Reinforcement Learning - Homework 3

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(1) Best Arm Identification

a) Let $\delta > 0$

First, by Hoeffding's inequality

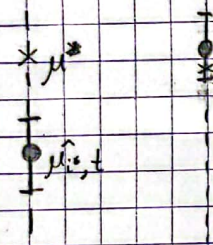
$$\mathbb{P}\left(|\hat{\mu}_{i,t} - \mu_i| > U\right) \leq 2e^{-2tU^2} = \delta \Leftrightarrow U(t,\delta) = \sqrt{\frac{1}{2t}\log\left(\frac{2}{\delta}\right)}$$

Let $\boxed{\delta' = \frac{6}{\pi^2}\frac{\delta}{K}\frac{1}{t^2}}$

$$\mathbb{P}\left(\bigcup_{i=1}^{K}\bigcup_{t=1}^{\infty}\left\{|\hat{\mu}_{i,t} - \mu_i| > U(t,\delta')\right\}\right) \leq \sum_{i=1}^{K}\sum_{t=1}^{\infty}\frac{6}{\pi^2}\frac{\delta}{K}\frac{1}{t^2}$$

$$= \frac{\delta}{\pi^2}\frac{6}{K}\left(\sum_{i=1}^{K}\right)\left(\sum_{t=1}^{\infty}\frac{1}{t^2}\right)$$

$$= \frac{\delta}{\pi^2}\frac{6}{K}\cdot K\cdot\frac{\pi^2}{6} = \delta$$

b) 

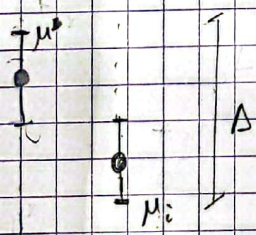
If $\exists j \in S / \left\{\hat{\mu}_{j,t} - U(t,\delta') \geq \hat{\mu}_{i^*,t} + U(t,\delta')\right\}$

then μ^* is outside the confidence interval

$\Rightarrow \mathbb{P}(i^* \text{ removed from } S) \leq \mathbb{P}(\mathcal{E}) \leq \delta$

$\Rightarrow \mathbb{P}(i^* \text{ remains on } S) \geq 1 - \delta$

c) Under $\mathcal{E}^c$, $\Delta_i = \mu^* - \mu_i \geq C_1 U(t,\delta')$

The worst case is 

then we need $C_1 = 4$ at least

Then, if $\Delta_i \geq 4U(t,\delta)$ the arm i will be removed

Analitically, since $\mathcal{E}^c$ holds $\mu^* \leq \hat{\mu}^* + U(t,\delta)$ and $\mu_i \geq \hat{\mu}_i - U(t,\delta)$

The arm elimination condition is

$$\hat{\mu}^* - U(t,\delta') \geq \hat{\mu}_i + U(t,\delta)$$

$$\Leftrightarrow \mu^* - U(t,\delta) - U(t,\delta') \geq \mu_i + U(t,\delta) + U(t,\delta')$$

$$\Leftarrow \mu^* - \mu_i \geq 4U(t,\delta')$$

(continue)

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(1) (c) $\Delta_i \geq 4\sqrt{\frac{1}{2t}\log\left(b^2t^2\right)}$ with $b^2 = \frac{9\pi^2 K}{6\delta} = \frac{\pi^2 K}{3\delta}$

$\Delta_i^2 \geq \frac{16}{2t}\log(b^2t^2) = \frac{16}{t}\log(bt) \Leftrightarrow \left(\frac{\Delta_i^2}{16}\right)\cdot t \geq \log(bt)$

a

Then, using Lambert W $t \geq \frac{1 + \sqrt{2u}}{a} + u$ with $u = \log(b/a) - 1$

$t \geq \frac{1 + \sqrt{2(\log(b/a)-1)}}{a} + \log(b/a) - 1$

$t \geq \frac{16}{\Delta_i^2}\left\{\sqrt{2\log\left(\frac{16}{\Delta_i^2}\pi\sqrt{\frac{K}{3\delta}}\right) - 2} + \log\left(\frac{16}{\Delta_i^2}\pi\sqrt{\frac{K}{3\delta}}\right)\right\}$ (*)

(d) The equation (*) is for the arm i and has complexity

$$\Theta\left(\frac{1}{\Delta_i^2}\log\left(\frac{1}{\Delta_i^2}\sqrt{\frac{K}{\delta}}\right)\right)$$ (simplifying constants and lower-order terms)

If we consider all the pulls we have a sum over arms $i \neq i^*$, so

$$\Theta\left(\sum_{i \neq i^*}\frac{1}{\Delta_i^2}\log\left(\frac{1}{\Delta_i^2}\sqrt{\frac{K}{\delta}}\right)\right)$$ is the sample complexity

(e) The algorithm won't work correctly. Between two optimal arms we have $\Delta = 0$, hence an infinit time for the removal of the arm, as given by equation (*).

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② Regret Minimization in RL

1) $\mathbb{P}\left(|\hat{r}_{nk}(s,a) - r_n(s,a)| \geq \beta^r_{nk}(s,a)\right) \leq 2e^{-2N_{nk}(s,a)\beta^r_{nk}(s,a)^2} = \delta_1$

$$\Leftrightarrow \beta^r_{nk}(s,a) = \sqrt{\frac{1}{2N_{nk}(s,a)} \log\left(\frac{2}{\delta_1}\right)} \quad (1)$$

$$\mathbb{P}\left(\|\hat{p}_{nk}(\cdot|s,a) - p_n(\cdot|s,a)\|_1 \geq \beta^p_{nk}(s,a)\right) \leq (2^S - 2)\, e^{-\frac{1}{2}N_{nk}(s,a)\beta^p_{nk}(s,a)^2} = \delta_2$$

$$\frac{1}{2}N_{nk}(s,a)\beta^p_{nk}{}^2 = \log\left(\frac{2^S - 2}{\delta_2}\right)$$

$$\Rightarrow \beta^p_{nk} = \sqrt{\frac{2}{N_{nk}(s,a)} \log\left(\frac{2^S - 2}{\delta_2}\right)} \quad (2)$$

- Let h, k, s, a be fixed and let $\delta' > 0$:

$$\mathbb{P}\left(|\hat{r}_{nk}(s,a) - r_n(s,a)| \geq \beta^r_{nk}(s,a) \text{ or } \|\hat{p}_{nk}(s,a) - p_n(\cdot|s,a)\|_1 \geq \beta^p_{nk}(s,a)\right)$$
$$\leq \mathbb{P}\left(|\hat{r}_{nk}(s,a) - r_n(s,a)| \geq \beta^r_{nk}(s,a)\right) + \mathbb{P}\left(\|\hat{p}_{nk}(s,a) - p_n(\cdot|s,a)\|_1 \geq \beta^p_{nk}(s,a)\right)$$
$$\leq \frac{\delta'}{2} + \frac{\delta'}{2} = \delta' \text{ for a proper } \beta^r_{nk}(s,a) \text{ and } \beta^p_{nk}(s,a)$$

$$\mathbb{P}\left(\forall k,h,s,a: |\hat{r}_{nk}(s,a) - r_n(s,a)| \geq \beta^r_{nk}(s,a) \text{ and } \|\hat{p}_{nk}(\cdot|s,a) - p_n(\cdot|s,a)\|_1 \geq \beta^p_{nk}(s,a)\right)$$
$$= 1 - \mathbb{P}\left(\bigcup_{k=1}^{K}\bigcup_{h=1}^{H}\bigcup_{s=1}^{S}\bigcup_{a=1}^{A}\left\{|\hat{r}_{nk}(s,a) - r_n(s,a)| \geq \beta^r_{nk}(s,a) \text{ or } \|\hat{p}_{nk}(\cdot|s,a) - p_n(\cdot|s,a)\|_1 \geq \beta^p_{nk}(s,a)\right\}\right)$$
$$\geq 1 - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{s=1}^{S}\sum_{a=1}^{A}\mathbb{P}\left(|\hat{r}_{nk}(s,a) - r_n(s,a)| \geq \beta^r_{nk}(s,a) \text{ or } \|\hat{p}_{nk}(\cdot|s,a) - p_n(\cdot|s,a)\|_1 \beta^p_{nk}(s,a)\right)$$

$$= 1 - \underbrace{KHSA\delta'}_{:=\frac{\delta}{2}} \Rightarrow \delta = 2KHSA\delta' \Rightarrow \frac{\delta'}{2} = \frac{\delta}{4KHSA}$$

$$\Rightarrow \beta^r_{nk}(s,a) = \sqrt{\frac{1}{2N_{nk}(s,a)} \log(8KHSA)}$$

$$\beta^p_{nk} = \sqrt{\frac{2}{N_{nk}(s,a)}\left[\log\left(\frac{(2^S-2)\,4KHSA}{\delta}\right)\right]}$$

then $\mathbb{P}(\neg\mathcal{E}) \geq 1 - \frac{\delta}{2}$

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(2) b) $Q_{nk}(s,a) = \hat{r}_{nk}(s,a) + b_{nk}(s,a) + \sum_{s'} \hat{p}_{nk}(s'|s,a) V_{n+1,k}(s')$ under $\mathcal{E}$

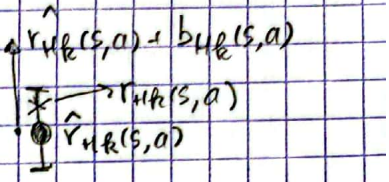
- Base case $h = H$

$$Q_{Hk}(s,a) = \hat{r}_{Hk}(s,a) + b_{Hk}(s,a) + \sum_{s'} \hat{p}_{Hk}(s'|s,a) \underbrace{V_{H+1,k}(s')}_{=0}$$

$$= \hat{r}_{Hk}(s,a) + b_{Hk}(s,a)$$

$$Q^*_H(s,a) = r_{Hk}(s,a)$$

$\Rightarrow Q_{Hk}(s,a) \geq Q^*_H(s,a)$ if $b_{Hk}(s,a) \geq \beta^r_{Hk}(s,a)$ $\Rightarrow$



- Inductive step

Assume $Q_{n+1,k}(s,a) \geq Q^*_{n+1}(s,a)$ $\forall s,a$

$$Q_{nk}(s,a) - Q^*_n(s,a) = \hat{r}_{nk}(s,a) + b_{nk}(s,a) + \sum_{s'} \hat{p}_{nk}(s'|s,a) \min\{H, \max_a Q_{n+1,k}(s,a)\}$$
$$- r_n(s,a) - \sum_{s'} p_{n+1}(s'|s,a) \max_a Q^*_{n+1}(s,a)$$
$$\geq \hat{r}_{nk}(s,a) + b_{nk}(s,a) - r_n(s,a)$$
$$+ \sum_{s'} \min\{H, \max_a Q_{n+1}(s',a)\} \left(\hat{p}_{nk}(s'|s,a) - p_n(s'|s,a)\right)$$
$$\geq \hat{r}_{nk}(s,a) + b_{nk}(s,a) - r_n(s,a)$$
$$- \sum_{s'} \min\{H, \max_a Q_{n+1}(s',a)\} |\hat{p}_{nk}(s'|s,a) - p_n(s'|s,a)|$$
$$\geq \hat{r}_{nk}(s,a) + b_{nk}(s,a) - r_n(s,a) - H \underbrace{\sum_{s'} |\hat{p}_{nk}(s'|s,a) - p_n(s'|s,a)|}_{\|\hat{p}_{nk}(s'|s,a) - p_n(s'|s,a)\|_1}$$
$$\geq \hat{r}_{nk}(s,a) + b_{nk}(s,a) - r_n(s,a) - H\beta^p_{nk}(s,a)$$
$$\geq -\beta^r_{nk}(s,a) + b_{nk}(s,a) - H\beta^p_{nk}(s,a) \geq 0$$
$$\geq -$$

$\Rightarrow b_{nk}(s,a) \geq \beta^r_{nk}(s,a) + H\beta^p_{nk}(s,a)$

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② 3) 1. $V_h^{\pi_k}(s) = r(s_{hk}, a_{hk}) + \sum_{s'} p_{hk}(s' | s_{hk}, a_{hk}) \underbrace{V_{h+1}^{\pi_k}(s')}_{= -\delta_{h+1,k}(s') + V_{h+1,k}(s')}$

$\Leftrightarrow \sum_{s'} p(s' | s_{hk}, a_{hk}) \left(V_{h+1,k}(s') - \delta_{h+1,k}(s') \right) =$

$= \mathbb{E}_{s' \sim p}\left(V_{h+1,k}(s') \right) - \mathbb{E}_{s' \sim p}\left[\delta_{h+1,k}(s') \right] = \mathbb{E}_{s' \sim p}\left[V_{h+1,k}(s') \right] - \delta_{h+1,k}(s_{h+1,k}) - m_{hk}$

$\Rightarrow V_h^{\pi_k}(s) = r(s_{hk}, a_{hk}) + \mathbb{E}_{s' \sim p}\left(V_{h+1,k}(s') \right) - \delta_{h+1,k}(s_{h+1,k}) - m_{hk}$

2. $V_{h,k}(s_{h,k}) = \min\left\{ H, \max_a Q_{h,k}(s_{h,k}, a) \right\} = \min\left\{ H, Q_{h,k}(s_{hk}, a_{hk}) \right\}$

$\leq Q_{h,k}(s_{h,k}, a_{hk})$

3. $\delta_{1k}(s_{1,k}) = V_{1k}(s_{1k}) - V_1^{\pi_k}(s_{1k})$

$\leq Q_{1k}(s_{1k}, a_{1k}) - r(s_{1k}, a_{1k}) - \mathbb{E}_p\left[V_{2k}(s') \right] + m_{1k} + \delta_{2k}(s_{2k})$

$\leq Q_{1k}(s_{1k}, a_{1k}) - r(s_{1k}, a_{1k}) - \mathbb{E}_p\left[V_{2k}(s') \right] + m_{1k}$

$+ Q_{2k}(s_{2k}, a_{2k}) - r(s_{2k}, a_{2k}) - \mathbb{E}_p\left[V_{2k}(s') \right] + m_{2k} + \delta_{3k}(s_{3k})$

$\vdots$

$\leq \sum_{h=1}^{H} Q_{hk}(s_{hk}, a_{hk}) - r(s_{hk}, a_{hk}) - \mathbb{E}_p\left[V_{hk}(s') \right] + m_{hk}$

4) $R(T) = \sum_{k=1}^{K} V_1^*(s_{1k}) - V_1^{\pi_k}(s_{1k}) \leq \sum_{k=1}^{T} \delta_{1k}(s_{1k})$ since $Q_{hk}(s,a) \geq Q_h^*(s,a) \; \forall s, a$

$\leq \sum_{k=1}^{K} \sum_{h=1}^{H} \left(Q_{hk}(s_{hk}, a_{hk}) - r(s_{hk}, a_{hk}) - \mathbb{E}_p\left[V_{h+1,k}(s') \right] + m_{hk} \right.$

$= \sum_{k=1}^{K} \sum_{h=1}^{H} \left\{ \hat{r}_{hk}(s_{hk}, a_{hk}) - r(s_{hk}, a_{hk}) + \sum_{s'} \left| \hat{p}_{hk}(s' | s_{hk}, a_{hk}) - p_{hk}(s' | s_{hk}, a_{hk}) \right| V_{h+1,k}(s') \right\} +$

$+ \sum_{hk} m_{hk} + \sum_{hk} b_{hk}(s_{hk}, a_{hk})$

using the bounds found of previous parts: $\leq \sum_{h=1}^{H} \sum_{k=1}^{K} \left\{ \left| \hat{r}_{hk}(s_{hk}, a_{hk}) - r(s_{hk}, a_{hk}) \right| + H \left\| \hat{p}_{hk}(\cdot | s_{hk}, a_{hk}) - p_h(\cdot | s_{hk}, a_{hk}) \right\|_1 \right\}$

$+ \sum_{hk} m_{hk} + \sum_{hk} b_{hk}(s_{hk}, a_{hk})$

with prob at least $1-\delta$ ↓

$\leq \sum_{h,k} \beta_{h,k}^r(s_{hk}, a_{hk}) + H \beta_{hk}^p(s_{h,k}, a_{hk}) + b_{hk}(s_{hk}, a_{hk}) + 2H\sqrt{KH \log\left(\frac{2}{\delta}\right)}$ → $\leq b_{hk}(s_{hk}, a_{hk})$

$\leq \sum_{hk} 2 b_{hk}(s_{hk}, a_{hk}) + 2H\sqrt{KH \log\left(\frac{2}{\delta}\right)}$

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5) Since $\sqrt{\cdot}$ is concave, by Jensen's inequality

$$\sum_{h=1}^{H}\sum_{s,a}\sqrt{N_{hk}(s,a)} = HSA\sum_{h=1}^{S}\sum_{s,a}\frac{1}{HSA}\sqrt{N_{hk}(s,a)}$$

$$\leq HSA\sqrt{\frac{1}{HSA}\sum_{h=1}^{H}\sum_{s,a}N_{hk}(s,a)} \qquad \left(\sum_{h=1}^{H}\sum_{s,a}N_{hk}(s,a) \leq K\right)$$

$$\leq \sqrt{HSA}\sqrt{\sum_{h=1}^{H}K} = H\sqrt{SAK} \qquad \left(\sum_{h=1}^{H}K = H\right)$$

Now we chose $b_{hk}(s_{hk},a_{hk}) = \beta^r_{hk}(s_{hk},a_{hk}) + H\beta^p_{hk}(s_{hk},a_{hk})$

Also since $S \gg 1 \Rightarrow \log(2^S-2) \approx \log 2^S = S\log 2$

$$R(T) \lesssim 2\sum_{hk}\sqrt{\frac{1}{2N_{hk}(s_{hk},a_{hk})}\log(8KHSA)} + H\sqrt{\frac{2}{N_{hk}(s_{hk},a_{hk})}\left[S\log 2 + \log\left(\frac{4KHSA}{\delta}\right)\right]}$$
$$+ 2H\sqrt{KH\log\left(\frac{2}{\delta}\right)}$$

$$\lesssim H^2S^2A + H\sqrt{SAK} + H\sqrt{S}\left(H^2S^2A + H\sqrt{SAK}\right) + 2H\sqrt{KH\log\left(\frac{2}{\delta}\right)}$$

$$\simeq H^2S\sqrt{AK}$$