STABLY REFLECTIVE HYPERBOLIC $\mathbb{Z}[\sqrt{2}]$-LATTICES OF RANK 4

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Abstract. In this paper we prove that the fundamental (Coxeter) polyhedron of a $\mathbb{Q}[\sqrt{2}]$-arithmetic reflection group in the three-dimensional Lobachevsky space contains an edge such that the distance between its framing faces is small enough. Using this fact we obtain a classification of stably reflective hyperbolic $\mathbb{Z}[\sqrt{2}]$-lattices of rank 4.

Keywords: stably reflective hyperbolic lattices, arithmetic hyperbolic reflection groups, fundamental polyhedra, compact Coxeter polyhedra, Vinberg's algorithm.

Contents

§ 1. Introduction

1.1. Preliminaries. Let $\mathbb{X}^n$ be one of the three spaces of constant curvature, that is, either the Euclidean space $\mathbb{E}^n$, or the n-dimensional sphere $\mathbb{S}^n$, or the n-dimensional (hyperbolic) Lobachevsky space $\mathbb{H}^n$.

Consider a convex polytope P in the space $\mathbb{X}^n$. If we act on P by the group Γ generated by reflections in the hyperplanes of its faces it can occur that the images of this polyhedron corresponding to different elements of Γ will cover the entire space $\mathbb{X}^n$ and will not overlap with each other. In this case we say that Γ is a *discrete reflection group*, and the polytope P is *the fundamental polyhedron* for Γ. If the polytope P is *bounded* (or, equivalently, *compact*), then the group Γ is called *a cocompact reflection group*, and if the polytope P has a *finite volume*, then the group Γ is called *cofinite* or a discrete group of *finite covolume*.

Which properties characterize such polyhedra P? For example, any two hyperplanes H_i and H_j bounding P either do not intersect or form a dihedral angle equal to π/n_{ij}, where $n_{ij} \in \mathbb{Z}$, $n_{ij} \geq 2$.

Such polyhedra are called *Coxeter polyhedra*, since the discrete reflection groups of finite covolume (hence their finite volume fundamental polyhedra) for $\mathbb{X}^n = \mathbb{E}^n, \mathbb{S}^n$ were determined and found by H. S. M. Coxeter in 1933 (see [11]).

In 1967 (see [30]), E. B. Vinberg developed his theory of discrete groups generated by reflections in the Lobachevsky spaces. He proposed new methods for studying hyperbolic reflection groups, in particular, a description of such groups in the form of the so-called Coxeter diagrams. He formulated and proved the arithmeticity criterion for hyperbolic reflection groups and constructed a number of various examples.

Suppose $\mathbb{F}$ is a totally real number field with the ring of integers $A = \mathbb{O}_{\mathbb{F}}$. For convenience we will assume that it is a principal ideal domain.

Definition 1.1. *A free finitely generated A-module L with an inner product of signature $(n, 1)$ is said to be a hyperbolic lattice if, for each non-identity embedding $\sigma\colon \mathbb{F} \to \mathbb{R}$, the quadratic space $L \otimes_{\sigma(A)} \mathbb{R}$ is positive definite.*

Suppose that L is a hyperbolic lattice. Then the vector space $\mathbb{E}^{n,1} = L \otimes_{\mathrm{id}(A)} \mathbb{R}$ is identified with the $(n + 1)$-dimensional real *Minkowski space*. The group $\Gamma = \mathcal{O}'(L)$ of integer (that is, with coefficients in A) linear transformations preserving the lattice L and mapping each connected component of the cone

$$\mathfrak{C} = \{v \in \mathbb{E}^{n,1} \mid (v, v) < 0\} = \mathfrak{C}^+ \cup \mathfrak{C}^-$$

into itself is a discrete group of motions of the *Lobachevsky space*. Here we mean the *vector model* $\mathbb{H}^n$ given as a set of points of the hyperboloid

$$\{v \in \mathbb{E}^{n,1} \mid (v, v) = -1\},$$

in the convex open cone $\mathfrak{C}^+$. The group of motions is $\mathrm{Isom}(\mathbb{H}^n) = \mathcal{O}'_{n,1}(\mathbb{R})$, which is the group of pseudoorthogonal transformations of the space $\mathbb{E}^{n,1}$ that leaves invariant the cone $\mathfrak{C}^+$.

It is known from the general theory of arithmetic discrete groups (A. Borel & Harish-Chandra [9] and G. Mostov & T. Tamagawa [18] in 1962) that if $\mathbb{F} = \mathbb{Q}$ and the lattice L is isotropic (that is, the quadratic form associated with it represents zero), then the quotient space $\mathbb{H}^n/\Gamma$ (the fundamental domain of Γ) is not compact, but is of finite volume (in this case we say that Γ is *a discrete subgroup of finite covolume*), and in all other cases it is compact. For $\mathbb{F} = \mathbb{Q}$, these assertions were first proved by B.A. Venkov in 1937 (see [29]).

Definition 1.2. *The groups Γ obtained in the above way and the subgroups of the group* $\mathrm{Isom}(\mathbb{H}^n)$ *that are commensurable*[1] *with them are called arithmetic discrete groups of the simplest type. The field $\mathbb{F}$ is called the definition field (or the ground field) of the group Γ (and all subgroups commensurable with it).*

A primitive vector e of a quadratic lattice L is called a *root* or, more precisely, a *k-root*, where $k = (e, e) \in A_{>0}$ if $2(e, x) \in kA$ for all $x \in L$. Every root e defines an *orthogonal reflection*

[1]Two subgroups Γ_1 and Γ_2 of some group are said to be commensurable if the group $\Gamma_1 \cap \Gamma_2$ is a subgroup of finite index in each of them.

(called a *k-reflection* if $(e,e)=k$) in the space $L\otimes_{\mathrm{id}(A)}\mathbb{R}$

$$\mathcal{R}_e : x \mapsto x - \frac{2(e,x)}{(e,e)}e,$$

which preserves the lattice L. In the hyperbolic case, R_e determines the reflection of the space $\mathbb{H}^n$ with respect to the hyperplane

$$H_e = \{x \in \mathbb{H}^n \mid (x,e) = 0\},$$

called the *mirror* of the reflection $\mathcal{R}_e$.

Definition 1.3. *A reflection $\mathcal{R}_e$ is called stable if $(e,e) \mid 2$ in A.*

Let L be a hyperbolic lattice over a ring of integers A. We denote by $\mathcal{O}_r(L)$ the subgroup of the group $\mathcal{O}'(L)$ generated by all reflections contained in it, and we denote by $S(L)$ the subgroup of $\mathcal{O}'(L)$ generated by all stable reflections.

Definition 1.4. *A hyperbolic lattice L is said to be reflective if the index $[\mathcal{O}'(L):\mathcal{O}_r(L)]$ is finite, and stably reflective if the index $[\mathcal{O}(L):S(L)]$ is finite.*

In 1972, Vinberg proposed an algorithm (see [31], [32]) that, given a lattice L, enables one to construct the fundamental polyhedron of the group $\mathcal{O}_r(L)$ and determine thereby the reflectivity of the lattice L.

A discrete reflection group of finite covolume is an arithmetic group with a ground field $\mathbb{F}$ (or an $\mathbb{F}$-*arithmetic reflection group*) if it is a subgroup of finite index in a group of the form $\mathcal{O}'(L)$, where L is some (automatically reflective) hyperbolic lattice over a totally real number field $\mathbb{F}$.

Now we formulate some fundamental theorems on the existence of arithmetic reflection groups and cocompact reflection groups in the Lobachevsky spaces.

Theorem 1.1 (Vinberg, 1984, see [33])**.**

(1) *Compact Coxeter polyhedra do not exist in the Lobachevsky spaces $\mathbb{H}^n$ for $n \geq 30$.*
(2) *Arithmetic reflection groups do not exist in the Lobachevsky spaces $\mathbb{H}^n$ for $n \geq 30$.*

It was also proved that there are no reflective hyperbolic $\mathbb{Z}$-lattices of rank $n+1 > 22$ (F. Esselmann, 1996, see [12]).

The next important result belongs to several authors.

Theorem 1.2. *For each $n \geq 2$, up to scaling, there are only finitely many reflective hyperbolic lattices. Similarly, up to conjugacy, there are only finitely many maximal arithmetic reflection groups in the spaces $\mathbb{H}^n$.*

The proof of this theorem is divided into the following stages:

- 1980, 1981 – V. V. Nikulin proved that there are only finitely many maximal arithmetic reflection groups in the spaces $\mathbb{H}^n$ for $n \geq 10$, see [20, 22];
- 2005 – D. D. Long, C. Maclachlan and A. W. Reid proved the finiteness of maximal arithmetic reflection groups in dimension $n = 2$, see [15];
- 2005 — I. Agol proved the finiteness in dimension $n = 3$, see [1];
- 2007 — V. V. Nikulin proved by induction the finiteness in the remaining dimensions $4 \leq n \leq 9$, see [25];
- 2007 — I. Agol, M. Belolipetsky, P. Storm, and K. Whyte independently proved the finiteness theorem for all dimensions by the *spectral method*, see [2] (see also the recent survey [4] of M. Belolipetsky).

The above results give the hope that all reflective hyperbolic lattices, as well as maximal arithmetic hyperbolic reflection groups can be classified.

1.2. Classification results.

1.2.1. *Lattices over* $\mathbb{Z}$. obtained the classification of unimodular reflective hyperbolic lattices over $\mathbb{Z}$.

In 1979, 1981 and 1984, V. V. Nikulin (see [19, 22, 23]) classified all 2-reflective hyperbolic $\mathbb{Z}$-lattices of rank not equal to 4, and after that in 2000 (see [24]) he found all reflective hyperbolic $\mathbb{Z}$-lattices of rank 3 with square free discriminants.

In 1998, (see [35] and [36]) E. B. Vinberg classified all 2-reflective hyperbolic $\mathbb{Z}$-lattices of rank 4. Subsequently, D. Allcock (2011, see [3]) classified at all all reflective hyperbolic $\mathbb{Z}$-lattices of rank 3.

In 1989–1993 in the papers [26, 27, 40] R. Scharlau and C. Walhorn presented a list of all maximal groups of the form $\mathcal{O}_r(L)$, where L is a reflective isotropic hyperbolic $\mathbb{Z}$-lattice of rank 4 or 5. A similar result was obtained in 2017 in the dissertation of I. Turkalj (see [28]) for $\mathbb{Z}$-lattices of rank 6. Finally, the author of this paper obtained a classification of all stably reflective anisotropic hyperbolic $\mathbb{Z}$-lattices of rank 4 (2016–2019, see [5, 6, 8]).

1.2.2. *Lattices over* $A \neq \mathbb{Z}$. Concerning reflective lattices over algebraic rings different from $\mathbb{Z}$, there is quite a small number of known results. In 1984, 1990 and 1992, V. O. Bugaenko obtained the classification of unimodular reflective hyperbolic lattices (see [10]). And in 2015, A. Mark (see [16, 17]) found reflective hyperbolic $\mathbb{Z}[\sqrt{2}]$-lattices of rank 3 with square free invariants.

A more complete history of the problem can be read in the recent survey [4] of M. Belolipetsky.

1.3. Notation. We introduce some notation:

1) $[C]$ is a quadratic lattice whose inner product in some basis is given by a symmetric matrix C;
2) $d(L) := \det C$ is the discriminant of the lattice $L = [C]$;
3) $L \oplus M$ is the orthogonal sum of the lattices L and M;
4) $[k]L$ is the quadratic lattice obtained from L by multiplying all inner products by $k \in \mathbb{Z}$;
5) $L^* = \{x \in L \otimes \mathbb{Q} \mid \forall y \in L \ \ (x, y) \in \mathbb{Z}\}$ is the *adjoint* lattice.

Let P be a compact acute-angled polyhedron in $\mathbb{H}^3$, E an edge of P, F_1 and F_2 the faces of P containing E, and u_3 and u_4 the unit outer normals to the faces F_3 and F_4 containing the vertices of E but not E itself.

Definition 1.5. *The faces* F_3 *and* F_4 *are called the framing faces of the edge* E*, and the number* $|(u_3, u_4)|$ *is its width.*

We associate with the edge E the set $\overline{\alpha} = (\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{14}, \alpha_{24})$, where α_{ij} is the angle between the faces F_i and F_j.

1.4. Main results of the paper. The main results of this paper are the following two assertions, the second of which is proved with the help of the first one.

Main Theorem 1. *The fundamental Coxeter polyhedron of any* $\mathbb{Q}[\sqrt{2}]$*-arithmetic reflection group in* $\mathbb{H}^3$ *has an edge of width less than* 4.14.

Actually a stronger result is obtained. Namely, it is proved that there is an edge of width less than $t_{\bar{\alpha}}$, where $t_{\bar{\alpha}} \leq 4.14$ is a number depending on the set $\bar{\alpha}$ of dihedral angles around this edge.

Main Theorem 2. *Any maximal stably reflective hyperbolic lattice of rank* 4 *over* $\mathbb{Z}[\sqrt{2}]$ *is isomorphic to one of the following seven lattices:*

No.	L	*# faces*	$d(L)$
1	$[-1-\sqrt{2}]\oplus[1]\oplus[1]\oplus[1]$	5	$-1-\sqrt{2}$
2	$[-1-2\sqrt{2}]\oplus[1]\oplus[1]\oplus[1]$	6	$-1-2\sqrt{2}$
3	$[-5-4\sqrt{2}]\oplus[1]\oplus[1]\oplus[1]$	5	$-5-4\sqrt{2}$
4	$[-11-8\sqrt{2}]\oplus[1]\oplus[1]\oplus[1]$	17	$-11-8\sqrt{2}$
5	$[-\sqrt{2}]\oplus[1]\oplus[1]\oplus[1]$	6	$-\sqrt{2}$
6	$\begin{bmatrix} 2 & -1 & -\sqrt{2} \\ -1 & 2 & \sqrt{2}-1 \\ -\sqrt{2} & \sqrt{2}-1 & 2-\sqrt{2} \end{bmatrix}\oplus[1]$	6	$-\sqrt{2}$
7	$[-7-5\sqrt{2}]\oplus[1]\oplus[1]\oplus[1]$	5	$-7-5\sqrt{2}$

(Here ,,# faces" denotes the number of faces of the fundamental Coxeter polyhedron for the group $\mathcal{O}_r(L)$, but not $S(L)$.)

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§ 2. The method of the outermost edge and proof of Main Theorem 1

2.1. Nikulin's method. In this paper we shall use *the method of the outermost edge*, which is a modification of *the method of narrow parts of polyhedra*, applied by V. V. Nikulin in his papers [20] and [24].

Definition 2.1. *A convex polyhedron in the space $\mathbb{H}^n$ is the intersection of finitely many halfspaces such that it has non-empty interior. A generalized convex polyhedron is the intersection of a family of halfspaces (possibly infinite) such that any ball intersects only finitely many their boundary hyperplanes.*

Definition 2.2. *A generalized convex polyhedron is called acute-angled if all its dihedral angles do not exceed $\pi/2$. A generalized convex polyhedron is called a Coxeter polyhedron if all its dihedral angles have the form π/k, where $k \in \{2, 3, \dots, \infty\}$.*

It is known that fundamental domains of discrete reflection groups are generalized *Coxeter polyhedra* (see the papers [11], [34], and [38]).

Here and throughout by faces of a polyhedron we mean its $(n-1)$-dimensional faces. The Gram matrix of a system of vectors $v_1, \dots, v_k$ will be denoted by $G(v_1, \dots, v_k)$.

In his earlier works (see Lemma 3.2.1 in [20] and the proof of Theorem 4.1.1 in [22]) V. V. Nikulin proved the following assertion[2].

Theorem 2.1. *Let P be an acute-angled convex polyhedron of finite volume in $\mathbb{H}^n$. Then there exists a face F such that*

$$\cosh \rho(F_1, F_2) \leq 7,$$

for any faces F_1 and F_2 of P adjacent to F, where $\rho(\,\cdot\,,\cdot\,)$ is the metric in the Lobachevsky space[3].

In the proof of this assertion, the face F was chosen as the outermost face from some fixed point O inside the polyhedron P. Notice that this theorem enables us to bound at once the absolute value of the inner product of outer normals to faces adjacent to the face F. Indeed,

[2] We present this assertion in a form convenient for us, although it was not formulated in this way anywhere.

[3] In Nikulin's papers, the squares of the lengths of the normals to faces are 2, therefore, in his works there is a bound $(\delta, \delta') \leq 14$.

if F_1 and F_2 intersect or are parallel, this value is equal to the cosine of the dihedral angle between these faces, and if these faces diverge, then it equals the hyperbolic cosine of the distance between them.

2.2. The method of the outermost edge. We have the following corollary of Theorem 2.1.

Proposition 2.1. *Each compact acute-angled polyhedron $P \subset \mathbb{H}^3$ contains an edge of width not greater than 7.*

Proof. Following V. V. Nikulin (see [22]), we consider an interior point O in P. Let E be *the outermost*[4] *edge from it*, and let F be a face containing this edge. Let E_1, E_2 be disjoint[5] edges of this face coming out from different vertices of E.

Let O' be the projection of O onto the face F. Note that O' is an interior point of this face, since otherwise the point O would lie outside of some dihedral angle adjacent to F (because the polyhedron P is acute-angled). Further, since E is the outermost edge of the polyhedron for O, then it follows from this and the three perpendiculars theorem that the distance between the point O' and the edge E is not less than the distance between this point and any other edge of the face F. This means that the edge E is the outermost edge to the point O' inside the polygon F and we can use Theorem 2.1.

Further, let F_3 and F_4 be the faces (with unit outer normals u_3 and u_4, respectively) of the polyhedron P framing the outermost edge E and containing the edges E_1 and E_2, respectively. Clearly, the distance between the faces is not greater than the distance between their edges. Therefore,

$$-(u_3, u_4) = \cosh \rho(F_3, F_4) \leq \cosh \rho(E_1, E_2) \leq 7.$$

The proposition is proved. ∎

Let now P be the fundamental polyhedron of the group $\mathcal{O}_r(L)$ for an anisotropic hyperbolic lattice L of rank 4. The lattice L is reflective if and only if the polyhedron P is compact (i. e., bounded).

Let E be an edge (of the polyhedron P) of width not greater than t. By Propositon 2.1 we can ensure that $t \leq 7$ (if we take the outermost edge from some fixed point O inside the polyhedron P). Let u_1, u_2 be the roots of the lattice L that are orthogonal to the faces containing the edge E and are the outer normals of these faces. Similarly, let u_3, u_4 be the roots corresponding to the framing faces. We denote these faces by F_1, F_2, F_3, and F_4, respectively. If $(u_3, u_3) = k$, $(u_4, u_4) = l$, then

$$|(u_3, u_4)| \leq t\sqrt{kl} \leq 7\sqrt{kl}. \tag{1}$$

Since we solve the classification problem for (1,2)-reflective lattices, we have to consider the fundamental polyhedra of arithmetic groups generated by 1- and 2-reflections. In this case we are given bounds on all elements of the matrix $G(u_1, u_2, u_3, u_4)$, because all the faces F_i are pairwise intersecting, excepting, possibly, the pair of faces F_3 and F_4. But if they do not intersect, then the distance between these faces is bounded by inequality (1). Thus, all entries of the matrix $G(u_1, u_2, u_3, u_4)$ are integer and bounded, so there are only finitely many possible matrices $G(u_1, u_2, u_3, u_4)$.

The vectors u_1, u_2, u_3, u_4 generate some sublattice L' of finite index in the lattice L. More precisely, the lattice L lies between the lattices L' and $(L')^*$, and

$$[(L')^* : L']^2 = |d(L')|.$$

[4]In an acute-angled polyhedron, the distance from the interior point to the face (of any dimension) is equal to the distance to the plane of this face.

[5]Note that we consider the case where the framing faces are divergent, since otherwise the absolute value of the inner product does not exceed one.

Hence it follows that $|d(L')|$ is divisible by $[L : L']^2$. Using this, in each case we shall find for a lattice L' all its possible extensions of finite index.

To reduce the enumeration of matrices $G(u_1, u_2, u_3, u_4)$ we shall use some additional considerations enabling us to get sharper bounds on the number $|(u_3, u_4)|$ than in inequality (1).

2.3. Bounds for the length of the edge E for a compact acute-angled polyhedron in $\mathbb{H}^3$. In this subsection P denotes a compact acute-angled polyhedron in the three-dimensional Lobachevsky space $\mathbb{H}^3$.

Keeping the assumptions and notation of the previous sections, we denote the vertices of the edge E by V_1 and V_2. The dihedral angles between the faces F_i and F_j will be denoted by α_{ij}.

Let E_1 and E_3 be the edges of the polyhedron P outgoing from the vertex V_1 and let E_2 and E_4 be the edges outgoing from V_2 such that the edges E_1 and E_2 lie in the face F_1. The length of the edge E is denoted by a, and the plane angles between the edges E_j and E are denoted by α_j (see Figure 1).

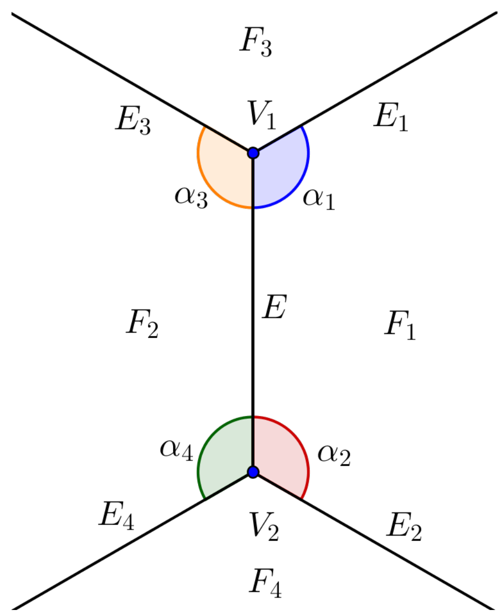

FIGURE 1. The outermost edge

Theorem 2.2 (BOGACHEV, 2019, [8]). *The length of the outermost edge satisfies the inequality*

$$a < \operatorname{arcsinh}\left(\frac{\cos(\alpha_{12}/2)}{\tan(\alpha_3/2)}\right) + \operatorname{arcsinh}\left(\frac{\cos(\alpha_{12}/2)}{\tan(\alpha_4/2)}\right).$$

Proof. Denote by O_1 and O_2 the orthogonal projections of the point O onto the faces F_1 and F_2, respectively. By the theorem of three perpendiculars, both points fall under the projection onto this edge E on the same point A, which is the projection of the point O onto this edge. Due to the fact that the polyhedron P is acute-angled, the point A is an inner point of the edge E.

Thus, we get a flat quadrilateral AO_1OO_2, in which $\angle A = \alpha_{12}$ (the dihedral angle between the faces F_1 and F_2), $\angle O_1 = \angle O_2 = \pi/2$, $AO_1 = a_1$, $AO_2 = a_2$ (see Figure 2).

In the limiting case where the point O is a point at infinity, the dihedral angle α_{12} is composed of the so-called *angles of parallelism* $\Pi(a_1)$ and $\Pi(a_2)$. In our case $O \in \mathbb{H}^3$, therefore,

$$\alpha_{12} < \Pi(a_1) + \Pi(a_2) = 2\arctan(e^{-a_1}) + 2\arctan(e^{-a_2}).$$

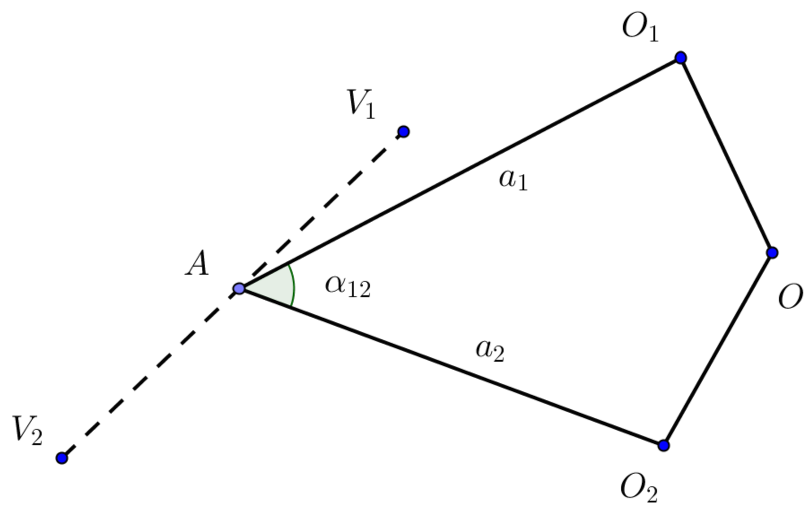

FIGURE 2. A quadrilateral AO_1OO_2

Denote by V_1I, V_2I, V_1J, V_2J the bisector of angles α_1, α_2, α_3, α_4, respectively. Let h_I and h_J be the distances from the points I and J to the edge E. Without loss of generality we can assume that $h_J \leqslant h_I$.

Since the edge E is the outermost edge for the point O, we have

$$\rho(O_1, E) \leq \rho(O_1, E_1), \rho(O_1, E_2), \qquad \rho(O_2, E) \leq \rho(O_2, E_3), \rho(O_2, E_4). \tag{2}$$

Then it is clear that $h_J \leq h_I \leq a_1$, $h_J \leq a_2$, since inequalities (2) imply that the points O_1 and O_2 lie inside flat angles vertical to the angles V_1IV_2 and V_1JV_2, respectively (the scan of faces around the edge E is represented in Figure 3).

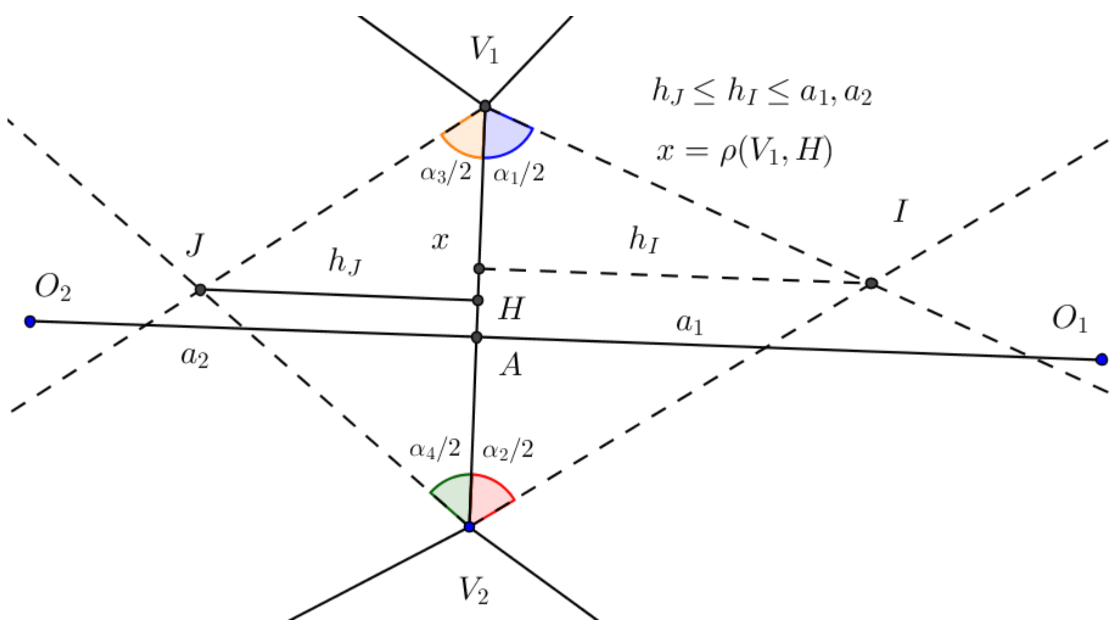

FIGURE 3. The scan

We have $\Pi(a_1), \Pi(a_2) \leq \Pi(h_J)$. It follows that $\arctan(e^{-h_J}) > \alpha_{12}/4$. Thus,

$$h_J < \ln\left(\cot\left(\frac{\alpha_{12}}{4}\right)\right).$$

We introduce the notation

$$A_0 := \tanh\big(\ln(\cot(\alpha_{12}/4))\big) = \cos\left(\frac{\alpha_{12}}{2}\right).$$

Then $\tanh h_J < A_0$. Let H be the projection of the point J onto the edge E and let $x = \rho(H, V_1)$.

From the right triangles V_1JH and V_2JH we find

$$\tanh h_J = \tan\left(\frac{\alpha_4}{2}\right)\sinh(a-x) = \tan\left(\frac{\alpha_3}{2}\right)\sinh x, \tag{3}$$

which implies that

$$\sinh x = \frac{\tanh h_J}{\tan(\alpha_3/2)} < \frac{A_0}{\tan(\alpha_3/2)}, \qquad \sinh(a-x) < \frac{A_0}{\tan(\alpha_4/2)}.$$

Hence,

$$a = x + (a-x) < \operatorname{arcsinh}\left(\frac{A_0}{\tan(\alpha_3/2)}\right) + \operatorname{arcsinh}\left(\frac{A_0}{\tan(\alpha_4/2)}\right),$$

which completes the proof. ∎

2.4. The proof of Main Theorem 1 and bounds on $|(u_3, u_4)|$. Let a polyhedron P be the fundamental polyhedron of a $\mathbb{Q}[\sqrt{2}]$-arithmetic cocompact reflection group in the three-dimensional Lobachevsky space and let E be the outermost edge of the polyhedron P. Consider the set of unit outer normals (u_1', u_2', u_3', u_4') to the faces F_1, F_2, F_3, F_4. Note that this vector system is linearly independent. Its Gram matrix is

$$G(u_1', u_2', u_3', u_4') = \begin{pmatrix} 1 & -\cos\alpha_{12} & -\cos\alpha_{13} & -\cos\alpha_{14} \\ -\cos\alpha_{12} & 1 & -\cos\alpha_{23} & -\cos\alpha_{24} \\ -\cos\alpha_{13} & -\cos\alpha_{23} & 1 & -T \\ -\cos\alpha_{14} & -\cos\alpha_{24} & -T & 1 \end{pmatrix},$$

where $T = |(u_3', u_4')| = \cosh\rho(F_3, F_4)$ in the case where the faces F_3 and F_4 diverge. Recall that otherwise $T \le 1$, and we do not need to consider this case separately.

Let $(u_1^*, u_2^*, u_3^*, u_4^*)$ be the basis dual to the basis (u_1', u_2', u_3', u_4'). Then u_3^* and u_4^* determine the vertices V_2 and V_1 in the Lobachevsky space. Indeed, the vector v_1 corresponding to the point $V_1 \in \mathbb{H}^3$ is uniquely determined (up to scaling) by the conditions $(v_1, u_1') = (v_1, u_2') = (v_1, u_3') = 0$. Note that the vector u_4^* satisfies the same conditions. Therefore, the vectors v_1 and u_4^* are proportional, hence,

$$\cosh a = \cosh\rho(V_1, V_2) = -(v_1, v_2) = -\frac{(u_3^*, u_4^*)}{\sqrt{(u_3^*, u_3^*)(u_4^*, u_4^*)}}.$$

It is known that $G(u_1^*, u_2^*, u_3^*, u_4^*) = G(u_1', u_2', u_3', u_4')^{-1}$, whence it follows that $\cosh a$ can be expressed in terms of the algebraic complements G_{ij} of the elements of the matrix $G = G(u_1', u_2', u_3', u_4')$:

$$\cosh a = -\frac{(u_3^*, u_4^*)}{\sqrt{(u_3^*, u_3^*)(u_4^*, u_4^*)}} = \frac{G_{34}}{\sqrt{G_{33}G_{44}}}.$$

Denote the right-hand side of the inequality from Theorem 2.2 by $F(\overline{\alpha})$, where $\overline{\alpha} = (\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{14}, \alpha_{24})$; then this theorem implies that $\cosh a < \cosh F(\overline{\alpha})$. It follows that

$$\frac{G_{34}}{\sqrt{G_{33}G_{44}}} < \cosh F(\overline{\alpha}). \tag{4}$$

For every $\overline{\alpha}$, in this way we obtain a linear inequality with respect to the number T.

Lemma 2.1. *The following relations are true:*
(i) $\alpha_{12} + \alpha_{23} + \alpha_{13} > \pi, \quad \alpha_{12} + \alpha_{24} + \alpha_{14} > \pi;$
(ii)

$$\cos\alpha_1 = \frac{\cos\alpha_{23} + \cos\alpha_{12}\cdot\cos\alpha_{13}}{\sin\alpha_{12}\cdot\sin\alpha_{13}}, \qquad \cos\alpha_2 = \frac{\cos\alpha_{24} + \cos\alpha_{12}\cdot\cos\alpha_{14}}{\sin\alpha_{12}\cdot\sin\alpha_{14}},$$
$$\cos\alpha_3 = \frac{\cos\alpha_{13} + \cos\alpha_{12}\cdot\cos\alpha_{23}}{\sin\alpha_{12}\cdot\sin\alpha_{23}}, \qquad \cos\alpha_4 = \frac{\cos\alpha_{14} + \cos\alpha_{12}\cdot\cos\alpha_{24}}{\sin\alpha_{12}\cdot\sin\alpha_{24}}.$$

Proof. To prove both parts of the lemma, we intersect each trihedral angle with the vertices V_1 and V_2 by spheres centered at these points. In the intersection we obtain spherical triangles the angles of which are the dihedral angles α_{ij}, and the lengths of their edges are the flat angles α_k. This implies at once the first assertion, and the second one follows from the dual cosine-theorem for these triangles (see, for example, [37, p. 71]). The lemma is proved. ■

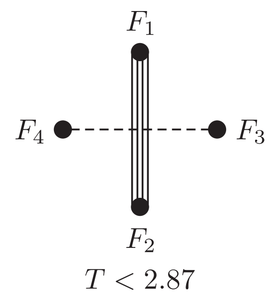

Figure 4. Coxeter diagram of the edge with angles $(\pi/6, \pi/2, \pi/2, \pi/2, \pi/2)$

It is known that the dihedral angles of the fundamental polyhedron of an arithmetic hyperbolic reflection group with a ground field $\mathbb{Q}[\sqrt{2}]$ can equal only $\pi/2$, $\pi/3$, $\pi/4$, $\pi/6$, and $\pi/8$.

It is easy to verify that, taking into account Lemma 2.1, (i), there are exactly 56 different (up to numbering) sets of angles $\overline{\alpha}$. For each such set $\overline{\alpha}$ inequality (4) gives some bound $T < t_{\overline{\alpha}}$.

For solving the 56 linear inequalities a program was compiled in the computer algebra system Sage[6], its code is available on the Internet[7].

The obtained results are presented in Table **??** in the form of a set of Coxeter diagrams for the faces F_1, F_2, F_3, F_4. The faces F_3 and F_4 will be connected by a dotted line, and the whole diagram will be signed by the relevant bound: $T < t_{\overline{\alpha}}$.

An example of how an edge diagram looks like for $\overline{\alpha} = (\pi/6, \pi/2, \pi/2, \pi/2, \pi/2)$ see in Figure 4. In this figure we see that $t_{(\pi/6,\pi/2,\pi/2,\pi/2,\pi/2)} = 2.87$.

Thus, we have proved the following theorem.

Theorem 2.3. *The fundamental polyhedron of every $\mathbb{Q}[\sqrt{2}]$-arithmetic cocompact reflection group in $\mathbb{H}^3$ contains an edge of width less than $t_{\overline{\alpha}}$, where $t_{\overline{\alpha}}$ is the number (depending on the set $\overline{\alpha}$), specified in the table, published in the Internet. Moreover,*

$$\max_{\overline{\alpha}}\{t_{\overline{\alpha}}\} = t_{(\pi/4,\pi/2,\pi/3,\pi/3,\pi/2)} = 4.14.$$

The numbers given in the Table were calculated on a computer with accuracy up to eight decimal places. The table shows them rounded up to the nearest hundredth, which is quite

[6]The Sage Developers, *the Sage Mathematics Software System (Version 7.6)*, SageMath, http://www.sagemath.org, 2017

[7]N. Bogachev, *Method of the outermost edge/bounds*, https://github.com/nvbogachev/OuterMostEdge/blob/master/bounds.sage/, 2017

enough for our purposes. *Note: the numbering of the faces on each diagram is the same as in the Figure 2.*

Let E be the edge (chosen in Theorem 2.3) of width less than $t_{\overline{\alpha}}$ for some $\overline{\alpha}$ in the fundamental polyhedron P of the group $\mathcal{O}_r(L)$, and let $(u_3, u_3) = k$, $(u_4, u_4) = l$. Then

$$|(u_3, u_4)| < t_{\overline{\alpha}} \cdot \sqrt{kl}. \tag{5}$$

Note that this bound is much better than estimate (1).

§ 3. Methods of testing a lattice for stably reflectivity

3.1. Vinberg's algorithm. As it was said before, in 1972, Vinberg suggested an algorithm of constructing the fundamental polyhedron of a hyperbolic reflection group. This algorithm is theoretically applicable to any hyperbolic reflection group, but practically it is efficient only for groups of the form $\mathcal{O}_r(L)$ (and also of the form $\mathcal{O}_r^{(2)}(L)$, etc.).

In this subsection we describe Vinberg's algorithm, following [31] and [32]. We pick a point $v_0 \in \mathbb{H}^n$, which we shall call a *a basic point*. The fundamental domain P_0 of its stabilizer $\mathcal{O}_r(L)_{v_0}$ is a polyhedral cone in $\mathbb{H}^n$. Let $H_1, \ldots, H_m$ be the sides of this cone and let $a_1, \ldots, a_m$ be the corresponding outer normals. We define the half-spaces

$$H_k^- = \{x \in \mathbb{E}^{n,1} \mid (x, a_k) \leq 0\}.$$

Then $P_0 = \bigcap_{j=1}^m H_j^-$.

There is the unique fundamental polyhedron P of the group $\mathcal{O}_r(L)$ contained in P_0 and containing the point v_0. Its faces containing v_0 are formed by the cone faces $H_1, \ldots, H_m$. The other faces $H_{m+1}, \ldots$ and the corresponding outer normals $a_{m+1}, \ldots$ are constructed by induction. Namely, for H_j we take a mirror such that the root a_j orthogonal to it satisfies the conditions:

1) $(v_0, a_j) < 0$;
2) $(a_i, a_j) \leq 0$ for all $i < j$;
3) the distance $\rho(v_0, H_j)$ is minimal subject to constraints 1) and 2).

The lengths of the roots of a lattice L satisfy the following condition.

Proposition 3.1. (Vinberg, see [33, Proposition 24]) *The squares of lengths of roots in the quadratic lattice L are divisors of the doubled last invariant factor of L.*

The fundamental polyhedron of a group $\mathcal{O}_r(L)$ is a Coxeter polyhedron and is determined by its Coxeter diagram. A polyhedron is of finite volume if and only if it is the convex span of a finite number of usual points or points at infinity of the space $\mathbb{H}^n$. Every vertex of a finite-volume Coxeter polyhedron corresponds either to an elliptic subdiagram or rank n (usual vertices) of its Coxeter diagram or to a parabolic subdiagram or rank $n-1$ (vertices at infinity). Thus, according to the Coxeter diagram, one can determine whether a polyhedron has a finite volume. For more details, see, for example, Vinberg's papers [30] and [34].

Some efforts to implement Vinberg's algorithm by using a computer have been made since the 1980s, but they all dealt with particular lattices, usually with an orthogonal basis. Such programs are mentioned, e.g., in the papers of Bugaenko (1992, see [10]), Scharlau and Walhorn (1992, see [27]), Nikulin (2000, see [24]), and Allcock (2011, see [3]). But the programs themselves have not been published; the only exception is Nikulin's paper, which contains a program code for lattices of several different special forms. The only known implementation published along with a detailed documentation is Guglielmetti's 2016 program `AlVin`[8], processing hyperbolic lattices with an orthogonal basis with square-free invariant factors over several ground fields. R. Guglielmetti used this program in his thesis (2017, see

[8]see https://rgugliel.github.io/AlVin

[13]) to classify reflective hyperbolic lattices with an orthogonal basis with small lengths of its elements. His program works fairly well in all dimensions in which reflective lattices exist.

In 2017, the author jointly with A. Yu. Perepechko created the program implementing Vinberg's algorithm for arbitrary hyperbolic $\mathbb{Z}$-lattices. This program `VinAl` is available on the Internet (see [39]), and one can find its detailed description in [7].

Vinberg's Algorithm for hyperbolic lattices over $\mathbb{Z}[\sqrt{d}]$. Since we also investigate the reflectivity of lattices over $\mathbb{Z}[\sqrt{2}]$, the author decided to write a program for Vinberg's Algorithm over quadratic fields. At the moment the author has a program for lattices over $\mathbb{Z}[\sqrt{2}]$, which requires some minor editing for each new lattice. This program enables one to investigate a lattice without orthogonal bases.

For lattices with an orthogonal basis was used the program `AlVin` of Guglielmetti mentioned above. In the nearest future we plan to merge the author's programs for lattices over $\mathbb{Z}[\sqrt{2}]$ with the `VinAl` project. Further work on the project that implements Vinberg's algorithm for arbitrary lattices over the quadratic fields $\mathbb{Z}[\sqrt{d}]$ is being carried out jointly with A. Yu. Perepechko.

3.2. The method of "bad" reflections. If we can construct the fundamental polyhedron of the group $O_r(L)$ for some reflective lattice L, then it is easy to determine whether it is (1,2)-reflective. One can consider the group Δ generated by the k-reflections for $k > 2$ (we shall call them "bad" reflections) in the sides of the fundamental polyhedron of the group $O_r(L)$. The following lemma holds (see [36]).

Lemma 3.1. *A lattice L is stably reflective if and only if it is reflective and the group Δ is finite.*

Actually, to prove that a lattice is not stably reflective, it is sufficient to construct only some part of the fundamental polyhedron containing an infinite subgroup generated by bad reflections.

3.3. Method of infinite symmetry. Recall that

$$\mathcal{O}'(L) = \mathcal{O}_r(L) \rtimes H,$$

where $H = \mathrm{Sym}(P) \cap \mathcal{O}'(L)$. If P is of infinite volume and has infinitely many faces, then the group H is infinite. To determine whether it is infinite or not, one can use the following lemma proved by V. O. Bugaenko in 1992 (see [10]).

Lemma 3.2. *Suppose H is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$. Then H is infinite iff there exists a subgroup of H without fixed point in $\mathbb{H}^n$.*

How can we find the set of fixed points?

Lemma 3.3. (Bugaenko, see [10, Lemma 3.2]) *Let η be an involutive trasnformation of a real vector space V. Then the set of its fixed points $Fix(\eta)$ is generated by vectors $e_j + \eta(e_j)$, where $\{e_j\}$ form a basis of V.*

Due to this lemma the proof of non-reflectivity of a lattice is the following. If we know a part of a polyhedron P for the group $\mathcal{O}_r(L)$, then we can find a few symmetries of its Coxeter diagram.

If these symmetries preserve the lattice L, then they generate the subgroup that preserves P. If this subgroup has no fixed points then $\mathcal{O}_r(L)$ is of infinite index in $\mathcal{O}'(L)$.

§ 4. Short list of candidate-lattices

4.1. Plan for finding a short list. Let P be the fundamental polyhedron of the group $S(L)$ for a hyperbolic lattice L of rank 4. This lattice is stably reflective if and only if P is compact. By Theorem 2.3, every such polyhedron contains an edge E of width less than $t_{\overline{\alpha}}$, where $t_{\overline{\alpha}} \leq 4.14$ is the number depending on the set $\overline{\alpha}$ of dihedral angles around this edge.

Let u_1, u_2, u_3, u_4 be the roots of the lattice L that are the outer normals to the faces F_1, F_2, F_3, F_4, respectively. These roots generate some sublattice

$$L' = [G(u_1, u_2, u_3, u_4)] \subset L.$$

Note that the elements of the Gram matrix $G(u_1, u_2, u_3, u_4)$ can assume only finitely many different values. Namely, the diagonal elements can equal only 1, 2, or $2+\sqrt{2}$, and the absolute values of the remaining elements g_{ij} must be strictly less than $\sqrt{g_{ii}g_{jj}}$, excepting $g_{34} = (u_3, u_4)$, whose absolute value is bounded by $t_{\overline{\alpha}}\sqrt{(u_3, u_3)(u_4, u_4)}$.

Thus, we obtain a finite list of matrices $G(u_1, u_2, u_3, u_4)$. We pick in these matrices the ones that define anisotropic lattices, and after that we find all their possible extensions.

In order to select only anisotropic lattices, we use a computer program[9], based on the results and methods formulated in § **??** of this paper.

We split the list of all anisotropic lattices into isomorphism classes and take only one representative of each class. So, now we obtain a substantially shorter list of anisotropic lattices that are pairwise non-isomorphic.

After that we find all their finite-index extensions and verify the resulting list of candidate-lattices on (1,2)-reflectivity using the methods described in § 3.

4.2. Short list of candidate-lattices. The final program that creates a list of numbers $t_{\overline{\alpha}}$ and then, using this list, displays all Gram matrices $G(u_1, u_2, u_3, u_4)$, is also available on the Internet[10].

As the output we obtain matrices G_1–G_{15}, for each of which we find all corresponding extensions.

To each Gram matrix G_k in our notation, there corresponds a lattice L_k that can have some other extensions. For each new anisotropic lattice (non-isomorphic to any previously found lattice) we introduce the notation $L(k)$, where k denotes its number:

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1-\sqrt{2} \\ 0 & 0 & -1-\sqrt{2} & 1 \end{pmatrix}, \qquad L_1 \simeq [-2(1+\sqrt{2})] \oplus [1] \oplus [1] \oplus [1];$$

A unique its extension is an extension of ,,index $\sqrt{2}$“

$$L(1) := [-(1+\sqrt{2})] \oplus [1] \oplus [1] \oplus [1].$$

$$G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1-\sqrt{2} \\ 0 & 0 & -1-\sqrt{2} & 2 \end{pmatrix}, \qquad L_2 \simeq [-(1+2\sqrt{2})] \oplus [1] \oplus [1] \oplus [1] := L(2);$$

[9]N. Bogachev, *Method of the outermost edge/is_anisotropic*, https://github.com/nvbogachev/OuterMostEdge/blob/master/is_anisotropic, 2017

[10]N. Bogachev, *Method of the outermost edge/CandidatesFor12Reflectivity*, https://github.com/nvbogachev/OuterMostEdge/blob/master/Is_equival, 2017

$$G_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1-\sqrt{2} \\ 0 & 0 & -1-\sqrt{2} & 2 \end{pmatrix}, \qquad L_3 \simeq \begin{bmatrix} 2 & -1-\sqrt{2} \\ -1-\sqrt{2} & 2 \end{bmatrix} \oplus [1] \oplus [1] := L(3);$$

$$G_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1-2\sqrt{2} \\ 0 & 0 & -1-2\sqrt{2} & 2 \end{pmatrix}, \ L_4 \simeq \begin{bmatrix} 2 & -1-2\sqrt{2} \\ -1-2\sqrt{2} & 2 \end{bmatrix} \oplus [1] \oplus [1] := L(4);$$

$$G_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 2 & -1-\sqrt{2} \\ 0 & -1 & -1-\sqrt{2} & 2 \end{pmatrix}, \ L_5 \simeq [-5-4\sqrt{2}] \oplus [1] \oplus [1] \oplus [1] := L(5);$$

$$G_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 2 & -1-2\sqrt{2} \\ 0 & -1 & -1-2\sqrt{2} & 2 \end{pmatrix}, \ L_6 \simeq [-11-8\sqrt{2}] \oplus [1] \oplus [1] \oplus [1] := L(6);$$

$$G_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2-2\sqrt{2} \\ 0 & 0 & -2-2\sqrt{2} & 2 \end{pmatrix}, \ L_7 = [G_7];$$

Its unique extension is an extension of „index $\sqrt{2}$“

$$L(7) := [-\sqrt{2}] \oplus [1] \oplus [1] \oplus [1].$$

$$G_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -\sqrt{2} \\ 0 & -1 & -\sqrt{2} & 2 \end{pmatrix}, \ L_8 = [G_8];$$

Its unique extension is an extension of „index $\sqrt{2}$“

$$L(8) := \begin{bmatrix} 2 & -1 & -\sqrt{2} \\ -1 & 2 & \sqrt{2}-1 \\ -\sqrt{2} & \sqrt{2}-1 & 2-\sqrt{2} \end{bmatrix} \oplus [1].$$

$$G_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -\sqrt{2}-1 \\ 0 & -1 & -\sqrt{2}-1 & 2 \end{pmatrix}, \ L_9 = [G_9];$$

Its unique extension is an extension of „index $\sqrt{2}$“

$$L(9) := \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -\sqrt{2} \end{bmatrix} \oplus [1].$$

$$G_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -\sqrt{2}-2 \\ 0 & -1 & -\sqrt{2}-2 & 2 \end{pmatrix}, \ L_{10} = [G_{10}];$$

Its unique extension is an extension of ,,index $\sqrt{2}$“

$$L(10) := \begin{bmatrix} 2 & -1-\sqrt{2} \\ -1-\sqrt{2} & 2 \end{bmatrix} \oplus [2+\sqrt{2}] \oplus [1].$$

$$G_{11} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -\sqrt{2}-1 \\ -1 & -1 & -\sqrt{2}-1 & 2 \end{pmatrix}, \ L_{11} \simeq [-7-6\sqrt{2}] \oplus [1] \oplus [1] \oplus [1] := L(11);$$

$$G_{12} = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & -2\sqrt{2}-1 \\ -1 & -1 & -2\sqrt{2}-1 & 2 \end{pmatrix}, \ L_{12} = [G_{12}];$$

Its unique extension is an extension of index 2

$$L(12) := [-7-5\sqrt{2}] \oplus [1] \oplus [1] \oplus [1].$$

$$G_{13} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -\sqrt{2} \\ -1 & -1 & -\sqrt{2} & 2 \end{pmatrix}, \ L_{13} = [G_{13}] := L(13);$$

$$G_{14} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -\sqrt{2}-1 \\ -1 & -1 & -\sqrt{2}-1 & 2 \end{pmatrix}, \ L_{14} = [G_{14}] := L(14);$$

$$G_{15} = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -\sqrt{2} \\ -1 & -1 & 2 & -\sqrt{2}-1 \\ -1 & -\sqrt{2} & -\sqrt{2}-1 & 2 \end{pmatrix}, \ L_{15} = [G_{15}] := L(15).$$

§ 5. Testing for stably reflectivity and proof of Main Theorem 2

Thus, we have 15 candidate lattices $L(1)$–$L(15)$. For each lattice $L(k)$ we will use Vinberg's algorithm of constructing the fundamental Coxeter polytope for the group $\mathcal{O}_r(L(k))$. After that it remains to apply Lemma 3.1.

5.1. Lattices with an orthogonal basis. First of all, we study candidate lattices with an orthogonal basis. We apply software implementation `AlVin` (which is due to R. Guglielmetti) of Vinberg's algorithm. This program is written for Lorentzian lattices associated with diagonal quadratic forms with square-free coefficients.

Proposition 5.1. *The lattice $L(1) = [-(1+\sqrt{2})] \oplus [1] \oplus [1] \oplus [1]$ is stably reflective.*

Proof. For $L(1)$ we apply Vinberg's algorithm. `AlVin` found first 5 roots:
$a_1 = (0, -1, 1, 0)$,

$a_2 = (0, 0, -1, 1)$,
$a_3 = (0, 0, 0, -1)$,
$a_4 = \big(1, 1+\sqrt{2}, 0, 0\big)$,
$a_5 = \big(1+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}\big)$.
The Gram matrix of these roots is equal to

$$G(a_1, a_2, a_3, a_4, a_5) = \begin{pmatrix} 2 & -1 & 0 & -\sqrt{2}-1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & -\sqrt{2}-1 \\ -\sqrt{2}-1 & 0 & 0 & \sqrt{2}+2 & 0 \\ 0 & 0 & -\sqrt{2}-1 & 0 & \sqrt{2}+2 \end{pmatrix}$$

It corresponds to a compact 3-dimensional Coxeter polytope. One can see, looking at this Gram matrix, that the group generated by "bad" reflections is trivial. Hence, the lattice $L(1)$ is reflective and stably reflective. ∎

Proposition 5.2. *The lattice $L(2) = [-(1+2\sqrt{2})] \oplus [1] \oplus [1] \oplus [1]$ is stably reflective.*

Proof. For $L(2)$ we apply Vinberg's algorithm. The program found 6 roots:
$a_1 = (0, -1, 1, 0)$,
$a_2 = (0, 0, -1, 1)$,
$a_3 = (0, 0, 0, -1)$,
$a_4 = \big(1, 1+\sqrt{2}, 0, 0\big)$,
$a_5 = \big(1+\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, 1\big)$.
$a_6 = \big(1+\sqrt{2}, 2+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}\big)$.
The Gram matrix of these roots equals

$$G(a_1, a_2, a_3, a_4, a_5, a_6) = \begin{pmatrix} 2 & -1 & 0 & -\sqrt{2}-1 & 0 & -1 \\ -1 & 2 & -1 & 0 & -\sqrt{2}-1 & 0 \\ 0 & -1 & 1 & 0 & -1 & -\sqrt{2}-1 \\ -\sqrt{2}-1 & 0 & 0 & 2 & -1 & -1 \\ 0 & -\sqrt{2}-1 & -1 & -1 & 2 & 0 \\ -1 & 0 & -\sqrt{2}-1 & -1 & 0 & 1 \end{pmatrix}$$

It corresponds to a compact 3-dimensional Coxeter polytope. One can see, looking at this Gram matrix, that the group generated by "bad" reflections is trivial. Hence, the lattice $L(2)$ is reflective and stably reflective.

∎

Proposition 5.3. *The lattice $L(5) = [-(5+4\sqrt{2})] \oplus [1] \oplus [1] \oplus [1]$ is stably reflective.*

Proof. For the lattice $L(5)$ we apply Vinberg's algorithm. The program found 5 roots:
$a_1 = (0, -1, 1, 0)$,
$a_2 = (0, 0, -1, 1)$,
$a_3 = (0, 0, 0, -1)$,
$a_4 = \big(1, 3+\sqrt{2}, 0, 0\big)$,
$a_5 = \big(1, 1+\sqrt{2}, 1+\sqrt{2}, 1\big)$.
The Gram matrix of these roots has the following form:

$$G(a_1, a_2, a_3, a_4, a_5) = \begin{pmatrix} 2 & -1 & 0 & -\sqrt{2}-3 & 0 \\ -1 & 2 & -1 & 0 & -\sqrt{2} \\ 0 & -1 & 1 & 0 & -1 \\ -\sqrt{2}-3 & 0 & 0 & 2\sqrt{2}+6 & 0 \\ 0 & -\sqrt{2} & -1 & 0 & 2 \end{pmatrix}$$

It corresponds to a compact 3-dimensional Coxeter polytope. One can see, looking at this Gram matrix, that the group generated by "bad" reflections is trivial. Hence, the lattice $L(5)$ is reflective and stably reflective.

■

Proposition 5.4. *The lattice* $L(6) = [-(11+8\sqrt{2})] \oplus [1] \oplus [1] \oplus [1]$ *is stably reflective.*

Proof. For the lattice $L(6)$ we apply Vinberg's algorithm. The program found 17 roots:
$a_1 = (0, -1, 1, 0)$,
$a_2 = (0, 0, -1, 1)$,
$a_3 = (0, 0, 0, -1)$,
$a_4 = \big(1, 2+\sqrt{2}, 2+\sqrt{2}, 1\big)$,
$a_5 = \big(1, 2+2\cdot\sqrt{2}, 1, 0\big)$,
$a_6 = \big(1, 2+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}\big)$,
$a_7 = \big(2+\sqrt{2}, 7+5\cdot\sqrt{2}, 3+3\cdot\sqrt{2}, 2+\sqrt{2}\big)$,
$a_8 = \big(1+2\cdot\sqrt{2}, 8+5\cdot\sqrt{2}, 4+3\cdot\sqrt{2}, 3+2\cdot\sqrt{2}\big)$,
$a_9 = \big(1+2\cdot\sqrt{2}, 8+6\cdot\sqrt{2}, 3+2\cdot\sqrt{2}, 2+2\cdot\sqrt{2}\big)$,
$a_{10} = \big(2+3\cdot\sqrt{2}, 13+9\cdot\sqrt{2}, 7+5\cdot\sqrt{2}, 2+\sqrt{2}\big)$,
$a_{11} = \big(4+2\cdot\sqrt{2}, 13+10\cdot\sqrt{2}, 9+6\cdot\sqrt{2}, 0\big)$,
$a_{12} = \big(4+4\cdot\sqrt{2}, 19+14\cdot\sqrt{2}, 9+6\cdot\sqrt{2}, 8+6\cdot\sqrt{2}\big)$,
$a_{13} = \big(4+4\cdot\sqrt{2}, 20+14\cdot\sqrt{2}, 11+8\cdot\sqrt{2}, 1\big)$,
$a_{14} = \big(4+2\cdot\sqrt{2}, 14+10\cdot\sqrt{2}, 6+4\cdot\sqrt{2}, 5+4\cdot\sqrt{2}\big)$,
$a_{15} = \big(4+3\cdot\sqrt{2}, 17+12\cdot\sqrt{2}, 8+5\cdot\sqrt{2}, 6+4\cdot\sqrt{2}\big)$,
$a_{16} = \big(4+3\cdot\sqrt{2}, 17+12\cdot\sqrt{2}, 9+7\cdot\sqrt{2}, 1+\sqrt{2}\big)$,
$a_{17} = \big(5+4\cdot\sqrt{2}, 22+15\cdot\sqrt{2}, 13+9\cdot\sqrt{2}, 1+\sqrt{2}\big)$.
The Gram matrix of this collection of roots corresponds to a compact 3-dimensional Coxeter polytope.
This matrix has the following main diagonal:

$$(2, 2, 1, 2, 2, 1, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1).$$

One can see, looking at this diagonal, that the group generated by "bad" reflections is trivial. Hence, the lattice $L(6)$ is reflective and stably reflective. ■

Proposition 5.5. *The lattice* $L(7) = [-\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$ *is stably reflective.*

Proof. For the lattice $L(6)$ we apply Vinberg's algorithm. The program found 6 roots:
$a_1 = (0, -1, 1, 0)$,
$a_2 = (0, 0, -1, 1)$,
$a_3 = (0, 0, 0, -1)$,
$a_4 = \big(1+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}, 0\big)$,
$a_5 = \big(1+\sqrt{2}, 2+\sqrt{2}, 0, 0\big)$.
$a_6 = \big(2+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}\big)$.
The Gram matrix of these roots is

$$G(a_1, a_2, a_3, a_4, a_5, a_6) = \begin{pmatrix} 2 & -1 & 0 & 0 & -\sqrt{2}-2 & 0 \\ -1 & 2 & -1 & -\sqrt{2}-1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -\sqrt{2}-1 \\ 0 & -\sqrt{2}-1 & 0 & \sqrt{2}+2 & 0 & 0 \\ -\sqrt{2}-2 & 0 & 0 & 0 & \sqrt{2}+2 & -\sqrt{2}-2 \\ 0 & 0 & -\sqrt{2}-1 & 0 & -\sqrt{2}-2 & 1 \end{pmatrix}$$

It corresponds to a compact 3-dimensional Coxeter polytope. One can see, looking at this Gram matrix, that the group generated by "bad" reflections is trivial. Hence, the lattice $L(6)$ is reflective and stably reflective. ∎

Proposition 5.6. *The lattice $L(11) = [-7-6\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$ is reflective, but not stably reflective.*

Proof. For the lattice $L(11)$ we apply Vinberg's algorithm. The program found 10 roots:
$a_1 = (0,\ -1,\ 1,\ 0)$,
$a_2 = (0,\ 0,\ -1,\ 1)$,
$a_3 = (0,\ 0,\ 0,\ -1)$,
$a_4 = \big(1,\ \sqrt{2}+1,\ \sqrt{2}+1,\ \sqrt{2}+1\big)$,
$a_5 = \big(1,\ \sqrt{2}+2,\ \sqrt{2}+1,\ 0\big)$,
$a_6 = \big(2\sqrt{2}+1,\ 6\sqrt{2}+7,\ 0,\ 0\big)$,
$a_7 = \big(\sqrt{2}+1,\ 3\sqrt{2}+5,\ \sqrt{2}+1,\ 1\big)$,
$a_8 = \big(\sqrt{2}+1,\ 3\sqrt{2}+4,\ \sqrt{2}+2,\ \sqrt{2}+2\big)$,
$a_9 = \big(4\sqrt{2}+6,\ 13\sqrt{2}+19,\ 7\sqrt{2}+12,\ 6\sqrt{2}+7\big)$,
$a_{10} = \big(2\sqrt{2}+2,\ 6\sqrt{2}+9,\ 2\sqrt{2}+3,\ 2\sqrt{2}+2\big)$.
The Gram matrix of this set of roots corresponds to a compact 3-dimensional Coxeter polytope. The main diagonal of this matrix equals

$$\{2, 2, 1, 2, 2, 2\sqrt{2}+10, 2, 1, 2\sqrt{2}+10\}.$$

It remains to see that the group generated by "bad" reflections with respect to mirrors H_{a_6} and $H_{a_{10}}$, is infinite, since the respective vertices of the Coxeter diagram are connected by the dotted edged. Hence, the lattice $L(11)$ is reflective, but not stably reflective. ∎

Proposition 5.7. *The lattice $L(12) = [-7-5\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$ is stably reflective.*

Proof. For the lattice $L(12)$ we apply Vinberg's algorithm. The program found 5 roots:
$a_1 = (0, -1, 1, 0)$,
$a_2 = (0, 0, -1, 1)$,
$a_3 = (0, 0, 0, -1)$,
$a_4 = \big(2-\sqrt{2}, 1+\sqrt{2}, 1, 0\big)$,
$a_5 = \big(1, 1+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}\big)$.
The Gram matrix of these roots is

$$G(a_1, a_2, a_3, a_4, a_5) = \begin{pmatrix} 2 & -1 & 0 & -\sqrt{2} & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & -\sqrt{2}-1 \\ -\sqrt{2} & -1 & 0 & 2 & 0 \\ 0 & 0 & -\sqrt{2}-1 & 0 & \sqrt{2}+2 \end{pmatrix}$$

It corresponds to a compact 3-dimensional Coxeter polytope. One can see, looking at this Gram matrix, that the group generated by "bad" reflections is trivial. Hence, the lattice $L(12)$ is reflective and stably reflective. ∎

5.2. The lattices with non-orthogonal basis.

Proposition 5.8. *The lattice $L(3) = \begin{bmatrix} 2 & -1-\sqrt{2} \\ -1-\sqrt{2} & 2 \end{bmatrix} \oplus [1] \oplus [1]$ is reflective, but not stably reflective.*

Proof. Notice that $L(3)$ is isomorphic to the lattice with coordinates

$$y = (y_0, y_1, y_2, y_3) \in \mathbb{Q}^4[\sqrt{2}]$$

and with inner product, given by the quadratic form

$$f(y) = -(2\sqrt{2} - 1)y_0^2 + y_1^2 + y_2^2 + y_3^2,$$

where

$$\sqrt{2}y_0 \in \mathbb{Z}[\sqrt{2}], \quad y_0 + \frac{y_0 + y_1}{\sqrt{2}} \in \mathbb{Z}[\sqrt{2}], \quad y_2, y_3 \in \mathbb{Z}[\sqrt{2}].$$

Our program `VinAl` finds 8 roots
$a_1 = (0,\ 0,\ 0,\ -1)$,
$a_2 = (0,\ 0,\ -1,\ 1)$,
$a_3 = \big(0,\ -\sqrt{2},\ 0,\ 0\big)$,
$a_4 = \big(1 + \sqrt{2},\ 3 + \sqrt{2},\ 0, 0\big)$,
$a_5 = \big(1 + \sqrt{2}/2,\ \sqrt{2}/2,\ \sqrt{2} + 1,\ 0\big)$,
$a_6 = \big(\sqrt{2} + 1,\ \sqrt{2} + 1,\ \sqrt{2} + 1,\ 1\big)$,
$a_7 = \big(\sqrt{2} + 1,\ 1,\ \sqrt{2} + 1,\ \sqrt{2} + 1\big)$,
$a_8 = \big(3\sqrt{2} + 4,\ 0,\ 4\sqrt{2} + 5,\ \sqrt{2} + 3\big)$,
the Gram matrix of which has the following form

$$\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & -1 & -1-\sqrt{2} & -3-\sqrt{2} \\
-1 & 2 & 0 & 0 & -1-\sqrt{2} & -\sqrt{2} & 0 & -2-3\sqrt{2} \\
0 & 0 & 2 & -2-3\sqrt{2} & -1 & -2-\sqrt{2} & -\sqrt{2} & 0 \\
0 & 0 & -2-3\sqrt{2} & 6+\sqrt{2} & -3-\sqrt{2} & 0 & -2-3\sqrt{2} & -18-13\sqrt{2} \\
0 & -1-\sqrt{2} & -1 & -3-\sqrt{2} & 1 & 0 & -1 & 0 \\
-1 & -\sqrt{2} & -2-\sqrt{2} & 0 & 0 & 2 & 0 & -2-3\sqrt{2} \\
-1-\sqrt{2} & 0 & -\sqrt{2} & -2-3\sqrt{2} & -1 & 0 & 2 & 0 \\
-3-\sqrt{2} & -2-3\sqrt{2} & 0 & -18-13\sqrt{2} & 0 & -2-3\sqrt{2} & 0 & 6+\sqrt{2}
\end{pmatrix}.$$

It is sufficient to consider the group generated by "bad" reflections with respect to mirrors H_{a_4} and H_{a_8}. Since these mirrors are divergent, this subgroup is infinite. ■

Proposition 5.9. *The lattice* $L(4) = \begin{bmatrix} 2 & -1-2\sqrt{2} \\ -1-2\sqrt{2} & 2 \end{bmatrix} \oplus [1] \oplus [1]$ *is reflective, but not stably reflective.*

Proof. Notice that $L(4)$ is isomorphic to the lattice with coordinates

$$y = (y_0, y_1, y_2, y_3) \in \mathbb{Q}^4[\sqrt{2}]$$

and with inner product, given by the quadratic form

$$f(y) = -(5 + 4\sqrt{2})y_0^2 + y_1^2 + y_2^2 + y_3^2,$$

where

$$\sqrt{2}y_0 \in \mathbb{Z}[\sqrt{2}], \quad \frac{y_0 + y_1}{\sqrt{2}} \in \mathbb{Z}[\sqrt{2}], \quad y_2, y_3 \in \mathbb{Z}[\sqrt{2}].$$

Our program `VinAl` finds 8 roots
$a_1 = (0,\ 0,\ 0,\ -1)$,
$a_2 = (0,\ 0,\ -1,\ 1)$,
$a_3 = \big(0,\ -\sqrt{2},\ 0,\ 0\big)$,
$a_4 = \big(1,\ 3 + \sqrt{2},\ 0, 0\big)$,
$a_5 = \big(\sqrt{2}/2,\ \sqrt{2}/2,\ \sqrt{2} + 1,\ 0\big)$,

$a_6 = \big(1,\, 1,\, \sqrt{2}+1,\, \sqrt{2}+1\big),$
$a_7 = \big(1,\, \sqrt{2}+1,\, \sqrt{2}+1,\, 1\big),$
$a_8 = \big(\sqrt{2}+2,\, 0,\, 4\sqrt{2}+5,\, \sqrt{2}+3\big),$
the Gram matrix of which has the following form

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & -1 & -1-\sqrt{2} & -3-\sqrt{2} \\ -1 & 2 & 0 & 0 & -1-\sqrt{2} & -\sqrt{2} & 0 & -2-3\sqrt{2} \\ 0 & 0 & 2 & -2-3\sqrt{2} & -1 & -2-\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & -2-3\sqrt{2} & 6+\sqrt{2} & -3-\sqrt{2} & 0 & -2-3\sqrt{2} & -18-13\sqrt{2} \\ 0 & -1-\sqrt{2} & -1 & -3-\sqrt{2} & 1 & 0 & -1 & 0 \\ -1 & -\sqrt{2} & -2-\sqrt{2} & 0 & 0 & 2 & 0 & -2-3\sqrt{2} \\ -1-\sqrt{2} & 0 & -\sqrt{2} & -2-3\sqrt{2} & -1 & 0 & 2 & 0 \\ -3-\sqrt{2} & -2-3\sqrt{2} & 0 & -18-13\sqrt{2} & 0 & -2-3\sqrt{2} & 0 & 6+\sqrt{2} \end{pmatrix}.$$

It is sufficient to consider the group generated by "bad" reflections with respect to mirrors H_{a_4} and H_{a_8}. Since these mirrors are divergent, this subgroup is infinite. ∎

Proposition 5.10. *The lattice* $L(8) := \begin{bmatrix} 2 & -1 & -\sqrt{2} \\ -1 & 2 & \sqrt{2}-1 \\ -\sqrt{2} & \sqrt{2}-1 & 2-\sqrt{2} \end{bmatrix} \oplus [1]$ *is stably reflective.*

Proof. Notice that $L(8)$ is isomorphic to the lattice with coordinates

$$y = (y_0, y_1, y_2, y_3) \in \mathbb{Q}^4[\sqrt{2}]$$

and with inner product, given by the quadratic form

$$f(y) = -\sqrt{2}\, y_0^2 + y_1^2 + y_2^2 + y_3^2,$$

where

$$y_0 \in \mathbb{Z}[\sqrt{2}], \quad -y_2 + \frac{y_1 + y_2}{\sqrt{2}} \in \mathbb{Z}[\sqrt{2}], \quad \sqrt{2}y_2, y_3 \in \mathbb{Z}[\sqrt{2}].$$

Our program `VinAl` finds 6 roots
$a_1 = \big(0,\, 0,\, 0,\, -\sqrt{2}\big),$
$a_2 = \big(0,\, 0,\, -\sqrt{2},\, 0\big),$
$a_3 = \big(0,\, -\sqrt{2}/2,\, 1+\sqrt{2}/2,\, 0\big),$
$a_4 = \big(1+\sqrt{2},\, 2+\sqrt{2},\, 0, 0\big),$
$a_5 = \big(1+\sqrt{2}, 0, 0, \sqrt{2}+2, \big),$
$a_6 = \big(2+\sqrt{2}, 2+\sqrt{2}, 0, 2+\sqrt{2}\big),$
the Gram matrix of which has the following form

$$\begin{pmatrix} 2 & 0 & 0 & 0 & -2-2\sqrt{2} & -2-2\sqrt{2} \\ 0 & 2 & -1-\sqrt{2} & 0 & 0 & 0 \\ 0 & -1-\sqrt{2} & 2+\sqrt{2} & -1-\sqrt{2} & 0 & -1-\sqrt{2} \\ 0 & 0 & -1-\sqrt{2} & 2+\sqrt{2} & -4-3\sqrt{2} & 0 \\ -2-2\sqrt{2} & 0 & 0 & -4-3\sqrt{2} & 2+\sqrt{2} & 0 \\ -2-2\sqrt{2} & 0 & -1-\sqrt{2} & 0 & 0 & 4+2\sqrt{2} \end{pmatrix}.$$

The subgroup generated by "bad" reflections is trivial. ∎

Proposition 5.11. *The lattice* $L(9) := \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -\sqrt{2} \end{bmatrix} \oplus [1]$ *is not stably reflective.*

Proof. Notice that $L(9)$ is isomorphic to the lattice with coordinates

$$y = (y_0, y_1, y_2, y_3) \in \mathbb{Q}^4[\sqrt{2}]$$

and with inner product, given by the quadratic form

$$f(y) = -\sqrt{2}\, y_0^2 + (3+\sqrt{2})y_1^2 + y_2^2 + y_3^2,$$

where

$$\sqrt{2}\, y_1 \in \mathbb{Z}[\sqrt{2}], \quad \frac{y_1 + y_2}{\sqrt{2}} \in \mathbb{Z}[\sqrt{2}], \quad y_1 - y_0,\ y_3 \in \mathbb{Z}[\sqrt{2}].$$

Our program `VinAl` finds 9 roots
$a_1 = (0,\ 0,\ 0,\ -1)$,
$a_2 = \big(0,\ 0,\ -\sqrt{2},\ 0\big)$,
$a_3 = \big(0,\ -\sqrt{2},\ 0,\ 0\big)$,
$a_4 = \big(1+\sqrt{2}, 0, 0, 2+\sqrt{2}\big)$,
$a_5 = \big(1+\sqrt{2}, 0, 2+\sqrt{2}, 0\big)$,
$a_6 = \big(2+\sqrt{2}, 0, 2+\sqrt{2}, 2+\sqrt{2}\big)$,
$a_7 = \big(1+\sqrt{2}, 1, 1+\sqrt{2}, 0\big)$,
$a_8 = \big(2+3\sqrt{2}/2, 1+\sqrt{2}/2, 1+\sqrt{2}/2, 2+\sqrt{2}\big)$,
$a_9 = \big(5+4\sqrt{2}, 2+\sqrt{2}, 0, 5+4\sqrt{2}\big)$,
the Gram matrix of which has the following form

$$\begin{pmatrix}
1 & 0 & 0 & -2-\sqrt{2} & 0 & -2-\sqrt{2} & 0 & -2-\sqrt{2} & -5-4\sqrt{2} \\
0 & 2 & 0 & 0 & -2-2\sqrt{2} & -2-2\sqrt{2} & -2-\sqrt{2} & -1-\sqrt{2} & 0 \\
0 & 0 & 6+2\sqrt{2} & 0 & 0 & 0 & -2-3\sqrt{2} & -5-4\sqrt{2} & -10-8\sqrt{2} \\
-2-\sqrt{2} & 0 & 0 & 2+\sqrt{2} & -4-3\sqrt{2} & 0 & -4-3\sqrt{2} & -1-\sqrt{2} & 0 \\
0 & -2-2\sqrt{2} & 0 & -4-3\sqrt{2} & 2+\sqrt{2} & 0 & 0 & -4-3\sqrt{2} & -18-13\sqrt{2} \\
-2-\sqrt{2} & -2-2\sqrt{2} & 0 & 0 & 0 & 4+2\sqrt{2} & -2-\sqrt{2} & -1-\sqrt{2} & -8-5\sqrt{2} \\
0 & -2-\sqrt{2} & -2-3\sqrt{2} & -4-3\sqrt{2} & 0 & -2-\sqrt{2} & 2 & -1-\sqrt{2} & -10-8\sqrt{2} \\
-2-\sqrt{2} & -1-\sqrt{2} & -5-4\sqrt{2} & -1-\sqrt{2} & -4-3\sqrt{2} & -1-\sqrt{2} & -1-\sqrt{2} & 2+\sqrt{2} & 0 \\
-5-4\sqrt{2} & 0 & -10-8\sqrt{2} & 0 & -18-13\sqrt{2} & -8-5\sqrt{2} & -10-8\sqrt{2} & 0 & 3+\sqrt{2}
\end{pmatrix}.$$

It is sufficient to consider the group generated by "bad" reflections with respect to mirrors H_{a_3}, H_{a_6} and H_{a_9}. Since the mirrors H_{a_6} and H_{a_9} are divergent, this subgroup is infinite.

■

Proposition 5.12. *The lattice* $L(10) := \begin{bmatrix} 2 & -1-\sqrt{2} \\ -1-\sqrt{2} & 2 \end{bmatrix} \oplus [2+\sqrt{2}] \oplus [1]$ *is not stably reflective.*

Proof. Notice that $L(10)$ is isomorphic to the lattice with coordinates

$$y = (y_0, y_1, y_2, y_3) \in \mathbb{Q}^4[\sqrt{2}]$$

and with inner product, given by the quadratic form

$$f(y) = -(1+2\sqrt{2})\, y_0^2 + (2+\sqrt{2})y_1^2 + y_2^2 + y_3^2,$$

where

$$\sqrt{2}\, y_0 \in \mathbb{Z}[\sqrt{2}], \quad \frac{y_0 + y_2}{\sqrt{2}} \in \mathbb{Z}[\sqrt{2}], \quad y_1, y_3 \in \mathbb{Z}[\sqrt{2}].$$

Our program `VinAl` finds 8 roots
$a_1 = (0,\ 0,\ 0,\ -1)$,
$a_2 = \big(0,\ 0,\ -\sqrt{2},\ 0\big)$,
$a_3 = \big(0,\ -\sqrt{2},\ 0,\ 0\big)$,
$a_4 = \big(\sqrt{2}/2,\ 0,\ 2+\sqrt{2}/2, 0\big)$,

$a_5 = \big(1+\sqrt{2}/2,\, 0, 1+\sqrt{2}/2,\, \sqrt{2}+2\big)$,
$a_6 = \big(1+\sqrt{2},\, 1+\sqrt{2},\, \sqrt{2}+1, 0\big)$,
$a_7 = \big(2+\sqrt{2}, 2+\sqrt{2}, 0, 2+\sqrt{2}\big)$,
$a_8 = \big(2+\sqrt{2},\, 1+2\sqrt{2}, 0, 0\big)$,
the Gram matrix of which has the following form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2-\sqrt{2} & 0 & -2-\sqrt{2} & 0 \\ 0 & 2 & 0 & -1-2\sqrt{2} & -1-\sqrt{2} & -2-\sqrt{2} & 0 & 0 \\ 0 & 0 & 4+2\sqrt{2} & 0 & 0 & -6-4\sqrt{2} & -8-6\sqrt{2} & -10-6\sqrt{2} \\ 0 & -1-2\sqrt{2} & 0 & 4+\sqrt{2} & 0 & 0 & -5-3\sqrt{2} & -5-3\sqrt{2} \\ -2-\sqrt{2} & -1-\sqrt{2} & 0 & 0 & 2+\sqrt{2} & -6-4\sqrt{2} & -5-4\sqrt{2} & -11-8\sqrt{2} \\ 0 & -2-\sqrt{2} & -6-4\sqrt{2} & 0 & -6-4\sqrt{2} & 2+\sqrt{2} & -2-\sqrt{2} & 0 \\ -2-\sqrt{2} & 0 & -8-6\sqrt{2} & -5-3\sqrt{2} & -5-4\sqrt{2} & -2-\sqrt{2} & 4+2\sqrt{2} & 0 \\ 0 & 0 & -10-6\sqrt{2} & -5-3\sqrt{2} & -11-8\sqrt{2} & 0 & 0 & 4+\sqrt{2} \end{pmatrix}.$$

It is sufficient to consider the group generated by "bad" reflections with respect to mirrors H_{a_3} and H_{a_8}. Since these mirrors are divergent, this subgroup is infinite. ■

Proposition 5.13. *The lattices $L(13)$, $L(14)$ and $L(15)$ are not reflective.*

Proof. Non-reflectivity of these lattices is determined by the method of infinite symmetry, described in §3.3. The implementation of this method is avalaible here https://github.com/nvbogachev/VinAlg-Z-sqrt-2-/blob/master/Infinite-Symm.py ■

Theorem 5.1. *Each maximal stably reflective Lorentzian lattice of rank 4 over $\mathbb{Z}[\sqrt{2}]$ is isomorphic to one of the following 7 lattices:*

No.	L	*# facets*	*Discriminant*
1	$[-1-\sqrt{2}]\oplus[1]\oplus[1]\oplus[1]$	5	$-1-\sqrt{2}$
2	$[-1-2\sqrt{2}]\oplus[1]\oplus[1]\oplus[1]$	6	$-1-2\sqrt{2}$
3	$[-5-4\sqrt{2}]\oplus[1]\oplus[1]\oplus[1]$	5	$-5-4\sqrt{2}$
4	$[-11-8\sqrt{2}]\oplus[1]\oplus[1]\oplus[1]$	17	$-11-8\sqrt{2}$
5	$[-\sqrt{2}]\oplus[1]\oplus[1]\oplus[1]$	6	$-\sqrt{2}$
6	$\begin{bmatrix} 2 & -1 & -\sqrt{2} \\ -1 & 2 & \sqrt{2}-1 \\ -\sqrt{2} & \sqrt{2}-1 & 2-\sqrt{2} \end{bmatrix}\oplus[1]$	6	$-\sqrt{2}$
7	$[-7-5\sqrt{2}]\oplus[1]\oplus[1]\oplus[1]$	5	$-7-5\sqrt{2}$

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