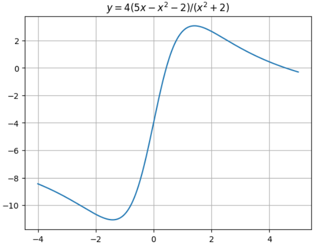
The plane with this system of labeling is often called the **Cartesian plane** in honor of the French mathematician Rene Descartes(1596-1650), who described this technique in his 1637 book Discourse on Method

Graph Functions

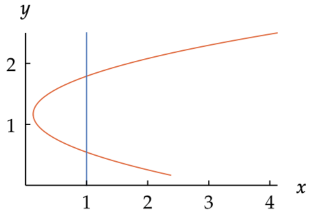
The graph of a function f is the set of points of the form $x, f(x)$ as x varies over the domain of f

Graph of a Function





Checking for a function: Vertical line test



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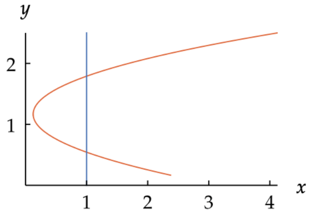


The line $x = 1$ intersects the curve at two points. That is that for each x value there are multiple y values which is contradicting to definition of a function

Vertical Line Test

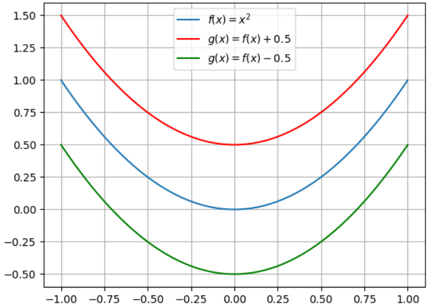
A set of points in the coordinate plane is the graph of some function if and only if every vertical line intersects the set in at most one point

Vertical Transformation

Shifting a graph up or down

Suppose f is a function and $a > 0$. Upshift g and Downshift h by

$$g(x) = f(x) + a \quad h(x) = f(x) - a$$

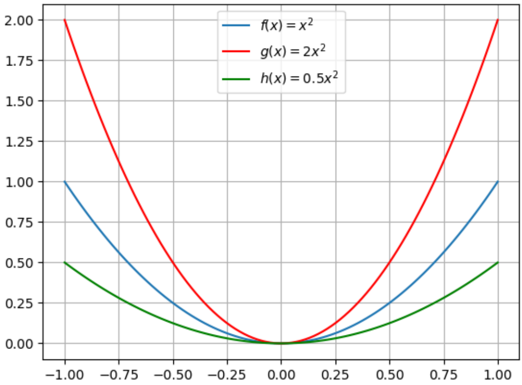


Vertical Transformation

Vertical Stretch

Suppose f is a function and $c > 0$. Define a function g by

$$g(x) = cf(x)$$

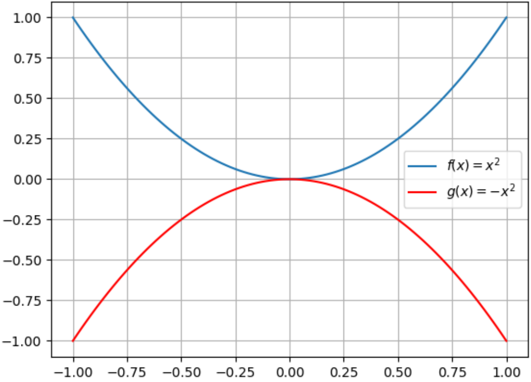


Vertical Transformation

Vertical Flipping

Vertical fliiping of $f(x)$ is

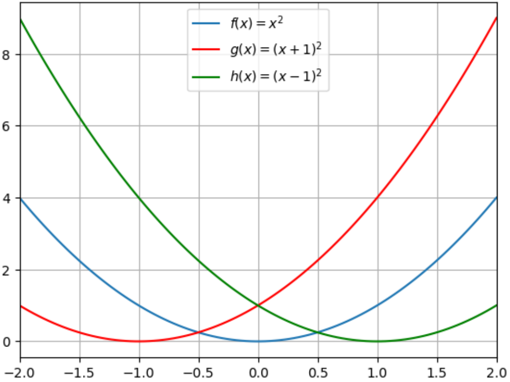
$$g(x) = -f(x)$$



Horizontal Transformation

Horizontal Shift

$$g(x) = f(x + a), h(x) = f(x - a)$$



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