Energy Conditions in Eternal Inflation

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We show that the eternal inflation scenario requires violations of the Null Energy Condition (NEC) in de Sitter space. Within the context of semiclassical gravity, NEC violations are absent if the quantum state of fields is assumed to be de Sitter invariant since the expectation value of the "quantum NEC operator" vanishes. We then discuss the necessity of defining a smeared NEC operator. We find that smeared energy-momentum fluctuations in de Sitter space violate the NEC with probability 1/2 and the magnitude of the violations depends on the spatial and temporal smearing scales, with the largest violations occurring on the smallest scales. In the case when both smearing scales are equal to the de Sitter horizon H^{-1}, the magnitude of the NEC violation is given by $\sim H^4/(2\pi)^2$. We also discuss NEC violations during slow roll inflation. Our results are consistent with what is obtained in the literature based on fluctuations of the scalar field except in the diffusive regime where quantum jumps are thought to be important. We discuss the need for a scheme to calculate the backreaction of such NEC violating quantum fluctuations on the metric.

I. INTRODUCTION

The idea underlying the inflationary scenario [1] is that there exists a scalar field (inflaton) that is displaced in some region from its value at the minimum of its potential, and that this field is evolving ("rolling") towards the minimum of the potential (for example as shown in Fig. 1). Under certain circumstances, the energy contribution of the displaced field is like that of a cosmological constant and this makes the spacetime inflate.

The idea behind the *eternal* inflationary scenario [2] is that the dynamics of the scalar field is actually governed by quantum field theory and therefore quantum effects make the scalar field fluctuate, occasionally causing the field to jump upward on the potential. These random "upward jumps" create regions with a larger cosmological constant, thus driving pockets of faster inflation within the ambient inflationary universe. For most inflaton potentials, there generically exists a regime where the "jumps" significantly affect or even dominate the scalar field classical evolution. In that regime, the universe will contain a statistical distribution of inflating regions, some of which will thermalize. Such a universe is termed a "multiverse".

In this paper we will examine the eternal inflation scenario more closely. We first examine the spacetime structure implied by the eternal inflation scenario along the lines of Ref. [3]. Our analysis, described in Sec. II and illustrated in more detail in Appendix A, shows that the Average Null Energy Condition (ANEC) and hence, also the Null Energy Condition (NEC), must be violated in the eternal inflation scenario. This conclusion was also reached by Borde and Vilenkin by examining the Friedmann-Robertson-Walker (FRW) cosmology equations [4].

At a classical level, it is trivial to show that violations of the NEC are not present in the dynamics of the inflaton. However, the eternal inflationary process is quantum and it is known that the NEC may be violated in quantum field theory [5]. Hence we search for NEC violations in the quantum field theory of the inflaton using the standard approach of linearized perturbations. (This is the same approximation used in the existing literature to describe eternal inflation scenarios.) Our search is limited to the case of de Sitter space with quantum fields in de Sitter invariant states. Within this restrictive set of conditions we show that there are no violations of the NEC that lead to eternal inflation within the *semiclassical* theory of gravity (Sec. III).

The semiclassical gravity case of Sec. III shows that there are no violations of the NEC in the *expectation value* of the energy-momentum tensor. The question arises if there still might be higher order fluctuations of the energy-momentum tensor that might provide NEC violations. In Sec. IV, we attempt to determine if such fluctuations are present in de Sitter space with scalar fields in the de Sitter invariant vacuum. The attempt to calculate fluctuations of the energy-momentum tensor at a spacetime point leads to an infinite answer. However, one may obtain finite results by using a smearing procedure. In Sec. V we find that violations of the NEC due to fluctuations of the smeared energy-momentum tensor are frequent (*i.e.* with probability one-half) in de Sitter space. The magnitude of NEC violation is calculated in Appendix B for a suitable class of smearing functions. If the spacetime smearing scales are set by the de Sitter expansion parameter H, the result is consistent with what is obtained in the literature based on
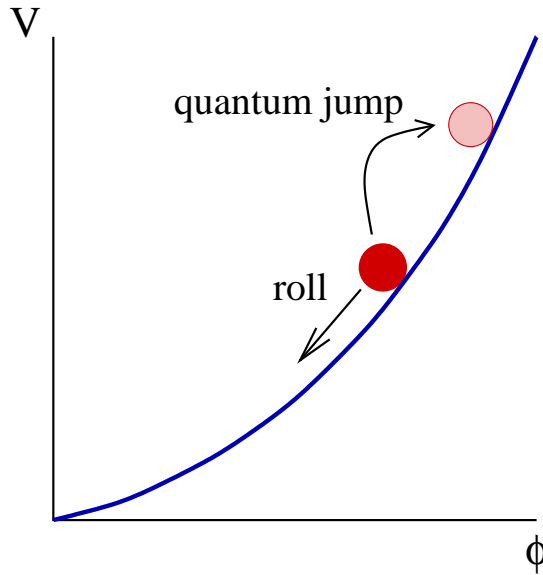
FIG. 1: A scalar field rolls down the potential slowly and causes inflation. Eternal inflation occurs if occasional quantum jumps in localized regions raise the scalar field to a higher value of the energy density, leading to regions of faster inflation.

fluctuations of the scalar field in the "deterministic" slow-roll parameter regime, where quantum jumps are negligible, but not in the diffusive regime where jumps are thought to be important. We discuss the present analysis in light of earlier work on the subject [2] in Sec. VI. Our results are summarized in the concluding section.

II.   NECESSITY OF VIOLATING THE NEC

In this section we will assume that a patch of spacetime satisfies FRW equations in regions of a few horizon size. We may write these equations in terms of scale factor a, energy density ρ, and pressure p as

$$\dot{\rho} = -3H(\rho + p), \tag{1}$$

$$\dot{H} = -4\pi G(\rho + p) + \frac{k}{a^2}. \tag{2}$$

We are mostly interested in the $k = 0$ case since this is relevant for eternal inflation. Then it is clear from Eq. (2) that if H is to increase and the curvature is zero or negative ($k \leq 0$), the term $\rho + p$ must be negative. If $k > 0$, H can increase as long as the dynamics is dominated by the spatial curvature term.

For a standard energy-momentum tensor $T_{\mu\nu}$ of a homogeneous fluid, $\rho + p < 0$ translates into

$$T_{\mu\nu}N^\mu N^\nu < 0 \tag{3}$$

for some null vector N^μ. In other words, the NEC must be violated in the patch of spacetime that satisfies Eqs. (1)-(2) and undergoes an upward jump in its local expansion rate H. (In Appendix A we illustrate the causal structure of the resulting spacetime and arrive to the same conclusion using the Raychaudhuri equation.)

In a semiclassical treatment, Einstein's equations are replaced by

$$G_{\mu\nu} = 8\pi G_N \langle T_{\mu\nu}^{ren} \rangle \tag{4}$$

where the right-hand side is the expectation value of the renormalized energy-momentum tensor and G_N is the physical Newton's gravitational constant. In this case the requirement for NEC violation becomes

$$\langle T_{\mu\nu}^{ren} \rangle N^\mu N^\nu < 0 \tag{5}$$

must hold at some spacetime point. Note that the violation of the NEC is only a necessary condition for eternal inflation and is not a sufficient condition.

III. POSSIBLE SOURCES OF NEC VIOLATIONS

Classically the NEC cannot be violated since, for a conventional scalar field ϕ, with arbitrary potential:

$$T_{\mu\nu}N^\mu N^\nu = (N^\mu \partial_\mu \phi)^2 \geq 0. \tag{6}$$

Quantum field theory, however, allows for the possibility of negative energy densities and violations of the NEC. To see if this occurs, define

$$\hat{O} = \hat{T}_{\mu\nu}N^\mu N^\nu = (N^\mu \partial_\mu \hat{\phi})^2 \tag{7}$$

where ˆ denotes a quantum operator. The null vector depends on the metric, which can also fluctuate in principle, but here we are restricting our attention to scalar field fluctuations in a fixed spacetime background and hence N^μ is a fixed null vector.

If we write $\hat{P} = N^\mu \partial_\mu \hat{\phi}$, then $\hat{O} = \hat{P}^\dagger \hat{P}$ and so it is clear that the operator $\hat{O}$ is non-negative. However, the unrenormalized expectation value of $\hat{O}$ will be divergent and will contain terms that are purely geometrical. These geometrical contributions can be absorbed by suitably rescaling and shifting the bare parameters of the theory (Newton's gravitational constant, the cosmological constant and coefficients of other geometrical terms in the action [8]), leaving behind the finite expectation value of the renormalized energy-momentum tensor operator, $T_{\mu\nu}^{ren}$. Then the question is whether the expectation value of $\hat{O}^{ren}$, denoted by $\langle \hat{O}^{ren} \rangle \equiv \langle s | \hat{O}^{ren} | s \rangle$ where $|s\rangle$ is the quantum state of the fields, can be negative.

To proceed further, we need to specify the quantum state $|s\rangle$ of the field. This in turn will depend on the evolution of the universe as well as on the initial conditions. The eternal inflation literature [2] commonly assumes a de Sitter universe with quantum field fluctuations calculated in the Bunch-Davies vacuum [8] and so it is of interest to consider this to be the state of the universe at early times:

$$|s\rangle = |0\rangle_{BD} . \tag{8}$$

This choice for the initial quantum state of the fields is also suggested by calculations in quantum cosmology [9–11].

Then, based on the tensorial properties of the expectation values,

$$\langle \hat{T}_{\mu\nu}^{ren} \rangle \propto g_{\mu\nu} . \tag{9}$$

Therefore,

$$\langle \hat{O}^{ren} \rangle \propto g_{\mu\nu} N^\mu N^\nu = 0 \tag{10}$$

since N^μ is a null vector in the unperturbed metric. So we conclude that NEC cannot be violated to linear order by scalar fields in de Sitter space when these fields are in a de Sitter invariant quantum ground state.

Observe that Eq. (7) says that $\langle \hat{O} \rangle$ measures fluctuations of the operator $\hat{P} \equiv N^\mu \partial_\mu \hat{\phi}$ since $\hat{O} = \hat{P}^\dagger \hat{P}$. The result in Eq. (10) does not say that these fluctuations vanish. Instead the result says that the (divergent) expectation value goes into renormalizing the various parameters that enter in the gravitational action. Once these parameters have been set to their measured values, Eq. (10) says that the residual value of $\langle \hat{O} \rangle$ that influences the metric and is to be placed on the right-hand side of Raychaudhuri's equation, is zero. (See the discussion at the end of Sec. 6.1 of Ref. [8] for an interpretation of the energy-momentum tensor renormalization procedure.)

IV. ENERGY-MOMENTUM TENSOR FLUCTUATIONS

In the semiclassical Einstein equations, the metric is completely determined by the *expectation* value of the energy-momentum tensor, which we have seen does not contain NEC violations. To see NEC violations, we must go beyond semiclassical gravity as given by Eq. (4). The next step is to consider if energy-momentum fluctuations (not just the expectation value) can violate the NEC.

A picture that is often used in the cosmological literature is to think of the energy-momentum tensor as being drawn from an ensemble of (tensorial) functions. Each realization of the energy-momentum tensor determines a metric via the Einstein equations. In this way, quantum fluctuations of the energy-momentum tensor lead to statistical fluctuations of the metric. In practice, however, a fixed spacetime background is generally assumed. Quantum fluctuations of the *field* are calculated in this fixed background and then the classical energy-momentum tensor is evaluated at the fluctuating values of the field. The perturbed values of the energy-momentum tensor are then fed into the Einstein equations which lead to perturbed values of the metric.

To understand the technical issues somewhat better, consider what is involved in calculating the expectation value of the energy-momentum tensor [8]. One starts with the gravitational action that contains all possible curvature invariants. Variation with respect to the metric then gives the classical equations of motion (see Eq. (6.53) of [8]),

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + S_{\mu\nu} = 8\pi GT_{\mu\nu} \tag{11}$$

where,

$$S_{\mu\nu} \equiv \Lambda g_{\mu\nu} + \alpha H_{\mu\nu}^{(1)} + \beta H_{\mu\nu}^{(2)} + \gamma H_{\mu\nu}^{(3)} + ... , \tag{12}$$

the $H_{\mu\nu}^{(i)}$ are certain tensors and the ellipsis represent other possible terms. (G is the bare value of Newton's gravitational constant, and α, β... are other bare parameters.) In a quantum theory, we treat the energy-momentum tensor as an operator and define a renormalized energy-momentum tensor via:

$$8\pi G_N \hat{T}_{\mu\nu}^{ren} = 8\pi G\hat{T}_{\mu\nu} - S_{\mu\nu} . \tag{13}$$

In the semiclassical theory $S_{\mu\nu}$ is a c-number and can be pulled out of any expectation value. We choose the various (bare) parameters so as to get the semiclassical Einstein equations with physical values of the parameters:

$$G_{\mu\nu} = 8\pi G_N \langle \hat{T}_{\mu\nu}^{ren} \rangle . \tag{14}$$

Now let us define the NEC operator

$$\hat{O} \equiv N^\mu N^\nu \hat{T}_{\mu\nu}^{ren} . \tag{15}$$

We wish to calculate the fluctuations of $\hat{O}$ and will restrict ourselves to a calculation in a fixed background. So for us $g_{\mu\nu}$ and $S_{\mu\nu}$ will be fixed c-number quantities determined by solving Eq. (14) and then inserting the solution into Eq. (12).

Note that the equation of motion for the metric, for any realization of $T_{\mu\nu}$ other than its expectation value, is now given by Eq. (11) and not by the Einstein equations. The usual Einstein equations are only valid after taking expectation values and do not apply to each individual realization of the energy-momentum tensor. So while we can certainly calculate $\langle \hat{O}^2 \rangle$, to determine its effect on the metric is a separate and more complicated issue. In the present paper we will restrict ourselves to an evaluation of $\langle \hat{O}^2 \rangle$ and not consider the effect of the fluctuations on the metric.

With the above limitations in mind, we will calculate the fluctuations of $\hat{O}$ on a fixed background by using

$$\langle (\hat{O} - \langle \hat{O} \rangle)^2 \rangle = \langle (: \hat{O} :)^2 \rangle \tag{16}$$

where $: \hat{O} :$ denotes normal ordering of the operator $\hat{O}$ with respect to the vacuum.

One final remark that we would like to make here is that a straightforward evaluation of $\langle (: \hat{O} :)^2 \rangle$ will give an infinite result because of the divergent term $\langle \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}'}^\dagger \rangle$ in the expansion in terms of creation ($\hat{a}_{\mathbf{k}}^\dagger$) and annihilation ($\hat{a}_{\mathbf{k}}$) operators. It may seem that this divergence should be removed by renormalization and this leads to:

$$\langle \hat{T}_{\mu\nu} \hat{T}_{\lambda\sigma} \rangle_{ren} = Cg_{\mu\nu}g_{\lambda\sigma} + \text{permutations} \tag{17}$$

where C is some constant, since the only quantity from which we can construct the right-hand side is the metric. Contracting this result with four null vectors gives us $\langle (: \hat{O} :)^2 \rangle_{ren} = 0$. However, it would be incorrect to interpret the renormalized expectation value on the left-hand side as a measure of the fluctuations of the NEC operator since the renormalization procedure involves a subtraction. The correct measure of the fluctuations is the unrenormalized value and it really is infinite since the two operators are taken at the same spacetime point. The procedure for getting finite results is to smear the operator $: \hat{O} :$ and then find fluctuations of the smeared operator. The physical basis behind using the smeared operator in this context is the belief that the metric responds to a smeared out energy-momentum tensor. On short smearing scales the metric behaves quantum mechanically while on larger scales it behaves more classically. In this context, it is worthwhile to recall that NEC violations are required on scales larger than the horizon scale for eternal inflation to occur (Sec. II and Appendix A).

V. VIOLATIONS OF THE NEC BY SMEARED ENERGY-MOMENTUM FLUCTUATIONS

From the formal point of view, the renormalized energy-momentum tensor $\hat{T}_{\mu\nu}^{ren}$ and the NEC $\hat{O}^{ren}$ are not operators but operator-valued distributions that become well-defined operators after averaging with a window (or smearing)

function $W(x)$, for instance

$$\hat{O}_W^{ren} \equiv \int d^4 x \sqrt{-g}\, W(x) \hat{O}^{ren} \tag{18}$$

where the smearing function has support only over a spatial extent R and temporal duration T. (Averaging over time as well as over space is necessary to avoid divergences in $\langle (\hat{O}_W^{ren})^2 \rangle$, as we show in Appendix B.) The operator of Eq. (18) may be interpreted as representing the NEC averaged over R and T. We note that a similar procedure of averaging is also necessary for the semiclassical formalism of stochastic inflation [2]. In that formalism, the phenomenological picture of "jumps" is derived for the scalar field $\tilde{\phi}$ smeared over super-horizon scales $R \gg H^{-1}$. The Fokker-Planck equation describing the probability distribution of $\tilde{\phi}$ is valid only for time steps $\Delta t \gg H^{-1}$. Smearing of the quantum field operator $\hat{\phi}$ in that formalism is a substitute description of the quantum-to-classical transition of the vacuum fluctuations of $\hat{\phi}$ that takes place on timescales of several Hubble times and on superhorizon distances [12].

The smearing procedure in Eq. (18) uses a particular window function $W(x)$ and thus breaks the de Sitter invariance of averaged operators. Therefore, the considerations of the previous section do not apply and we may expect to find finite NEC-violating fluctuations of the averaged operator $\hat{O}_W^{ren}$. We shall now estimate these fluctuations during inflation and investigate the conditions for violations of the smeared NEC. The expectation value cannot violate the NEC, as we have seen above. However, if the dispersion σ of the averaged NEC defined by

$$\sigma^2[\hat{O}_W] \equiv \langle (\hat{O}_W)^2 \rangle - \langle \hat{O}_W \rangle^2 \tag{19}$$

is greater than its expectation value

$$\mu \equiv \langle \hat{O}_W \rangle, \tag{20}$$

then the NEC is likely to be frequently violated by its fluctuations.

First we need to specify the quantum state in which to compute expectation values (see Appendix B for details). The usual framework for computing energy density fluctuations from inflation assumes that the spacetime is approximately de Sitter and that the scalar field ϕ is in a state with a spatially homogeneous expectation value $\phi_0(\eta)$ and small fluctuations $\delta\phi(x,\eta) \ll \phi_0(\eta)$ superimposed on it,

$$\phi(x,\eta) = \phi_0(\eta) + \delta\phi(x,\eta) \tag{21}$$

where η denotes the conformal time. The homogeneous component $\phi_0(\eta)$ is a solution of the slow-roll equation of motion

$$\dot{\phi} \equiv -(H\eta)\frac{d\phi}{d\eta} = -\frac{1}{3H}\frac{dV(\phi)}{d\phi}. \tag{22}$$

Here the overdot denotes the derivative with respect to proper time. In Appendix B we show that there exists a suitable coherent state $|\phi_0(\eta)\rangle$ of the quantum field ϕ that corresponds to the classically rolling inflaton field, $\phi_0(\eta)$. Fluctuations of the NEC in that state can be computed using Eq. (21) where $\delta\phi$ is a quantum field in its vacuum state and $\phi_0(\eta)$ is a c-number contribution.

The technical aspects of the calculation of the smeared fluctuations are described in the Appendix B. Here we will only consider the results when the spatial and temporal smearing scales are chosen to be equal as in eternal inflation, even though fluctuations on smaller temporal scales occur with larger amplitudes,

$$R = T = (\varepsilon H)^{-1} \tag{23}$$

where ε is a free parameter and R, T are physical (not comoving) scales. The smearing window is taken to be a product of a time-dependent and a space-dependent smearing profile and corresponds to a rectangle in the $\eta - r$ plane. We show in Appendix B that the relative magnitude of the NEC fluctuations is

$$\frac{\sigma^2}{\mu^2} \sim \frac{H^4 \max\left(c_1^2 \varepsilon^2, c_1'^2 \varepsilon^4\right)}{\left(2\pi\dot{\phi}_0\right)^2} + \frac{c_2^2 H^8 \varepsilon^8}{\left(2\pi\dot{\phi}_0\right)^4}. \tag{24}$$

Numerical evaluations show that the constants $c_{1,2}$ are of order 1 for typical window profiles. It follows that the fluctuations are significant when

$$\left|\dot{\phi}_0\right| \lesssim \frac{H^2}{2\pi}\varepsilon \max\left(c_1, \varepsilon c_1', \varepsilon\sqrt{c_2}\right) \tag{25}$$

and negligible otherwise.

Note that large values of ε correspond to smearing the energy-momentum tensor over a very small 4-volume, which naturally leads to large fluctuations. However, for emergence of eternal inflation we need NEC violations on scales of order H^{-1} and therefore we must choose $\varepsilon \lesssim 1$. With this choice, the condition of Eq. (25) can only be satisfied in the fluctuation-dominated regime of inflation, where the potential is so flat that the typical magnitude of jumps $\Delta\phi \sim H/(2\pi)$ during one e-folding time $\Delta t = H^{-1}$ is much larger than the change in the field $\dot{\phi}_0 \Delta t$ due to the classical roll. In this regime $\left| \dot{\phi}_0 \right| \ll H^2$ and thus there is a range of ε for which the condition of Eq. (25) holds. The fluctuations of the NEC averaged over the corresponding range of scales dominate the expectation value and may provide frequent violations of the NEC.

It is interesting to compare the result of Eq. (24) with standard calculations of energy density fluctuations in inflation. The standard approach is to compute fluctuations of the field ϕ itself to first order, which is equivalent to taking only the first term in Eq. (24). However, the second term in Eq. (24) is larger than the first when Eq. (25) is satisfied, i.e. in the fluctuation-dominated regime. It is reassuring that we find possible violations of the NEC in the same regime where eternal inflation is expected to occur.

We also note that the calculation of the magnitude σ^2 of the NEC fluctuations holds also in pure de Sitter space where $\dot{\phi}_0 = 0$. In that case, the expectation value of the NEC operator is 0 while the fluctuations are non-zero and violate the NEC approximately with probability 1/2.

VI. DISCUSSION

Based on the results of Sec. IV, it appears that NEC violations within the semiclassical approximation (where the spacetime background is classical and fixed) can only arise due to departures of the quantum state of the field from de Sitter invariance, or departures of the spacetime from de Sitter. We first discuss possible sources of such departures.

One possibility is that the rolling state of the field might cause the spacetime to depart from exact de Sitter and that these departures would be sufficient to lead to NEC violations. This possibility seems difficult to analyze since we would need to consider quantum fluctuations in "near de Sitter" spacetimes to see if these can lead to violations of the NEC.

One can also imagine a situation in which the spacetime may be de Sitter, but the state of the quantum fields may be more complicated and may not share the de Sitter symmetry. This may be another avenue worth investigating for NEC violations, especially in view of the analysis treating the quantum state of the scalar field as a "squeezed" state [14, 16] since squeezed quantum states are known to have regions of negative energy density. On the other hand, it may be possible that the Bunch-Davies vacuum is an "attractor" state for quantum fields in de Sitter space [15] and even if the initial state of the fields is arbitrary, time evolution may restore de Sitter invariance [16]. Also, for the inflation to be eternal, the scalar fields must evolve into the squeezed state every time there is an upward quantum jump, so that the NEC can be violated in future jumps.

It is worthwhile to describe why we do not find any semiclassical NEC violations especially since it is known that quantum field theory can easily provide negative energy densities [5] (though the amount and duration of the negative energy is constrained [17]). There are two differences between the current problem and that which is often analyzed that make it possible to understand why we do not see any NEC violations in the expectation value of the renormalized energy-momentum tensor.

The first difference is that the quantum field theory calculations commonly evaluate violations of the weak energy condition (WEC) and not of the NEC. In the WEC, the energy-momentum tensor is contracted with time-like vectors, u^μ and so one can construct scalars that do not vanish, namely $u^\mu u_\mu$. In our problem, the only vector we have at our disposal is a null vector, and so there is no relevant non-vanishing scalar that we can construct from it. Another way of saying this is that the WEC is sensitive to negative values of the energy density while the NEC is sensitive only to the energy density plus the pressure. Hence it might be possible that energy density and pressure fluctuations are correlated so that the NEC is not violated.

The second important difference is that the work on violations of the WEC uses a quantum state of the free scalar fields that is a superposition of many Fock states. The negative energy contributions arise due to interference terms between the different Fock states. While these arguments are well presented in the literature [18], it is worthwhile to repeat them here. For a scalar field in flat spacetime, consider the state

$$|s\rangle = N(|0\rangle + \varepsilon|2_k\rangle) \tag{26}$$

where N is a normalization factor, ε is a parameter and only one momentum state (k) is excited. Then the negative energy density terms arise due to matrix elements such as

$$\langle 2_k | \hat{a}_k^\dagger \hat{a}_k^\dagger | 0 \rangle . \tag{27}$$

These are the interference terms between the $|0\rangle$ and the $|2_k\rangle$ state, and these terms would be absent if $\varepsilon = 0$. In our case, we are seeking violations of the NEC in the $|0\rangle$ state alone and there are no interference terms present.

In Sec. V we proposed that NEC violations can be obtained by going beyond the semiclassical approximation. Then it is necessary to smear the energy-momentum tensor of the scalar field. We have shown in Sec. V that there exists a range of scalar field values ϕ_0 where fluctuations of the smeared energy-momentum tensor on super-horizon scales are significant and lead to frequent NEC violations. This range of ϕ_0 coincides with the regime where the quantum "jumps" dominate the evolution of ϕ, which is required for eternal inflation. In the traditional description of inflation, the quantum field ϕ smeared on super-horizon scales is treated as a classical field. If super-horizon scale smearing is an accurate phenomenological description of the quantum-to-classical transition of the vacuum fluctuations in de Sitter space, these fluctuations would indeed seem to provide the necessary NEC violations.

Underlying our calculation is the assumption that fluctuations of the smeared $T_{\mu\nu}$ have an effect on the metric that we can heuristically write down as the usual FRW equations, as in the work on eternal inflation [2]. However, strictly speaking, if one is to consider effects of smeared fluctuations on the metric, one would also need to construct a smeared version of Einstein's equations—a formalism that has not yet been worked out (see, however, [13] for an approach based on quantum statistical mechanics). Disregarding this difficulty, we can consider a homogeneous field ϕ that is rolling down some potential, as is standard in inflationary theory. The Hubble constant H is given by the (classical) Friedman-Robertson-Walker (FRW) equation:

$$H^2 = \frac{8\pi G}{3} V(\phi) \tag{28}$$

and the curvature term is absent since inflation makes the universe flat. Now we know that the field ϕ and its energy density have quantum fluctuations about their classical value. Hence, the field ϕ will sometimes undergo a quantum jump to a higher level of the potential as shown in Fig. 1. As long as the potential energy continues to dominate, Eq. (28) implies that the expansion rate in the region of the upward jump will also increase. This causes faster inflation and potentially eternal inflation. An assumption is that Eq. (28) is sufficient to describe this process and, for example, the region in which the field jumps up does not become curvature dominated or enter a contracting phase. Clearly an unsatisfactory feature of the heuristic argument is that it only concerns itself with the spacetime before and after the fluctuation. The evolution of the spacetime during the fluctuation itself is not considered. In terms of Fig. 3, this corresponds to discussing the state of the universe at times η_a and η_b but without considering what happens at epochs between η_P and η_Q. The metric and the fields at η_P will play a role in determining their state at η_b and so it is questionable to simply assume that Eq. (28) holds at η_b with a larger value of the (potential) energy density. There are important questions that are not addressed in this picture: does the metric evolve continuously through the fluctuation? What role does the conservation of energy-momentum play in the process? Are regions that jump upwards, surrounded by regions that jump downwards? So the heuristic argument, while suggestive, is certainly not sufficient by itself to show that eternal inflation can occur.

Finally we discuss the case of Minkowski spacetime by setting $\Lambda = 0$ in Fig. 2. Could Minkowski spacetime be unstable towards eternal inflation due to upward jumps? There is a crucial difference between Minkowski spacetime and any other FRW spacetime, namely that the radius of the horizon in Minkowski spacetime is infinite whereas in other cosmologies it is finite. Therefore for Minkowski spacetime to be unstable to eternal inflation, NEC-violating fluctuation would have to survive for an infinitely long time. The probability for this to happen vanishes and hence Minkowski spacetime is stable towards eternal inflation.

VII. CONCLUSIONS

In this paper we have examined the spacetime structure of eternal inflation and found that the NEC needs to be violated. This conclusion agrees with the earlier analysis by Borde and Vilenkin [4]. Next we showed that violations of the NEC are not present in de Sitter spacetime when the quantum fields are in the (de Sitter invariant) Bunch-Davies vacuum if we work in the semiclassical theory. This leads us to conclude that eternal inflation is only possible in the semiclassical theory when the initial quantum state is prepared so as to contain departures from de Sitter invariance. Further, for inflation to be eternal, every upward quantum jump must provide for these departures.

We then demonstrated that NEC violations are possible due to quantum fluctuations of the energy-momentum tensor. Since these fluctuations involve a product of energy-momentum tensors, it is not possible to work out the backreaction of such fluctuations on the spacetime in semiclassical theory. Indeed, NEC violations are only relevant within the semiclassical theory where there is a definite relation between the NEC operator and the properties of the spacetime via the Raychaudhuri equation. In theories of quantum gravity (*e.g.* string theory) it might be possible to carry a similar analysis further, though the presence of other fields (*e.g.* the dilaton) means that the setting for the
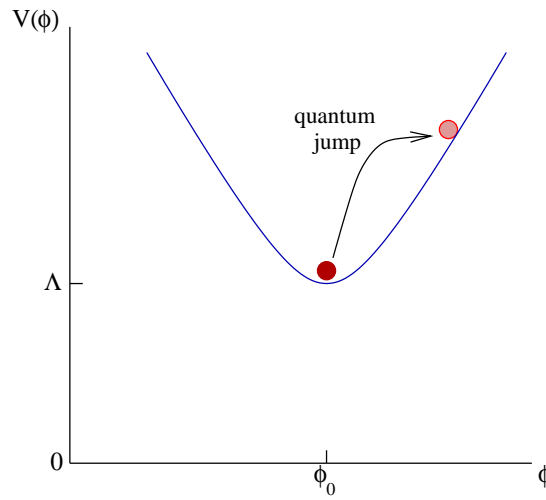
FIG. 2: The analysis in this paper can also be applied to a classical scalar field sitting at the bottom of the potential which has non-zero energy density Λ causing de Sitter expansion. The quantum perturbations of the field about the minimum are in the Bunch-Davies vacuum. Smeared quantum fluctuations of the energy-momentum tensor violate the NEC with probability half. In the heuristic picture, these violations would correspond to localized quantum jumps of the scalar field to higher values of the potential, causing faster inflation in those regions and an instability of de Sitter expansion towards eternal inflation.

analysis would be significantly different, and there may well be violations of the NEC even at the classical level that could lead to eternal inflation.

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FIG. 3: A spacetime diagram for an episode of eternal inflation. The diagram has been drawn for exponentially expanding space in which the solid line represents the minimal antitrapped two-sphere (angular coordinates have been suppressed). The dashed portion of the line and question marks between the points P and Q represents the unknown position of the MAS during a fluctuation. The two-spheres denoted by points in the shaded regions are antitrapped, while those in the unshaded regions are normal. The arrow denotes a future directed, radially ingoing, affinely parametrized null geodesic from point a in the old (slow) inflating region to point b in the new (fast) inflating region.

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APPENDIX A: CAUSAL STRUCTURE OF INFLATING SPACETIME

In this Appendix we shall consider the causal structure of a spacetime undergoing an upward fluctuation of the expansion rate H, as defined in Sec. II, and demonstrate the necessity of NEC violations using the Raychadhuri equation.

Consider an inflating spacetime. To simplify the analysis we shall assume spherical symmetry. Two-dimensional spheres centered at the origin of the coordinates in such a spacetime will either be "normal"– that is, ingoing null geodesics will be converging and outgoing rays will be diverging – or else they will be "antitrapped" – both ingoing and outgoing null geodesics will be diverging. Let the radius of the minimal antitrapped sphere be H^{-1}. In the eternal inflation scenario, regions of spacetime are produced that expand at a faster rate (larger H) and hence have a smaller Minimal Antitrapped Sphere (MAS). Denote the radius of this smaller minimal antitrapped sphere by $(H')^{-1}$.

A diagram for an episode ("upward jump") of eternal inflation is shown in Fig. 3. At early (conformal) times η_a, the spacetime is de Sitter and the MAS slopes along the inward-directed light-cone. Then there is a quantum fluctuation from η_P until η_Q, after which the spacetime is again de Sitter but the MAS has a smaller physical radius.

During the transition between points P and Q, the position of the MAS is determined by the dynamics of the fluctuation of H. However, one can show that in a flat or open FRW spacetime with growing H the MAS must be space-like. We sketch the derivation here. Consider a general FRW metric

$$ds^2 = dt^2 - \frac{a(t)^2}{1 - kr^2}dr^2 - a(t)^2 r^2 d\Omega^2 \tag{A1}$$

(the values $k = -1$, 0, 1 correspond to the open, flat and closed geometry as usual). In this metric, the trajectories of radial null rays satisfy $a^2(dr/dt)^2 = 1 - kr^2$. Assume that $H(t) \equiv \dot{a}/a$ is an arbitrary function of time. A congruence of inward-directed radial null rays $\mathbf{r}(t)$ with $dr/dt < 0$ diverges if the area $A = \pi r^2 a^2$ of the corresponding spherical

shell increases with time. We find that the radius r_{mas} of the MAS satisfies

$$aH = \sqrt{r_{\text{MAS}}^{-2} - k}. \tag{A2}$$

The MAS is space-like if $(a\dot{r}_{\text{MAS}})^2 > 1 - k r_{\text{MAS}}^2$. Differentiating Eq. (A2) with respect to time, we find that the MAS is space-like if $\dot{H} > k/a^2$. Therefore, for $k \leq 0$ the MAS is space-like whenever H grows. In closed geometry ($k = 1$), the MAS is space-like for large enough fluctuations of H. Note that the condition for the MAS to be space-like is exactly the same as the condition for the NEC violations obtained from Eq. 2. Since a is growing quickly during inflation, the flat ($k = 0$) FRW geometry is a good approximation; we shall limit ourselves to this case below.

Now consider ingoing null rays that are initially within the MAS of physical radius H^{-1}. (Definitions of trapped surfaces and other background material may be found in [6, 7].) Some of these rays can propagate to points beyond the later minimal antitrapped sphere of radius $(H')^{-1}$. An example of such a ray is the one that goes from point a to point b in Fig. 3. At point a the divergence Θ of these null rays is negative, while at point b it is positive. However, the Raychaudhuri equation yields

$$\frac{d\Theta}{d\tau} \leq -R_{\mu\nu} N^\mu N^\nu \tag{A3}$$

where $R_{\mu\nu}$ is the Ricci tensor and N^μ is the null geodesic parametrized by the affine parameter τ. For Θ to have changed from negative to positive, we necessarily have

$$\int_a^b R_{\mu\nu} N^\mu N^\nu d\tau < 0 , \tag{A4}$$

which requires that

$$R_{\mu\nu} N^\mu N^\nu < 0 \tag{A5}$$

at least at some points in the spacetime.

If the classical Einstein equations hold, then the requirement in Eq. (A5) becomes:

$$T_{\mu\nu} N^\mu N^\nu < 0. \tag{A6}$$

This says that the NEC must be violated. The condition corresponding to Eq. (A4) becomes

$$\int_a^b T_{\mu\nu} N^\mu N^\nu d\tau < 0 \tag{A7}$$

which says that the Averaged Null Energy Condition (ANEC) must be violated.

Finally, note that the violations of NEC must occur on the spacelike boundary PQ in Fig. 3. This region must extend all the way to the horizon of the slowly inflating region, otherwise null rays originating at P would diverge without encountering any NEC violating region. Hence NEC violations must occur on scales $\gtrsim H^{-1}$.

APPENDIX B: FLUCTUATIONS OF THE SMEARED NEC OPERATOR

In this Appendix we calculate the fluctuations of the averaged NEC operator for a nearly massless ($m \ll H$) scalar field in de Sitter spacetime and for smoothing scales R, T within the physically relevant range $R, T \ll m^{-1}$ for which the mass can be neglected. We show that the dispersion σ^2 of the NEC fluctuations is independent of the window profiles up to factors of order 1.

We consider the de Sitter spacetime with the metric

$$ds^2 = (H\eta)^{-2} \left[-d\eta^2 + d\mathbf{r}^2 \right], \tag{B1}$$

where

$$\eta = -H^{-1} e^{-Ht} = -\left(H a\left(t \right) \right)^{-1} \tag{B2}$$

is the conformal time. The field is quantized using the mode expansion

$$\hat{\phi}\left(\mathbf{r}, t \right) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left(\hat{a}_{\mathbf{k}} \psi_k\left(t \right) e^{i\mathbf{k}\cdot\mathbf{r}} + H.c. \right) \tag{B3}$$

where $k \equiv |\mathbf{k}|$ and the mode functions ψ_k for the Bunch-Davies vacuum are expressed through the Hankel functions of the 2nd kind [8]

$$\psi_k(\eta) = \frac{\sqrt{\pi}}{2} H \eta^{3/2} H_\nu^{(2)}(k\eta), \quad \nu^2 = \frac{9}{4} - \frac{m^2}{H^2}. \tag{B4}$$

In the massless case ($m = 0$) these mode functions become

$$\psi_k(\eta) = iHe^{-ik\eta} \frac{1 + ik\eta}{k^{3/2}\sqrt{2}}. \tag{B5}$$

With this simple form the calculations are tractable and we shall restrict ourselves to the massless case in this Appendix.

We choose a null vector N^μ with spatial component parallel to a constant unit vector,

$$N^\mu = \bar{N}(H\eta)^2 [1, \hat{\mathbf{n}}]. \tag{B6}$$

The constant vector field configuration was chosen for simplicity and amenability to analytic calculation. We have performed numerical evaluations for radial inward- and outward-directed vector fields $\hat{\mathbf{n}}$ and obtained qualitatively similar results. The time-dependent normalization of the null vector N^μ is chosen so that N^μ is affinely parametrized, while the constant $\bar{N}$ will be fixed later. The normal-ordered operator describing the NEC is

$$\hat{O}_N(\eta, \mathbf{r}) \equiv N^\mu N^\nu : \hat{T}_{\mu\nu} := \frac{(H\eta)^4}{2} : \left(\partial_\eta \hat{\phi} - \hat{\mathbf{n}} \cdot \nabla \hat{\phi}\right)^2 : \tag{B7}$$

We may perform normal ordering here in lieu of renormalization, because $\langle \hat{O}_N \rangle = 0$ in the vacuum state. The window function $W(\eta, \mathbf{r})$ for averaging the NEC operator (see Eq. (18)) is chosen to be the product of window functions that describe smoothing in space and time on scales R and T respectively. A possible choice is

$$W(\eta, \mathbf{r}) = \frac{1}{\sqrt{-g}} \frac{a_0^4}{R^3 \tau} W_\eta \left(\frac{|\eta - \eta_0|}{a_0^{-1}\tau}\right) W_r \left(\frac{|\mathbf{r} - \mathbf{r}_0|}{a_0^{-1}R}\right). \tag{B8}$$

Here

$$a_0 \equiv a(\eta_0), \quad \tau \equiv H^{-1}\tanh(HT) \tag{B9}$$

and we have explicitly introduced the reference point $(\eta_0, \mathbf{r}_0)$ assuming that $\eta_0 \neq 0$, since $\eta = 0$ is a singular point of the conformal coordinate system. (In our coordinates $\sqrt{-g} = (H\eta)^{-4} = a^4$.) The normalization of the window and of the null vector N^μ must be chosen to make the final result after averaging independent of η_0; we shall see below that it requires $\bar{N} = a_0$. The parameter τ is defined so that averaging in conformal time η with the window $a_0\tau^{-1}W_\eta\left(a_0\tau^{-1}|\eta - \eta_0|\right)$ corresponds to a window with proper time duration $2T$. This can be seen from the relation in Eq. (B2). Note that $H\tau < 1$ always holds. Likewise, the spatial window in Eq. (B8) is such that in the neighborhood of $\eta = \eta_0$ the proper length corresponding to the spatial averaging is R.

We shall not need to introduce specific window function profiles $W_{r,\eta}(\zeta)$ but merely assume that they are sufficiently smooth, normalizable functions satisfying

$$W_{r,\eta}(\zeta) \ll 1 \quad (\zeta \gtrsim 1). \tag{B10}$$

Normalization means that

$$\int W(\eta, \mathbf{r}) \sqrt{-g} d^3\mathbf{r} d\eta = 1. \tag{B11}$$

Under these assumptions our results will not depend on the choice of window profiles.

We define the averaged NEC operator, following Eq. (18), by

$$\hat{O}_W(\eta_0, \mathbf{r}_0) = \int \hat{O}_N(\eta, \mathbf{r}) W(\eta, \mathbf{r}) \sqrt{-g} d^3\mathbf{r} d\eta. \tag{B12}$$

Our goal is to compute the mean μ and the variance σ^2 of $\hat{O}_W$, as defined by Eqs. (19)-(20), in a suitable quantum state $|\phi_0(\eta)\rangle$ that describes slow-roll inflation. The case of exact de Sitter space is included as a special case. So we

first define the slowly rolling classical field as a coherent quantum state, $|\phi_0(\eta)\rangle$, such that the expectation value of the operator $\hat{\phi}(\eta, \mathbf{r})$ in this state is equal to the classical function $\phi_0(\eta)$:

$$\langle\phi_0(\eta)|\hat{\phi}|\phi_0(\eta)\rangle = \phi_0(\eta) \tag{B13}$$

We may explicitly choose the state $|\phi_0(\eta)\rangle$, by taking a coherent state of the zero mode defined in the usual manner,

$$|\phi_0(\eta)\rangle = \exp(s(\eta)\hat{a}_0^\dagger - s^*(\eta)\hat{a}_0)|0\rangle \equiv \hat{D}_0(s)|0\rangle. \tag{B14}$$

where $s(\eta)$ is to be determined.

The unitary "displacement operator" $\hat{D}_0(s)$ acts only on the $k = 0$ mode and satisfies

$$\hat{D}_0^\dagger(s)\hat{a}_\mathbf{k}\hat{D}_0(s) = \hat{a}_\mathbf{k} + s\delta^{(3)}(\mathbf{k}). \tag{B15}$$

Using this relation and the defining requirement in Eq. (B13), we find that $s(\eta)$ must satisfy

$$s(\eta)\psi_0(\eta) + s^*(\eta)\psi_0^*(\eta) = (2\pi)^{3/2}\phi_0(\eta), \tag{B16}$$

where ψ_0 is the $\mathbf{k} = 0$ mode function in Eq. (B3) and $\phi_0(\eta)$ is the classically rolling inflaton solution.

The mode expansion of Eq. (B3) now gives

$$\hat{D}_0^\dagger\hat{\phi}\hat{D}_0 = \phi_0(\eta) + \hat{\phi} \tag{B17}$$

and, since the "displacement operator" $\hat{D}_0$ is unitary, we may replace $\hat{D}_0^\dagger F(\hat{\phi})\hat{D}_0 = F(\hat{D}_0^\dagger\hat{\phi}\hat{D}_0)$ and obtain

$$\langle\phi_0(\eta)|F(\hat{\phi})|\phi_0(\eta)\rangle = \langle 0|F(\phi_0(\eta) + \delta\hat{\phi})|0\rangle. \tag{B18}$$

Therefore any expectation values in the state $|\phi_0(\eta)\rangle$ can be equivalently found from Eq. (21) by treating $\phi_0(\eta)$ as a classical component and $\delta\hat{\phi}$ as a quantum field in its vacuum state $|0\rangle$. We shall use this classical and quantum decomposition in what follows.

In slow-roll inflation the classical expectation value $\phi_0(\eta)$ is a slowly changing function of time and therefore the time derivative of ϕ_0 is approximately constant,

$$\dot{\phi}_0 = -H\eta\frac{\partial\phi_0}{\partial\eta} \approx \text{const.} \tag{B19}$$

where we have used Eq. (B2).

To evaluate NEC fluctuations, as in Eq. (19), we shall need two-point functions of $\hat{O}_N$, $\langle\hat{O}_N(x)\hat{O}_N(x')\rangle$, in the quantum state $|\phi_0(\eta)\rangle$. Our strategy for the calculation is to convert rapidly oscillating integrals into integrals of slowly changing functions of magnitude that can be estimated regardless of the specific window function profiles.

In the averaging procedure of Eq. (B12), the coordinate $x \equiv (\eta, \mathbf{r})$ will vary around the fiducial center $(\eta_0, \mathbf{r}_0)$ specified by the window functions of Eqs. (B8). After some algebra, we obtain

$$\langle\hat{O}_N(x)\hat{O}_N(x')\rangle_{\phi_0} - \langle\hat{O}_N(x)\rangle_{\phi_0}\langle\hat{O}_N(x')\rangle_{\phi_0} = T_1 + T_2 \tag{B20}$$

where

$$T_1 \equiv 4\chi_0(\eta)\chi_0(\eta')\langle\hat{\chi}(x)\hat{\chi}(x')\rangle \tag{B21}$$

$$T_2 \equiv \langle : (\hat{\chi}(x))^2 :: (\hat{\chi}(x'))^2 :\rangle \tag{B22}$$

and we have denoted

$$\hat{\chi}(x) \equiv N^\mu\partial_\mu\hat{\phi}(x), \quad \chi_0(\eta) \equiv N^\mu\partial_\mu\phi_0(\eta), \tag{B23}$$

and all expectation values in Eqs. (B21) and (B22) should be computed in the vacuum state, $|0\rangle$. We use a mode expansion for the operator $\hat{\chi}(x)$,

$$\hat{\chi}(x) = \int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}(\hat{a}_\mathbf{k}\chi_\mathbf{k}(\eta)e^{i(\mathbf{k}\cdot\mathbf{r}-k\eta)} + H.c.), \tag{B24}$$

where

$$\chi_{\mathbf{k}}(\eta) \equiv \bar{N}a^{-2}e^{ik\eta}\left[\frac{\partial\psi_k(\eta)}{\partial\eta} - i\mathbf{k}\cdot\hat{\mathbf{n}}\psi_k(\eta)\right] \tag{B25}$$

where $\bar{N}$ is the normalization of the null vector N^μ.

We now restrict our attention to the massless case and use Eq. (B5),

$$\chi_{\mathbf{k}}(\eta) = i\frac{H\bar{N}a^{-2}}{\sqrt{2k}}\left[k\eta - i\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\left(1 + ik\eta\right)\right]. \tag{B26}$$

Next we shall estimate the smeared version of the two terms of the RHS of Eq. (B20) separately. For example, the smeared version of T_1 is

$$T_1^S = \int d^3x d\eta\sqrt{-g}\int d^3x' d\eta'\sqrt{-g'}W(\eta', x')W(\eta, x)T_1 \tag{B27}$$

Now T_1 contains

$$\langle\hat{\chi}(x)\hat{\chi}(x')\rangle = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\chi_{\mathbf{k}}(\eta)\chi_{\mathbf{k}}^*(\eta')e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-ik(\eta-\eta')}. \tag{B28}$$

and so the spatial dependence only enters in the exponential factor in the integrand. Therefore it is convenient to introduce the Fourier transformed window profile

$$w_r(k) \equiv \int d^3\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}}W_r(|\mathbf{r}|) \tag{B29}$$

in terms of which the spatial integrals can be done using

$$\int d^3\mathbf{r}\, e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)}W_r\left(\frac{|\mathbf{r}-\mathbf{r}_0|}{a_0^{-1}R}\right) = (a_0^{-1}R)^3 w_r(a_0^{-1}Rk) \tag{B30}$$

The integrations over conformal time in T_1^S contain the oscillating factors $\exp(ik\eta)$ as well as slowly changing prefactors entering through the scale factor and χ_0. We use Eq. (B19) to obtain

$$\chi_0 = \bar{N}a^{-1}\dot{\phi}_0. \tag{B31}$$

where $\dot{\phi}_0$ will be taken to be constant.

The smearing integrals in Eq. (B27) now lead to

$$T_1^S = 4\bar{N}^4\dot{\phi}_0^2 H^2\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2k}\left|w_r\left(\frac{Rk}{a_0}\right)P_1(k)\right|^2, \tag{B32}$$

where

$$P_1(k) \equiv \int d\eta\frac{(H\eta)^3}{\tau a_0^{-1}}e^{-ik\eta}W_\eta\left(\frac{\eta - \eta_0}{\tau a_0^{-1}}\right)$$
$$\times\left[k\eta - i\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}\left(1 + ik\eta\right)\right]. \tag{B33}$$

Note that $a_0^{-1} = -H\eta_0$, and the dependence of the integrals on the temporal location, η_0, can be removed by multiplying k by the scale factor a_0^{-1} if we also choose $\bar{N} = a_0$. The result is the same as if we substituted $a_0 = 1$, $\eta_0 = -H^{-1}$ in all expressions, and we will take these values in the rest of this appendix.

We find it convenient to estimate the oscillating integral of Eq. (B33) in terms of the Fourier transformed temporal window function

$$w_\eta(\omega) \equiv \int du\, e^{-i\omega u}W_\eta(u) \tag{B34}$$

Then we find

$$\int\frac{d\eta}{\tau}e^{-ik\eta}W_\eta\left(\frac{\eta - \eta_0}{\tau}\right) = e^{-ik\eta_0}w_\eta(k\tau) \tag{B35}$$

The expression for $P_1(k)$ can now be written in terms of derivatives with respect to k

$$P_1(k) = \left(1 + \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\right) k H^3 \frac{\partial^4}{\partial k^4} \left(e^{-ik\eta_0} w_\eta(k\tau)\right)$$
$$- \left(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\right) H^3 \frac{\partial^3}{\partial k^3} \left(e^{-ik\eta_0} w_\eta(k\tau)\right). \tag{B36}$$

The real-space window functions, $W_{\eta,r}$ are assumed to be smooth. (For example, one can specifically think of Gaussian window functions in real space, and then they will also be Gaussian in Fourier space.) So the Fourier space window functions satisfy

$$w_{r,\eta}(\zeta) \approx 1 \quad (\zeta \ll 1), \tag{B37}$$

$$w_{r,\eta}(\zeta) \ll 1 \quad (\zeta \gtrsim 1), \tag{B38}$$

while, at intermediate values of ζ, $w_{r,\eta}$ is of order 1. Therefore, we may assume that the derivatives of the Fourier profiles $w_\eta(\zeta)$ are also generically of order 1 and decay quickly at large values of the argument. Derivatives of $w_\eta(k\tau)$ with respect to k introduce multiplicative factors of τ, while derivatives of $\exp(-ik\eta_0)$ introduce multiplicative factors of $-i\eta_0 = iH^{-1}$.

Then the magnitude of $P_1(k)$ for $k\tau \leq 1$ is estimated from Eq. (B36) as

$$P_1(k) \sim \left(c_1 k H^{-1} + c_1'\right), \quad k\tau < 1 \tag{B39}$$

where c_1 and c_1' are quartic polynomials in $H\tau$ which depend on the choice of window functions. For $k\tau > 1$, the temporal window function is small (Eq. (B38)) and $P_1(k) \sim 0$.

After thus eliminating the oscillatory integrals, we can find the magnitude of the first term in Eq. (B20). The integration in Eq. (B32) is effectively performed over a spherical 3-dimensional domain $|\mathbf{k}| \lesssim L^{-1}$ where $L \equiv \max(\tau, R)$. Typically, $k \sim L^{-1}$, so the integrand in Eq. (B32) is of order

$$L \left(c_1 L^{-1} H^{-1} + c_1'\right)^2 \sim L \max\left(c_1'^2, c_1^2 (HL)^{-2}\right), \tag{B40}$$

and therefore we may estimate the leading order contribution to Eq. (B32)

$$T_1^S \sim \frac{H^2 \dot{\phi}_0^2}{(2\pi)^2 L^2} \max\left(c_1'^2, c_1^2 (HL)^{-2}\right). \tag{B41}$$

We should note that this estimate hinges on the fast decay of the Fourier transformed windows $w_{r,\eta}(\zeta)$, which is ensured only when the real-space window profiles $W_{r,\eta}$ are sufficiently smooth. The factors c_1, c_1' are window-dependent but are typically of order 1 since the windows are normalized and Eqs. (B37), (B38) hold. (Explicit expressions for c_1, c_1' through the window functions $w_{r,\eta}(\zeta)$ can be obtained but will not be useful for our purposes.)

Having presented the estimation of T_1^S in detail, we now sketch the calculations needed for T_2^S. In the expression for T_2, Eq. (B22), only expressions containing $\hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}'}^\dagger$ will give a nonzero contribution. In terms of the Fourier transformed spatial window function, we are left with a 6-dimensional integral

$$T_2^S = 2 \int \frac{d^3\mathbf{k} \, d^3\mathbf{k}'}{(2\pi)^6} \frac{H^4}{4kk'} \left| w_r(R(\mathbf{k} + \mathbf{k}')) P_2(\mathbf{k}, \mathbf{k}') \right|^2, \tag{B42}$$

where

$$P_2(\mathbf{k}, \mathbf{k}') \equiv \int d\eta \frac{(H\eta)^4}{\tau} e^{-i(k+k')\eta} W_\eta \left(\frac{\eta - \eta_0}{\tau}\right) \times$$
$$\left[k\eta - i\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}(1 + ik\eta)\right] \left[k'\eta - i\hat{\mathbf{k}}' \cdot \hat{\mathbf{n}}(1 + ik'\eta)\right]. \tag{B43}$$

The window function W_η constrains this integral to $|\mathbf{k}| \lesssim \tau^{-1}$ and $|\mathbf{k}'| \lesssim \tau^{-1}$, while the w_r further limits the integration in Eq. (B42) to a slab-shaped domain $|\mathbf{k} + \mathbf{k}'| \lesssim R^{-1}$. The 6-volume of the resulting region is of order $\tau^{-3} \min(\tau^{-3}, R^{-3}) = (L\tau)^{-3}$.

We can now write P_2 in terms of derivatives, corresponding to Eq. (B36), and then estimate the derivatives as in Eq. (B39) to get

$$P_2(\mathbf{k}, \mathbf{k}') \;=\; -\frac{kk'}{H^2}(1 + \hat{\mathbf{k}} \cdot \hat{\mathbf{n}})(1 + \hat{\mathbf{k}}' \cdot \hat{\mathbf{n}})\, c_2 \tag{B44}$$

$$-\left[\frac{k}{H}\hat{\mathbf{k}}' \cdot \hat{\mathbf{n}}(1 + \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}) + \frac{k'}{H}\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}(1 + \hat{\mathbf{k}}' \cdot \hat{\mathbf{n}})\right] c_2' \tag{B45}$$

$$-\; \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\, \hat{\mathbf{k}}' \cdot \hat{\mathbf{n}}\, c_2'' \tag{B46}$$

where $k\tau < 1$ has been assumed and c_2, c_2' and c_2'' are polynomials of order 6, 5 and 4 respectively in $H\tau$.

Most of the volume in the k integrals in Eq. (B42) is for $k \sim \tau^{-1}$ and so we estimate P_2 as

$$P_2 \;\sim\; \frac{k^2}{H^2}c_2 + \frac{k}{H}c_2' + c_2''$$

$$\sim\; \frac{1}{H^2\tau^2}p_6 \tag{B47}$$

where p_6 is a polynomial of order 6 in $H\tau$. The integrand of Eq. (B42) is

$$\sim H^4\tau^2|P_2|^2 \sim \frac{p_6^2}{\tau^2} \tag{B48}$$

leading to

$$T_2^S \sim \frac{p_6^2}{(2\pi)^4}\frac{1}{L^3\tau^5} \tag{B49}$$

Here the polynomial p_6 in $H\tau$ depends on the window functions but, for smooth functions, the coefficients can be expected to be of order 1.

The result in Eq. (B49) already shows that without temporal smearing (with $\tau = 0$) the integral would diverge. This is because the volume of the domain $|\mathbf{k}+\mathbf{k}'| \lesssim R^{-1}$ is infinite if $\tau = 0$. Thus 4-dimensional smearing is necessary to obtain a finite result.

We now combine the estimates for T_1^S and T_2^S (Eqs. (B41), (B49)) to get an estimate for the variance of the NEC operator (Eq. (19))

$$\sigma^2 \sim \frac{H^2\dot{\phi}_0^2}{(2\pi)^2 L^2}\max\left(c_1'^2, c_1^2\,(HL)^{-2}\right) + \frac{p_6^2}{(2\pi)^4}\frac{1}{L^3\tau^5} \tag{B50}$$

There are three length scales in the problem, T, R and H^{-1}, and the result depends on their relative ordering. For example, if we consider fluctuations on small (sub-horizon) time scales but over large length scales, as is relevant to "island cosmology" [20]

$$\sigma^2 \sim \frac{H^2\dot{\phi}_0^2}{(2\pi)^2 R^2} + \frac{p_6^2}{(2\pi)^4}\frac{1}{R^3T^5}\;,\quad T \ll H^{-1} < R \tag{B51}$$

and the second term dominates since $T \sim \tau$ (see Eq. (B9)) is smaller than the other length scales. Note that the amplitude for NEC violating fluctuations is largest for small T and, in fact, diverges as $T \to 0$.

In the context of eternal inflation, all three length scales are taken to be of the same magnitude. So we set $R = T = L = (\varepsilon H)^{-1}$ as in Eq. (23). From Eq. (B9) we estimate $\tau \approx H^{-1}$ for $\epsilon \approx 1$. This gives

$$\sigma^2 \sim \frac{H^4\dot{\phi}_0^2\max\left(c_1^2\varepsilon^2, c_1'^2\varepsilon^4\right)}{(2\pi)^2} + \frac{c_2^2H^8\varepsilon^8}{(2\pi)^4}. \tag{B52}$$

The quantity we need for Sec. V is therefore

$$\frac{\sigma^2}{\mu^2} \sim \frac{H^4\max\left(c_1^2\varepsilon^2, c_1'^2\varepsilon^4\right)}{\left(2\pi\dot{\phi}_0\right)^2} + \frac{c_2^2H^8\varepsilon^8}{\left(2\pi\dot{\phi}_0\right)^4}. \tag{B53}$$