Image Denoising - Homework 3

8.4) $\boxed{MSE} = \mathbb{E}[\|P-\hat{P}\|^2] = \int \mathbb{P}(P)\|P-\hat{P}\|^2 dP$

$$= \int\left(\int \mathbb{P}(P|\tilde{P})\mathbb{P}(\tilde{P})\,d\tilde{P}\right)\|P-\hat{P}\|^2 dP$$

$$= \boxed{\int \mathbb{P}(\tilde{P})\int \mathbb{P}(P|\tilde{P})\,\|P-\hat{P}\|^2\, d\tilde{P}\, dP}$$

8.5) $MMSE(\tilde{P}) = \mathbb{E}[\|P-\hat{P}\|^2/\tilde{P}] = \int \mathbb{P}(P|\tilde{P})\|P-\hat{P}\|^2 dP$

$$\frac{\partial MMSE(\tilde{P})}{\partial \hat{P}} = \int \mathbb{P}(P|\tilde{P})\frac{\partial}{\partial \hat{P}}\|P-\hat{P}\|^2 dP$$

$$= -2\int \mathbb{P}(P|\tilde{P}).(P-\hat{P})\,dP$$

Setting the derivative to zero we get

$$\int \mathbb{P}(P|\tilde{P})P\,dP = \int \mathbb{P}(P|\tilde{P})\hat{P}\,dP.$$

$$\int \mathbb{P}(P|\tilde{P})P\,dP = \hat{P}\underbrace{\int \mathbb{P}(P|\tilde{P})\,dP}_{=1}$$

$$\Rightarrow \hat{P} = \int \mathbb{P}(P|\tilde{P})P\,dP$$

$$\boxed{\hat{P} = \int \frac{\mathbb{P}(\tilde{P}|P)\,\mathbb{P}(P)}{\mathbb{P}(\tilde{P})}P\,dP}$$

Image Denoising   Homework 3

8.1

$$\underbrace{\tilde{P}}_{\text{noisy patch}} = \underbrace{P}_{\text{clean patch}} + \underbrace{N}_{\text{noise}}$$

$$Cov(\tilde{P}) = E[\tilde{P}\tilde{P}^T] - E[\tilde{P}]E[\tilde{P}]^T$$

$E[\tilde{P}] = E[P] = \bar{P}$ by definition of $\bar{P}$

$$E[\tilde{P}\tilde{P}^T] = E[(P+N)(P+N)^T] = E[PP^T] + \underbrace{E[NN^T]}_{=\sigma^2 I}$$

$$\Rightarrow \boxed{Cov(\tilde{P})} = E[PP^T] + \sigma^2 I - E[P]E[P]^T$$
$$= \boxed{Cov(P) + \sigma^2 I}$$

8.3 a)

$$\hat{P}_1 - \bar{P} = (C_{\tilde{P}} - \sigma^2 I)C_{\tilde{P}}^{-1}(\tilde{P} - \bar{P}) \quad , \quad C_{\tilde{P}} = U\Lambda U^T \ (SVD)$$
$$= (U\Lambda U^T - \sigma^2 I)U\Lambda^{-1}U^T(\tilde{P} - \bar{P})$$
$$= (U\Lambda - \sigma^2 U)U^T U\Lambda^{-1}U^T(\tilde{P} - \bar{P})$$
$$= U(\Lambda - \sigma^2 I)\Lambda^{-1}U^T(\tilde{P} - \bar{P})$$

$$\Rightarrow U^T(\hat{P}_1 - \bar{P}) = \underbrace{(\Lambda - \sigma^2 I)\Lambda^{-1}}_{= D \text{ diagonal s.t. } D_{ii} = \frac{\lambda_i^2 - \sigma^2}{\lambda_i^2}} U^T(\tilde{P} - \bar{P})$$

$\Rightarrow$ D applies a linear shrinkage of the principal components based on the noisy patch $\tilde{P}$ $\rightarrow$ empirical Wiener

b) Analogously $U_1^T(\hat{P}_2 - \bar{P}^1) = \underbrace{(\Lambda_1 - \sigma^2 I)\Lambda_1^{-1}}_{D \text{ diagonal s.t. } D_{ii} = \frac{\lambda_i^{1\,2} - \sigma^2}{\lambda_i^{1\,2}}} U_1^T(\tilde{P} - \bar{P}^1)$

where $C_P^1 = U_1\Lambda_1 U_1^T$

$\Rightarrow$ D_1 applies a linear shrinkage of the principle components based on the previous step output $\rightarrow$ oracle Weiner

Note that this operators work directly on the principal components domain and the classical Weiner filter in the frequency domain