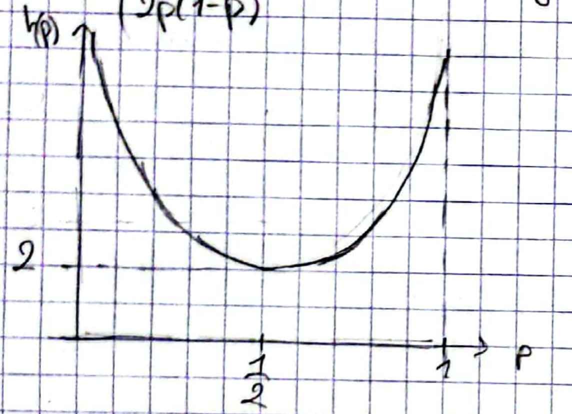
Detection Theory - Homework 2 Nicolás Violante

- $X_1, \dots, X_\ell$ independent random variables, $0 \leq X_i \leq 1$, iid
- $S_\ell := \sum_{i=1}^{\ell} X_i$, $p := \mathbb{E}\left[\frac{S_\ell}{\ell}\right]$

1) $h(p) = \begin{cases} \frac{1}{1-2p} \log\left(\frac{1-p}{p}\right) & \text{if } 0 < p < \frac{1}{2} \\ \frac{1}{2p(1-p)} & \text{if } \frac{1}{2} \leq p < 1 \end{cases}$



2) X r.v / $a \leq X \leq b$. $\lambda > 0$

- $\frac{b-x}{b-a} + \frac{x-a}{b-a} = 1$

- Since $a \leq X \leq b \Rightarrow \frac{b-x}{b-a} \geq 0$ and $\frac{x-a}{b-a} \geq 0$

$$e^{\lambda\left(\underbrace{\frac{b-x}{b-a} \cdot a + \frac{x-a}{b-a} \cdot b}_{x}\right)} \leq \frac{(b-x)}{b-a} \cdot e^{\lambda a} + \frac{(x-a)}{b-a} \cdot e^{\lambda b} \quad \text{by convexity}$$

$$\Rightarrow e^{\lambda x} \leq \left(\frac{b-x}{b-a}\right) e^{\lambda a} + \left(\frac{x-a}{b-a}\right) e^{\lambda b}$$

$$\Rightarrow \mathbb{E}(e^{\lambda x}) \leq \left(\frac{b-\mathbb{E}(x)}{b-a}\right) e^{\lambda a} + \left(\frac{\mathbb{E}(x)-a}{b-a}\right) e^{\lambda b}$$

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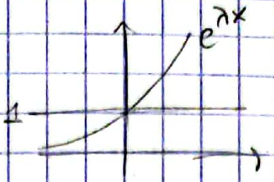
3) a) $\mathbb{1}_{x \geq 0} \leq e^{\lambda x}$



- At $x=0$: $e^{\lambda \cdot 0} = 1$
- if $\lambda > 0$ and $x > 0$ then $\lambda x > 0$ which implies that $e^{\lambda x} > 1$

b) $\mathbb{1}_{\{S_\ell - \mathbb{E}(S_\ell) - \ell t \geq 0\}} \geq e^{\lambda(S_\ell - \mathbb{E}(S_\ell) - \ell t)}$

Taking expectation

$$\underbrace{\mathbb{E}\left(\mathbb{1}_{\{S_\ell - \mathbb{E}(S_\ell) - \ell t \geq 0\}}\right)}_{= \mathbb{P}(S_\ell - \underbrace{\mathbb{E}(S_\ell)}_{\ell p} - \ell t \geq 0)} \geq \underbrace{\mathbb{E}\left(e^{\lambda(S_\ell - \mathbb{E}(S_\ell) - \ell t)}\right)}_{= e^{-\lambda(\underbrace{\mathbb{E}(S_\ell)}_{=\ell p} + \ell t)} \, \mathbb{E}(e^{\lambda S_\ell})}$$

$$\Rightarrow \mathbb{P}(S_\ell \geq \ell(p+t)) \geq e^{-\lambda \ell (p+t)} \, \mathbb{E}\left(e^{\lambda \sum_{i=1}^{\ell} x_i}\right)$$

$$= e^{-\lambda \ell (p+t)} \, \mathbb{E}\left(\prod_{i=1}^{\ell} e^{\lambda x_i}\right)$$

$$= e^{-\lambda \ell (p+t)} \prod_{i=1}^{\ell} \mathbb{E}\left(e^{\lambda x_i}\right)$$

X_i are independent

4) $p_i = \mathbb{E}(X_i)$

a) $\prod_{i=1}^{\ell} \mathbb{E}(e^{\lambda X_i}) \leq \prod_{i=1}^{\ell} 1 - \mathbb{E}(X_i) + \mathbb{E}(X_i) e^{\lambda}$ by part b)

$= \prod_{i=1}^{\ell} 1 - p_i + p_i e^{\lambda}$

b) If X_i are Bernoulli then $X_i \in \{0, 1\}$, and the equality holds

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5) $\left(\prod_{i=1}^{\ell} a_i\right)^{1/\ell} \leq \frac{1}{\ell}\sum_{i=1}^{\ell} a_i \qquad a_i \geq 0$

- $\frac{1}{\ell} \geq 0$ and $\sum_{i=1}^{\ell} \frac{1}{\ell} = 1$

By Jensen's inequality $\log \sum_{i=1}^{\ell} \frac{1}{\ell} a_i \geq \sum_{i=1}^{\ell} \frac{1}{\ell} \log a_i$

$\log\left(\prod_{i=1}^{\ell} a_i\right)^{1/\ell} = \frac{1}{\ell}\sum_{i=1}^{\ell} \log a_i$ $\Rightarrow$

$\Rightarrow \log \sum_{i=1}^{\ell} \frac{1}{\ell} a_i \geq \log\left(\prod_{i=1}^{\ell} a_i\right)^{1/\ell} \Rightarrow \sum_{i=1}^{\ell} \frac{1}{\ell} a_i \geq \left(\prod_{i=1}^{\ell} a_i\right)^{1/\ell}$

6) $\prod_{i=1}^{\ell} \mathbb{E}(e^{\lambda X_i}) \leq \left\{\left(\prod_{i=1}^{\ell} (1-p_i+p_i e^{\lambda})\right)^{1/\ell}\right\}^{\ell}$

$\leq \left\{\frac{1}{\ell}\sum_{i=1}^{\ell} (1-p_i+p_i e^{\lambda})\right\}^{\ell}$

$= \left(1 - \sum_{i=1}^{\ell} \frac{\mathbb{E}(X_i)}{\ell} + e^{\lambda}\sum_{i=1}^{\ell} \frac{\mathbb{E}(X_i)}{\ell}\right)^{\ell}$

$= \left(1 - \mathbb{E}\left(\frac{S_\ell}{\ell}\right) + e^{\lambda}\,\mathbb{E}\left(\frac{S_\ell}{\ell}\right)\right)^{\ell} = (1-p+pe^{\lambda})^{\ell}$

7) Combining 3) and 6)

$P(S_\ell \geq (p+t)\ell) \leq e^{-\lambda(p+t)\ell}(1-p+pe^{\lambda})^{\ell} = \left[e^{-\lambda(p+t)}(1-p+pe^{\lambda})\right]^{\ell}$

Taking log on the right side: $\ell\left[-\lambda(p+t) + \log(1-p+pe^{\lambda})\right] =: f(\lambda)$

$\Rightarrow$ we derivate $\frac{\partial f}{\partial \lambda} = \ell\left[-(p+t) + \frac{pe^{\lambda}}{1-p+pe^{\lambda}}\right] = 0$

$\Leftrightarrow (p+t)(1-p+pe^{\lambda}) = pe^{\lambda}$

if $\lambda = \log\frac{(1-p)(p+t)}{(1-p-t)p}$ $\qquad$ • $pe^{\lambda} = \frac{\cancel{p}(1-p)(p+t)}{(1-p-t)\cancel{p}} = \frac{(1-p)(p+t)}{1-p-t}$

• $(p+t)(1-p+pe^{\lambda}) = (p+t)(1-p)\left(1 + \frac{(p+t)}{1-p-t}\right) = \frac{(p+t)(1-p)}{1-p-t}$

$\Rightarrow$ the upper-bound is minimized for this value of λ

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7) $\lambda = \log \frac{(1-p)(p+t)}{(1-p-t)p}$ for $0 < t < 1-p$

$\lambda > 0 \iff (1-p)(p+t) > (1-p-t)p \iff p + t - p^2 - \cancel{pt} > \cancel{p} - p^2 - \cancel{tp}$

$\iff t > 0$ ✓

8) $G(t,p) = \frac{p+t}{t^2} \log\frac{(p+t)}{p} + \frac{1-p-t}{t^2} \log\frac{(1-p-t)}{1-p}$

- $t^2 \frac{\partial G}{\partial t} = t^2 \left\{ \frac{(t^2 - (p+t)2t)}{t^4} \log\frac{(p+t)}{p} + \frac{\cancel{(p+t)}}{t^2} \frac{\cancel{p}}{\cancel{(p+t)}} \cdot \frac{1}{\cancel{p}} \right.$

$\left. + \frac{-t^2 - (1-p-t)2t}{t^4} \log\frac{(1-p-t)}{1-p} - \frac{\cancel{(1-p-t)}}{t^2} \frac{\cancel{(1-p)}}{\cancel{(1-p-t)}} \cdot \frac{1}{\cancel{1-p}} \right\}$

$= \left[1 - 2\frac{(p+t)}{t}\right] \log\left(\frac{p+t}{p}\right) - \left[1 + 2\left(\frac{1-p-t}{t}\right)\right] \log\left(\frac{1-p-t}{1-p}\right)$

$= -\left[1 - \frac{2(p+t)}{t}\right] \log\left(1 - \frac{t}{t+p}\right) - \left[-1 + \frac{2(1-p)}{t}\right] \log\left(1 - \frac{t}{1-p}\right)$

$= \left[1 - 2\left(\frac{1-p}{t}\right)\right] \log\left(1 - \frac{t}{1-p}\right) - \left[1 - \frac{2(p+t)}{t}\right] \log\left(1 - \frac{t}{t+p}\right)$

$= H\left(\frac{t}{1-p}\right) - H\left(\frac{t}{t+p}\right)$

- $H(x) = \left(1 - \frac{2}{x}\right) \log(1-x)$

The McLaurin expansion of $\log(1-x)$ is $-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$

$H(x) = \left(1 - \frac{2}{x}\right) \sum_{i \geq 1} \frac{-x^i}{i} = \sum_{i \geq 1} -\frac{x^i}{i} + 2 \sum_{i \geq 1} \frac{x^{i-1}}{i} = 2 + \sum_{i \geq 2} \left(\frac{2}{i+1} - \frac{1}{i}\right) x^i$

- For $0 < x < 1$: the terms $\left(\frac{2}{i+1} - \frac{1}{i}\right) > 0$ and so $H'(x) > 0 \Rightarrow$ it's increasing

- $\frac{\partial G(t,p)}{\partial t} = \frac{1}{t^2} \left[H\left(\frac{t}{1-p}\right) - H\left(\frac{t}{t+p}\right)\right] > 0 \iff \frac{t}{1-p} > \frac{t}{t+p}$ since H increases

$\iff t + p > 1 - p$

$\iff t > 1 - 2p$

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8) when $t = 1-2p \Rightarrow \frac{\partial G}{\partial t} = 0$

$$G(1-2p, p) = \frac{p + 1-2p}{(1-2p)^2} \log\left(\frac{p+1-2p}{p}\right) + \frac{(1-p-1+2p)}{(1-2p)^2} \log\left(\frac{1-p-1+2p}{1-p}\right)$$

$$= \frac{(1-p)}{(1-2p)^2} \log\left(\frac{1-p}{p}\right) + \frac{p}{(1-2p)^2} \log\left(\frac{p}{1-p}\right)$$

$$= \frac{(1-p-p)}{(1-2p)^2} \log\left(\frac{1-p}{p}\right) = \frac{1}{1-2p} \log\left(\frac{1-p}{p}\right)$$

9) $1-2p \leq 0$

Since G is increasing (on t) the minimum is attained when $t \to 0$