Detection Theory - Homework 1

Nicolás Violante

- 30 students with independent and uniform birthdays
- C_n : # of tuples of students with the same birthday
- $\mathbb{P}_n = \mathbb{P}(C_n \geq 1)$
- p_n : prob. of at least one n-tuple and no (n+1)-tuple

$p_n = \mathbb{P}(C_n \geq 1 \text{ and } C_{n+1} = 0)$

1) a) $\mathbb{P}_n = \mathbb{P}(C_n \geq 1) = 1 - \mathbb{P}(C_n = 0)$

$$
\begin{aligned}
\mathbb{P}(C_n = 0) &= \mathbb{P}(C_n = 0, C_{n-1} \geq 1) + \mathbb{P}(C_n = 0, C_{n-1} = 0) \\
&= p_{n-1} + \mathbb{P}(C_n = 0, C_{n-1} = 0, C_{n-2} \geq 1) + \mathbb{P}(C_n = 0, C_{n-1} = 0, C_{n-2} = 0) \\
&= p_{n-1} + p_{n-2} + \mathbb{P}(C_n = 0, C_{n-1} = 0, C_{n-2} = 0) \\
&\;\vdots \\
&= p_{n-1} + p_{n-2} + \cdots + p_1 + \underbrace{\mathbb{P}(C_n = 0, C_{n-1} = 0, \ldots, C_1 = 0)}_{=0}
\end{aligned}
$$

$$\Rightarrow \boxed{\mathbb{P}_n = 1 - \sum_{i=1}^{n-1} p_i}$$

b) Since $\mathbb{P}_n = 1 - \sum_{i=1}^{n-1} p_i$ and $\mathbb{P}_{n-1} = 1 - \sum_{i=1}^{n-2} p_i \Rightarrow \mathbb{P}_n - \mathbb{P}_{n-1} = -p_{n-1}$

$$\Rightarrow \boxed{\mathbb{P}_n = \mathbb{P}_{n-1} - p_{n-1}}$$

2) Let's define

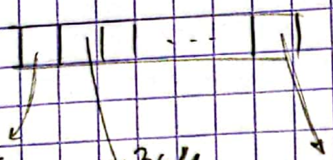
- A_n = students $i_1, \ldots, i_n$ have the same birthday

$$\boxed{\mathbb{E}C_n} = \mathbb{E}\sum_{1 \leq i_1 < i_2 \cdots < i_n \leq 30} \mathbb{1}_{A_n} = \sum_{1 \leq i_1 < \cdots \leq 30} \mathbb{E}[\mathbb{1}_{A_n}] = \binom{30}{n} \mathbb{P}(A_n) = \boxed{\binom{30}{n} \frac{1}{365^{n-1}}}$$

- $\mathbb{E}C_2 = 435 \cdot \frac{1}{365} \simeq 1{,}192$
- $\mathbb{E}C_2 = 4060 \cdot \frac{1}{365^2} \simeq 0{,}0305$
- $\mathbb{E}C_2 = 27405 \cdot \frac{1}{365^3} \simeq 5{,}64 \times 10^{-4}$

Detection Theory - Homework 1 Nicolás Violante

3) a) $P(C_2 = 0)$: no pair of students with the same birthday



365 options for birthday

364 options

336 options

$\Rightarrow$ how many forms of having no pair of students with the same birthday?

$\hookrightarrow 365 \cdot 364 \cdot \ldots \cdot 337 \cdot 336$

Now we need to divide by the total number of birthdays combinations: 365^{30}

$$\Rightarrow \boxed{P(C_2 = 0) = \frac{365 \cdot 364 \cdot \ldots \cdot 337 \cdot 336}{365^{30}} = \frac{365!}{335!} \frac{1}{365^{30}}}$$

We can approximate this by Stirling's formula: $\log n! = n \log n - n$

$$\Rightarrow P(C_2 = 0) \simeq \frac{1}{365^{30}} e^{1788,5 - 1627,7} \simeq 0,281 \quad (\sim 0,294 \text{ in hw})$$

b) $P_2 = P(C_2 \geq 1) = 1 - P(C_2 = 0) \simeq 0,719 \quad (\sim 0,706 \text{ in hw})$

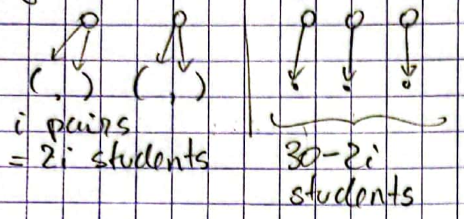
Detection Theory - Homework 1 Nicolás Virolante

4) $p_2 = P(C_2 \geq 1, C_3 = 0) = \sum_{i=1}^{15} P(C_2 = i, C_3 = 0)$

$$P(C_2 = i, C_3 = 0) = \frac{\#\{\text{valid configurations}\}}{365^{30}}$$

$\#\{\text{valid configs}\} = \#\{\text{ways of choose pairs}\}\#\{\text{pairs have valid birthdays}\}$

- If $C_2 = i$ and $C_3 = 0$ then the birthdays in the pairs are unique.



We can assign valid birthdays in:

$$\underbrace{365 \cdot 364}_{i \text{ times}} \cdot \underbrace{363 \cdot 362 \cdot 361}_{30-2i \text{ times}}$$

i times + 30-2i times = 30-i times

$\Rightarrow$ we can write it as $\sum_{k=0}^{29-i} (365-k)$

- Now we need to choose i pairs out of 30 students

$$\Rightarrow \underbrace{\binom{30}{2}\binom{28}{2}\binom{26}{2}}_{i \text{ terms}} \cdots = \prod_{j=1}^{i} \binom{32-2j}{2}$$

but with this strategy there are "equivalent" case, for example $\{(1,2),(3,4)\} \sim \{(3,4),(1,2)\}$.

Generally, given i pairs there are $i!$ possibilities

$\Rightarrow$ our final number is $\dfrac{\prod_{j=1}^{i}\binom{32-2j}{2}}{i!}$

Finally we get

$$\boxed{p_2 = \frac{1}{365^{30}} \sum_{i=1}^{15} \frac{\prod_{j=1}^{i}\binom{32-2j}{2}}{i!} \prod_{k=0}^{29-i}(365-k)}$$

Escaneado con CamScanner

Detection Theory - Homework 1 Nicolás Violante

5) $p_2 \simeq 0{,}678$

6) $\boxed{\mathbb{P}_3} = \mathbb{P}_2 - p_2 = 0{,}706 - 0{,}678 = \boxed{0{,}028}$

7) $p_3 = \mathbb{P}(C_3 \geq 1, C_3 = 0) = \sum_{i=1}^{10} \mathbb{P}(C_3 = i, C_4 = 0)$

$$\mathbb{P}(C_3 = i, C_4 = 0) = \frac{\#\{\text{ways of choose triplets}\}\,\#\{\text{triplets have valid birthdays}\}}{365^{30}}$$

- There are $\dfrac{\prod_{j=1}^{i}\binom{33-3j}{3}}{i!}$ ways of choosing the triplets

- If $C_3 = i$ and $C_4 = 0 \Rightarrow$ triplet's birthdays are unique and there are at most pairs of different birthdays outside the triplets

We distinguish two cases (we sum the results of each one)

1) Unique birthday outside the triplets

$\underbrace{(\ ,\ ,\ )\,(\ ,\ ,\ )}_{i \text{ triplets}} \;\Big|\; \underbrace{\ \ \ \ \ \ }_{30-3i \text{ students}} \Rightarrow \underbrace{365 \cdot 364}_{i \text{ terms}} \cdot \underbrace{363 \cdot 362 \cdot 361}_{30-3i \text{ terms}} = \prod_{k=0}^{29-2i}(365-k)$

2) ℓ pairs of students with the same birthday outside the triplets

$\Rightarrow$ we have at most $\ell = \left[\frac{30-3i}{2}\right]$ pairs

$\Rightarrow$ we can choose the pairs in $\dfrac{\prod_{m=1}^{\ell}\binom{30-3i+2-2m}{2}}{\ell!}$ ways

$\underbrace{(\ ,\ ,\ )\,(\ ,\ ,\ )}_{i \text{ terms}} + \underbrace{(\ ,\ )\ (\ ,\ )}_{\ell \text{ terms}} + \underbrace{\ \ \ \ }_{30-3i-2\ell \text{ terms}} = 30-2i-\ell \text{ terms}$

$\Rightarrow$ we can choose the birthdays in $\sum_{k=0}^{29-2i-\ell}(365-k)$ ways

$$\Rightarrow p_3 = \frac{1}{365^{30}} \sum_{i=1}^{10} \frac{\prod_{j=1}^{i}\binom{33-3j}{3}}{i!} \left[\prod_{k=0}^{29-2i}(365-k) + \sum_{\ell=1}^{\left[\frac{30-3i}{2}\right]} \frac{\prod_{m=1}^{\ell}\binom{30-3i+2-2m}{2}}{\ell!} \sum_{k=0}^{29-2i-\ell}(365-k) \right]$$

8)

9)

$$p_n = \frac{1}{365^{30}} \sum_{i=1}^{\left[\frac{30}{n}\right]} \frac{\prod_{j=1}^{i}\binom{30+n-nj}{n}}{i!} \left\{ \prod_{k=0}^{29-(n-1)i}(365-k) + \sum_{r=2}^{n-1} \sum_{\ell=1}^{\left[\frac{30-ni}{r}\right]} \frac{\prod_{m=1}^{\ell}\binom{30-ni+r-rm}{r}}{\ell!} \prod_{k=0}^{29-ri-(r-1)\ell}(365-k) \right\}$$

Detection Theory - Homework 1 Nicolás Violante

10) • $\boxed{\mathbb{P}_{30}} = \mathbb{P}(C_{30} \geq 1) = \mathbb{P}(C_{30} = 1) = \frac{365}{365^{30}} = \boxed{\frac{1}{365^{29}}}$

↳ All students have the same birthday

• $\boxed{\mathbb{E}C_{30}} = \binom{30}{30} \cdot \frac{1}{365^{29}} = \boxed{\frac{1}{365^{29}}}$

• $\mathbb{P}_{29} = \mathbb{P}(C_{29} \geq 1) = \underbrace{\mathbb{P}(C_{29} \geq 1, C_{30} = 0)}_{= \mathbb{P}(C_{29} = 1, C_{30} = 0)} + \underbrace{\mathbb{P}(C_{29} \geq 1, C_{30} = 1)}_{= \mathbb{P}(C_{30} = 1) = \frac{1}{365^{29}}}$

one student has a different birthday than the rest

30×364

30: ways of chosing the student

364: ways of chosing the birthday of the other 29 students

$\Rightarrow \mathbb{P}(C_{29} = 1, C_{30} = 0) = \frac{30 \times 64}{365^{29}}$

$\Rightarrow \boxed{\mathbb{P}_{29} = \frac{30 \times 64 + 1}{365^{29}}}$

• $\boxed{\mathbb{E}C_{29}} = \binom{30}{29} \cdot \frac{1}{365^{28}} = \boxed{\frac{30}{365^{28}}}$

11) $n = 2$ $\quad \frac{\mathbb{E}C_2 - \mathbb{P}_2}{\mathbb{P}_2} = \frac{1{,}192 - 0{,}706}{0{,}706} = 0{,}6884$

$n = 3$ $\quad \frac{\mathbb{E}C_3 - \mathbb{P}_3}{\mathbb{P}_3} = \frac{0{,}0347 - 0{,}0285}{0{,}0285} = 0{,}2175$

$n = 4$ $\quad \frac{\mathbb{E}C_4 - \mathbb{P}_4}{\mathbb{P}_4} = \frac{5{,}6 \times 10^{-4} - 5{,}3 \times 10^{-4}}{5{,}3 \times 10^{-4}} = 0{,}0566$

$\vdots$

$n = 29$ $\quad \frac{\mathbb{E}C_{29} - \mathbb{P}_{29}}{\mathbb{P}_{29}} = \frac{30/365^{28} - (30 \times 364 + 1)/365^{29}}{(30 \times 364 + 1)/365^{29}} = 2{,}65 \times 10^{-3}$

$n = 30$ $\quad \frac{\mathbb{E}C_{30} - \mathbb{P}_{30}}{\mathbb{P}_{30}} = 0$ since $\mathbb{E}C_{30} = \mathbb{P}_{30}$

12) Because of Markov's inequality: $\mathbb{P}_n \leq \mathbb{E}C_n \ \forall n$

Low values of $\mathbb{E}C_n$ are good upper bounds of $\mathbb{P}_n$

Since $\mathbb{E}C_{n+1} \leq \mathbb{E}C_n$ and $\mathbb{E}C_4 = 5{,}3 \times 10^{-4}$, we have good upper bounds for the cases where $n \geq 4$.