

Conditional Probability

$$P(A|X) = \frac{P(X|A)P(A)}{P(X|A)P(A) + P(X|A^c)P(A^c)}$$

Special Random Variables

Discrete Distributions

Bernoulli Distribution

$$X \sim \text{Bernoulli}(p), 0 < p < 1$$

PMF:

$$P(X = i) = p^i(1-p)^{1-i}, i \in \{0, 1\}$$

Binomial Distribution

$$X \sim \text{Binomial}(n, p), 0 < p < 1$$

PMF:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

CDF:

$$P(X \leq i) = \sum_{k=0}^i P(X = k)$$

$$E[X] = np, \text{Var}(X) = np(1-p)$$

Poisson

$$X \sim \text{Poisson}(\lambda), \lambda > 0$$

Poisson distribution is applicable:

The number of meteors greater than 1 meter diameter that strike earth in a year. The number of patients arriving in an emergency room between 10 and 11 pm

PMF:

$$P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

CDF:

$$P(X \leq i) = e^{-\lambda} \sum_{k=0}^i \frac{\lambda^k}{k!}$$

$$E[X] = \lambda, \text{Var}(X) = \lambda.$$

Hypergeometric

$$X \sim \text{Hypergeometric}(N, M, n)$$

N good and M bad. Sample size n randomly chosen w/o replacement. Bin of transistors.

PMF:

$$P(X = i) = \frac{\binom{N}{i} \binom{M}{n-i}}{\binom{N+M}{n}}$$

CDF:

$$P(X \leq i) = \sum_{k=0}^i \frac{\binom{N}{k} \binom{M}{n-k}}{\binom{N+M}{n}}$$

$$E[X] = \frac{nN}{N+M}, \text{Var}(X) = \frac{N}{N+M} \frac{M}{N+M}$$

$$\text{Var}(X) = np(1-p) \left(1 - \frac{n-1}{N+M-1}\right)$$

PMF given two independent binomial random variables, X, Y with respective parameters (n, p) and (m, p) .

$$P(X = i | X + Y = k) = \frac{\binom{n}{i} \binom{m}{k-i}}{\binom{n+m}{k}}$$

Continuous Distributions

Uniform Distribution

$$X \sim \text{Uniform}(\alpha, \beta)$$

PDF:

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

CDF:

$$P(a < X < b) = \frac{b-a}{\beta-\alpha}$$

$$E[X] = \frac{\alpha+\beta}{2}, \text{Var}(X) = \frac{(\beta-\alpha)^2}{12}$$

Normal Distribution

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

PDF:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

$$E[X] = \mu, \text{Var}(X) = \sigma^2$$

$$Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$$

$$\Phi(x) = P(Z \leq x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$\Phi(-x) = 1 - \Phi(x), x > 0$$

CDFs:

$$P(X < b) = \Phi\left(\frac{b-\mu}{\sigma}\right)$$

$$P(a < X < b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Sums of NRVs are also normal with $\mu = \sum \mu_i$, $\sigma^2 = \sum \sigma_i^2$. Linear transformations also produce normally distributed variables.

$$a + bX \sim \mathcal{N}(a + b\mu, b^2\sigma^2)$$

Exponential Distribution

$$X \sim \text{Exponential}(\lambda), \lambda > 0$$

PDF:

$$f(x) = \lambda e^{-\lambda x}, x \geq 0$$

CDF:

$$P(X \leq i) = 1 - e^{-\lambda x}, x \geq 0$$

$$E[X] = \lambda^{-1}, \text{Var}(X) = \lambda^{-2}$$

Exponential distribution is memoryless, adding conditions does not change the distribution. Useful for probability you will have to wait x time for a bus that on average you have to wait y for.

Chi-Square Distribution

Z is a list of n independent standard normal variables.

$$X = \sum_i^n Z_i^2$$

$$X \sim \chi_n^2$$

$$E[X] = n, \text{Var}(X) = 2n. \text{ CDF:}$$

$$P(X > \chi_{\alpha,n}^2) = \alpha, \alpha \in [0, 1]$$

$$P(X \leq \chi_{\alpha,n}^2) = 1 - \alpha$$

If $n = 1$:

$$X \sim \chi_1^2$$

$$P(X < x) = P(-\sqrt{x} < Z < \sqrt{x}) \\ = 2\Phi(\sqrt{x}) - 1$$

If $n = 2$:

$$X \sim \chi_2^2 \sim \text{Exponential}\left(\frac{1}{2}\right)$$

t-Distribution

Z std nrv ind to, χ_n^2

$$T_n = \frac{Z}{\sqrt{\chi_n^2/n}} \sim t_n$$

$$E[T_n] = 0, n > 1 \quad \text{Var}(T_n) = \frac{n}{n-2}, n > 2$$

$$P\{T_n > t_{\alpha,n}\} = \alpha$$

$$P\{T_n \leq t_{\alpha,n}\} = 1 - \alpha$$

$$P\{T_n > -t_{\alpha,n}\} = 1 - \alpha$$

$$-t_{\alpha,n} = t_{1-\alpha,n}$$

F-Distribution

χ_n^2 ind. of χ_m^2

$$Y = \frac{\chi_n^2/n}{\chi_m^2/m} \sim F_{n,m}$$

$$E[Y] = \frac{m}{m-2}, m > 2$$

$$\text{Var}(Y) = 2 \left(\frac{m}{m-2} \right)^2 \frac{n+m-2}{n(m-4)}, m > 4$$

$$P\{Y > F_{\alpha,n,m}\} = \alpha$$

$$P\{Y \leq F_{\alpha,n,m}\} = 1 - \alpha$$

$$F_{1-\alpha,m,n} = \frac{1}{F_{\alpha,n,m}}$$

Distributions of Sampling Statistics

Sample Mean

$$\bar{X} = \frac{X_1 + \cdots + X_n}{n}$$

$$E[\bar{X}] = \mu, \text{Var}(\bar{X}) = \frac{\sigma^2}{n}$$

The Central Limit Theorem

X_1, \dots, X_n are ind., id. dist. r.v. each with mean μ and variance $\sigma^2 R$. For n large,

$$P\left\{\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} < x\right\} \approx P\{Z < x\}$$

where Z is a std. n.r.v.

Distribution of Sample Mean

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim Z$$

Distribution of Sample Variance

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$$

Distribution of Sample Mean and Sample Variance

$$\sqrt{n} \frac{\bar{X} - \mu}{S} \sim t_{n-1}$$

Sampling From a Finite Population

Population of N elements. p is proportion possessing a certain trait. X_i is 1 if i th member has trait, 0 else.

Assume population size much greater than sample size. Then

$$E[X] = np, \text{Var}(X) = np(1-p)$$

$$E[\bar{X}] = p, \text{Var}(\bar{X}) = p(1-p)/n$$

Parameter Estimation

Maximum Likelihood Estimators

Given sample $X_1 \dots X_n$ with joint distribution f known except for some parameter θ . Define likelihood function:

$$L(\theta) = f_{\theta}(x_1, \dots, x_n)$$

If X_i are independent,

$$L(\theta) = f_{\theta}(x_1) \cdots f_{\theta}(x_n).$$

To find $\hat{\theta}_{MLE}$ maximize $L(\theta)$.

Bernoulli

$$\hat{p}_{MLE} = \bar{X}$$

Binomial

$$\hat{p}_{MLE} = \bar{X}$$

Poisson

$$\hat{\lambda}_{MLE} = \bar{X}$$

Normal

When μ and σ unknown:

$$\hat{\mu}_{MLE} = \bar{X}$$

$$\hat{\sigma}_{MLE} = \sqrt{\frac{\sum (X_i - \bar{X})^2}{n}}$$

When only σ unknown:

$$\hat{\sigma}_{MLE}^2 = \frac{\sum (X_i - \mu)}{n}$$

Interval Estimates

2-Sided Confidence Intervals

$$[\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}]$$

Upper Confidence Intervals

$$[\bar{x} - z_{\alpha} \frac{\sigma}{\sqrt{n}}, \infty]$$

Lower Confidence Intervals

$$[-\infty, \bar{x} + z_{\alpha} \frac{\sigma}{\sqrt{n}}]$$

Normal Mean with Unknown Variance

$$[\bar{x} \pm t_{\alpha/2, n-1} \frac{s}{\sqrt{n}}]$$