Image Denoising

4.1) $X \sim p(\lambda)$ $\boxed{E[X]} = \sum_{n\geq 1} n\, \mathbb{P}(X=n) = \sum_{n\geq 1} n \frac{\lambda^n e^{-\lambda}}{n!}$

$= e^{-\lambda} \sum_{n\geq 1} \frac{\lambda^n}{(n-1)!} \cdot \frac{\lambda^{-1}}{\lambda^{-1}} = \lambda e^{-\lambda} \underbrace{\sum_{n\geq 1} \frac{\lambda^{n-1}}{(n-1)!}}_{= e^{\lambda}}$

$= \lambda e^{-\lambda} e^{\lambda} = \boxed{\lambda}$

$Var(X) = E(X^2) - E(X)^2 = E(X^2) - \lambda^2$

$E[X(X-1)] = E[X^2 - X] = E[X^2] - \lambda$

$E[X(X-1)] = \sum_{n\geq 2} \frac{n(n-1)\lambda^n e^{-\lambda}}{n!} = \sum_{n\geq 2} \frac{\lambda^n e^{-\lambda}}{(n-2)!}$

$= e^{-\lambda} \lambda^2 \underbrace{\sum_{n\geq 2} \frac{\lambda^{n-2}}{(n-2)!}}_{e^{\lambda}} = \lambda^2$

$\Rightarrow E[X^2] = \lambda^2 + \lambda$

$\Rightarrow \boxed{Var(X)} = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2} = \boxed{\lambda}$

4.2) $X_1, \ldots, X_n$ independent poissen with parameter λ_i

Define $Y = \sum_{i=1}^{n} X_i$, we prove for two r.v

$\mathbb{P}(Y=n) = \mathbb{P}(X_1 + X_2 = n) = \sum_{i=0}^{n} \mathbb{P}(X_1 = n-i, X_2 = i)$

$= \sum_{i=0}^{n} \frac{\lambda_1^{n-i} e^{-\lambda_1}}{(n-i)!} \frac{\lambda_2^i e^{-\lambda_2}}{i!}$

$= \sum_{i=0}^{n} e^{-(\lambda_1+\lambda_2)} \frac{\lambda_1^{n-i} \lambda_2^{i}}{(n-i)!\, i!} \cdot \frac{n!}{n!}$

$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{i=0}^{n} \binom{n}{i} \lambda_1^{n-i} \lambda_2^{i}$

$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^n$ Then $Y \sim p(\lambda_1+\lambda_2)$

4.3) Linear variance model $g(t) = c^2 t$

$$f(u) = \int_0^u \frac{c}{\sqrt{t}}\, dt = \frac{c\sqrt{t}}{1/2}\Big|_0^u = 2c\sqrt{u}$$

Then, if we apply this f to $\tilde{u}_i$, by construction

$$f(\tilde{u}) = f(u) + \frac{c}{\sqrt{u}} \sqrt{u}\, n = f(u) + cn$$

4.5) $D_{inf} = \arg\min_{D} E[\|U - DO\|^2]$ is given by

$$a(i) = \frac{|\langle U, G_i\rangle|^2}{|\langle U, G_i\rangle|^2 + \sigma^2} \quad \text{and MSE is} \quad \sum_{i=1}^{M} \frac{|\langle U, G_i\rangle|^2 \sigma^2}{|\langle U, G_i\rangle|^2 + \sigma^2}$$

a)

$$E\left[\left\|\sum_{i=1}^{M} \langle U, G_i\rangle G_i - \sum_{i=1}^{M} a(i) \langle O, G_i\rangle G_i\right\|^2\right] =$$

$$= E\left[\left\|\sum_{i=1}^{M} \langle U, G_i\rangle G_i - \sum_{i=1}^{M} a(i) \langle U + N, G_i\rangle G_i\right\|^2\right]$$

$$= E\left[\left\|\sum_{i=1}^{M} \langle U, G_i\rangle G_i - \sum_{i=1}^{M} a(i) \langle U, G_i\rangle G_i - \sum_{i=1}^{M} a(i) \langle N, G_i\rangle G_i\right\|^2\right]$$

$$= E\left[\left\|\sum_{i=1}^{M} (1 - a(i)) \langle U, G_i\rangle G_i - \sum_{i=1}^{M} a(i) \langle N, G_i\rangle G_i\right\|^2\right]$$

(Cross terms involving $E[\langle N, G_i\rangle]$ are zero)

$$= E\left[\left\|\sum_{i=1}^{M} (1 - a(i)) \langle U, G_i\rangle G_i\right\|^2\right] + E\left[\left\|\sum_{i=1}^{M} a(i) \langle N, G_i\rangle G_i\right\|^2\right]$$

$$= \left\|\sum_{i=1}^{M} (1 - a(i)) \langle U, G_i\rangle G_i\right\|^2 + E\left[\sum_{i,j} a(i) a(j) \langle N, G_i\rangle \langle N, G_j\rangle \underbrace{\langle G_i, G_j\rangle}_{= \begin{cases} 1 & i = j \\ 0 & \text{o.w.} \end{cases}}\right]$$

$$= \sum_{i,j=1}^{M} (1 - a(i))(1 - a(j)) \langle U, G_i\rangle \langle U, G_j\rangle \langle G_i, G_j\rangle + E\left[\sum_{i=1}^{M} a^2(i) \langle N, G_i\rangle^2\right]$$

$$= \sum_{i=1}^{M} (1 - a(i))^2 \langle U, G_i\rangle^2 + E\left[\sum_{i=1}^{M} a^2(i) \langle N, G_i\rangle^2\right]$$

$$= \sum_{i=1}^{M} (1 - a(i))^2 \langle U, G_i\rangle^2 + \sum_{i=1}^{M} a^2(i) \underbrace{E[\langle N, G_i\rangle^2]}_{= \sigma^2}$$

$$E[\|U-D\tilde{U}\|^2] = \sum_{i=1}^{M} (1-a(i))^2 \langle U, G_i\rangle^2 + \sum_{i=1}^{M} a^2(i)\,\sigma^2 \quad \text{is convex}$$

$$\frac{\partial E[\|U-D\tilde{U}\|^2]}{\partial a(j)} = -2(1-a(j))\langle U,G_j\rangle^2 + 2\sigma^2 a(j) = 0$$

$$a(j)\left(2\sigma^2 + 2\langle U,G_j\rangle^2\right) = 2\langle U,G_j\rangle^2$$

$$\boxed{a(j) = \frac{\langle U,G_j\rangle^2}{\langle U,G_j\rangle^2+\sigma^2}} \rightarrow \text{is the minimum}$$

b)

$$E[\|U-D\tilde{U}\|^2] = \sum_{i=1}^{M} a(i)^2\left(\sigma^2 + \langle U,G_i\rangle^2\right) - 2a(i)\langle U,G_i\rangle^2 + \langle U,G_i\rangle^2$$

$$= \sum_{i=1}^{M} \frac{\langle U,G_j\rangle^4 \cancel{(\sigma^2+\langle U,G_i\rangle^2)}}{(\sigma^2+\langle U,G_j\rangle^2)^{\cancel{2}}} - \frac{2\langle U,G_j\rangle^4}{\langle U,G_j\rangle^2+\sigma^2} + \langle U,G_i\rangle^2$$

$$= \sum_{i=1}^{M} \frac{\cancel{\langle U,G_j\rangle^4} - 2\cancel{\langle U,G_j\rangle^4} + \sigma^2\langle U,G_i\rangle^2 + \cancel{\langle U,G_j\rangle^4}}{\sigma^2+\langle U,G_j\rangle^2}$$

$$= \sum_{i=1}^{M} \frac{\sigma^2\langle U,G_j\rangle^2}{\langle U,G_j\rangle^2+\sigma^2}$$

4b) D_{inf} with $a(i) = \begin{cases} 1 & \text{if } \langle U,G_i\rangle^2 \geq c\sigma^2 \\ 0 & \text{otherwise} \end{cases}$ Let $S = \{ i \in \{1,\dots,M\} : a(i)=1\}$ and $|S|$ its cardinal.

$$MSE = \sum_{i\in S}\sigma^2 + \sum_{i\notin S}\langle U,G_i\rangle^2 \leq \sum_{i\in S} c\sigma^2 + \sum_{i\notin S}\langle U,G_i\rangle^2$$

$$= \sum_{i=1}^{M} \min(\langle U,G_i\rangle^2, c\sigma^2)$$

Image denoising

4.7 Let $X, Y \in \mathbb{R}^N$

$$DCT(X)_k = 2\alpha_k \sum_{j=0}^{N-1} X_j \cos\left[\pi\left(j+\frac{1}{2}\right)\frac{k}{N}\right] \qquad \alpha_k = \begin{cases} \sqrt{1/(4N)} & k=0 \\ \sqrt{1/(2N)} & k=1,\dots,N-1 \end{cases}$$

1.

Let's write this as an inner product

Consider the vectors $v_k = 2\alpha_k \begin{pmatrix} \cos\left[\pi(0+1/2)\frac{k}{N}\right] \\ \vdots \\ \cos\left[\pi(N-1+1/2)\frac{k}{N}\right] \end{pmatrix}$

Now we'll prove that the vectors v_k form an orthonormal basis, thus DCT is an isometry

$$\langle v_k, v_\ell \rangle = \sum_{j=0}^{N-1} 4\alpha_k\alpha_\ell \cos\left[\pi\left(j+\frac{1}{2}\right)\frac{k}{N}\right]\cos\left[\pi\left(j+\frac{1}{2}\right)\frac{\ell}{N}\right]$$

$$= \sum_{j=0}^{N-1} 2\alpha_k\alpha_\ell\left[\cos\left(\frac{\pi}{N}\left(j+\frac{1}{2}\right)(k+\ell)\right) + \cos\left(\frac{\pi}{N}\left(j+\frac{1}{2}\right)(k-\ell)\right)\right]$$

$$= \sum_{j=0}^{N-1} \cancel{2}\alpha_k\alpha_\ell\left[\frac{e^{i\left[\frac{\pi}{N}\left(j+\frac{1}{2}\right)(k+\ell)\right]} + e^{-i\left[\frac{\pi}{N}\left(j+\frac{1}{2}\right)(k+\ell)\right]}}{\cancel{2}} + \frac{e^{i\pi\left(j+\frac{1}{2}\right)(k-\ell)} + e^{-i\pi\left(j+\frac{1}{2}\right)(k-\ell)}}{\cancel{2}}\right]$$

$$= \sum_{j=0}^{N-1} \alpha_k\alpha_\ell\left[e^{\frac{i\pi(k+\ell)}{2N}}\left(e^{\frac{i\pi(k+\ell)j}{N}} + e^{-\frac{i\pi(k+\ell)(1+j)}{N}}\right) + e^{\frac{i\pi(k-\ell)}{2N}}\left(e^{\frac{i\pi(k-\ell)j}{N}} + e^{-\frac{i\pi(k-\ell)(1+j)}{N}}\right)\right]$$

$$= \alpha_k\alpha_\ell\left[e^{\frac{i\pi(k+\ell)}{2N}}\left(\sum_{j=0}^{N-1} e^{\frac{i\pi(k+\ell)j}{N}} + \sum_{j=0}^{N-1} e^{-\frac{i\pi(k+\ell)(1+j)}{N}}\right) + e^{\frac{i\pi(k-\ell)}{2N}}\left(\sum_{j=0}^{N-1} e^{\frac{i\pi(k-\ell)j}{N}} + \sum_{j=0}^{N-1} e^{-\frac{i\pi(k+\ell)(1+j)}{N}}\right)\right]$$

$$= \alpha_k\alpha_\ell\left[e^{\frac{i\pi}{2N}(k+\ell)}\left(\sum_{j=0}^{N-1} e^{\frac{i\pi(k+\ell)j}{N}} + \sum_{j=-N}^{-1} e^{\frac{i\pi(k+\ell)j}{N}}\right) + e^{\frac{i\pi(k-\ell)}{2N}}\left(\sum_{j=0}^{N-1} e^{\frac{i\pi(k-\ell)j}{N}} + \sum_{j=-N}^{-1} e^{\frac{i\pi(k-\ell)j}{N}}\right)\right]$$

$$\langle v_k, v_\ell \rangle = \alpha_k \alpha_\ell \left[e^{\frac{i\pi(k+\ell)}{2N}} \sum_{j=-N}^{N-1} e^{\frac{i\pi(k+\ell)j}{N}} + e^{\frac{i\pi(k-\ell)}{2N}} \sum_{j=-N}^{N-1} e^{\frac{i\pi(k-\ell)j}{N}} \right]$$

$$= \alpha_k \alpha_\ell \left[e^{\frac{i\pi(k+\ell)}{2N}} \left(\frac{e^{-i\pi(k+\ell)j} - e^{i\pi(k+\ell)j}}{1 - e^{\frac{i\pi(k+\ell)j}{N}}} \right) + e^{\frac{i\pi(k-\ell)}{2N}} \left(\frac{e^{-i\pi(k-\ell)j} - e^{i\pi(k-\ell)j}}{1 - e^{\frac{i\pi(k-\ell)j}{N}}} \right) \right]$$

only when both geometric sum are valid

- If $k=0$ and $\ell=0$:

$$\langle v_k, v_\ell \rangle = \alpha_0^2 \left[2N + 2N \right] = \frac{4N}{4N} = 1$$

- If $k \neq \ell$:

$\langle v_k, v_\ell \rangle = 0$ (exponentials take values 1 or -1 depending on $k+\ell$, $k-\ell$ odd or even)

- If $k = \ell \neq 0$

$$\langle v_k, v_\ell \rangle = \frac{1}{2N}(0 + 2N) = 1$$

Then the DCT is orthogonal
The same reasoning leads to IDCT being orthogonal as well as it's defined with the the same basis.

b) Let A be the DCT matrix and B the IDCT matrix

Then $A_{ij} = 2\alpha_i \cos\left[\pi\left(j+\frac{1}{2}\right)\frac{i}{N}\right]$

$B_{ij} = 2\beta_j \cos\left[\pi\left(i+\frac{1}{2}\right)\frac{j}{N}\right]$ with $\beta_j = \begin{cases} \sqrt{1/4N} & j=0 \\ \sqrt{1/2N} & \text{o.w} \end{cases}$

$\alpha_i = \begin{cases} \sqrt{1/4N} & i=0 \\ \sqrt{1/2N} & \text{o.w} \end{cases}$

Note that $A_{ji} = B_{ij} \; \forall i,j$ then $A^T = B$

Since $A^T A = I$ and $B^T B = I$ we conclude that $B = A^{-1}$

With the same argument $B^{-1} = A$.

4.8 $\underset{\sum_k \alpha_k = 1}{\operatorname{argmin}} \sum_k \alpha_k^2 \sigma_k^2$ The Lagrangean is

$$L = \sum_k \alpha_k^2 \sigma_k^2 + \lambda\left(\sum_k \alpha_k - 1\right) = 0$$

Then $\frac{\partial L}{\partial \alpha_j} = 2\alpha_j \sigma_j^2 - \lambda = 0 \quad \forall j$

4.9 In this case, using Parseval we have that

$$\sigma_k^2 = \| \sigma^2 \rho_{P_k} \|^2 = \sigma^4 \| \rho_{P_k} \|^2$$

Then

$$\alpha_k = \frac{\cancel{\sigma^{-4}} \| \rho_{P_k} \|^{-2}}{\sum_j \cancel{\sigma^{-4}} \| \rho_{P_j} \|^{-2}} = \frac{\| \rho_{P_k} \|^{-2}}{\sum_j \| \rho_{P_j} \|^{-2}}$$

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