Linear Algebra

Lecture 6: Convex Sets

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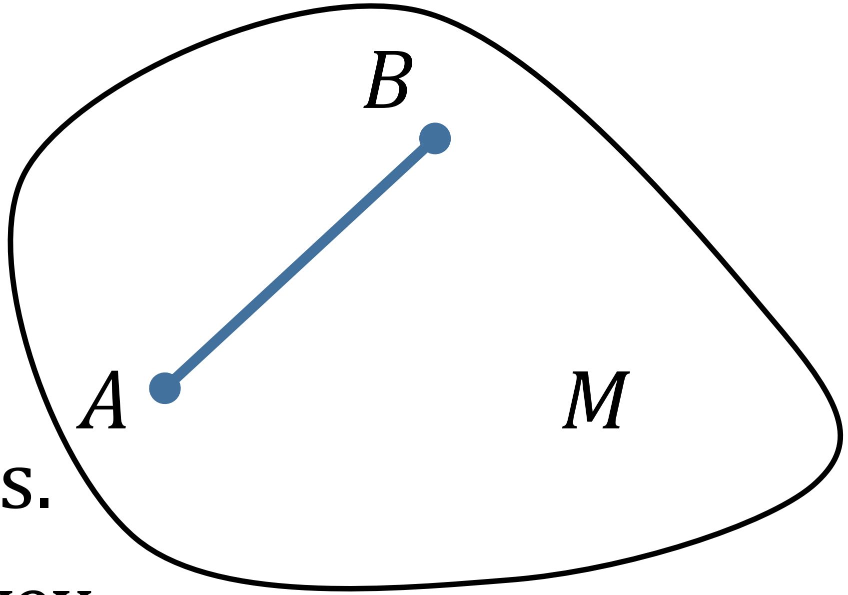
Convex Sets

Suppose $\mathbb{A}$ is an affine space.

$AB = [A, B] = \{\lambda A + (1-\lambda)B \mid 0 \leq \lambda \leq 1\}$ is a segment.

M is convex if:



Planes are convex sets.
If M_1 and M_2 are convex, then $M_1 \cap M_2$ is also convex.

Convex Hull

A convex linear combination of points in $\mathbb{A}$ is their barycentric combination with non-negative coefficients.

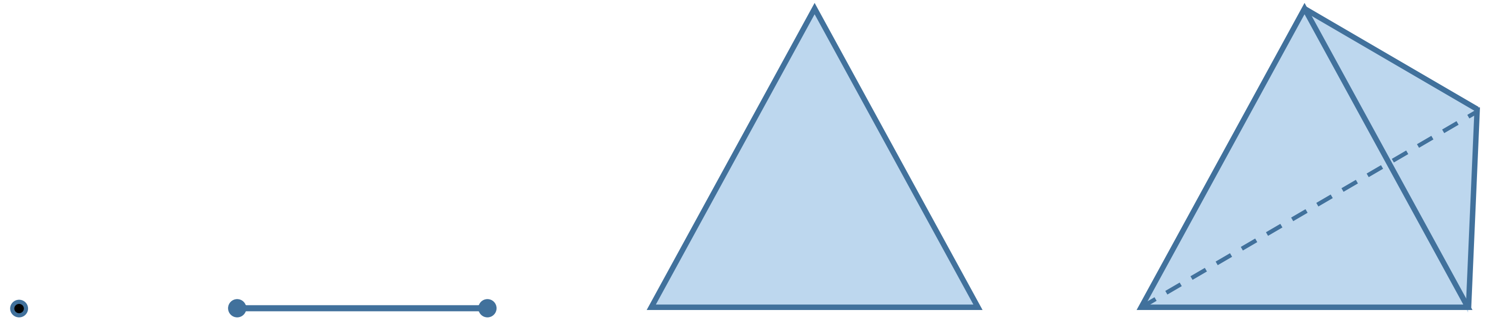
For $\forall\, A_0, A_1, \dots, A_k \in M$, where M is convex, M also contains every convex combination $\sum_{j=0}^{k} \lambda_j A_j$.

For any $M \subset \mathbb{A}$, the set $\mathrm{conv}(M)$ of all convex combinations of points in M is also convex. It is a convex hull of M.

Simplex

A convex hull of a set of affinely independent points $A_0, A_1, \dots, A_k \in \mathbb{A}$ is a k-dim simplex or a k-simplex.

0-simplex is a point, 1-simplex is a segment, 2-simplex is a triangle, etc.



$k = 0$ $\quad$ $k = 1$ $\quad$ $k = 2$ $\quad$ $k = 3$

Neighborhoods and Interior Points

An ε-neighborhood of $A \in \mathbb{E}^n$ is an open ball $B(A, \varepsilon) = \{X \in \mathbb{E}^n \mid \rho(A, X) < \varepsilon\}$.

$A \in M$ is an interior point of the set M if $B(A, \varepsilon) \subset M$ for some $\varepsilon > 0$.

A set $\mathrm{int}(M)$ of all interior points $M \subset \mathbb{E}^n$ is called the interior of M.

If $\mathrm{int}(M) \neq \emptyset$ for a convex set M, then M is a convex body.

Convex Body

Suppose M is convex. Then

$$\mathrm{int}(\mathrm{M}) \neq \emptyset \Leftrightarrow \mathrm{aff}(M) = \mathbb{E}^n.$$

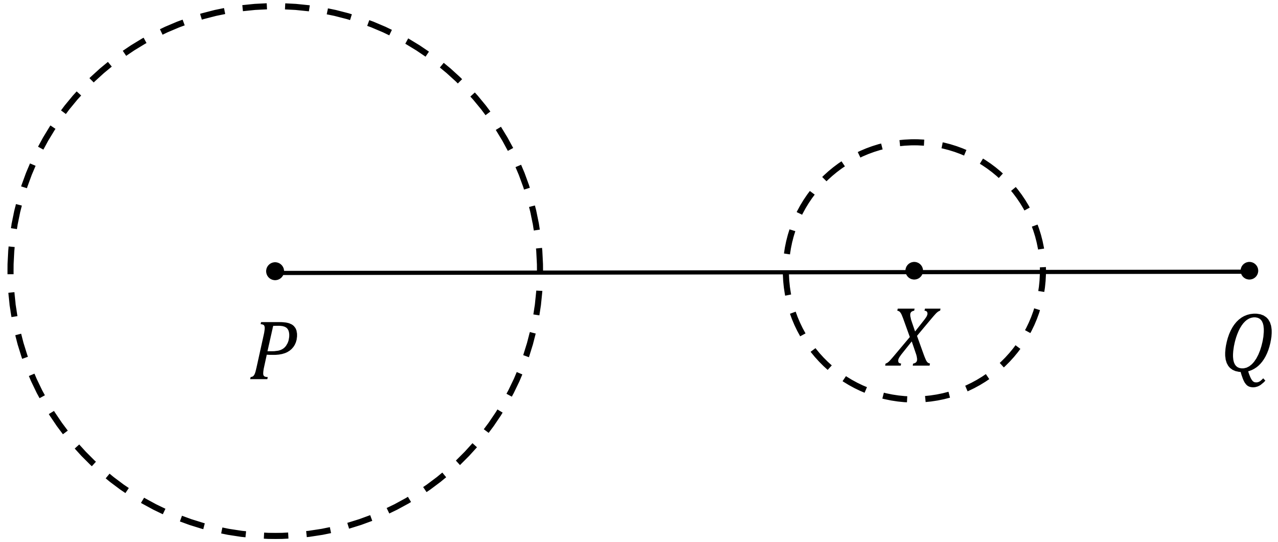
Proof: If $\mathrm{aff}(M) = \mathbb{E}^n$, then M has $n + 1$ affinely independent points. It implies that M contains a simplex and a small ball inside.

The converse is obvious.

Neighborhoods and Interior Points

Suppose $P \in \mathrm{int}(M), Q \in M$, and M is convex. Then for any $X \in (P, Q)$, we have $X \in \mathrm{int}(M)$.

Proof: If $P \in \mathrm{int}(M)$ with a ball $B(P, \varepsilon)$, and $\overrightarrow{\boldsymbol{QX}} = \lambda\, \overrightarrow{\boldsymbol{QP}}$, then $H_Q^{\lambda}(B(P, \varepsilon))$ is the $(\lambda\varepsilon)$-neighborhood of $X \in (P, Q)$.



Hyperplanes and Half-Spaces

Suppose f is affinely-linear function on $\mathbb{E}^n$.

Then we define a hyperplane

$$H_f := \{x \in \mathbb{E}^n \mid f(x) = 0\}$$

and half-spaces

$$H_f^+ := \{x \in \mathbb{E}^n \mid f(x) \geq 0\}$$

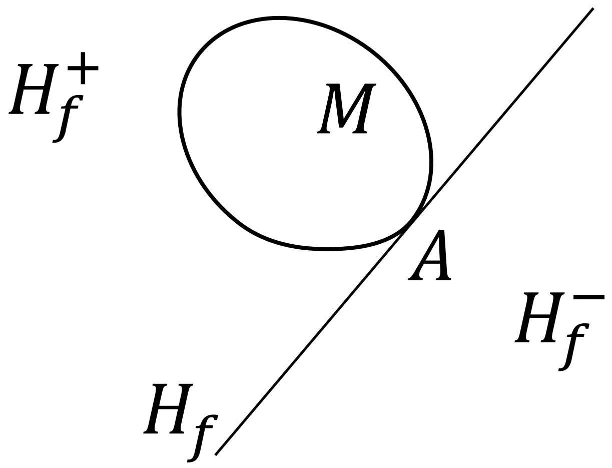
$$H_f^- := \{x \in \mathbb{E}^n \mid f(x) \leq 0\}.$$

Hyperplanes and Half-Spaces

Boundary points of M are the points from $\mathrm{clos}(M) \setminus \mathrm{int}(M)$. The boundary of M is

$$\partial M = \mathrm{clos}(M) \setminus \mathrm{int}(M).$$

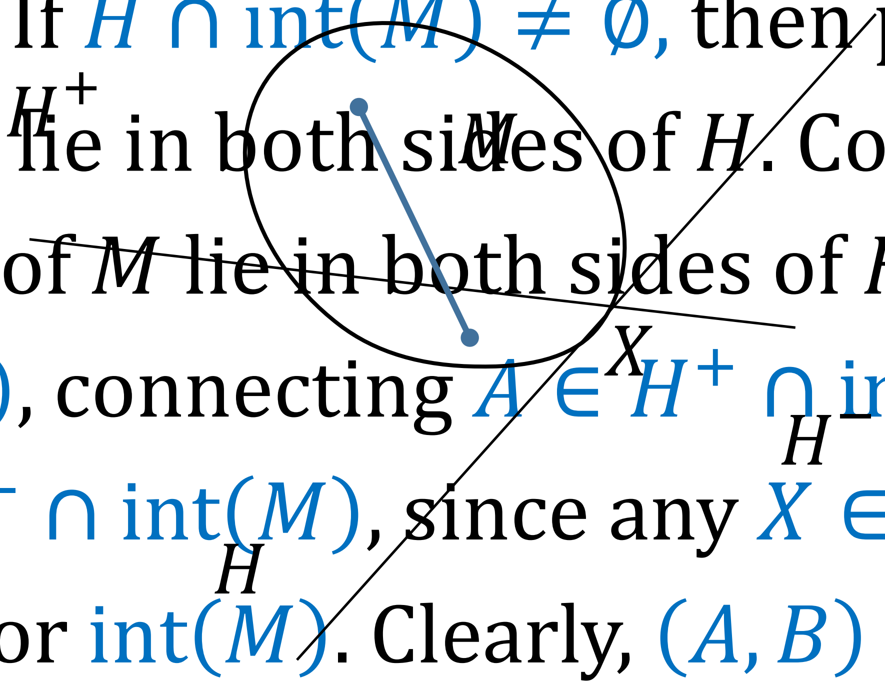
H_f is supporting for a closed convex body M if $M \subset H_f^+$ and $\exists A \in M$, s.t. $A \in H_f$.



Supporting Hyperplanes

H that passes through a point $X \in \partial M$ of a closed convex body M, is a supporting hyperplane iff $H \cap \text{int}(M) = \emptyset$.

Proof: If $H \cap \text{int}(M) \neq \emptyset$, then points of $\text{int}(M)$ lie in both sides of H. Conversely, if points of M lie in both sides of H, then $\exists (A, B)$, connecting $A \in H^+ \cap \text{int}(M)$ and $B \in H^- \cap \text{int}(M)$, since any $X \in M$ is a limit point for $\text{int}(M)$. Clearly, $(A, B) \cap H \neq \emptyset$.



The Separation Theorem

For every point $X \in \partial M$ for a closed convex body $M \subset \mathbb{E}^n$ there exists a supporting hyperplane $H \ni X$.

Proof:

Let us prove by induction on $k \leq n - 1$, that there exists a k-dimensional plane through X that does not intersect $\mathrm{int}(M)$.

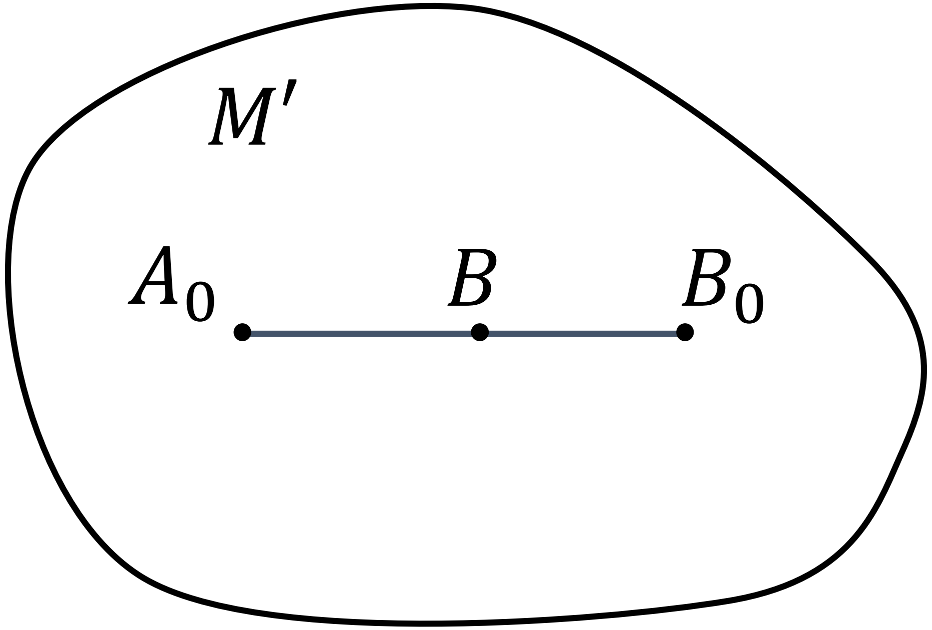
The Separation Theorem

For $k = 0$ this plane is X. Assume that we have a $(k - 1)$-dim plane P with the required conditions.

Pick any $(k + 1)$-dim space S', containing P and $A_0 \in \mathrm{int}(M)$. Let us find our k-dim plane.

The Separation Theorem

$M' = M \cap S'$ is a convex body in S'. Clearly, $\text{int}(M) \cap S' \subset \text{int}(M')$. Conversely, $\forall B \in \text{int}(M')$ is a point of (A_0, B_0), where $B_0 \in M' \subset M$. Hence, $B \in \text{int}(M)$.



The Separation Theorem

Hence, $B \in \mathrm{int}(M)$. Therefore,

$$\mathrm{int}(M') = \mathrm{int}(M) \cap S'.$$

It follows that $P \cap \mathrm{int}(M') = \emptyset$. Then it remains to prove that $S' \supset$ a supporting hyperplane of M' that contains P.

We change the notation:

$$S' = S, \qquad M' = M, \qquad k + 1 = n.$$

The Separation Theorem

Let P be a $(n-2)$–dim plane through the point $X \in \partial M$, such that $P \cap \text{int}(M) \neq \emptyset$.

If a hyperplane $H \supset \text{P}$, then P divides H into 2 half-spaces H' and H''. If $H' \cap \text{int}(M) = \emptyset$ and $H'' \cap \text{int}(M) = \emptyset$, we are done.

Since H' and H'' can not intersect $\text{int}(M)$ simultaneously, we may assume that H' intersects $\text{int}(M)$ while H'' does not.

The Separation Theorem

Let us rotate H around P, clockwise.