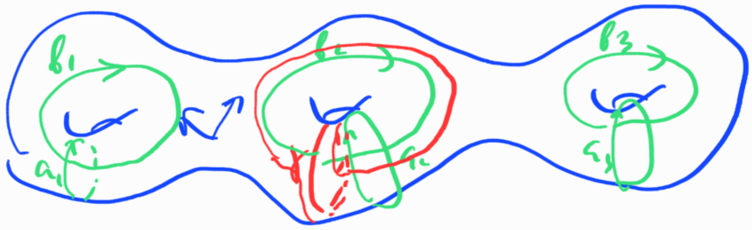
16.04.2021

Группа Торелли

S_g



$$Mod_g = \pi_0 \, Diff^+(S_g)$$

$T_\gamma \in Mod_g$

$a_1 \; a_2 \; \cancel{b_2} \; a_2 \; a_3 \; b_3$

$T_g \subset Mod_g$

$$\boxed{\begin{aligned} T_\gamma a_1 &= a_1 \\ &\vdots \\ T_\gamma b_3 &= b_3 \\ T_\gamma b_2 &= b_2 + a_2 \end{aligned}}$$

$$Mod_g \curvearrowright H = H_1(S_g)$$

$$-(b_1, a_1) = (a_1, b_1) = 1$$
$$(a_2, b_2) = 1$$
$$(a_3, b_3) = 1$$

$$M = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

Утв.

$Mod_g \curvearrowright H \qquad (a, b) = (T \cdot a, T \cdot b)$

$$\underline{Mod_g \xrightarrow{p} Sp(2g)}$$

Утв. p – эпиморфизм $\quad \det A = 1 \quad S^T = S$

$$Sp(2g) = \left\langle \begin{pmatrix} A^T & 0 \\ 0 & A^{-1} \end{pmatrix}, \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}, \right.$$

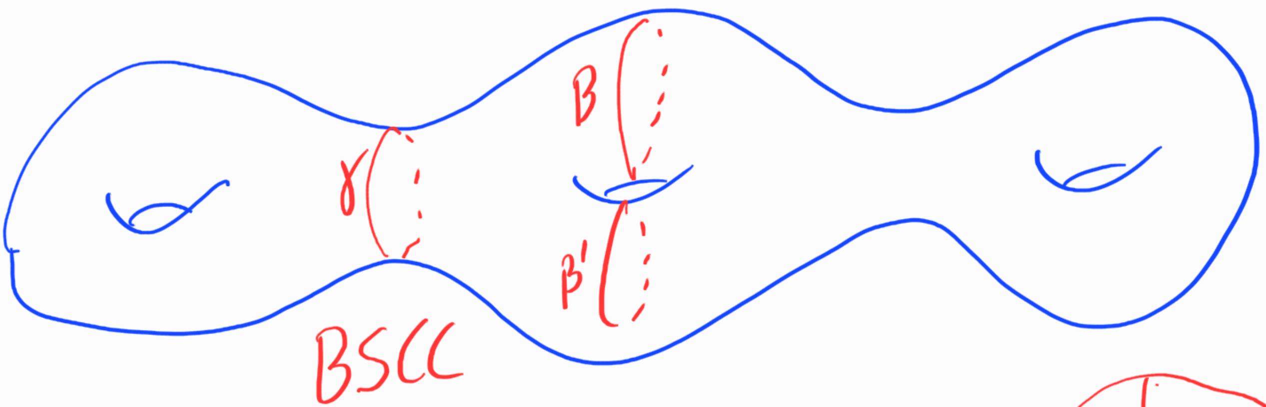
$$\left.\left\{\left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 1 & & & 0 & \\ & & 1 & & & 0 \\ \hline -1 & & & 0 & & \\ & 0 & & & 1 & \\ & & 0 & & & 1 \end{array}\right)\right\}\right\rangle$$

$$T_{a_i} : b_i \longmapsto b_i + a_i$$

$$T_{a_i+b_j} T_{b_j}^{-1} T_{a_i}^{-1} : \begin{array}{l} a_j \longmapsto a_j - a_i \\ b_i \longmapsto b_i + b_j \end{array}$$

$$0 \longrightarrow \underset{\text{Группа Торелли}}{\underbrace{T_g}} \longrightarrow Mod_g \longrightarrow Sp(2g) \longrightarrow 0$$

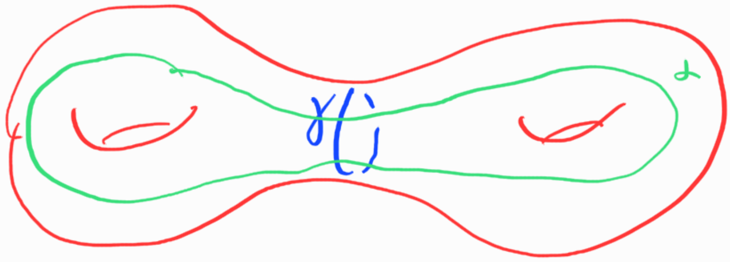
Powell '77



$$T_g = \left\langle \left\{ T_\gamma,\ T_\beta T_{\beta'}^{-1} \right\} \right\rangle$$

$T_1 = 1$

при $g \geq 3$ T_g конечно порождена.



$\mathrm{Mod}_g \ni T_\alpha$

$$\boxed{T_{\gamma'} = T_\alpha T_\gamma T_\alpha^{-1}} \qquad \gamma' = T_\alpha \cdot \gamma$$

$$(T_g)_{ab} = \underline{\mathbb{Z}}^{\circ} \oplus \mathbb{Z}_2^{\circ}$$

Гомоморфизм Джонсона

$S_{g,1}$ $\quad \mathrm{Mod}_{g,1} > T_{g,1}$

1) $T_{g,1} \longrightarrow \mathrm{Hom}(H, H \wedge H)$

$T \in T_{g,1}$

$\pi_1(S_{g,1}) = \langle a_1, b_1, \dots, a_g, b_g \rangle$

$\alpha \in \pi_1(S_{g,1})$

$x_T(\alpha) = (T \cdot \alpha)\,\alpha^{-1} \in \pi_1(S_{g,1})$

$[x_T(\alpha)] \in \pi_1'$

$$[\pi_1,\pi_1]/[\pi_1,[\pi_1,\pi_1]] \cong H\wedge H$$

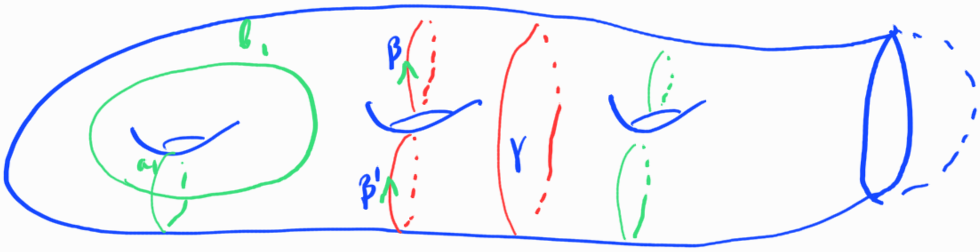
$$[a_i,b_j] \longmapsto [a_i]\wedge[b_j]$$

$$T \longmapsto \big([\alpha]\longmapsto [X_T(\alpha)]\big)$$

$$H \longrightarrow H\wedge H$$

2)
$$\begin{tikzcd}
 & \mathrm{Hom}(H, H\wedge H) \arrow[r] & H\otimes(H\wedge H) \\
T_{g,1} \arrow[ur] \arrow[urr, bend right, "\tau"'] & &
\end{tikzcd}$$

$$\underline{\mathrm{Im}\,\tau = H\wedge H\wedge H} = \Lambda^3 H$$



$$\underbrace{\sum(a_i\wedge b_i)}\wedge[\beta]$$

$$\tau(T_\gamma) = 0$$

$$\tau(T_\beta T_{\beta'}^{-1}) = \underbrace{(a_1\wedge b_1)\wedge \overset{\substack{-[\beta'] \\ \parallel}}{[\beta]}}$$

3) $\pi_1(U'S_g) \longrightarrow T_{g,1} \twoheadrightarrow T_g$ $\quad \underline{\boxed{S_{g,1}\to S_g}}$

$$H \xhookrightarrow{\wedge(\sum_i a_i\wedge b_i)} \Lambda^3 H \longrightarrow \Lambda^3 H/H$$

Утв. $T_g \longrightarrow \Lambda^3 H / H$

$$(T_g)_{ab} = \Lambda^3 H / H$$

ядро Джонсона K_g

$$1 \to K_g \longrightarrow T_g \longrightarrow \Lambda^3 H / H \to 1$$

Утв. $K_g = \langle T_\gamma \mid \gamma - BSCC \rangle$

$\dim H_1 K_g^{\otimes \mathbb{C}} < \infty$ $g \geq 4$??

Гомологии групп

G $\mathbb{C}$

$$\mathbb{C}\{[g_0 | g_1 | \ldots | g_n]\} = C_n(G)$$

$\rho : G \to \mathbb{C}^\times$

$$\partial : C_n(G) \to C_{n-1}(G)$$

$$[g_0 | g_1 | \ldots | g_n] \longmapsto \rho(g_0)[g_1 | \ldots | g_n] -$$

$$- [g_0 g_1 | g_2 | \ldots | g_n] +$$

$$+ [g_0 | g_1 g_2 | \ldots | g_n] - \ldots$$

$$+ (-1)^{\circ}[g_0 | g_1 | \ldots | g_{n-1} g_n] +$$

$$\ldots + (-1)^{\circ}[g_0 | \ldots | g_{n-1}]$$

$\partial^2 = 0$ $\quad H_*(G; \mathbb{C}_\rho)$

$\underline{K(G,1)}$ $\quad H^*(G; \mathbb{C}_\rho)$

$$V_n^k = \{\rho \in \{G_{ab} \to \mathbb{C}^\times\} \mid \dim H^k(G; \check{\mathbb{C}}_\rho) \geq n\}$$

$\bigsqcup (\mathbb{C}^\times)^d$ $\quad \ker[C^k \to C^{k+1}]$

$\mathbb{C}[G_{ab}]$ $\quad$ ~~A~~CC

$$R_n^k \longleftrightarrow V_n^k$$

$$R_1^1(G) \subseteq 0 \Leftrightarrow \dim \underbrace{H_1(G')}_{\mathbb{C} G_{ab}} < \infty$$

T_g

$K_g \supset T_g'$

$\underline{V_1^1(G)}$ – конечно $\Rightarrow$ $H_1(G')$ – к/м

$\forall K \supset G'$

$H_1(K)$ – к/м

21.04.21

Assoz. grad. alg. Lie

$$G = G_1 \supset \underbrace{[G,G]}_{= G_2} \supset \underbrace{[G,[G,G]]}_{= G_3} \supset \ldots$$

$$\oplus\, G_i / G_{i+1} = \underbrace{gr_{\bullet} G}$$

$$gr_i G \times gr_j G \xrightarrow{[\cdot,\cdot]} gr_{i+j} G$$

$gr_{\bullet} G \otimes \mathbb{C}$

$$T_g \hookrightarrow Mod_g \longrightarrow Sp(2g)$$

$$(T_g)_{ab} \curvearrowleft Sp(2g)$$

$$\overset{\shortparallel}{\Lambda^3 H / H}$$

$$Sp(2g) \subset Sp_{\mathbb{C}}(2g)$$

$$Sp_{\mathbb{C}}(2g) \curvearrowright (T_g)_{ab} \otimes \mathbb{C} \cong gr_1 T_g$$

$$Sp_{\mathbb{C}}(2g) \curvearrowright gr_{\bullet} T_g \otimes \mathbb{C}$$

$$s[a,b] = [sa, sb]$$

гр. $T_g \otimes \mathbb{C}$ $\qquad sp(2g)$-модуль

$$\Lambda^3 H / H$$

$$t_i = \begin{array}{c} \\ i \\ \\ \\ \end{array}\!\!\begin{array}{c} \quad i \\ \left(\begin{array}{c|c} 0 \;\; & \\ \;\; 1 & \\ \quad \ddots\; 0 & \\ \hline & 0 \\ & \;\; -1 \\ & \quad\; 0 \end{array}\right) \end{array} \qquad \begin{aligned} t_i \cdot a_i &= a_i \\ t_i \cdot b_i &= -b_i \end{aligned}$$

$$[t_i, t_j] = 0$$

$$h \subset sp(2g)$$

$$n^- \oplus h \oplus n^+ = sp(2g)$$

$$h^+ = \langle T_i, \underset{i<j}{S_{ij}}, F_{ij} \rangle$$

$$\left\{ \begin{aligned} T_i \cdot b_i &= a_i \\ S_{ij} \cdot a_j &= a_i \\ S_{ij} \cdot b_i &= -b_j \\ F_{ij} \cdot b_i &= a_i \\ F_{ij} \cdot b_j &= a_j \end{aligned} \right. \qquad i<j$$

$$[h, g] = \alpha(h) g \qquad \mathfrak{g} = sp(2g)$$

$$g \in \mathfrak{g}_\alpha$$

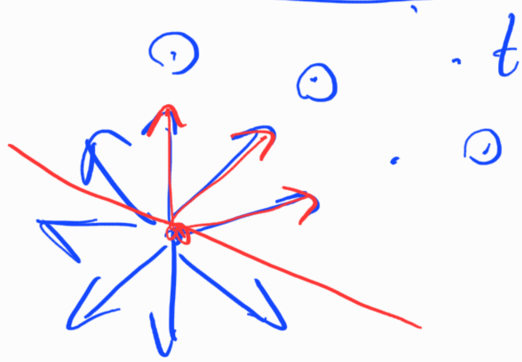
$$\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$$

$$\alpha \in \boxed{h^*}$$

$$\underset{i,j}{\pm t^i \pm t^j}; \pm 2 t^i$$

$\boxed{t^i - t^j,\quad t^i + t^j,\quad 2t^i \qquad i<j}$ — полож. корни

$t_i < t_j \quad i<j$



$t_1 - t_2 \quad t_2 - t_3 \quad \dots \quad t_{n-1} - t_n \quad 2t_n$

$\mathfrak{g} \curvearrowright V$

$V = \bigoplus_{\alpha} V_\alpha \quad \alpha \in \mathfrak{h}^*$

$\alpha \in \mathbb{Z}\langle t^i \rangle$

$\lambda_i = t^1 + t^2 + \dots + t^i, \quad i \le g$

$\alpha = (\alpha_1 \dots \alpha_s)$

$\alpha_1 \lambda_1 + \dots + \alpha_s \lambda_s$

$V(\alpha) = V(\alpha_1 \lambda_1 + \dots + \alpha_s \lambda_s)$

$V(\lambda_1) = V$

$v_1 \wedge v_2 \mapsto (v_1, v_2)$

$V(\lambda_2) = \mathrm{Ker}[V \wedge V \to \mathbb{C}]$

$V(\lambda_3) = \mathrm{Ker}[V \wedge V \wedge V \to V]$

$\boxed{V(2\lambda_2)}$ $\qquad v_1 \wedge v_2 \wedge v_3 \longmapsto (v_1 - v_2) v_3 + \sim$

$n = n_1 + \dots + n_k$

$V^{\otimes n} \supset W$

$\mathfrak{gl}(n)$

$sp(2n)$

$so(n)$

$$\Lambda^2 \mathfrak{gr}_1 T_g \xrightarrow[\beta]{[\cdot,\cdot]} \mathfrak{gr}_2 T_g$$

Main

$$\Lambda^2 V(\lambda_3) \cong \begin{cases} V(2\lambda_2) + V(0), & g = 3 \\ \vdots & g = 4, 5 \\ V(\lambda_6) + V(\lambda_4) + V(\lambda_4 + \lambda_2) + V(2\lambda_2) + V(\lambda_0) + V(\lambda_2) & g \geq 6 \end{cases}$$

T. Main $\quad \ker \beta = V(2\lambda_2) + V(\lambda_0)$

$$R^k_i(G) = \{\alpha \in H^1(G) \mid \dim H^k(H^*(G), u_\alpha) \geq i\}$$

$$u_\alpha = (\alpha \cdot) : H^n(G) \to H^{n+1}(G)$$

$$H^{k-1}(G) \xrightarrow{\alpha\cdot} H^k(G) \xrightarrow{\alpha\cdot} H^{k+1}(G)$$

$$\underbrace{R^1_1(G)}_{V^1_1(G)} \ni \alpha \qquad \exists \underset{H^1(G)}{\beta} \neq c\alpha \qquad \alpha\beta = 0$$

$\boxed{T \subset U^1_1(G) \subset R^1_1(G)}$

I. $R^1_1(T_g) = \begin{cases} \{0\} & \text{при } g \geq 4 \\ H^1(T_g) & \text{при } g = 3 \end{cases}$

$$(gr_1 T_g)^* \cong H^1 T_g$$

$$\begin{array}{c} (gr_1 T_g) \\ \wr\wr \\ (T_g)_{ab} \\ \wr\wr \\ H_1(T_g) \end{array}$$

Sullivan, Lambe

$H_1 T_g < \infty$

$$0 \to (gr_2 T_g)^* \xrightarrow{\beta^*} (\Lambda^2 gr_1 T_g)^*$$

$$(\Lambda^2 gr_1 T_g)^* = \Lambda^2 H^1 T_g \xrightarrow{\smile} H^2 T_g$$

$$a \smile b = 0$$

$$\Downarrow$$

$$a, b \in R^1_1(T_g)$$

$$(V(2\lambda_2) + V(0))^* \xrightarrow{\sim} (\Lambda^2 V(\lambda_3))^* \ni x \wedge y$$

$g = 3 \Rightarrow \Lambda^2 H^1 T_g \in \mathrm{Ker}(\smile)$

$g \geq 4$ $\quad Sp(2g) \quad V(\lambda_3)^* = V(\lambda_3)$

$$V \to \Lambda^3 V \to V(\lambda_3)$$

$$V(\lambda_3) = \Lambda^3 V / V$$

$$x_1 \wedge x_2 \wedge x_3 \mapsto (x_1, x_2)\, x_3 + \dots$$

$$\left(\sum_{i=1}^{g} a_i \wedge b_i\right) \wedge \cdot$$

$$[a_1 \wedge a_2 \wedge b_3] \subset V(\lambda_3)$$

$v = [a_1 \wedge a_2 \wedge a_3] \in V(\lambda_3)$

$v \notin R_1^1(T_g)$

$v' \wedge v \notin \mathrm{Im}\ \ V(2\lambda_2) + V(0)$

$R_1^1(T_g) \subset V(\lambda_3)$ $\quad\lambda_3$ — старший вес $V(\lambda_3)$

<u>Утв.</u> $R_1^1(T_g) \neq \{0\} \Rightarrow R_1^1(T_g) \ni v \in V(\lambda_3)_{\lambda_3} = \mathbb{C}\langle a_1 \wedge a_2 \wedge a_3\rangle$

G — разр. группа Ли $\quad (\mathfrak{h} \oplus \mathfrak{n}^+)$

$G \curvearrowright X$ — полное мн-е $\quad (R_1^1(T_g))$

Тогда у G на X есть неподв. точка

$$h v = (\underbrace{t^1 + t^2 + t^3}_{\in \mathfrak{h}^*})(h)\, v, \qquad t^1 + t^2 + t^3 = \lambda_3$$

$$t_1 v = t_1 a_1 \wedge a_2 \wedge a_3 + a_1 \wedge t_1 a_2 \wedge a_3 + a_1 \wedge a_2 \wedge t_1 a_3 = v, \qquad t_1 a_1 = a_1,\ t_1 a_2 = 0,\ t_1 a_3 = 0$$

$t_2 v = v$

$t_3 v = v$

$t_k v = 0 \qquad k \geq 4$

$V = [a_1 \wedge a_2 \wedge a_3]$

$\exists w: w \wedge v \in V(2\lambda_2) + V(0)$??

<u>План:</u>

1) $R(T_g) \subseteq \{0\}$ $\qquad V = V_1' \qquad R = R_1'$

2) $TC_1 V(X) \subseteq R(X)$

3)* $TC_1 V(T_g) = 0 \Rightarrow V(T_g)$ — конечное

4) $V(X)$ — конечно $\Rightarrow H^1(X^{abf}, \mathbb{C})$ — к/м

$T_g \qquad K_g \qquad X \to$

5)* $A \overset{T_g}{\leftrightarrows} K_g \Rightarrow H^1(A; \mathbb{C})$ — к/м

6)? $R(K) = 0 \Rightarrow \widehat{H^1 K}$ — к/м

$H^1 K$ — модуль над $\mathbb{C}G_{ab}$

$I < \mathbb{C}G_{ab} \to \mathbb{C}$

$g \mapsto 1$

$x_1 \sim x_n \sim$

$x_{N_1} - x_{N_2} \in I^M$

$Sp(2g) \quad V \ni a_1 \ldots a_g, b_1 \ldots b_g$ — фунд. представл.

$\cup$

$h \qquad h_i \cdot a_i = a_i$

$\qquad h_i \cdot b_i = -b_i$

$V^{\otimes n}$ $V^{\wedge n}$ – предст.

$$V^{\wedge 2} = V(\lambda_2) \oplus V(0)$$

$$V^{\wedge 2} \longrightarrow \mathbb{C}$$

$$(x \wedge y) \longmapsto (x \cdot y)$$

$$V^{\wedge 3} = V(\lambda_3) \oplus \underset{\overset{\parallel}{V}}{V(\lambda_1)}$$

$$(x \wedge y \wedge z) \longmapsto (x \cdot y) z + (y \cdot z) x + (z \cdot x) y$$

$$V(2\lambda_2) = \ker [Sym^2 \Lambda^2 V \to \dots]$$

$a_{i_1} \otimes a_{i_2} \otimes b_{j_1} \otimes \dots$

Пример

$$h_1 (a_1 \wedge b_2) = a_1 \wedge b_2$$

$$h_2 (a_1 \wedge b_2) = -a_1 \wedge b_2$$

$$\underset{h^*}{\underset{\cap}{t^1 - t^2}} \qquad h(a_1 \wedge b_2) = ((t^1 - t^2) h)\, a_1 \wedge b_2$$

$$V(\lambda_3)$$

$$\underset{v, w}{\cup}$$

$$\hat{v} \vee \hat{w} = 0 \in H^1(T_g) \qquad R^!(T_g)$$

$$\underline{v \wedge w} \in \Lambda^2 V(\lambda_3) = \dots \oplus V(2\lambda_2) \oplus V(0)$$

$$p : \Lambda^2 V(\lambda_3) \longrightarrow (\dots)$$

$$v \wedge w \in V(2\lambda_2) + V(0)$$

$V(\lambda_3)$ $\underline{a_1 \wedge a_2 \wedge a_3} =: V_0$

<u>утв</u> $\operatorname{Im}(v \wedge \cdot) \cap \left(V(2\lambda_2) \oplus V(0)\right) \ni w$

n_+ $\qquad n_+ \cdot w = 0$

$$u_0 = \sum_{k=3}^{g} (a_1 \wedge a_2 \wedge a_k) \wedge (a_1 \wedge a_2 \wedge b_k)$$

$$2t^1 + 2t^2 = 2\lambda_2$$

$$\underline{V_0 \wedge w = u_0}$$

$$V_0 \wedge u_0 \in \Lambda^3 V(\lambda_3)$$

$\neq 0$ при $g \geq 4$

$$R(T_g) \leq \{0\}$$

$$V_m^k(X) = \{\rho \in H^1(X; \mathbb{C}^\times) \mid \dim H^k(X; \mathbb{C}_\rho) \geq m\}$$

<u>утв.</u> X имеет конечный k-остов $\Rightarrow V_m^k(X)$ - алг.

$k+1$

$$\operatorname{Hom}_{\mathbb{C}H_1 X}\left(C_{k-1}(\tilde{X}), \mathbb{C}_\rho\right) \xrightarrow{d^k} H_{\mathbb{C}H_1(X)}\left(C_k(\tilde{X}), \mathbb{C}_\rho\right)$$

$\mathbb{C}H_1(X)$ - модуль

$$d^k \in \operatorname{Mat}^{n_{k-1}, n_k}(\mathbb{C}H_1(X))$$

$d^{k+1} \in \ldots$

$$H^k(X; \mathbb{C}_p) = \ker d^{k+1} / \operatorname{Im} d^k$$

$$\operatorname{rk} H^k(X; \mathbb{C}_p) = -(\operatorname{rk} d^k + \operatorname{rk}^{k+1}) + c_k$$

m

$$\underline{\operatorname{rk} d^k + \operatorname{rk} d^{k+1} \leq c_k + m}$$

$$V^k_m = \bigcup_{a+b=c_k+m} V_{a,b}$$

$$C_{k+1}(X) \xrightarrow{\partial_k} C_k(X)$$

$\mathbb{C}H_1(X)$ – модули

$\operatorname{Im} \partial_k$ – к/п

$\tilde{X} = X^k \cup \{$конечное число $k+1$-клеток$\}$

I. $TC_\eta V^k_m(X) \subseteq R^k_m(X; \mathbb{C}_\eta)$

(Libgober'02)

$$H^k(X; \mathbb{C}_\eta) \smile H^\ell(X; \mathbb{C}_{\eta'}) \in H^{k+\ell}(X; \mathbb{C}_{\eta\eta'})$$

$$H^1(X; \mathbb{C}_1) \smile H^\ell(X; \mathbb{C}_\eta) \in H^{\ell+1}(X; \mathbb{C}_\eta)$$

$a \in$

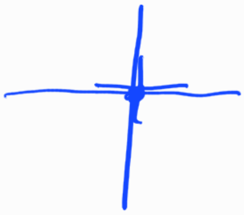
$$(a \smile \cdot)^2 = 0$$

$$\cdots \to H^{\ell}(X;\mathbb{C}_\eta) \xrightarrow{a\cup} H^{\ell+1}(X;\mathbb{C}_\eta) \to \cdots$$

(Aomoto)

$$R^k_m(X;\mathbb{C}_\eta) = \{a \in H^1(X;\mathbb{C}_1) \mid \dim H^k(H^*(X;\mathbb{C}_\eta); d_a) \geq m\}$$

$$TC_x \overset{V(I)}{V}$$



$$\underset{\parallel}{I} \quad z_1 z_2 = 0 \qquad x = (0,0)$$

$$\mathfrak{M}_x = \mathbb{C}[z_1, z_2] z_1 + \mathbb{C}[z_1, z_2] z_2$$

$$TC_x V = \mathrm{spec}\,(z_1 z_2)$$

$$x \rightsquigarrow \mathfrak{M}_x$$

$$TC_x V = \mathrm{Spec} \bigoplus_n \left(I \cap \mathfrak{M}_x^n \big/ I \cap \mathfrak{M}_x^{n+1} \right)$$

$$\subset \bigoplus \mathfrak{M}_x^n / \mathfrak{M}_x^{n+1} \cong \mathbb{C}[z_1, z_2]$$

$$I \cap \mathfrak{M}_x^1 = I$$

$$I \cap \mathfrak{M}_x^2 = I$$

$$I \cap \mathfrak{M}_x^3 = I \setminus \mathbb{C} z_1 z_2$$

$$I \cap \mathfrak{M}_x^4 = I \setminus \mathbb{C} z_1 z_2 \quad z_1^2 z_2 \quad z_1 z_2^2$$

$$I \cap \mathfrak{M}_x^2 / I \cap \mathfrak{M}_x^3 = \underline{\mathbb{C} z_1 z_2}$$

$$\bigcup_n V(p_{n,1}, \dots, p_{n,k})$$

- $R' > TC_1 V'$
- V' – конечное $\Longrightarrow H'K$ – к.п.

Dwyer – Freed '86

I. X $\quad \tilde{X}$ – св. аб. накрытие

V – конечно $\Longleftrightarrow H_1(\tilde{X}; \mathbb{C})$ – к.п.

$$\begin{tikzcd} \pi_1 X \arrow[r, "f"] \arrow[d] & \mathbb{Z}^k \\ H_1 X \arrow[ur] & \end{tikzcd}$$

$\{ f \mid H_1(\tilde{X}_f ; \mathbb{C}) - \text{к.п.} \}$

инв. Dwyer – Freed

$\triangleright$ $H_1(X, \mathbb{C}_\rho)$

$H_1(\tilde{X})$

$C_k(\tilde{X}) = \mathbb{C}\mathbb{Z}^k\{x_1 \ldots x_m\}$

$C_k(X) = \mathbb{C}\{x_1 \ldots x_m\}$

$$\begin{tikzcd} C_k(\tilde{X}) \arrow[r, "\partial_k"] \arrow[d, "\otimes \mathbb{C}_\rho"] & C_{k-1}(\tilde{X}) \arrow[d] \\ C_k(X) \arrow[r, "\partial_k(\rho)"] & C_{k-1}(X) \end{tikzcd}$$

$g \in \pi_1 X \qquad g \cdot x \sim \rho(g) x$

$C_*(\tilde{X}) \otimes_{\mathbb{C}\mathbb{Z}^k} \mathbb{C}_\rho = C_*(X)$

$$\boxed{H_*(C_*(\tilde{X}), \partial_*) \otimes_{\mathbb{C}\mathbb{Z}^k} \mathbb{C}_\rho = H_*(C_*(X), \partial_*(\rho))}$$

$$V = \{ p \mid H_1(C_*(X); \partial_*(p)) \neq 0 \}$$

$$\| $$

$$H_1(C_*(\tilde{X}); \partial_*) \underset{\mathbb{C}\mathbb{Z}^k}{\otimes} \mathbb{C}_p$$

<u>Утв.</u> $\mathcal{M}$ — к/п. модуль над $\mathbb{C}\mathbb{Z}^k$

$\mathcal{M}$ — к/м над $\mathbb{C} \Longleftrightarrow \operatorname{supp} \mathcal{M}$ конечен

$$\operatorname{supp} \mathcal{M} = \{ p \mid \mathcal{M} \otimes \mathbb{C}_p \neq 0 \}$$

$$\operatorname{Ann} \mathcal{M} = \{ a \in \mathbb{C}\mathbb{Z}^k \mid a\mathcal{M} = 0 \}$$

<u>Пример.</u> 1) $\mathcal{M} = \mathbb{C}[x^{\pm 1}]\{a, b\} / (p(x)a, q(x)b)$

$$\operatorname{Ann} \mathcal{M} = (p) \cap (q) \qquad \{x^n \cdot a\}$$

$$\operatorname{Supp} \mathcal{M} = \operatorname{Zero}(p, q) \qquad \{x^n \cdot b\}$$

$x - p$

$$\operatorname{loc}_p(\mathcal{M}) = \left\{ \frac{m}{s},\ \begin{matrix} m \in \mathcal{M} \\ s \in \mathbb{C}[x^{\pm 1}] \\ s(p) \neq 0 \end{matrix} \right\} \Big/ \ \frac{m_1}{s_1} = \frac{m_2}{s_2} \text{ когда } m_1 s_2 = m_2 s_1$$

$$\frac{r_1(x)}{s_1(x)} a + \frac{r_2(x)}{s_2(x)} b \sim 0$$

$$\text{П} \begin{cases} p(p) \neq 0 \\ q(p) \neq 0 \end{cases} \qquad \begin{matrix} a \sim \dfrac{p \cdot a}{p} = 0 \\ b \sim \dfrac{q \cdot b}{q} = 0 \end{matrix}$$

$\Rightarrow p \notin \operatorname{Supp} M$

2) $M = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]\{a\} / (p(x_1)a)$

$\operatorname{Ann} M = (p)$

$\operatorname{Supp} M = \operatorname{zero}(p)$

$p = (p_1, p_2)$

$s \,\vdots\, p$

$a \sim 0 \Leftrightarrow \exists s \quad \frac{sa}{s} = 0$

$s(p_1, p_2) \neq 0$

$p(p_1) \neq 0$

$\underline{\operatorname{supp} M = \operatorname{zero}(\operatorname{Ann} M)}$

D $\Rightarrow$ $M = \mathbb{C}\{m_1 \ldots m_k\}$

$\{m_i, x_1 m_i, x_1^2 m_i \ldots\}$

$p_{1,i}(x_1) m_i = 0$

$\forall m_i \quad p_1(x_1) m_i = 0 \qquad p_1(x_1) \in \operatorname{Ann} M$

$x_2 \qquad p_2(x_2)$

x_3

$\vdots \qquad \vdots$

$\operatorname{zero}(\operatorname{Ann} M)$ - конечно

$\Leftarrow$ $\dim_{\mathbb{C}} M = \infty$

$\{x_1^n m\}$ – л/нез.

$\forall p_1 \ \exists p_2 \dots p_k : (p_1 p_2 \dots p_k) \in \operatorname{Supp} M$

$\triangleleft$

$H_1(\tilde{X})$ – к/м над $\mathbb{C}$ $\iff$ $\operatorname{Supp} H_1(\tilde{X})$ $= V^1(X)$ конечен

Утв. $TC_1 V^1 \subseteq R^1$ $\quad V^1 = V_1^1$

I. (Libgober'2002)

$TC_\eta V_m^k \subseteq R_m^k(\eta)$

$$H^*(X, \mathbb{C}_\eta) \cup H^*(X, \mathbb{C}_0) \longrightarrow H^*(X, \mathbb{C}_{\eta 0})$$

$\eta = 1$ $\quad H^*(X; \mathbb{C})$ $\quad H^1(X; \mathbb{C}^*)$

$a \in H^1(X, \mathbb{C})$

$$H^k(X; \mathbb{C}_\eta) \xrightarrow{\cup a} H^{k+1}(X; \mathbb{C}_\eta)$$

Аомото

$D_{\eta, a}^*$

$$\underset{\substack{\parallel \\ zero(I)}}{\mathcal{A}} \ni x = 0$$

$\underbrace{in\, I}$ — идеал начальных членов

$$TC_x \mathcal{A} = zero(in\, I)$$

$$C^k(x) \xrightarrow{d^k(\rho)} C^{k+1}(x)$$

$$\underbrace{\frac{\partial d^k(\cdot)}{\partial a}(\eta) = \delta^k(\eta, a)}$$

$$\underbrace{H^k(x; \mathbb{C}_\eta) \xrightarrow{\delta^k(\eta, a)} H^{k+1}(x; \mathbb{C}_\eta)}$$

I. $\delta^k(\eta, a) = \cdot \smile a$

$$\theta \in H^1(G; \mathbb{C}_\eta)$$

$$a \in H^1(G; \mathbb{C})$$

$$d_\rho \theta[g_1 | g_2] = \rho(g_1)\theta[g_2] - \theta[g_1 g_2] + \theta[g_1]$$

$$\frac{\partial d_\rho}{\partial a} = \underbrace{a(g_1)\theta[g_2]}_{\substack{\parallel \\ a \smile \theta[g_1 g_2]}}$$

V_m^k

$$C^{k-1}(X) \xrightarrow{d^{k-1}(\eta)} C^k(X) \xrightarrow{d^k(\eta)} C^{k+1}(X)$$

$$m \le \operatorname{rk} H^k(X; \mathbb{C}_\eta) = -\operatorname{rk} d^k(\eta) - \operatorname{rk} d^{k-1}(\eta) + \underbrace{+ \dim C^k(X)}_{} = m_0$$

$$\underbrace{\operatorname{rk} d^k(\eta) + \operatorname{rk} d^{k-1}(\eta) \le m_0 - m}$$

$$(a, b) : \quad a + b = m_0 - m$$

$$\bigvee_{(a,b)} \left(\operatorname{rk} d^k(\eta) \le a \quad \text{u} \quad \operatorname{rk} d^{k-1}(\eta) \le b \right)$$

$$V_{a, d^k} = \{ \eta \mid \operatorname{rk} d^k(\eta) \le a \}$$

$$V_{b, d^{k-1}} = \ldots$$

$$V_m^k = \bigcap_{(a,b)} \left(V_{a,d^k} \cup V_{b,d^{k-1}} \right)$$

$$\mathcal{I} = \sum_{a,b} \mathcal{I}_{a,d^k} \cdot \mathcal{I}_{b,d^{k-1}}$$

$$C^k(x) \xrightarrow{d^k(p)} C^{k+1}(x)$$

$e_1 \cdots e_{s_1}$ $\qquad$ $f_1 \cdots f_{s_2}$

$$e_1 \cdots e_r \xmapsto{d^k(\eta)} f_1 \cdots f_r$$

$$e_{r+1} \cdots e_{s_1} \longmapsto 0$$

$p = \eta + v$

$$\underline{d^k(p)} = \left(\begin{array}{cccc|ccc} 1+F_{11} & F_{12} & \cdots & F_{1r} & F_{1r+1} & \cdots & \\ \vdots & 1+F_{22} & & & & & \\ \vdots & & & 1+F_{rr} & F_{rr+1} & \cdots & \\ \hline \vdots & & & & F_{r+1r+1} & & \\ \vdots & & & & \vdots & \ddots & \\ & & & & \vdots & & \end{array}\right)$$

$$\delta^k : \frac{\operatorname{Ker} d^k(\eta)}{\operatorname{Im} d^{k-1}(\eta)} \longrightarrow \frac{\operatorname{Ker} d^{k+1}(\eta)}{\operatorname{Im} d^k(\eta)}$$

$$\hat{\delta}^k : \operatorname{Ker} d^k(\eta) \longrightarrow \operatorname{Coker} d^k(\eta)$$

$$\hat{\delta}^k(\eta, a) = \begin{pmatrix} \frac{\partial}{\partial a} F_{r+1\, r+1} & & \\ & \ddots & \\ & & \frac{\partial}{\partial a} F_{s_1+1\, s_2+1} \end{pmatrix}$$

$$R^k_m(\eta) = \operatorname{zero}(I)$$

$$I = \sum I_{a, \delta^k} \cdot I_{b, \delta^{k-1}}$$

$$g^{u}_{a-u}\, I_{a,d^k} = g^{u}_{a-u}\, I_{a-u,\delta^k}$$

$$TC_\eta V^k_m \leq R^k_m(\eta) \qquad \triangleleft$$

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1) $\underline{TC_1} V \leq R^1 \subseteq \{0\} \overset{?}{\Longrightarrow} V$ – конечное

2) $R^1 \subseteq \{0\} \Longrightarrow \dots \sim$

3) $H^1 K_y$ к/м $\underset{\dots}{\Longrightarrow}$ K_y к/п
