Группа Торелли

16.04.2021

S_g

$\mathrm{Mod}_g = \pi_0 \mathrm{Diff}^+(S_g)$ $\quad$ $T_\gamma \in \mathrm{Mod}_g$

$a_1 \quad a_2 \quad \cancel{b_2} \quad a_2 \quad a_3 \quad b_3$

$T_g \subset \mathrm{Mod}_g$

$\mathrm{Mod}_g \curvearrowright H = H_1(S_g)$

$T_\gamma a_1 = a_1$

$\vdots$

$T_\gamma b_3 = b_3$

$T_\gamma b_2 = b_2 + a_2$

$-(b_1, a_1) = (a_1, b_1) = 1$

$(a_2, b_2) = 1$

$(a_3, b_3) = 1$

$$M = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

Утв. $\mathrm{Mod}_g \curvearrowright H$ $\quad (a, b) = (T\cdot a, T\cdot b)$

$$\mathrm{Mod}_g \xrightarrow{p} Sp(2g)$$

Утв. p – эпиморфизм $\quad \det A = 1 \quad S^T = S$

$$Sp(2g) = \left\langle \begin{pmatrix} A^T & 0 \\ 0 & A^{-1} \end{pmatrix}, \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}, \right.$$

$$\left\{\left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 1 & & & 0 & \\ & & 1 & & & 0 \\ \hline -1 & & & 0 & & \\ & 0 & & & 1 & \\ & & 0 & & & 1 \end{array}\right)\right\}\rangle$$

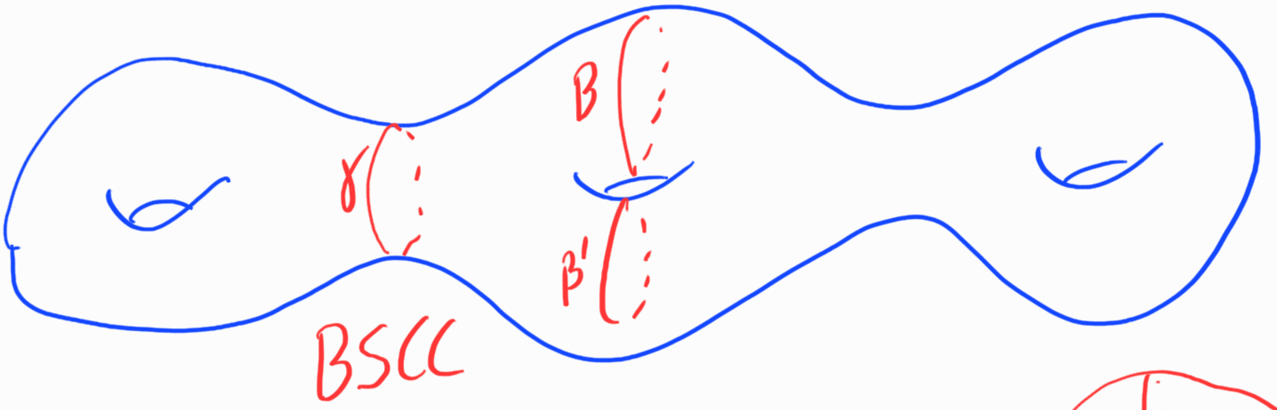
$$T_{a_i} : b_i \longmapsto b_i + a_i$$

$$T_{a_i+b_j} T_{b_j}^{-1} T_{a_i}^{-1} : \begin{array}{l} a_j \longmapsto a_j - a_i \\ b_i \longmapsto b_i + b_j \end{array}$$

$$0 \to \underset{\sim}{T_g} \to \mathrm{Mod}_g \to Sp(2g) \to 0$$

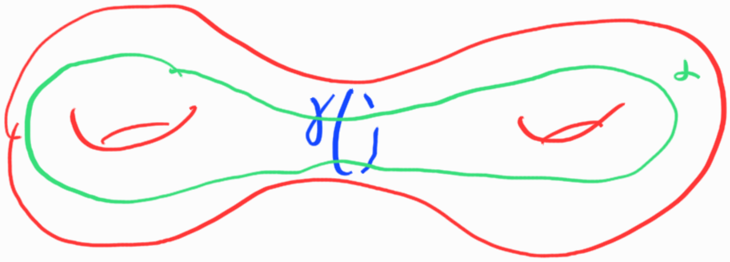
Группа Торелли

Powell '77



$$T_g = \langle \{ T_\gamma , T_\beta T_{\beta'}^{-1} \} \rangle \qquad T_1 = 1$$

при $g \geq 3$ T_g конечно порождена.



Mod_g
$\rotatebox{90}{$\in$}$
T_α

$$\boxed{\underbrace{T_{\gamma'}} = T_\alpha \underbrace{T_\gamma} T_\alpha^{-1}} \quad \underbrace{\gamma'} = \underbrace{T_\alpha \cdot \gamma}$$

$$(T_g)_{ab} = \underline{\mathbb{Z}} \oplus \mathbb{Z}_2$$

Гомоморфизм Джонсона

$S_{g,1}$ $\quad \mathrm{Mod}_{g,1} \supset T_{g,1}$

1) $T_{g,1} \longrightarrow \mathrm{Hom}(H, H \wedge H)$

$T \in T_{g,1}$

$\pi_1(S_{g,1}) = \langle a_1, b_1, \ldots, a_g, b_g \rangle$

$\alpha \in \pi_1(S_{g,1})$

$x_T(\alpha) = (T \cdot \alpha)\,\alpha^{-1} \in \pi_1(S_{g,1})$

$[x_T(\alpha)] \in \pi_1'$

$$[\pi_1,\pi_1]/[\pi_1,[\pi_1,\pi_1]] \cong H\wedge H$$

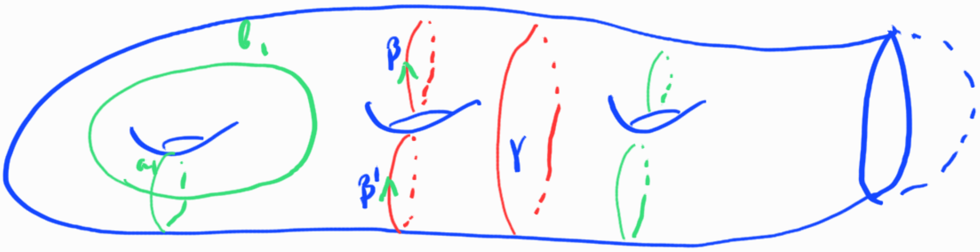
$$[a_i, b_j] \longmapsto [a_i]\wedge[b_j]$$

$$T \longmapsto \left([\alpha] \longmapsto [X_T(\alpha)]\right)$$

$$H \longrightarrow H\wedge H$$

2)
$$\begin{tikzcd}
 & \mathrm{Hom}(H, H\wedge H) \arrow[r] & H\otimes(H\wedge H) \\
\mathcal{T}_{g,1} \arrow[ur] \arrow[urr, bend right, "\tau"'] & &
\end{tikzcd}$$

$$\underline{\mathrm{Im}\,\tau = H\wedge H\wedge H = \Lambda^3 H}$$



$$\underbrace{\sum(a_i\wedge b_i)}\wedge[\beta]$$

$$\tau(T_\gamma) = 0$$

$$\tau(T_\beta T_{\beta'}^{-1}) = \underbrace{(a_1\wedge b_1)\wedge \overset{-[\beta']}{\overset{\|}{[\beta]}}}$$

3) $\pi_1(U'S_g) \longrightarrow \mathcal{T}_{g,1} \twoheadrightarrow \mathcal{T}_g$ $\quad \boxed{S_{g,1} \to S_g}$

$$H \xhookrightarrow{\wedge(\sum_i a_i\wedge b_i)} \Lambda^3 H \longrightarrow \Lambda^3 H/H$$

Утв. $T_g \longrightarrow \Lambda^3 H / H$

$$(T_g)_{abt} = \Lambda^3 H / H$$

Ядро Джонсона K_g

$$1 \to \underset{\sim}{K_g} \longrightarrow T_g \longrightarrow \Lambda^3 H/H \to 1$$

Утв. $K_g = \langle T_\gamma \mid \gamma - BSCC \rangle$

$\dim H_1 K_g^{\otimes\mathbb{C}} < \infty \quad g \geq 4 \quad ??$

Гомологии групп

$G \quad \mathbb{C}$

$$\mathbb{C}\{[g_0|g_1|\dots|g_n]\} = C_n(G) \qquad \rho : G \to \mathbb{C}^\times$$

$$\partial : C_n(G) \to C_{n-1}(G)$$

$$[g_0|g_1|\dots|g_n] \longmapsto \overset{\rho(g_0)}{\curvearrowright}[g_1|\dots|g_n] -$$

$$- [g_0g_1|g_2|\dots|g_n] +$$

$$+ [g_0|g_1g_2|\dots|g_n] \dots$$

$$+ (-1)^n[g_0|g_1|\dots|g_{n-1}g_n] +$$

$$\dots + (-1)^n[g_0|\dots|g_{n-1}]$$

$\partial^2 = 0$ $\quad$ $H_*(G;\mathbb{C}_\rho)$

$\underline{K(G,1)}$ $\quad$ $H^*(G;\mathbb{C}_\rho)$

$$V^k_n = \{\rho \in \{G_{ab} \to \mathbb{C}^\times\} \mid \dim H^k(G;\check{\mathbb{C}}_\rho) \geq n\}$$

$\bigsqcup (\mathbb{C}^\times)^d$ $\quad$ $\ker[C^k \to C^{k+1}]$

$\mathbb{C}[G_{ab}]$ $\quad$ ACC

$$R^k_n \longleftrightarrow V^k_n$$

$$R^1_1(G) \subseteq 0 \Leftrightarrow \dim \underbrace{\widehat{\cancel{H}_1(G')}}_{\mathbb{C}G_{ab}} < \infty$$

$T_g \nearrow$

$K_g \supset T_g'$

$\underline{V^1_1(G)}$ - конечно $\Rightarrow H_1(G')$ - к/м

$\forall K \supset G'$

$H_1(K)$ - к/м

21.04.21

Ассоц. град. алг. Ли

$$G = G_1 \supset \underset{\substack{\shortparallel \\ G_2}}{[G,G]} \supset \underset{\substack{\shortparallel \\ G_3}}{[G,[G,G]]} \supset \dots$$

$$\oplus\; G_i / G_{i+1} = \underbrace{gr_\bullet G}$$

$$gr_i G \times gr_j G \xrightarrow{[\cdot,\cdot]} gr_{i+j} G$$

$$gr_\bullet G \otimes \mathbb{C}$$

$$T_g \hookrightarrow \mathrm{Mod}_g \longrightarrow Sp(2g)$$

$$\underset{\substack{\shortparallel \\ \Lambda^3 H / H}}{(T_g)_{ab}} \curvearrowleft Sp(2g)$$

$$Sp(2g) \subset Sp_{\mathbb{C}}(2g)$$

$$Sp_{\mathbb{C}}(2g) \curvearrowright (T_g)_{ab} \otimes \mathbb{C} = gr_1 T_g$$

$$Sp_{\mathbb{C}}(2g) \curvearrowright gr_\bullet T_g \otimes \mathbb{C}$$

$$s[a,b] = [sa, sb]$$

гр. $T_g \otimes \mathbb{C}$ $sp(2g)$-модули

$$\Lambda^3 H / H$$

$$t_i = \left(\begin{array}{c|c} \begin{matrix} 0 & & & \\ & \overset{i}{1} & & \\ & & 0 & \\ & & & 0 \end{matrix} & \\ \hline & \begin{matrix} 0 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{matrix} \end{array}\right)$$

$t_i \cdot a_i = a_i$

$t_i \cdot b_i = -b_i$

$[t_i, t_j] = 0$

$h \subset sp(2g)$

$n^- \oplus h \oplus n^+ = sp(2g)$

$h^+ = \langle T_i, \underset{i<j}{S_{ij}}, F_{ij} \rangle$

$$\begin{cases} T_i \cdot b_i = a_i \\ S_{ij} \cdot a_j = a_i \\ S_{ij} \cdot b_i = -b_j \\ F_{ij} \cdot b_i = a_i \\ F_{ij} \cdot b_j = a_j \end{cases} \quad i<j$$

$[h, \underset{\mathfrak{g}_\alpha}{\overset{\cap}{g}}] = \alpha(h) g$ $\mathfrak{g} = sp(2g)$

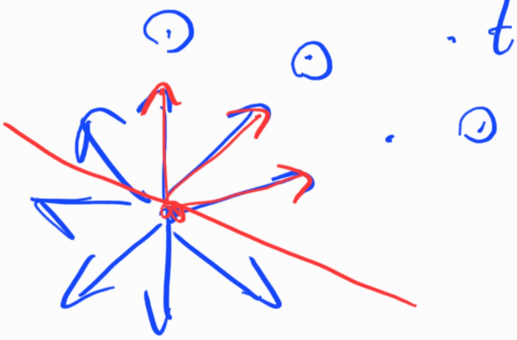
$\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$

$\alpha \in \boxed{h^*}$

$\underset{i,j}{\pm t^i \pm t^j}$; $\pm 2t^i$

$$\boxed{t^i - t^j,\quad t^i + t^j,\quad 2t^i \atop i<j}$$ — полож. корни

$t_i < t_j \quad i<j$



$\mathfrak{g} \curvearrowright V$

$V = \bigoplus_{\alpha} V_\alpha \quad \alpha \in \mathfrak{h}^*$

$\alpha \in \mathbb{Z}\langle t^i \rangle$

$\lambda_i = t^1 + t^2 + \dots + t^i, \quad i \le g$

$\alpha = (\alpha_1 \dots \alpha_s)$

$\boxed{\alpha_1 \lambda_1 + \dots + \alpha_s \lambda_s}$

$V(\alpha) = V(\alpha_1 \lambda_1 + \dots + \alpha_s \lambda_s)$

$V(\lambda_1) = V$

$v_1 \wedge v_2 \mapsto (v_1, v_2)$

$V(\lambda_2) = \mathrm{Ker}[V \wedge V \longrightarrow \mathbb{C}]$

$V(\lambda_3) = \mathrm{Ker}[V \wedge V \wedge V \longrightarrow V]$

$\boxed{V(2\lambda_2)}$ $\qquad v_1 \wedge v_2 \wedge v_3 \longmapsto (v_1 - v_2) v_3 + \sim$

$n = n_1 + \dots + n_k$

$V^{\otimes n} \supset W$

$\mathfrak{gl}(n)$

$\mathfrak{sp}(2n)$

$\mathfrak{so}(n)$

$$\Lambda^2 \mathrm{gr}_1 T_g \xrightarrow[\beta]{[\cdot,\cdot]} \mathrm{gr}_2 T_g$$

$\boxed{\text{Main}}$

$$\Lambda^2 V(\lambda_3) \cong \begin{cases} V(2\lambda_2) + V(0), & g = 3 \\ \vdots & g = 4, 5 \\ V(\lambda_6) + V(\lambda_4) + V(\lambda_4 + \lambda_2) + V(2\lambda_2) + \\ \quad + V(\lambda_2) \qquad + V(\lambda_0) & g \geq 6 \end{cases}$$

$\underline{\text{T. Main}}$ $\quad \underline{\operatorname{Ker} \beta = V(2\lambda_2) + V(\lambda_0)}$

$$R^k_i(G) = \{\alpha \in H^1(G) \mid \dim H^k(H^*(G), u_\alpha) \geq i\}$$

$$u_\alpha = (\alpha \cdot) : H^n(G) \to H^{n+1}(G)$$

$$H^{k-1}(G) \xrightarrow{\alpha\cdot} H^k(G) \xrightarrow{\alpha\cdot} H^{k+1}(G)$$

$\underbrace{R^1_1(G)}_{V^1_1(G)} \ni \alpha \quad \exists \underset{\underset{H^1(G)}{\cap}}{\beta} \neq c\alpha \quad \alpha\beta = 0$

$\boxed{T G,\ U^!_1(G) \subset R^!_1(G)}$

I. $R^!_1(T_g) = \begin{cases} \{0\} & \text{при } g \geq 4 \\ H^1(T_g) & \text{при } g = 3 \end{cases}$

$$\begin{tikzcd}
(gr_1 T_g)^* \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\wr\wr"] & H^1 T_g \\
(T_g)_{ab} \arrow[d, phantom, "\wr\wr"] & \\
H_1(T_g) &
\end{tikzcd}$$

Sullivan, Lambe

$H_1 T_g < \infty$

$$\begin{tikzcd}
0 \arrow[r] & (gr_2 T_g)^* \arrow[r, "\beta^*"] & (\Lambda^2 gr_1 T_g)^* \arrow[d, equal] & \\
& & \Lambda^2 H^1 T_g \arrow[r, "\smile"] & H^2 T_g
\end{tikzcd}$$

$$a \smile b = 0$$
$$\Downarrow$$
$$a, b \in R^!_1(T_g)$$

$(V(2\lambda_2) + V(0))^* \xrightarrow{\sim} (\Lambda^2 V(\lambda_3))^* \ni x \wedge y$

$g = 3 \Rightarrow \Lambda^2 H^1 T_g \in \ker(\smile)$

$g \geq 4$ $\quad Sp(2g) \quad V(\lambda_3)^* = V(\lambda_3)$

$$\begin{tikzcd} V \arrow[r, "\left(\sum_{i=1}^{g} a_i\wedge b_i\right)\wedge\cdot"'] & \Lambda^3 V \arrow[r, "x_1\wedge x_2\wedge x_3 \mapsto (x_1,x_2)\,x_3+\dots"] & V(\lambda_3) \arrow[d, phantom, "\parallel"] \\ & & \Lambda^3 V/V \end{tikzcd}$$

$$[a_1\wedge a_2\wedge b_3]\subset V(\lambda_3)$$

$v=[a_1\wedge a_2\wedge a_3]\in V(\lambda_3)$

$v\notin R^1_1(T_g)$

$v'\wedge v\notin \mathrm{Im}\ V(2\lambda_2)+V(0)$

$R^1_1(T_g)\subset V(\lambda_3)$ λ_3 – старший вес $V(\lambda_3)$

<u>Утв.</u> $R^1_1(T_g)\neq\{0\}\Rightarrow R^1_1(T_g)\ni v\in V(\lambda_3)_{\lambda_3}$

$V(\lambda_3)_{\lambda_3} = \mathbb{C}\langle a_1\wedge a_2\wedge a_3\rangle$

G – разр. группа Ли $(h\oplus n^+)$

$G\curvearrowright X$ – полное мн-е $(R^1_1(T_g))$

Тогда у G на X есть неподв. точка

$$hv=\overset{\in h^*}{\underset{=\lambda_3}{(t^1+t^2+t^3)}}(h)\,v$$

$$t_1v=\overset{=a_1}{t_1a_1}\wedge a_2\wedge a_3+a_1\wedge \cancel{t_1a_2}\wedge a_3+a_1\wedge a_2\wedge\cancel{t_1a_3}=v$$

$t_2v=v$

$t_3v=v$

$t_kv=0$ $k\geq 4$

$V = [a_1 \wedge a_2 \wedge a_3]$

$\exists w: w \wedge v \in V(2\lambda_2) + V(0)$??

План:

1) $R(T_g) \subseteq \{0\}$ $\qquad V = V_1' \qquad R = R_1'$

2) $TC_1 V(X) \subseteq R(X)$

3)* $TC_1 V(T_g) = 0 \Rightarrow V(T_g)$ – конечное

4) $V(\underset{\nearrow T_g}{X})$ – конечно $\Rightarrow H^1(\underset{\nearrow K_g}{X^{abf}}, \mathbb{C})$ – к/м $\qquad X \to$

5)* $A \overset{T_g}{\lessgtr} K_g \Rightarrow H^1(A; \mathbb{C})$ – к/м

6)? $R(\mathbb{C}) = 0 \Rightarrow \widehat{H^1 K}$ – к/м

$H^1 K$ – модуль над $\mathbb{C}G_{ab}$

$I < \mathbb{C}G_{ab} \to \mathbb{C}$

$g \mapsto 1$

$x_1 \cdot \sim \quad x_n \cdot \sim$

$x_{N_1} - x_{N_2} \in I^M$

$Sp(2g)$ $\qquad V$ – фунд. представл. $\ni a_1 \dots a_g, b_1 \dots b_g$

$\cup$

$h \qquad h_i \cdot a_i = a_i$

$\qquad h_i \cdot b_i = -b_i$

$V^{\otimes n}$ $V^{\wedge n}$ – предст.

$$V^{\wedge 2} = V(\lambda_2) \oplus V(0)$$

$$V^{\wedge 2} \longrightarrow \mathbb{C}$$

$$(x \wedge y) \longmapsto (x \cdot y)$$

$$V^{\wedge 3} = V(\lambda_3) \oplus V(\overset{\overset{V}{\|}}{\lambda_1})$$

$$(x \wedge y \wedge z) \longmapsto (x \cdot y) z + (y \cdot z) x + (z \cdot x) y$$

$$V(2\lambda_2) = \ker\left[\mathrm{Sym}^2 \Lambda^2 V \to \dots\right]$$

$$a_{i_1} \otimes a_{i_2} \otimes b_{j_1} \otimes \dots$$

<u>Пример</u>

$$h_1(a_1 \wedge b_2) = a_1 \wedge b_2$$

$$h_2(a_1 \wedge b_2) = -a_1 \wedge b_2$$

$$\underline{t^1 - t^2} \qquad h(a_1 \wedge b_2) = ((t^1 - t^2) h)\, a_1 \wedge b_2$$

$$\|$$

$$h^*$$

$$V(\lambda_3)$$

$$\rotatebox{90}{$\in$}$$

$$v, w$$

$$\in \mu^1(T_g)$$

$$\hat{v} \vee \hat{w} = 0 \qquad R^!(T_g)$$

$$\underline{v \wedge w} \in \Lambda^2 V(\lambda_3) = \dots \oplus V(2\lambda_2) \oplus V(0)$$

$$p: \Lambda^2 V(\lambda_3) \longrightarrow (\dots)$$

$$v \wedge w \in V(2\lambda_2) + V(0)$$

$V(\lambda_3)$ $\qquad \underline{a_1\wedge a_2\wedge a_3} =: V_0$

<u>утв</u> $\mathrm{Im}(v\wedge\cdot)\cap\big(V(2\lambda_2)\oplus V(0)\big)\ni w$

n_+ $\qquad n_+\cdot w=0$

$$u_0=\sum_{k=3}^{g}(a_1\wedge a_2\wedge a_k)\wedge(a_1\wedge a_2\wedge b_k)$$

$$2t^1+2t^2=2\lambda_2$$

$$\boxed{V_0\wedge w=u_0}$$

$V_0\wedge u_0\in\Lambda^3V(\lambda_3)$

$\neq 0$ при $g\geq 4$

$R(T_g)\subseteq\{0\}$

$V_m^k(X)=\{\rho\in H^1(X;\mathbb{C}^*)\mid \dim H^k(X;\mathbb{C}_\rho)\geq m\}$

<u>утв.</u> X имеет конечный k-остов $\Rightarrow V_m^k(X)$ - алг.

$\boxed{k+1}$

$$\mathrm{Hom}_{\mathbb{C}H_1X}\big(C_{k-1}(\tilde X),\mathbb{C}_\rho\big)\xrightarrow{d^k}H_{\mathbb{C}H_1(X)}\big(C_k(\tilde X),\mathbb{C}_\rho\big)$$

$\mathbb{C}H_1(X)$ - модуль

$$d^k\in\mathrm{Mat}^{n_{k-1}\,n_k}\big(\mathbb{C}H_1(X)\big)$$

$$d^{k+1}\in\ \ldots$$

$$H^k(X; C_p) = \ker d^{k+1} / \mathrm{Im}\, d^k$$

$$\mathrm{rk}\, H^k(X; \mathbb{C}_p) = -(\mathrm{rk}\, d^k + \mathrm{rk}^{k+1}) + c_k$$

$m \Leftarrow$

$$\underline{\mathrm{rk}\, d^k + \mathrm{rk}\, d^{k+1} \leq c_k + m}$$

$$V_m^k = \bigcup_{a+b=c_k+m} V_{a,b}$$

$$\begin{tikzcd} C_{k+1}(X) \arrow[r, "\partial_k"] & C_k(X) \\ {} \arrow[u] \arrow[ur] & \end{tikzcd}$$

$\mathbb{C}H_1(X)$ – модули

$\mathrm{Im}\, \partial_k$ – к/п

$\tilde{X} = X^k \cup \{$конечное число $k+1$-клеток$\}$

I. $TC_\eta V_m^{k(X)} \subseteq R_m^k(X; \mathbb{C}_\eta)$

(Libgober'02)

$$H^k(X; \mathbb{C}_\eta) \smile H^l(X; \mathbb{C}_{\eta'}) \in H^{k+l}(X; \mathbb{C}_{\eta\eta'})$$

$$H^1(X; \mathbb{C}_1) \smile H^l(X; \mathbb{C}_\eta) \in H^{l+1}(X; \mathbb{C}_\eta)$$

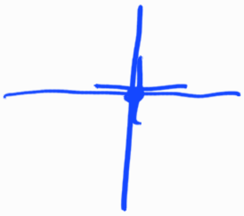
$a^{\in}$ $(a \smile \cdot)^2 = 0$

$$\cdots \to H^{\ell}(X;\mathbb{C}_\eta) \xrightarrow{a\cup} H^{\ell+1}(X;\mathbb{C}_\eta) \to \cdots$$

(Aomoto)

$$R^k_m(X;\mathbb{C}_\eta) = \{a \in H^1(X;\mathbb{C}_1) \mid \dim H^k(H^*(X;\mathbb{C}_\eta); d_a) \geq m\}$$

$$TC_x \overset{\swarrow V(I)}{V}$$



$$\overset{I}{\overset{\shortparallel}{z_1 z_2}} = 0 \qquad x = (0,0)$$

$$\mathcal{M}_x = \mathbb{C}[z_1, z_2] z_1 + \mathbb{C}[z_1, z_2] z_2$$

$$TC_x V = \operatorname{spec}(z_1 z_2)$$

$$x \rightsquigarrow \mathcal{M}_x$$

$$TC_x V = \operatorname{Spec} \bigoplus_n \left(I \cap \mathcal{M}_x^n / I \cap \mathcal{M}_x^{n+1} \right)$$

$$\subset \bigoplus \underset{\underset{\mathbb{C}[z_1, z_2]}{\shortparallel\wr}}{\mathcal{M}_x^n / \mathcal{M}_x^{n+1}}$$

$$I \cap \mathcal{M}_x^1 = I$$

$$I \cap \mathcal{M}_x^2 = I$$

$$I \cap \mathcal{M}_x^3 = I \setminus \mathbb{C} z_1 z_2$$

$$I \cap \mathcal{M}_x^4 = I \setminus \mathbb{C} z_1 z_2 \quad z_1^2 z_2 \quad z_1 z_2^2$$

$$I \cap \mathcal{M}_x^2 / I \cap \mathcal{M}_x^3 = \underline{\mathbb{C} z_1 z_2}$$

$$\bigcup_n V(p_{n,1} \cdots p_{n k})$$