Conditional Probability

$$P(A \mid X) = \frac{P(X \mid A)P(A)}{P(X \mid A)P(A) + P(X \mid A^{\complement})P(A^{\complement})}$$

Special Random Variables

Discrete Distributions

Bernoulli Distribution

$$X \sim Bernoulli(p), 0 < p < 1$$

PMF:

$$P(X = i) = p^i(1-p)^{1-i}, i \in \{0, 1\}$$

Binomial Distribution

$$X \sim Binomial(n, p), 0 < p < 1$$

PMF:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

CDF:

$$P(X \leq i) = \sum_{k=0}^{i} P(X = k)$$

$E[X] = np$, $Var(X) = np(1-p)$

Poisson

$$X \sim Poisson(\lambda), \lambda > 0$$

Poisson distribution is applicable:

The number of meteors greater than 1 meter diameter that strike earth in a year. The number of patients arriving in an emergency room between 10 and 11 pm

PMF:

$$P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

CDF:

$$P(X \leq i) = e^{-\lambda} \sum_{k=0}^{i} \frac{\lambda^k}{k!}$$

$E[X] = \lambda$, $Var(X) = \lambda$.

Hypergeometric

$$X \sim Hypergeometric(N, M, n)$$

N good and M bad. Sample size n randomly chosen w/o replacement. Bin of transistors.

PMF:

$$P(X = i) = \frac{\binom{N}{i}\binom{M}{n-i}}{\binom{N+M}{n}}$$

CDF:

$$P(X \leq i) = \sum_{k=0}^{i} \frac{\binom{N}{i}\binom{M}{n-i}}{\binom{N+M}{n}}$$

$E[X] = \frac{nN}{N+M}$, $Var(X_i) = \frac{N}{N+M}\frac{M}{N+M}$

$$Var(X) = np(1-p)\left(1 - \frac{n-1}{N+M-1}\right)$$

PMF given two independent binomial random variables, X, Y with respective parameters (n, p) and (m, p).

$$P(X = i \mid X + Y = k) = \frac{\binom{n}{i}\binom{m}{k-i}}{\binom{n+m}{k}}$$

Continuous Distributions

Uniform Distribution

$$X \sim Uniform(\alpha, \beta)$$

PDF:

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

CDF:

$$P(a < X < b) = \frac{b-a}{\beta - \alpha}$$

$E[X] = \frac{\alpha+\beta}{2}$, $Var(X) = \frac{(\beta-\alpha)^2}{12}$

Normal Distribution

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

PDF:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-(x-\mu)^2/2\sigma^2}$$

$E[X] = \mu$, $Var(X) = \sigma^2$

$$Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

$$\Phi(x) = P(Z \leq x) = \int_{-\infty}^{x} \frac{1}{2\pi} \mathrm{e}^{-\frac{z^2}{2}} \, dz$$

$$\Phi(-x) = 1 - \Phi(x), \; x > 0$$

CDFs:

$$P(X < b) = \Phi\left(\frac{b-\mu}{\sigma}\right)$$

$$P(a < X < b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Sums of NRVs are also normal with $\mu = \sum \mu_i$, $\sigma^2 = \sum \sigma_i^2$. Linear transformations also produce normally distributed variables.

$$a + bX \sim \mathcal{N}(a + b\mu, b^2\sigma^2)$$

Exponential Distribution

$$X \sim Exponential(\lambda), \; \lambda > 0$$

PDF:

$$f(x) = \lambda \mathrm{e}^{-\lambda x}, \; x \geq 0$$

CDF:

$$P(X \leq i) = 1 - \mathrm{e}^{-\lambda x}, \; x \geq 0$$

$E[X] = \lambda^{-1}$, $Var(X) = \lambda^{-2}$

Exponential distribution is memoryless, adding conditions does not change the distribution. Useful for probability you will have to wait x time for a bus that on average you have to wait y for.

Chi-Square Distribution

Z is a list of n *independent* standard normal variables.

$$X = \sum_{i}^{n} Z_i^2$$

$$X \sim \chi_n^2$$

$E[X] = n$, $Var(X) = 2n$. CDF:

$$P(X > \chi_{\alpha,n}^2) = \alpha, \; \alpha \in [0, 1]$$

$$P(X \leq \chi_{\alpha,n}^2) = 1 - \alpha$$

If $n = 1$:

$$X \sim \chi_1^2$$

$$\begin{aligned} P(X < x) &= P(-\sqrt{x} < Z < \sqrt{x}) \\ &= 2\Phi(\sqrt{x}) - 1 \end{aligned}$$

If $n = 2$:

$$X \sim \chi_2^2 \sim Exponential\left(\frac{1}{2}\right)$$

t-Distribution

Z std nrv ind to, χ_n^2

$$T_n = \frac{Z}{\sqrt{\chi_n^2/n}} \sim t_n$$

$E[T_n] = 0, n > 1 \quad Var(T_n) = \frac{n}{n-2}, n > 2$

$$P\{T_n > t_{\alpha,n}\} = \alpha$$

$$P\{T_n \le t_{\alpha,n}\} = 1 - \alpha$$

$$P\{T_n > -t_{\alpha,n}\} = 1 - \alpha$$

$$-t_{\alpha,n} = t_{1-\alpha,n}$$

F-Distribution

χ_n^2 ind. of χ_m^2

$$Y = \frac{\chi_n^2/n}{\chi_m^2/m} \sim F_{n,m}$$

$$E[Y] = \frac{m}{m-2}, m > 2$$

$$Var(Y) = 2\left(\frac{m}{m-2}\right)^2 \frac{n+m-2}{n(m-4)}, m > 4$$

$$P\{Y > F_{\alpha,n,m}\} = \alpha$$

$$P\{Y \le F_{\alpha,n,m}\} = 1 - \alpha$$

$$F_{1-\alpha,\mathbf{m},n} = \frac{1}{F_{\alpha,n,m}}$$

Distributions of Sampling Statistics

Sample Mean

$$\bar{X} = \frac{X_1 + \cdots X_n}{n}$$

$E[\bar{X}] = \mu, Var(\bar{X}) = \frac{\sigma^2}{n}$

The Central Limit Theorem

$X_1, \dots X_n$ are ind., id. dist. r.v. each with mean μ and variance $\sigma^2 R$. For n large,

$$P\left\{\frac{X_1 + \cdots X_n - n\mu}{\sigma\sqrt{n}} < x\right\} \approx P\{Z < x\}$$

where Z is a std. n.r.v.

Distribution of Sample Mean

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim Z$$

Distribution of Sample Variance

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$$

Distribution of Sample Mean and Sample Variance

$$\sqrt{n}\frac{\bar{X} - \mu}{S} \sim t_{n-1}$$

Sampling From a Finite Population

Population of N elements. p is proportion posessing a certain trait. X_i is 1 if ith member has trait, 0 else. Assume population size much greater than sample size. Then

$$E[X] = np, Var(X) = np(1-p)$$

$$E[\bar{X}] = p, Var(\bar{X}) = p(1-p)/n$$

Parameter Estimation

Maximum Likelihood Estimators

Given sample $X_1 \dots X_n$ with joint distribution f known except for some parameter θ. Define likelihood function:

$$L(\theta) = f_\theta(x_1, \dots x_n)$$

If X_i are independent,

$$L(\theta) = f_\theta(x_1) \cdots f_\theta(x_n).$$

To find $\hat{\theta}_{MLE}$ maximize $L(\theta)$.

Bernoulli

$$\hat{p}_{MLE} = \bar{X}$$

Binomial

$$\hat{p}_{MLE} = \bar{X}$$

Poisson

$$\hat{\lambda}_{MLE} = \bar{X}$$

Normal

When μ and σ unknown:

$$\hat{\mu}_{MLE} = \bar{X}$$

$$\hat{\sigma}_{MLE} = \sqrt{\frac{\sum(X_i - \bar{X})^2}{n}}$$

When only σ unknown:

$$\hat{\sigma^2}_{MLE} = \frac{\sum(X_i - \mu)}{n}$$

Interval Estimates

2-Sided Confidence Intervals

$$[\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}]$$

Upper Confidence Intervals

$$[\bar{x} - z_\alpha\frac{\sigma}{\sqrt{n}}, \infty]$$

Lower Confidence Intervals

$$[-\infty, \bar{x} + z_\alpha\frac{\sigma}{\sqrt{n}}]$$

Normal Mean with Unknown Variance

$$[\bar{x} \pm t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}]$$