FROM GEOMETRY TO ARITHMETICITY OF COMPACT HYPERBOLIC COXETER POLYTOPES

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Abstract. In this paper we prove that any compact hyperbolic Coxeter polyhedron in the three-dimensional Lobachevsky space contains an edge such that the distance between its framing faces is small enough. Also, we provide some applications of this theorem.

Keywords: compact Coxeter polyhedra, stably reflective hyperbolic lattices, arithmetic hyperbolic reflection groups, fundamental polyhedra.

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§ 1. Introduction

Let $\mathbb{X}^n$ be one of the three spaces of constant curvature, that is, either the Euclidean space $\mathbb{E}^n$, or the n-dimensional sphere $\mathbb{S}^n$, or the n-dimensional (hyperbolic) Lobachevsky space $\mathbb{H}^n$.

Consider a convex polytope P in the space $\mathbb{X}^n$. If we act on P by the group Γ generated by reflections in the hyperplanes of its faces it can occur that the images of this polyhedron corresponding to different elements of Γ will cover the entire space $\mathbb{X}^n$ and will not overlap with each other. In this case we say that Γ is a *discrete reflection group*, and the polytope P is *the fundamental polyhedron* for Γ. If the polytope P is *bounded* (or, equivalently, *compact*), then the group Γ is called *a cocompact reflection group*, and if the polytope P has a *finite volume*, then the group Γ is called *cofinite* or a discrete group of *finite covolume*.

Which properties characterize such polyhedra P? For example, any two hyperplanes H_i and H_j bounding P either do not intersect or form a dihedral angle equal to π/n_{ij}, where $n_{ij} \in \mathbb{Z}$, $n_{ij} \geq 2$.

Such polyhedra are called *Coxeter polyhedra*, since the discrete reflection groups of finite covolume (hence their finite volume fundamental polyhedra) for $\mathbb{X}^n = \mathbb{E}^n, \mathbb{S}^n$ were determined and found by H. S. M. Coxeter in 1933 (see [11]).

In 1967 (see [31]), E. B. Vinberg developed his theory of discrete groups generated by reflections in the Lobachevsky spaces. He proposed new methods for studying hyperbolic reflection groups, in particular, a description of such groups in the form of the so-called Coxeter diagrams. He formulated and proved the arithmeticity criterion for hyperbolic reflection groups and constructed a number of various examples.

Thus, a *hyperbolic reflection group* Γ is a discrete subgroup of the isometry group $\mathrm{Isom}(\mathbb{H}^n)$ of hyperbolic Lobachevsky n-space which is generated by reflections in the faces of a *hyperbolic Coxeter polyhedron.*

In his earlier works (see Lemma 3.2.1 in [20] and the proof of Theorem 4.1.1 in [22]) V. V. Nikulin proved the following assertion[1].

Theorem 1.1 (V. V. Nikulin, 1980)**.** *Let P be an acute-angled convex polyhedron of finite volume in $\mathbb{H}^n$. Then there exists a face F such that*

$$\cosh \rho(F_1, F_2) \leq 7,$$

for any faces F_1 and F_2 of P adjacent to F, where $\rho(\,\cdot\,,\cdot\,)$ is the metric in the Lobachevsky space[2].

In the proof of this assertion, the face F was chosen as the outermost face from some fixed point O inside the polyhedron P. Notice that this theorem enables us to bound at once the absolute value of the inner product of outer normals to faces adjacent to the face F. Indeed, if F_1 and F_2 intersect or are parallel, this value is equal to the cosine of the dihedral angle between these faces, and if these faces diverge, then it equals the hyperbolic cosine of the distance between them.

Let P be a compact acute-angled polyhedron in $\mathbb{H}^n$, E an edge (of dimension 1) of P, $F_1, \ldots, F_{n-1}$ the facets of P containing E, and u_n and u_{n+1} the unit outer normals to the facets F_n and F_{n+1} containing the vertices of E but not E itself.

Definition 1.1. *The faces F_n and F_{n+1} are called the framing faces of the edge E, and the number $|(u_n, u_{n+1})|$ is its width.*

For $n = 3$, we associate with the edge E the set $\overline{\alpha} = (\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24})$, where α_{ij} is the angle between the faces F_i and F_j.

The main results of this paper are the following two assertions.

Main Theorem 1. *Each compact Coxeter polyhedron in the hyperbolic Lobachevsky space $\mathbb{H}^3$ contains an edge of width less than $t_{\overline{\alpha}}$, where $t_{\overline{\alpha}}$ is the number depending on the set $\overline{\alpha}$ and*

$$\max_{\overline{\alpha}}\{t_{\overline{\alpha}}\} = t_{(\pi/5,\pi/2,\pi/3,\pi/3,\pi/2)} = 5.75.$$

Main Theorem 2. *Each right-angled compact Coxeter polyhedron in the Lobachevsky space $\mathbb{H}^n$ of dimension $n \geq 3$ contains an edge, such that an inner product between the unit outer normals u and v to the framing faces of this edge satisfies the inequality $|(u, v)| < 2$.*

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[1] We present this assertion in a form convenient for us, although it was not formulated in this way anywhere.

[2] In Nikulin's papers, the squares of the lengths of the normals to faces are equal to (-2), therefore, in his works there is a bound $(\delta, \delta') \leq 14$.

§ 2. Proof of Main Theorem 1

2.1. Weak version of Main Theorem 1. We have the following corollary of Theorem **??**.

Proposition 2.1. *Each compact acute-angled polyhedron $P \subset \mathbb{H}^3$ contains an edge of width not greater than* 7*.*

Proof. Following V. V. Nikulin (see [22]), we consider an interior point O in P. Let E be *the outermost*[3] *edge from it*, and let F be a face containing this edge. Let E_1, E_2 be disjoint[4] edges of this face coming out from different vertices of E.

Let O' be the projection of O onto the face F. Note that O' is an interior point of this face, since otherwise the point O would lie outside of some dihedral angle adjacent to F (because the polyhedron P is acute-angled). Further, since E is the outermost edge of the polyhedron for O, then it follows from this and the three perpendiculars theorem that the distance between the point O' and the edge E is not less than the distance between this point and any other edge of the face F. This means that the edge E is the outermost edge to the point O' inside the polygon F and we can use Theorem **??**.

Further, let F_3 and F_4 be the faces (with unit outer normals u_3 and u_4, respectively) of the polyhedron P framing the outermost edge E and containing the edges E_1 and E_2, respectively. Clearly, the distance between the faces is not greater than the distance between their edges. Therefore,

$$-(u_3, u_4) = \cosh \rho(F_3, F_4) \leq \cosh \rho(E_1, E_2) \leq 7.$$

The proposition is proved. ∎

2.2. Bounds for the length of the outermost edge for a compact acute-angled polyhedron in $\mathbb{H}^3$. In this subsection P denotes a compact acute-angled polyhedron in the three-dimensional Lobachevsky space $\mathbb{H}^3$. Following V. V. Nikulin (see [22]), we consider an interior point O in P. Let E be *the outermost*[5] *edge from it.*

We denote the vertices of the edge E by V_1 and V_2. The dihedral angles between the faces F_i and F_j will be denoted by α_{ij}.

Let E_1 and E_3 be the edges of the polyhedron P outgoing from the vertex V_1 and let E_2 and E_4 be the edges outgoing from V_2 such that the edges E_1 and E_2 lie in the face F_1. The length of the edge E is denoted by a, and the plane angles between the edges E_j and E are denoted by α_j (see Figure 1).

Denote by $V_1 I$, $V_2 I$, $V_1 J$, $V_2 J$ the bisector of angles α_1, α_2, α_3, α_4, respectively. Let h_I and h_J be the distances from the points I and J to the edge E.

Theorem 2.1 (Bogachev, 2019, [8])**.** *If $h_J \leqslant h_I$, then the length of the outermost edge satisfies the inequality*

$$a < \operatorname{arcsinh}\left(\frac{\cos(\alpha_{12}/2)}{\tan(\alpha_3/2)}\right) + \operatorname{arcsinh}\left(\frac{\cos(\alpha_{12}/2)}{\tan(\alpha_4/2)}\right).$$

Proof. Denote by O_1 and O_2 the orthogonal projections of the point O onto the faces F_1 and F_2, respectively. By the theorem of three perpendiculars, both points fall under the projection onto this edge E on the same point A, which is the projection of the point O onto this edge. Due to the fact that the polyhedron P is acute-angled, the point A is an inner point of the edge E.

[3]In an acute-angled polyhedron, the distance from the interior point to the face (of any dimension) is equal to the distance to the plane of this face.

[4]Note that we consider the case where the framing faces are divergent, since otherwise the absolute value of the inner product does not exceed one.

[5]In an acute-angled polyhedron, the distance from the interior point to the face (of any dimension) is equal to the distance to the plane of this face.

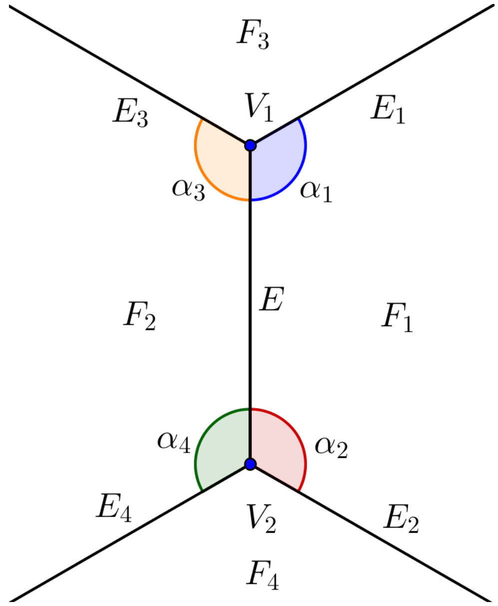

FIGURE 1. The outermost edge

Thus, we get a flat quadrilateral AO_1OO_2, in which $\angle A = \alpha_{12}$ (the dihedral angle between the faces F_1 and F_2), $\angle O_1 = \angle O_2 = \pi/2$, $AO_1 = a_1$, $AO_2 = a_2$ (see Figure 2).

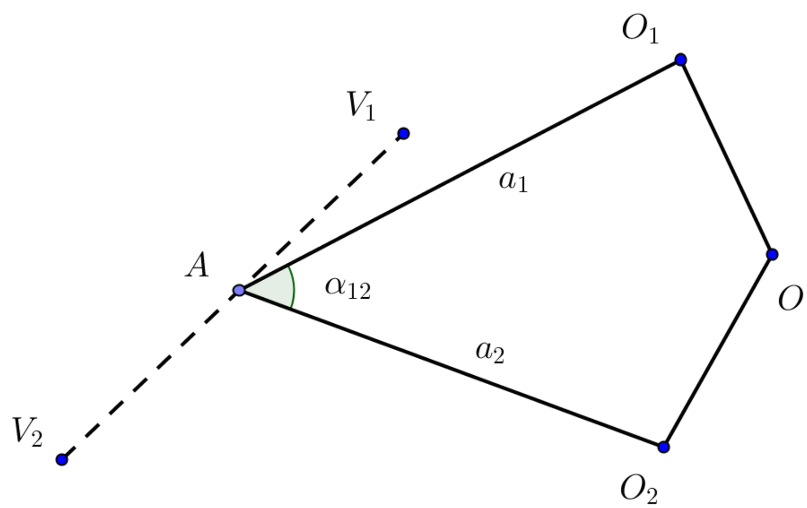

FIGURE 2. A quadrilateral AO_1OO_2

In the limiting case where the point O is a point at infinity, the dihedral angle α_{12} is composed of the so-called *angles of parallelism* $\Pi(a_1)$ and $\Pi(a_2)$. In our case $O \in \mathbb{H}^3$, therefore,

$$\alpha_{12} < \Pi(a_1) + \Pi(a_2) = 2\arctan(e^{-a_1}) + 2\arctan(e^{-a_2}).$$

Since the edge E is the outermost edge for the point O, we have

$$\rho(O_1, E) \leq \rho(O_1, E_1), \rho(O_1, E_2), \qquad \rho(O_2, E) \leq \rho(O_2, E_3), \rho(O_2, E_4). \tag{1}$$

Then it is clear that $h_J \leq h_I \leq a_1$, $h_J \leq a_2$, since inequalities (1) imply that the points O_1 and O_2 lie inside flat angles vertical to the angles V_1IV_2 and V_1JV_2, respectively (the scan of faces around the edge E is represented in Figure 3).

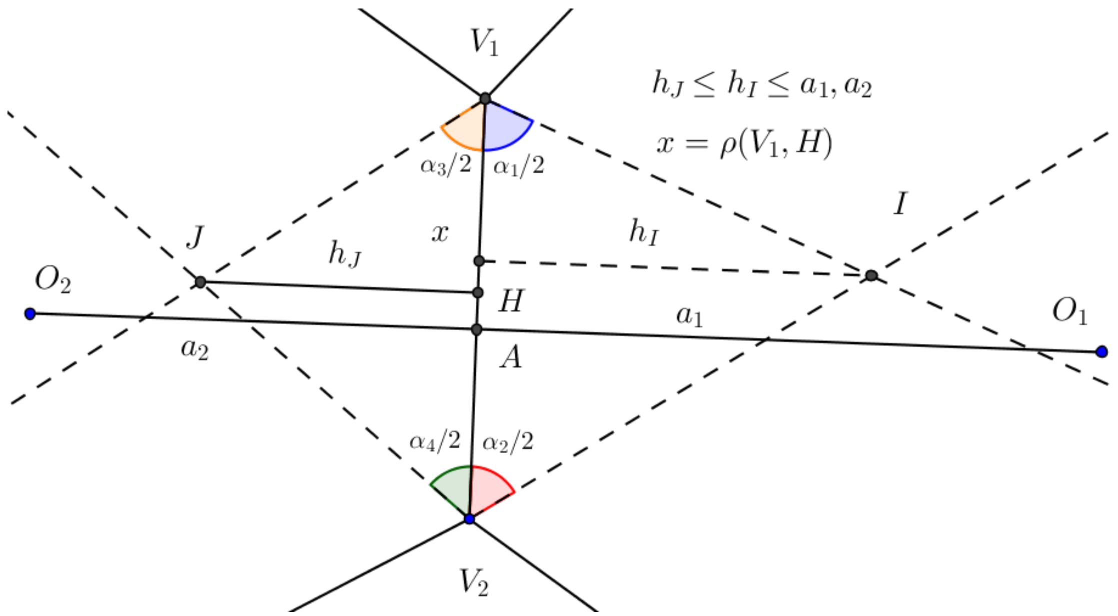

FIGURE 3. The scan

We have $\Pi(a_1), \Pi(a_2) \leq \Pi(h_J)$. It follows that $\arctan(e^{-h_J}) > \alpha_{12}/4$. Thus,

$$h_J < \ln\left(\cot\left(\frac{\alpha_{12}}{4}\right)\right).$$

We introduce the notation

$$A_0 := \tanh\big(\ln(\cot(\alpha_{12}/4))\big) = \cos\left(\frac{\alpha_{12}}{2}\right).$$

Then $\tanh h_J < A_0$. Let H be the projection of the point J onto the edge E and let $x = \rho(H, V_1)$.

From the right triangles V_1JH and V_2JH we find

$$\tanh h_J = \tan\left(\frac{\alpha_4}{2}\right)\sinh(a-x) = \tan\left(\frac{\alpha_3}{2}\right)\sinh x, \tag{2}$$

which implies that

$$\sinh x = \frac{\tanh h_J}{\tan(\alpha_3/2)} < \frac{A_0}{\tan(\alpha_3/2)}, \qquad \sinh(a-x) < \frac{A_0}{\tan(\alpha_4/2)}.$$

Hence,

$$a = x + (a-x) < \operatorname{arcsinh}\left(\frac{A_0}{\tan(\alpha_3/2)}\right) + \operatorname{arcsinh}\left(\frac{A_0}{\tan(\alpha_4/2)}\right),$$

which completes the proof. ∎

The following statement will be useful sometimes, however it is not efficient in general.

Corollary 1. *The length of the outermost edge satisfies the inequality*

$$a < \max\left\{\operatorname{arcsinh}\left(\frac{\cos(\alpha_{12}/2)}{\tan(\alpha_1/2)}\right) + \operatorname{arcsinh}\left(\frac{\cos(\alpha_{12}/2)}{\tan(\alpha_2/2)}\right),\right.$$
$$\left.\operatorname{arcsinh}\left(\frac{\cos(\alpha_{12}/2)}{\tan(\alpha_3/2)}\right) + \operatorname{arcsinh}\left(\frac{\cos(\alpha_{12}/2)}{\tan(\alpha_4/2)}\right).\right\}.$$

Proposition 2.2. *The following formulas hold:*

$$\tanh h_I = \frac{c_1 c_2 \sinh a}{\sqrt{c_1^2 + c_2^2 + 2c_1 c_2 \cosh a}}, \quad \tanh h_J = \frac{c_3 c_4 \sinh a}{\sqrt{c_3^2 + c_4^2 + 2c_3 c_4 \cosh a}},$$

where $c_j := \tan(\alpha_j/2)$.

Proof. Using (2) , we have that

$$c_3 \sinh x = c_4(\sinh a \cosh x - \sinh x \cosh a).$$

This implies that

$$\sinh x (c_3 + c_4 \cosh a) = c_4 \sinh a \cosh x,$$

and therefore we have

$$\tanh x = \frac{c_4 \sinh a}{c_3 + c_4 \cosh a}.$$

Then it remains to use (2) again:

$$\tanh h_J = c_3 \sinh x = c_3 \sinh\left(\operatorname{arctanh}\left(\frac{c_4 \sinh a}{c_3 + c_4 \cosh a}\right)\right) = \frac{c_3 c_4 \sinh a}{\sqrt{c_3^2 + c_4^2 + 2c_3 c_4 \cosh a}}.$$

Similarly one can prove the formula for h_I. ∎

Proposition 2.3. *In the above notation,* $h_J \leq h_I$ *if and only if*

$$2c_1 c_2 c_3 c_4 (c_1 c_2 - c_3 c_4) \cosh a \geq c_3^2 c_4^2 (c_1^2 + c_2^2) - c_1^2 c_2^2 (c_3^2 + c_4^2).$$

Proof. Since $h_I, h_J \geq 0$, then $h_I \geq h_J$ if and only if

$$h_I^2 \geq h_J^2 \Longleftrightarrow \frac{c_1^2 c_2^2}{c_1^2 + c_2^2 + 2c_1 c_2 \cosh a} \geq \frac{c_3^2 c_4^2}{c_3^2 + c_4^2 + 2c_3 c_4 \cosh a}.$$

∎

Lemma 2.1. *The following relations are true:*
(i) $\alpha_{12} + \alpha_{23} + \alpha_{13} > \pi, \quad \alpha_{12} + \alpha_{24} + \alpha_{14} > \pi$;
(ii)

$$\cos \alpha_1 = \frac{\cos \alpha_{23} + \cos \alpha_{12} \cdot \cos \alpha_{13}}{\sin \alpha_{12} \cdot \sin \alpha_{13}}, \qquad \cos \alpha_2 = \frac{\cos \alpha_{24} + \cos \alpha_{12} \cdot \cos \alpha_{14}}{\sin \alpha_{12} \cdot \sin \alpha_{14}},$$
$$\cos \alpha_3 = \frac{\cos \alpha_{13} + \cos \alpha_{12} \cdot \cos \alpha_{23}}{\sin \alpha_{12} \cdot \sin \alpha_{23}}, \qquad \cos \alpha_4 = \frac{\cos \alpha_{14} + \cos \alpha_{12} \cdot \cos \alpha_{24}}{\sin \alpha_{12} \cdot \sin \alpha_{24}}.$$

Proof. To prove both parts of the lemma, we intersect each trihedral angle with the vertices V_1 and V_2 by spheres centered at these points. In the intersection we obtain spherical triangles the angles of which are the dihedral angles α_{ij}, and the lengths of their edges are the flat angles α_k. This implies at once the first assertion, and the second one follows from the dual cosine-theorem for these triangles (see, for example, [38, p. 71]). The lemma is proved. ∎

2.3. The proof of Main Theorem 1. Let a polyhedron P be a compact Coxeter polyhedron in the three-dimensional hyperbolic Lobachevsky space and let E be the outermost edge of the polyhedron P. Consider the set of unit outer normals (u_1, u_2, u_3, u_4) to the faces F_1, F_2, F_3, F_4. Note that this vector system is linearly independent. Its Gram matrix is

$$G(u_1, u_2, u_3, u_4) = \begin{pmatrix} 1 & -\cos \alpha_{12} & -\cos \alpha_{13} & -\cos \alpha_{14} \\ -\cos \alpha_{12} & 1 & -\cos \alpha_{23} & -\cos \alpha_{24} \\ -\cos \alpha_{13} & -\cos \alpha_{23} & 1 & -T \\ -\cos \alpha_{14} & -\cos \alpha_{24} & -T & 1 \end{pmatrix},$$

where $T = |(u_3, u_4)| = \cosh \rho(F_3, F_4)$ in the case where the faces F_3 and F_4 diverge. Recall that otherwise $T \leq 1$, and we do not need to consider this case separately.

Let $(u_1^*, u_2^*, u_3^*, u_4^*)$ be the basis dual to the basis (u_1, u_2, u_3, u_4). Then u_3^* and u_4^* determine the vertices V_2 and V_1 in the Lobachevsky space. Indeed, the vector v_1 corresponding to the point $V_1 \in \mathbb{H}^3$ is uniquely determined (up to scaling) by the conditions $(v_1, u_1) = (v_1, u_2) = (v_1, u_3) = 0$. Note that the vector u_4^* satisfies the same conditions. Therefore, the vectors v_1 and u_4^* are proportional, hence,

$$\cosh a = \cosh \rho(V_1, V_2) = -(v_1, v_2) = -\frac{(u_3^*, u_4^*)}{\sqrt{(u_3^*, u_3^*)(u_4^*, u_4^*)}}.$$

It is known that $G(u_1^*, u_2^*, u_3^*, u_4^*) = G(u_1, u_2, u_3, u_4)^{-1}$, whence it follows that $\cosh a$ can be expressed in terms of the algebraic complements G_{ij} of the elements of the matrix $G = G(u_1, u_2, u_3, u_4)$:

$$\cosh a = -\frac{(u_3^*, u_4^*)}{\sqrt{(u_3^*, u_3^*)(u_4^*, u_4^*)}} = \frac{G_{34}}{\sqrt{G_{33}G_{44}}}.$$

Denote the right-hand side of the inequality from Theorem 2.1 by $F(\overline{\alpha})$, and the terms in the right-hand side of the Corollary 1 by $F_{1,2}(\overline{\alpha})$ and $F_{3,4}(\overline{\alpha})$, where

$$\overline{\alpha} = (\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{14}, \alpha_{24}).$$

By the Corollary 1 we have that $\cosh a < \max\{\cosh F_{1,2}(\overline{\alpha}), \cosh F_{3,4}(\overline{\alpha})\}$. If $h_J \leq h_I$, then the Theorem 2.1 implies that

$$\cosh a < \cosh F(\overline{\alpha}) = \cosh F_{3,4}(\overline{\alpha}).$$

It follows that

$$\frac{G_{34}}{\sqrt{G_{33}G_{44}}} < \cosh F(\overline{\alpha}). \tag{3}$$

For every $\overline{\alpha}$, in this way we obtain a linear inequality with respect to the number T.

We have

$$G_{34} = T(1 - \cos^2 \alpha_{12}) + (\cos \alpha_{12} \cos \alpha_{13} \cos \alpha_{24} + \cos \alpha_{12} \cos \alpha_{14} \cos \alpha_{23} + \\ + \cos \alpha_{13} \cos \alpha_{14} + \cos \alpha_{23} \cos \alpha_{24}) = T \cdot \sin^2 \alpha_{12} + g(\overline{\alpha}) < \cosh F(\overline{\alpha}) \cdot \sqrt{G_{33}G_{44}}, \tag{4}$$

where

$$g(\overline{\alpha}) := \cos \alpha_{12} \cos \alpha_{13} \cos \alpha_{24} + \cos \alpha_{12} \cos \alpha_{14} \cos \alpha_{23} + \cos \alpha_{13} \cos \alpha_{14} + \cos \alpha_{23} \cos \alpha_{24}.$$

Hence, the following statement holds.

Theorem 2.2. *Each compact Coxeter polyhedron in the hyperbolic Lobachevsky space $\mathbb{H}^3$ contains an edge of width less than*

$$W(\overline{\alpha}) := \frac{\cosh F(\overline{\alpha}) \cdot \sqrt{G_{33}G_{44}} - g(\overline{\alpha})}{\sin^2 \alpha_{12}}.$$

Thus, it remains to prove that

$$\max_{\overline{\alpha}} W(\overline{\alpha}) = W(\pi/5, \pi/2, \pi/3, \pi/3, \pi/2) = 4.89.$$

Taking into account Lemma 2.1, (i), we can see that only one or two angles α_{ij} can be equal to π/k, where $k \geq 6$.

Proposition 2.4. *Suppose $\alpha_{12} = \frac{\pi}{k}$, where $k \geq 6$. Then $|(u_3, u_4)| < 2\sqrt{2}$.*

Proof. Due to Lemma 2.1, (i), we can see that in this case all other $\alpha_{ij} = \pi/2$. Then

$$T = |(u_3, u_4)| < W\left(\frac{\pi}{k}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right).$$

In this case we have that $F(\overline{\alpha}) = F_{1,2}(\overline{\alpha}) = F_{3,4}(\overline{\alpha})$.

Let us now calculate:

$$g\left(\frac{\pi}{k}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) = 0, \tag{5}$$

$$\sqrt{G_{33}G_{44}} = \sin^2\left(\frac{\pi}{k}\right), \tag{6}$$

$$F\left(\frac{\pi}{k}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) = 2\operatorname{arcsinh}\left(\frac{\cos(\pi/2k)}{\tan(\pi/4)}\right). \tag{7}$$

We have

$$\cosh F\left(\frac{\pi}{k}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) = \cosh\left(2\operatorname{arcsinh}\left(\frac{\cos(\pi/2k)}{\tan(\pi/4)}\right)\right) = 2\cos\left(\frac{\pi}{2k}\right)\sqrt{1+\cos^2\left(\frac{\pi}{2k}\right)}.$$

It implies that

$$T < W(\overline{\alpha}) = \frac{2\cos\left(\frac{\pi}{2k}\right)\sqrt{1+\cos^2\left(\frac{\pi}{2k}\right)}\sin^2\left(\frac{\pi}{k}\right)}{\sin^2\left(\frac{\pi}{k}\right)} = 2\cos\left(\frac{\pi}{2k}\right)\sqrt{1+\cos^2\left(\frac{\pi}{2k}\right)} \leq 2\sqrt{2}.$$

∎

Now we can assume that $\alpha_{12} \geq \pi/5$. Then only one or two angles among remaining α_{ij} can be equal to π/k, where $k \geq 6$. Without loss of generality, we suppose that $\alpha_{13} = \frac{\pi}{k}$, where $k \geq 6$. Then $\alpha_{12} = \alpha_{23} = \pi/2$.

Proposition 2.5. *Suppose $\alpha_{13} = \frac{\pi}{k}$, where $k \geq 6$. Then $|(u_3, u_4)| < 4.86$.*

Proof. As it was said before, $\alpha_{12} = \alpha_{23} = \pi/2$, hence $\overline{\alpha} = \left(\frac{\pi}{2}, \frac{\pi}{k}, \alpha_{14}, \frac{\pi}{2}, \alpha_{24}\right)$.
In this case we use the Corollary 1: $F(\overline{\alpha}) = \max\{F_{1,2}(\overline{\alpha}), F_{3,4}(\overline{\alpha})\}$.
Let us now calculate:

$$g\left(\frac{\pi}{2}, \frac{\pi}{k}, \alpha_{14}, \frac{\pi}{2}, \alpha_{24}\right) = \cos\left(\frac{\pi}{k}\right)\cos\alpha_{14}, \tag{8}$$

$$\sqrt{G_{33}G_{44}} = \sin\left(\frac{\pi}{k}\right)\sqrt{1-\cos^2\alpha_{14}-\cos^2\alpha_{24}}, \tag{9}$$

$$F_{1,2}\left(\frac{\pi}{2}, \frac{\pi}{k}, \alpha_{14}, \frac{\pi}{2}, \alpha_{24}\right) = \operatorname{arcsinh}\left(\frac{\cos\left(\frac{\pi}{4}\right)}{\tan\left(\frac{\pi}{4}\right)}\right) + \operatorname{arcsinh}\left(\frac{\cos\left(\frac{\pi}{4}\right)}{\tan\left(\frac{\alpha_2}{2}\right)}\right), \tag{10}$$

$$F_{3,4}\left(\frac{\pi}{2}, \frac{\pi}{k}, \alpha_{14}, \frac{\pi}{2}, \alpha_{24}\right) = \operatorname{arcsinh}\left(\frac{\cos\left(\frac{\pi}{4}\right)}{\tan\left(\frac{\pi}{2k}\right)}\right) + \operatorname{arcsinh}\left(\frac{\cos\left(\frac{\pi}{4}\right)}{\tan\left(\frac{\alpha_4}{2}\right)}\right). \tag{11}$$

Then

$$\cosh F_{1,2}(\overline{\alpha}) = \cosh\left(\operatorname{arcsinh}(1/\sqrt{2}) + \operatorname{arcsinh}(\cot(\alpha_2/2)/\sqrt{2})\right) =$$
$$= \frac{1}{2}\cot\left(\frac{\alpha_2}{2}\right) + \sqrt{1+\frac{\cot^2\left(\frac{\alpha_2}{2}\right)}{2}}\sqrt{\frac{3}{2}} = \frac{\cos\left(\frac{\alpha_2}{2}\right)+\sqrt{3}\sqrt{1+\sin^2\left(\frac{\alpha_2}{2}\right)}}{2\sin\left(\frac{\alpha_2}{2}\right))} \tag{12}$$

and

$$\cosh F_{3,4}(\overline{\alpha}) \leq \frac{1}{2}\cot\left(\frac{\alpha_4}{2}\right)\cot\left(\frac{\pi}{2k}\right) + \sqrt{1+\frac{\cot^2\left(\frac{\alpha_4}{2}\right)}{2}}\sqrt{1+\frac{\cot^2\left(\frac{\pi}{2k}\right)}{2}} =$$
$$= \frac{\cos\left(\frac{\alpha_4}{2}\right)\cos\left(\frac{\pi}{2k}\right)}{2\sin\left(\frac{\alpha_4}{2}\right)\sin\left(\frac{\pi}{2k}\right)} + \frac{1}{2\sin\left(\frac{\pi}{2k}\right)\sin\left(\frac{\alpha_4}{2}\right)}\sqrt{1+\sin^2\left(\frac{\alpha_4}{2}\right)}\sqrt{1+\sin^2\left(\frac{\pi}{2k}\right)}. \tag{13}$$

We have

$$\begin{aligned}
W_{1,2}(\overline{\alpha}) &= \cosh F_{1,2}(\overline{\alpha})\sqrt{G_{33}G_{44}} - g(\overline{\alpha}) \leq \\
&\leq \frac{\sqrt{\sin^2\alpha_{14} - \cos^2\alpha_{24}}}{2\sin\left(\frac{\alpha_2}{2}\right)} \cdot \sin\left(\frac{\pi}{k}\right)\left(\cos\left(\frac{\alpha_2}{2}\right) + \sqrt{3}\sqrt{1+\sin^2\left(\frac{\alpha_2}{2}\right)}\right) = \\
&= \frac{\sqrt{\sin^2\alpha_{14} - \cos^2\alpha_{24}}}{\sqrt{2}\sqrt{2\sin^2\left(\frac{\alpha_2}{2}\right)}} \cdot \sin\left(\frac{\pi}{k}\right)\left(\cos\left(\frac{\alpha_2}{2}\right) + \sqrt{3}\sqrt{1+\sin^2\left(\frac{\alpha_2}{2}\right)}\right) \leq \\
&\leq \frac{\sqrt{\sin^2\alpha_{14} - \cos^2\alpha_{24}}}{\sqrt{2}\sqrt{1-\cos\alpha_2}} \cdot \sin\left(\frac{\pi}{12}\right)\left(1 + \sqrt{3}\sqrt{1+\sin^2\left(\frac{\pi}{4}\right)}\right) \leq \frac{0.808\sqrt{\sin^2\alpha_{14} - \cos^2\alpha_{24}}}{\sqrt{2}\sqrt{1-\cos\alpha_2}} = \\
&= \frac{0.808\sin\alpha_{14}\sqrt{\sin^2\alpha_{14} - \cos^2\alpha_{24}}}{\sqrt{2}\sqrt{\sin\alpha_{14} - \cos\alpha_{24}}} = 0.808\sin\alpha_{14}\frac{\sqrt{\sin\alpha_{14} + \cos\alpha_{24}}}{\sqrt{2}} \leq 0.808 < 1
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
W_{3,4}(\overline{\alpha}) &= \cosh F_{3,4}(\overline{\alpha})\sqrt{G_{33}G_{44}} - g(\overline{\alpha}) = \frac{\cos\left(\frac{\alpha_4}{2}\right)\cos\left(\frac{\pi}{2k}\right)}{2\sin\left(\frac{\alpha_4}{2}\right)\sin\left(\frac{\pi}{2k}\right)}\sin\left(\frac{\pi}{k}\right)\sqrt{1-\cos^2\alpha_{14}-\cos^2\alpha_{24}} + \\
&+ \frac{\sqrt{1+\sin^2\left(\frac{\alpha_4}{2}\right)}\sqrt{1+\sin^2\left(\frac{\pi}{2k}\right)}}{2\sin\left(\frac{\pi}{2k}\right)\sin\left(\frac{\alpha_4}{2}\right)}\sin\left(\frac{\pi}{k}\right)\sqrt{1-\cos^2\alpha_{14}-\cos^2\alpha_{24}} - \cos\left(\frac{\pi}{k}\right)\cos\alpha_{14} = \\
&= \frac{\sqrt{\sin^2\alpha_{24}-\cos^2\alpha_{14}}}{\sin\left(\frac{\alpha_4}{2}\right)} \cdot \left(\cos\left(\frac{\alpha_4}{2}\right)\cos^2\left(\frac{\pi}{2k}\right) + \sqrt{1+\sin^2\left(\frac{\alpha_4}{2}\right)}\sqrt{1+\sin^2\left(\frac{\pi}{2k}\right)}\cos\left(\frac{\pi}{2k}\right)\right) = \\
&= \frac{\sqrt{2}\sqrt{\sin^2\alpha_{24}-\cos^2\alpha_{14}}}{\sqrt{2\sin^2\left(\frac{\alpha_4}{2}\right)}} \cdot \left(\cos\left(\frac{\alpha_4}{2}\right)\cos^2\left(\frac{\pi}{2k}\right) + \sqrt{1+\sin^2\left(\frac{\alpha_4}{2}\right)}\sqrt{1+\sin^2\left(\frac{\pi}{2k}\right)}\cos\left(\frac{\pi}{2k}\right)\right) \leq \\
&\leq \frac{\sqrt{2}\sqrt{\sin^2\alpha_{24}-\cos^2\alpha_{14}}}{\sqrt{2\sin^2\left(\frac{\alpha_4}{2}\right)}} \cdot \left(1 + \sqrt{\frac{3}{2}}\sqrt{\frac{6-\sqrt{3}}{4}}\right) \leq \frac{2.43\sqrt{2}\sqrt{\sin^2\alpha_{24}-\cos^2\alpha_{14}}}{\sqrt{1-\cos\alpha_4}} = \\
&= \frac{2.43\sqrt{2}\sin\alpha_{24}\sqrt{\sin^2\alpha_{24}-\cos^2\alpha_{14}}}{\sqrt{\sin\alpha_{24}-\cos\alpha_{14}}} = 2.43\sqrt{2}\sin\alpha_{24}\sqrt{\sin\alpha_{24}+\cos\alpha_{14}} \leq 2.43\cdot 2 = 4.86
\end{aligned} \tag{15}$$

Thus, by the Corollary 1 we have

$$W(\overline{\alpha}) \leq \max\{W_{1,2}(\overline{\alpha}), W_{3,4}(\overline{\alpha})\} = 4.86.$$

∎

If no α_j is equal to π/k for $k \geq 6$, then these angles can equal only $\pi/2$, $\pi/3$, $\pi/4$, and $\pi/5$. It is easy to verify that, taking into account Lemma 2.1, (i), there are exactly 65 different (up to numbering) sets of angles $\overline{\alpha}$. For each such set $\overline{\alpha}$ inequality (3) gives some bound $T < t_{\overline{\alpha}}$.

For solving the 65 linear inequalities a program was compiled in the computer algebra system Sage[6], its code is available on the Internet[7].

[6]The Sage Developers, *the Sage Mathematics Software System (Version 7.6)*, SageMath, http://www.sagemath.org, 2017

[7]N. Bogachev, *Method of the outermost edge/bounds*, https://github.com/nvbogachev/OuterMostEdge/blob/master/bounds.sage/, 2017

$$\{\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 4.47435394669, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi\right) : 2.5489020017,$$
$$\left(\frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 5.74850431686, \left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 2.87989690675,$$
$$\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 3.2539035981, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 2.22302677317,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 2.10188770571, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 2.75,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi\right) : 1.97474489796, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi\right) : 2.52791559843,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 2.10188770571, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 1.81509375961,$$
$$\left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.56693021181, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi\right) : 1.98833391178,$$
$$\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 4.74221754159, \left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 2.52791559843,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi\right) : 1.70090228705, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi\right) : 1.95489616647,$$
$$\left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi\right) : 1.99675008972, \left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 4.34436709886,$$
$$\left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi\right) : 2.06066021599, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 2.84239033426,$$
$$\left(\frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 3.16384601713, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi\right) : 1.93085548347,$$
$$\left(\frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 3.81026927342, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 2.22302677317,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.4270508654, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi\right) : 2.48710392108,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.4270508654, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 2.27647264254,$$
$$\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.5, \left(\frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 4.74221754159,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 1.73457880864, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 2.51627091247,$$
$$\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 3.28829441428, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 2.41369537584,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi\right) : 1.44879249939, \left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 3.57915779781,$$

$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi\right) : 2.5489020017, \left(\frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 4.06538413177,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 2.56693021181, \left(\frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 4.89809388899,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 2.62864418595, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 2.87989690675,$$
$$\left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 2.27647264254, \left(\frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 3.81026927342,$$
$$\left(\frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 3.57915779781, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi\right) : 1.93085548347,$$
$$\left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi\right) : 2.51627091247, \left(\frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 5.74850431686,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 2.48710392108, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.14913653544,$$
$$\left(\frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 3.16384601713, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi\right) : 2.5701250409,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 2.56693021181, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi\right) : 1.74633372518,$$
$$\left(\frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 4.9748737038, \left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi\right) : 1.11670745356,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 2.58378331506, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 2.56771335352,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 1.85873212921, \left(\frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 4.34436709886,$$
$$\left(\frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 4.13728430574, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi\right) : 1.73457880864,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi\right) : 1.46379945432, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi\right) : 2.48710392108,$$
$$\left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 2.5489020017, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 2.48710392108,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi\right) : 2.33670832644, \left(\frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 3.61636139176,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 2.91421362489, \left(\frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 4.89809388899,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 2.5489020017, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 1.54507828603,$$
$$\left(\frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 4.13728430574, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 2.51627091247,$$
$$\left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 2.96352558412, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi\right) : 1.98833391178,$$

$$\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 4.61615637433, \left(\frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 4.34436709886,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 2.84239033426, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi\right) : 2.58378331506,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi\right) : 2.51627091247, \left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 3.57915779781,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 2.56771335352, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 2.49627278453,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi\right) : 1.99675008972, \left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 3.16384601713,$$
$$\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 3.71622096906, \left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 2.9414272637,$$
$$\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 4.74221754159, \left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 4.34436709886,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.62864418595, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 2.4270508654,$$
$$\left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 2.58378331506, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi\right) : 2.28023909986,$$
$$\left(\frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 4.47435394669, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.33670832644,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 2.5701250409, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi\right) : 1.46379945432,$$
$$\left(\frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 4.86617838887, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi\right) : 2.06066021599,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi\right) : 1.73457880864, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi\right) : 1.93085548347,$$
$$\left(\frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 3.71622096906, \left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 1.70090228705,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 2.22302677317, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi\right) : 1.86602534022,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi\right) : 2.14913653544, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 2.52791559843,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi\right) : 1.11670745356, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 2.9414272637,$$
$$\left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi\right) : 1.44879249939, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 2.49627278453,$$
$$\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 3.61636139176, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 2.06066021599,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi\right) : 2.56771335352, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 2.06066021599,$$
$$\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 3.71622096906, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.28023909986,$$

$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 2.9414272637, \left(\frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 4.20894257029,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi\right) : 1.98833391178, \left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 3.50079052417,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi\right) : 1.93085548347, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 1.70090228705,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 2.41369537584, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi\right) : 2.00987348014,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 2.4270508654, \left(\frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 3.81026927342,$$
$$\left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 2.91421362489, \left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi\right) : 2.56771335352,$$
$$\left(\frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 3.61636139176, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 2.07990316453,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi\right) : 1.70090228705, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi\right) : 1.97474489796,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi\right) : 2.41369537584, \left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 1.74633372518,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 2.96352558412, \left(\frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 4.20894257029,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi\right) : 2.52791559843, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi\right) : 2.33670832644,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi\right) : 1.95489616647, \left(\frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 3.81026927342,$$
$$\left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 2.22302677317, \left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 4.47435394669,$$
$$\left(\frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 3.28829441428, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 2.9414272637,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi\right) : 2.00987348014, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 2.75,$$
$$\left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 2.07990316453, \left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 3.28829441428,$$
$$\left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi\right) : 1.98833391178, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi\right) : 2.49627278453,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.28023909986, \left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi\right) : 1.74633372518,$$
$$\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 4.06538413177, \left(\frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.8090169067,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 1.85873212921, \left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 3.61636139176,$$

$$\left(\frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 3.2539035981, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi\right) : 2.5701250409,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi\right) : 1.46379945432, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 2.00987348014,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 2.00987348014, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 2.87989690675,$$
$$\left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi\right) : 1.95489616647, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi\right) : 1.95489616647,$$
$$\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 3.50079052417, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.0,$$
$$\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 3.16384601713, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 1.81509375961,$$
$$\left(\frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.70710696339, \left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.62864418595,$$
$$\left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 1.73457880864, \left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 1.85873212921,$$
$$\left(\frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 4.20894257029, \left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 4.86617838887,$$
$$\left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.33670832644, \left(\frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 3.50079052417,$$
$$\left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 2.5701250409, \left(\frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 3.07193143557,$$
$$\left(\frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 2.7735613141, \left(\frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 4.74221754159,$$
$$\left(\frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 4.47435394669, \left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 3.28829441428,$$
$$\left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi\right) : 2.58378331506, \left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi\right) : 1.46379945432,$$
$$\left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 1.54507828603, \left(\frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 4.20894257029,$$
$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi\right) : 2.49627278453, \left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 3.71622096906,$$
$$\left(\frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 4.61615637433, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 1.74633372518,$$
$$\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 3.57915779781, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 2.62864418595,$$

$$\left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 2.84239033426, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi\right) : 2.28023909986,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi\right) : 2.84239033426, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi\right) : 2.41369537584,$$
$$\left(\frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi\right) : 3.50079052417, \left(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 2.87989690675,$$
$$\left(\frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi\right) : 2.07990316453, \left(\frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi\right) : 1.85873212921,$$
$$\left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 2.7735613141, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi\right) : 2.07990316453,$$
$$\left(\frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 4.9748737038, \left(\frac{1}{3}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi\right) : 3.07193143557,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{5}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi\right) : 2.27647264254, \left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{5}\pi, \frac{1}{2}\pi\right) : 2.27647264254,$$
$$\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi\right) : 2.56693021181\}$$

The maximal number in this list equals 5.74850431686. Thus, combining Propositions 2.4, 2.5, and this list, we obtain Main Theorem 1.

§ 3. Proof of Main Theorem 2

3.1. Auxiliary results.

Theorem 3.1. *The distance from the point $e_0 \in \mathbb{H}^n$, where $(e_0, e_0) = -1$, to the plane*
$$H_{e_1,\ldots,e_k} := \{x \in \mathbb{H}^n \colon x \in \langle e_1, \ldots, e_k\rangle^\perp\}$$
can be calculated by the formula
$$\sinh^2 \rho(e_0, H_{e_1,\ldots,e_k}) = \sum_{i,j} \overline{g_{ij}} y_i y_j,$$
where $\overline{g_{ij}}$ are the elements of the inverse matrix $G^{-1} = G(e_1, \ldots, e_k)^{-1}$, and $y_j = -(e_0, e_j)$ for all $1 \le j \le k$.

Proof. Let f be the orthogonal projection e_0 to the plane $H_{e_1,\ldots,e_k}$. It is the intersection of the plane $H_{e_1,\ldots,e_k}$ with the straight line ℓ passing through the point e_0 and perpendicular to $H_{e_1,\ldots,e_k}$. Since ℓ and $H_{e_1,\ldots,e_k}$ are orthogonal, the intersection of their defining subspaces $\langle \ell\rangle$ and $\langle H_{e_1,\ldots,e_k}\rangle$ is a one-dimensional hyperbolic subspace $\langle f'\rangle$. Hence the sections of these subspaces by the subspace $\langle f'\rangle^\perp$ are orthogonal to each other.

It follows that
$$\langle \ell\rangle = \langle f'\rangle \oplus \langle h'\rangle,$$
where $h' \perp \langle H_{e_1,\ldots,e_k}\rangle$.

It remains to observe that the points $0, f, f'$ lie in the one-dimensional hyperbolic subspace $\langle f'\rangle$, hence $f = cf'$, where a positive constant number c can be found from the condition
$$(f, f) = c^2(f', f') = -1.$$

Then the distance from the point $e_0 \in \mathbb{H}^n$, where $(e_0, e_0) = -1$, to the plane $H_{e_1,\ldots,e_k}$ equals the distance from this point to its orthogonal projection, that is,
$$\cosh \rho(e_0, H_{e_1,\ldots,e_k}) = \cosh \rho(e_0, f) = -(e_0, f),$$

whence we find that

$$\cosh^2 \rho(e_0, H_{e_1,\ldots,e_k}) = c^2(e_0, f')^2 = -\frac{(e_0, f')^2}{(f', f')} = -(f', f').$$

Let us prove the following lemma.

Lemma 3.1. *The following equation holds:*

$$\cosh \rho(e_0, H_{e_1,\ldots,e_k}) = \sqrt{\frac{|\det G(e_0, e_1, \ldots, e_k)|}{|\det G(e_1, \ldots, e_k)|}}$$

Proof. Indeed,

$$\det G(e_0, e_1, \ldots, e_k) = \det G(f', e_1, \ldots, e_k) = (f', f') \det G(e_1, \ldots, e_k).$$

Taking into account that $(f', f') < 0$, we have

$$\frac{|\det G(e_0, e_1, \ldots, e_k)|}{|\det G(e_1, \ldots, e_k)|} = -(f', f') = \cosh^2 \rho(e_0, H_{e_1,\ldots,e_k}),$$

which completes the proof of the lemma. ∎

It remains to show that

$$-(f', f') = 1 + \overline{y}^T G^{-1} \overline{y},$$

where $G = G(e_1, \ldots, e_k)$, $\overline{y} = (y_1, \ldots, y_k)^T \in \mathbb{R}^k$. We observe that

$$f' = e_0 + \lambda_1 e_1 + \ldots + \lambda_k e_k,$$

and $(f', e_j) = 0$ for all $1 \leq j \leq k$. From these orthogonality conditions we obtain that the column $\overline{\lambda} = (\lambda_1, \ldots, \lambda_k)^T$ is a solution to the system of linear equations with the matrix G:

$$G\overline{\lambda} = \overline{y}.$$

Then $\overline{\lambda} = G^{-1}\overline{y}$. Therefore,

$$(f', f') = (e_0, e_0) - 2(\overline{\lambda}, \overline{y}) + \overline{\lambda}^T G^{-1} \overline{\lambda} = -1 - \overline{y}^T G^{-1} \overline{y},$$

whence we have that

$$\sinh^2 \rho(e_0, H_{e_1,\ldots,e_k}) = \cosh^2 \rho(e_0, H_{e_1,\ldots,e_k}) - 1 = -(f', f') - 1 = \overline{y}^T G^{-1} \overline{y},$$

as required. ∎

3.2. Proof of Main Theorem 2. Let P be a compact right-angled Coxeter polytope in $\mathbb{H}^n$, and let E be the outermost edge from some point e_0.

The Gram matrix has the form

$$G(e_0, e_1, e_2, \ldots e_{n+1}) = \begin{pmatrix} -1 & -y_1 & \ldots & -y_n & -y_{n+1} \\ -y_1 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -y_n & 0 & 0 & 1 & -f \\ -y_{n+1} & 0 & 0 & -f & 1 \end{pmatrix},$$

where

$$y_j = -(e_0, e_j) = -\sinh \rho(e_0, H_j).$$

Moreover, we can assume that $y_n \leq y_{n+1}$ and that the inequalities have the following form:

$$y_n \leq y_{n+1} \leq y_1, y_2, \ldots, y_{n-1}.$$

Then

$$\det G(e_0, e_1, \ldots, e_{n+1}) = (y_1^2 + \ldots + y_{n-1}^2 + 1) f^2 - 2 y_n y_{n+1} f - (y_1^2 + \ldots + y_{n+1}^2 + 1) = 0,$$

i.e.,

$$f = \frac{y_n y_{n+1} + \sqrt{y_n^2 y_{n+1}^2 + AB}}{A},$$

where

$$A := y_1^2 + \ldots + y_{n-1}^2 + 1, \quad B := y_1^2 + \ldots + y_{n+1}^2 + 1.$$

Therefore,

$$2y_n y_{n+1} \leq y_n^2 + y_{n+1}^2 < A \leq B, \quad B/A = 1 + y_n y_{n+1}/A < 1.5$$

and

$$f < 0.5 + \sqrt{0.25 + 1 + 1} = 2.$$

§ 4. Appendix: Arithmetic hyperbolic reflection groups

4.1. Definitions and preliminaries. Suppose $\mathbb{F}$ is a totally real number field with the ring of integers $A = \mathbb{O}_{\mathbb{F}}$. For convenience we will assume that it is a principal ideal domain.

Definition 4.1. *A free finitely generated A-module L with an inner product of signature $(n, 1)$ is said to be a hyperbolic lattice if, for each non-identity embedding $\sigma\colon \mathbb{F} \to \mathbb{R}$, the quadratic space $L \otimes_{\sigma(A)} \mathbb{R}$ is positive definite.*

Suppose that L is a hyperbolic lattice. Then the vector space $\mathbb{E}^{n,1} = L \otimes_{\mathrm{id}(A)} \mathbb{R}$ is identified with the $(n+1)$-dimensional real *Minkowski space*. The group $\Gamma = \mathcal{O}'(L)$ of integer (that is, with coefficients in A) linear transformations preserving the lattice L and mapping each connected component of the cone

$$\mathfrak{C} = \{v \in \mathbb{E}^{n,1} \mid (v, v) < 0\} = \mathfrak{C}^+ \cup \mathfrak{C}^-$$

into itself is a discrete group of motions of the *Lobachevsky space*. Here we mean the *vector model* $\mathbb{H}^n$ given as a set of points of the hyperboloid

$$\{v \in \mathbb{E}^{n,1} \mid (v, v) = -1\},$$

in the convex open cone $\mathfrak{C}^+$. The group of motions is $\mathrm{Isom}(\mathbb{H}^n) = \mathcal{O}'_{n,1}(\mathbb{R})$, which is the group of pseudoorthogonal transformations of the space $\mathbb{E}^{n,1}$ that leaves invariant the cone $\mathfrak{C}^+$.

It is known from the general theory of arithmetic discrete groups (A. Borel & Harish-Chandra [9] and G. Mostov & T. Tamagawa [18] in 1962) that if $\mathbb{F} = \mathbb{Q}$ and the lattice L is isotropic (that is, the quadratic form associated with it represents zero), then the quotient space $\mathbb{H}^n/\Gamma$ (the fundamental domain of Γ) is not compact, but is of finite volume (in this case we say that Γ is *a discrete subgroup of finite covolume*), and in all other cases it is compact. For $\mathbb{F} = \mathbb{Q}$, these assertions were first proved by B.A. Venkov in 1937 (see [30]).

Definition 4.2. *The groups Γ obtained in the above way and the subgroups of the group* $\mathrm{Isom}(\mathbb{H}^n)$ *that are commensurable*[8] *with them are called arithmetic discrete groups of the simplest type. The field $\mathbb{F}$ is called the definition field (or the ground field) of the group Γ (and all subgroups commensurable with it).*

A primitive vector e of a quadratic lattice L is called a *root* or, more precisely, a *k-root*, where $k = (e, e) \in A_{>0}$ if $2(e, x) \in kA$ for all $x \in L$. Every root e defines an *orthogonal reflection* (called a *k-reflection* if $(e, e) = k$) in the space $L \otimes_{\mathrm{id}(A)} \mathbb{R}$

$$\mathcal{R}_e : x \mapsto x - \frac{2(e, x)}{(e, e)} e,$$

[8]Two subgroups Γ_1 and Γ_2 of some group are said to be commensurable if the group $\Gamma_1 \cap \Gamma_2$ is a subgroup of finite index in each of them.

which preserves the lattice L. In the hyperbolic case, R_e determines the reflection of the space $\mathbb{H}^n$ with respect to the hyperplane

$$H_e = \{x \in \mathbb{H}^n \mid (x, e) = 0\},$$

called the *mirror* of the reflection $\mathcal{R}_e$.

Definition 4.3. *A reflection $\mathcal{R}_e$ is called stable if $(e, e) \mid 2$ in A.*

Let L be a hyperbolic lattice over a ring of integers A. We denote by $\mathcal{O}_r(L)$ the subgroup of the group $\mathcal{O}'(L)$ generated by all reflections contained in it, and we denote by $S(L)$ the subgroup of $\mathcal{O}'(L)$ generated by all stable reflections.

Definition 4.4. *A hyperbolic lattice L is said to be reflective if the index $[\mathcal{O}'(L) : \mathcal{O}_r(L)]$ is finite, and stably reflective if the index $[\mathcal{O}(L) : S(L)]$ is finite.*

In 1972, Vinberg proposed an algorithm (see [32], [33]) that, given a lattice L, enables one to construct the fundamental polyhedron of the group $\mathcal{O}_r(L)$ and determine thereby the reflectivity of the lattice L.

A discrete reflection group of finite covolume is an arithmetic group with a ground field $\mathbb{F}$ (or an $\mathbb{F}$-*arithmetic reflection group*) if it is a subgroup of finite index in a group of the form $\mathcal{O}'(L)$, where L is some (automatically reflective) hyperbolic lattice over a totally real number field $\mathbb{F}$.

Now we formulate some fundamental theorems on the existence of arithmetic reflection groups and cocompact reflection groups in the Lobachevsky spaces.

Theorem 4.1 (Vinberg, 1984, see [34])**.**

(1) *Compact Coxeter polyhedra do not exist in the Lobachevsky spaces $\mathbb{H}^n$ for $n \geq 30$.*
(2) *Arithmetic reflection groups do not exist in the Lobachevsky spaces $\mathbb{H}^n$ for $n \geq 30$.*

It was also proved that there are no reflective hyperbolic $\mathbb{Z}$-lattices of rank $n + 1 > 22$ (F. Esselmann, 1996, see [12]).

The next important result belongs to several authors.

Theorem 4.2. *For each $n \geq 2$, up to scaling, there are only finitely many reflective hyperbolic lattices. Similarly, up to conjugacy, there are only finitely many maximal arithmetic reflection groups in the spaces $\mathbb{H}^n$.*

The proof of this theorem is divided into the following stages:

- 1980, 1981 – V. V. Nikulin proved that there are only finitely many maximal arithmetic reflection groups in the spaces $\mathbb{H}^n$ for $n \geq 10$, see [20, 22];
- 2005 – D. D. Long, C. Maclachlan and A. W. Reid proved the finiteness of maximal arithmetic reflection groups in dimension $n = 2$, see [15];
- 2005 — I. Agol proved the finiteness in dimension $n = 3$, see [1];
- 2007 — V. V. Nikulin proved by induction the finiteness in the remaining dimensions $4 \leq n \leq 9$, see [25];
- 2007 — I. Agol, M. Belolipetsky, P. Storm, and K. Whyte independently proved the finiteness theorem for all dimensions by the *spectral method*, see [2] (see also the recent survey [4] of M. Belolipetsky).

The above results give the hope that all reflective hyperbolic lattices, as well as maximal arithmetic hyperbolic reflection groups can be classified.

4.2. Appendix A: Applications to classifying reflective hyperbolic lattices. Let now P be the fundamental polyhedron of the group $\mathcal{O}_r(L)$ for an anisotropic hyperbolic lattice L of rank 4. The lattice L is reflective if and only if the polyhedron P is compact (i. e., bounded).

Let E be an edge (of the polyhedron P) of width not greater than t. By Propositon 2.1 we can ensure that $t \leq 7$ (if we take the outermost edge from some fixed point O inside the polyhedron P). Let u_1, u_2 be the roots of the lattice L that are orthogonal to the faces containing the edge E and are the outer normals of these faces. Similarly, let u_3, u_4 be the roots corresponding to the framing faces. We denote these faces by F_1, F_2, F_3, and F_4, respectively. If $(u_3, u_3) = k$, $(u_4, u_4) = l$, then

$$|(u_3, u_4)| \leq t\sqrt{kl} \leq 7\sqrt{kl}. \tag{16}$$

Since we solve the classification problem for (1,2)-reflective lattices, we have to consider the fundamental polyhedra of arithmetic groups generated by 1- and 2-reflections. In this case we are given bounds on all elements of the matrix $G(u_1, u_2, u_3, u_4)$, because all the faces F_i are pairwise intersecting, excepting, possibly, the pair of faces F_3 and F_4. But if they do not intersect, then the distance between these faces is bounded by inequality (16). Thus, all entries of the matrix $G(u_1, u_2, u_3, u_4)$ are integer and bounded, so there are only finitely many possible matrices $G(u_1, u_2, u_3, u_4)$.

The vectors u_1, u_2, u_3, u_4 generate some sublattice L' of finite index in the lattice L. More precisely, the lattice L lies between the lattices L' and $(L')^*$, and

$$[(L')^* : L']^2 = |d(L')|.$$

Hence it follows that $|d(L')|$ is divisible by $[L : L']^2$. Using this, in each case we shall find for a lattice L' all its possible extensions of finite index.

To reduce the enumeration of matrices $G(u_1, u_2, u_3, u_4)$ one can use Main Theorem 1 and Theorem 2.2 enabling us to get much sharper bounds on the number $|(u_3, u_4)|$ than in inequality (16).

4.3. Appendix B: Upper bounds of degrees of ground fields of quasi-arithmetic reflection groups in $\mathbb{H}^3$. Suppose $d = [\mathbb{F} : \mathbb{Q}]$ is a degree of the ground field $\mathbb{F}$ for an arithmetic hyperbolic reflection group Γ in $\mathbb{H}^3$. Following Nikulin [26], we consider our outermost edge E in the fundamental Coxeter polytope of Γ. The faces $F_1, \ldots, F_{n+1}$ around this edge E give the opportunity to construct the edge polyhedron (or edge chamber). Using the fact that all quasi-arithmetic simplices are arithmetic, we can see that Nukilin's argument work for quasi-arithmetic Coxeter polytopes.

Nikulin's bounds for d were obtained via Fekete's polynomials. Taking into account that those polynomials were considered for the constant $t = 2 \cdot 7 = 14$, we can naively put in them our maximal constant $t_m = 2 \cdot 5.75 = 11.5$.

Then one can see that the upper bound improves from 25 to 23 in case of quasi-arithmetic cocompact reflection groups in $\mathbb{H}^3$.

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