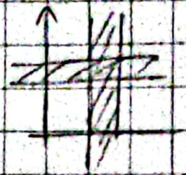
Ex 1

1)

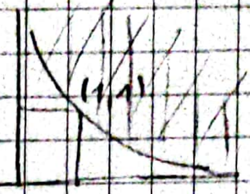


- $\{x_i : \alpha_i \leq x_i\}$ is a half-space
- $\{x_i : x_i \leq \beta_i\}$ is a halfspace

$\Rightarrow$ their intersection is convex

Then the intersection $\{x \in \mathbb{R}^n / \alpha_i \leq x_i \leq \beta_i, i \in \{1, \dots, n\}\}$ is also convex

2) $C = \{x \in \mathbb{R}^2_+ / x_1 x_2 \geq 1\}$



Let $(x_1, x_2), (y_1, y_2) \in C$

$\theta(x_1, x_2) + (1-\theta)(y_1, y_2) = (\theta x_1 + (1-\theta) y_1, \theta x_2 + (1-\theta) y_2)$

$$[\theta x_1 + (1-\theta) y_1][\theta x_2 + (1-\theta) y_2] = \theta^2 x_1 x_2 + (1-\theta)^2 y_1 y_2 + \theta(1-\theta) \frac{(x_1 y_2 + x_2 y_1)}{2} \cdot 2$$

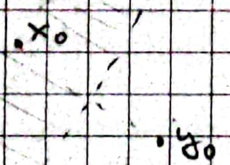
$$\geq \theta^2 + (1-\theta)^2 + 2(1-\theta)\theta \sqrt{x_1 x_2 y_1 y_2}$$

$$= \cancel{\theta^2} + 1 + \cancel{\theta^2} - \cancel{2\theta} + \cancel{2\theta} - \cancel{2\theta^2} = 1$$

Then C is convex

3) $C = \{x : \|x - x_0\|_2 \leq \|x - y\|_2 \ \forall y \in S\}$

Let $y_0 \in S$ and $C_{y_0} = \{x : \|x - x_0\|_2 \leq \|x - y_0\|_2\}$



We have $\|x - x_0\|_2^2 \leq \|x - y_0\|_2^2$

$\Leftrightarrow \cancel{\|x\|_2^2} + \|x_0\|_2^2 - 2x_0^T x \leq \cancel{\|x\|_2^2} + \|y_0\|_2^2 - 2y_0^T x$

$(y_0 - x_0)^T x \leq \frac{\|y_0\|^2 - \|x_0\|^2}{2}$

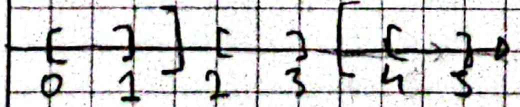
$\downarrow$

half-space (convex)

Then C is convex since it's the intersection of convex sets

4) $C = \{x : d(x, S) \leq d(x, T)\}$ is not convex, let's see an example on $\mathbb{R}$

$S = [0, 1] \cup [4, 5]$ and $T = [2, 3]$



Then $C = (-\infty, \frac{3}{2}] \cup [\frac{7}{2}, +\infty)$ is not convex

5) $C = \{x : x + S_2 \subseteq S_1\} = \{x : x + s_2 \in S_1 \ \forall s_2 \in S_2\}$

First we'll prove that $\{x : x + s_2 \in S_1\}$ for a given s_2 is convex

Let $x_1, x_2 \in \{x : x + s_2 \in S_1\} \Rightarrow x_1 + s_2 \in S_1 \quad x_2 + s_2 \in S_1$

Since S_1 is convex. $\theta(x_1 + s_2) + (1-\theta)(x_2 + s_2) \in S_1 \quad \forall \theta \in [0,1]$

$\theta(x_1 + s_2) + (1-\theta)(x_2 + s_2) = \theta x_1 + (1-\theta)x_2 + s_2 \in S_1$

$\Rightarrow \quad \theta x_1 + (1-\theta)x_2 \in \{x : x + s_2 \in S_1\} \Rightarrow$ this set is convex

Finally C is convex since it's intersection of convex sets.

Ex2) 1) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$

$\frac{\partial f}{\partial x_1}(x_1, x_2) = x_2$ $\qquad \frac{\partial f}{\partial x_2}(x_1, x_2) = x_1$

$\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2) = 1$ $\qquad \frac{\partial^2 f}{\partial x_1^2}(x_1, x_2) = \frac{\partial^2}{\partial x_2^2} f(x_1, x_2) = 0$

$\nabla^2 f(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad \begin{vmatrix} -s & 1 \\ 1 & -s \end{vmatrix} = s^2 - 1 = 0$

then the eigenvalues are $\begin{cases} s = 1 \\ s = -1 \end{cases}$

Then • $\nabla^2 f$ is not positive semi-definite $\Rightarrow f$ is not convex

• $\nabla^2 f$ is not negative semi-definite $\Rightarrow f$ is not concave

↳ We could also see that the eigenvalues of f and $-f$ are the same, since $\nabla^2(-f)(x_1, x_2) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and $-f$ is not convex

2) $f(x_1, x_2) = \frac{1}{x_1 x_2}$ on $\mathbb{R}^2_{++}$

$\frac{\partial f}{\partial x_1}(x_1, x_2) = \frac{-1}{x_2 x_1^2}$ $\qquad \frac{\partial^2 f}{\partial x_1^2}(x_1, x_2) = \frac{2}{x_2 x_1^3}$ $\qquad \frac{\partial^2 f}{\partial x_2^2}(x_1, x_2) = \frac{2}{x_1 x_2^3}$

$\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2) = \frac{1}{x_1^2 x_2^2}$

$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} \frac{2}{x_2 x_1^3} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{pmatrix} = \frac{1}{x_1 x_2} \begin{pmatrix} \frac{2}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & \frac{2}{x_2^2} \end{pmatrix}$$

• $\mathrm{tr}(\nabla^2 f) = \frac{2}{x_2 x_1^3} + \frac{2}{x_1 x_2^3} \geq 0 \Rightarrow \lambda_1 + \lambda_2 \geq 0$ (λ_1, λ_2 eigenvalues)

• $\det(\nabla^2 f) = \frac{1}{x_1 x_2}\left(\frac{4}{x_1^2 x_2^2} - \frac{1}{x_1^2 x_2^2}\right) = \frac{3}{x_1^3 x_2^3} \geq 0 \Rightarrow \lambda_1 \lambda_2 \geq 0$

then $\lambda_1 \geq 0$ and $\lambda_2 \geq 0 \Rightarrow \nabla^2 f \succeq 0 \Rightarrow f$ is convex (strictly)

Also f is not concave

3) $f(x_1,x_2)=\frac{x_1}{x_2}$ on $\mathbb{R}^2_{++}$ (dom f is convex)

$\frac{\partial f}{\partial x_1}(x_1,x_2)=\frac{1}{x_2}$ $\quad \frac{\partial^2 f}{\partial x_1^2}(x_1,x_2)=0$ $\quad \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1,x_2)=-\frac{1}{x_2^2}$

$\frac{\partial f}{\partial x_2}(x_1,x_2)=-\frac{x_1}{x_2^2}$ $\quad \frac{\partial^2 f}{\partial x_2^2}(x_1,x_2)=2\frac{x_1}{x_2^3}$

$$\nabla^2 f(x_1,x_2)=\begin{pmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{pmatrix}$$

- $tr(\nabla^2 f(x_1,x_2))=\frac{2x_1}{x_2^3}>0 \Rightarrow \lambda_1+\lambda_2>0$ (λ_1,λ_2 eigenvalues of $\nabla^2 f$)

- $det(\nabla^2 f(x_1,x_2))=-\frac{1}{x_2^4}<0 \Rightarrow \lambda_1\lambda_2<0$

Then λ_1 and λ_2 have opposite sign (one positive, one negative)

$\Rightarrow f$ is not convex and not concave

4) $f(x_1,x_2) = x_1^{\alpha} x_2^{1-\alpha}$ $\quad 0 \leq \alpha \leq 1$ on $\mathbb{R}^2_{++}$ (dom f is convex)

$\frac{\partial f}{\partial x_1}(x_1,x_2) = \alpha x_1^{\alpha-1} x_2^{1-\alpha}$; $\frac{\partial^2 f}{\partial x_1^2}(x_1,x_2) = \alpha(\alpha-1) x_1^{\alpha-2} x_2^{1-\alpha}$

$\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1,x_2) = \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha}$

$\frac{\partial f}{\partial x_2}(x_1,x_2) = (1-\alpha) x_1^{\alpha} x_2^{-\alpha}$, $\frac{\partial^2 f}{\partial x_2^2}(x_1,x_2) = -\alpha(1-\alpha) x_1^{\alpha} x_2^{-(\alpha+1)}$

$$\nabla f^2(x_1,x_2) = \begin{pmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^{1-\alpha} & \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \\ \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} & -\alpha(1-\alpha) x_1^{\alpha} x_2^{-(\alpha+1)} \end{pmatrix}$$

$$= \alpha(1-\alpha) x_1^{\alpha} x_2^{-\alpha} \begin{pmatrix} -\frac{x_2}{x_1^2} & \frac{1}{x_1} \\ \frac{1}{x_1} & -\frac{1}{x_2} \end{pmatrix}$$

if $0 < \alpha < 1$:

$tr(\nabla f^2) = \alpha(1-\alpha) x_1^{\alpha} x_2^{-\alpha} \left(-\frac{x_2}{x_1^2} - \frac{1}{x_2} \right) = -\alpha(1-\alpha) x_1^{\alpha} x_2^{-\alpha} \left(\frac{x_2}{x_1^2} + \frac{1}{x_2} \right) < 0$

$\det(\nabla f^2) = 0$; $\left.\begin{array}{l} \lambda_1 + \lambda_2 < 0 \\ \lambda_1 \lambda_2 = 0 \end{array}\right\}$ one eigenvalue is negative and the other is 0

$\Rightarrow$ f is not convex

f is concave

• if $\alpha = 0$ or $\alpha = 1$:

We have $\left\{\begin{array}{ll} f(x_1,x_2) = x_1 & (\alpha = 1) \\ f(x_1,x_2) = x_2 & (\alpha = 0) \end{array}\right.$ $\Rightarrow f$ is affine $\rightarrow$ f is convex

f is concave

Convex Optimization: $\boxed{Ex2}$ optionals

1) $f(x_1,x_2) = x_1x_2$ on $\mathbb{R}^2_{++}$ $\Rightarrow$ we saw it's not convex and not concave

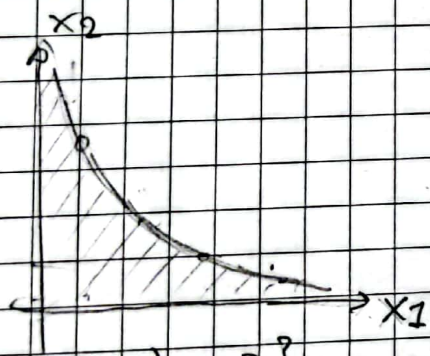
- Let $S_\alpha = \{x \in \mathbb{R}^2_{++} : x_1x_2 \geq \alpha\}$
  - If $\alpha \leq 0$: then $S_\alpha = \mathbb{R}^2_{++}$ which is a convex set
  - If $\alpha > 0$: Let $x, y \in S_\alpha$ and $\theta \in [0,1]$

$\theta x + (1-\theta)y = (\theta x_1 + (1-\theta)y_1, \theta x_2 + (1-\theta)y_2)$

$$[\theta x_1 + (1-\theta)y_1][\theta x_2 + (1-\theta)y_2] = \theta^2 x_1x_2 + (1-\theta)^2 y_1y_2 + \theta(1-\theta)[x_1y_2 + x_2y_1]$$
$$\geq \theta^2\alpha + (1-\theta)^2\alpha + 2\theta(1-\theta)\underbrace{|\alpha|}_{=\alpha}$$
$$= \alpha(\cancel{\theta^2} + (1 + \cancel{\theta^2} - \cancel{2\theta} + \cancel{2\theta} - \cancel{2\theta^2})$$
$$= \alpha$$

Then S_α is convex

Since S_α is convex $\forall \alpha \in \mathbb{R}$ $\Rightarrow$ f is quasiconcave.

- 

Let $S_\alpha = \{x \in \mathbb{R}^2_{++} : x_1x_2 \leq \alpha\}$

$x = (1, \alpha)$ and $y = (\alpha, 1) \in S_\alpha$

but $p = \frac{x+y}{2} = \left(\frac{1+\alpha}{2}, \frac{1+\alpha}{2}\right)$

$p_1p_2 = \frac{(1+\alpha)^2}{4} = 25$ for $\alpha = 9$

then $p_1p_2 > \alpha$ for $\alpha = 9$

$\Rightarrow S(\alpha=9)$ is not convex

Thus f is not quasiconvex

Convex Optimization: Ex 2 | optionals

2) $f(x_1,x_2) = \frac{1}{x_1 x_2}$ on $\mathbb{R}^2_{++}$ is convex and not concave

Since f is convex $\rightarrow f$ is quasiconvex

Let $S_\alpha = \{ x \in \mathbb{R}^2_{++} : \frac{1}{x_1 x_2} \geq \alpha \}$

- If $\alpha \leq 0$: $S_\alpha = \mathbb{R}^2_{++}$ which is convex
- If $\alpha > 0$: $\frac{1}{x_1 x_2} \geq \alpha \iff \frac{1}{\alpha} \geq x_1 x_2$ which is not convex (same reasoning as previous ex)

$\Rightarrow f$ is not quasiconcave

3) $f(x_1,x_2) = \frac{x_1}{x_2}$ on $\mathbb{R}^2_{++}$ is not convex and not concave

- Quasiconvex?

Let $S_\alpha = \{ x \in \mathbb{R}^2_{++} : \frac{x_1}{x_2} \leq \alpha \}$

- If $\alpha \leq 0$: $S_\alpha = \emptyset$ which is convex
- If $\alpha > 0$: $\frac{x_1}{x_2} \leq \alpha \iff x_1 \leq \alpha x_2$

Let $x, y \in S_\alpha$

$\theta x + (1-\theta) y = (\theta x_1 + (1-\theta) y_1, \theta x_2 + (1-\theta) y_2)$

$\theta \underbrace{x_1}_{\leq \alpha x_2} + (1-\theta) \underbrace{y_1}_{\leq \alpha y_2} \leq \theta \alpha x_2 + (1-\theta) \alpha y_2 = \alpha [\theta x_2 + (1-\theta) y_2]$

$\Rightarrow S_\alpha$ is convex for $\alpha > 0$

Then f is quasiconvex

- Quasiconcave? Let $S_\alpha = \{ x \in \mathbb{R}^2_{++} : \frac{x_1}{x_2} \geq \alpha \}$

If $\alpha \leq 0$: $S_\alpha = \mathbb{R}^2_{++}$ which is convex

If $\alpha > 0$: Let $x, y \in S_\alpha$

$\theta \underbrace{x_1}_{\geq \alpha x_2} + (1-\theta) \underbrace{y_2}_{\geq \alpha y_2} \geq \alpha(\theta x_2 + (1-\theta) y_2) \Rightarrow S_\alpha$ is convex for $\alpha > 0$

Then f is quasiconcave

Convex Optimization: EX2 | optionals

4) $f(x_1,x_2) = x_1^{\alpha} x_2^{1-\alpha}$ $\quad 0 \le \alpha \le 1$ on $\mathbb{R}_{++}^2$

- f is concave $\Rightarrow$ f is quasiconcave
- If $\alpha = 0$ or $\alpha = 1$ $\Rightarrow$ f is convex $\Rightarrow$ f quasiconvex for $\alpha = 0$, $\alpha = 1$

If $0 < \alpha < 1$ $\Rightarrow$ f is not convex

Let $S_\beta = \{x \in \mathbb{R}_{++}^2 : f(x_1, x_2) \le \beta\}$

Then if we take $\alpha = \frac{1}{2}$ $\quad f(x_1,x_2) = \sqrt{x_1 x_2}$

Then $\sqrt{x_1 x_2} \le \beta \Leftrightarrow x_1 x_2 \le \beta^2 \Rightarrow S_\beta$ not convex

Thus f is not quasiconvex

Ex3) 1) $f(X) = tr(X^{-1})$ on S^n_{++}

Let t, V / $X + tV \in S^n_{++}$

$$tr((X+tV)^{-1}) = tr\left(\left[X^{1/2}(I + t\overbrace{X^{-1/2} V X^{-1/2}}^{\hat{V}})X^{1/2}\right]^{-1}\right)$$

$$= tr\left(X^{-1/2}(I + t\hat{V})^{-1}X^{-1/2}\right)$$

$$= tr\left(X^{-1}(I + t\hat{V})^{-1}\right)$$

$\hat{V}$ diagonalizable $\begin{cases} QQ^T = I \\ D \text{ diagonal.} \end{cases}$

$$= tr\left(X^{-1}(QQ^T + tQDQ^T)^{-1}\right)$$

$$= tr\left(X^{-1}\left[Q(Q^T + tDQ^T)\right]^{-1}\right)$$

$$= tr\left(X^{-1}\left[Q(I + tD)Q^T\right]^{-1}\right)$$

$$= tr\left(X^{-1}Q(I + tD)^{-1}Q^T\right)$$

$$= tr\left(Q^T X^{-1} Q (I + tD)^{-1}\right)$$

$$= \sum_{i=1}^{n} (Q^T X^{-1} Q)_{ii} \frac{1}{1 + td_i} = g(t)$$

$$g'(t) = -\sum_{i=1}^{n} (Q^T X^{-1} Q)_{ii} \frac{d_i}{(1 + td_i)^2}$$

$$g''(t) = \sum_{i=1}^{n} (Q^T X^{-1} Q)_{ii} \frac{2d_i^2}{(1 + td_i)^3} \geq 0 \Rightarrow f \text{ is convex}$$

2) $f(X, y) = y^T X^{-1} y$ $\quad$ $\text{dom} f = S^n_{++} \times \mathbb{R}^n$

$$\frac{1}{2} y^T X^{-1} y = \sup_z \left(y^T z - \frac{1}{2} z^T X z\right)$$

Let $g(X, y, z) = y^T z - \frac{1}{2} z^T X z$

For a fixed z we have:

- $g_1(y) = y^T z$ is affine, thus convex (on y)
- $g_2(X) = -\frac{1}{2} z^T X z$ is affine, thus convex (on X)
- Then $y^T z - \frac{1}{2} z^T X z$ is convex (sum of convex)

Finally, using the pointwise supremum property, we have that $\frac{1}{2} y^T X^{-1} y$ is convex, then $f(X, y)$ is convex

Convex Optimization Ex3

3) $f(X) = \sum_{i=1}^{n} \sigma_i(X)$ on S^n

Let Q be an orthogonal matrix, $X = U\Delta V^T$ SVD

$$|tr(QX)| = |tr(QU\Lambda V^T)| = |tr(V^T Q U \Lambda)|$$
$$= \left|\sum_{i=1}^{n} (V^T Q U)_{ii} \sigma_i(X)\right|$$
$$\leq \sum_{i=1}^{n} |\underbrace{V^T Q U}_{\text{is orthogonal}}|_{ii}\, \sigma_i(X) \leq \sum_{i=1}^{n} \sigma_i(X) \quad \forall Q \text{ orthogonal}$$

But if we choose $\tilde{Q} = VU^T$

$$|tr(VU^T X)| = \left|\sum_{i=1}^{n} 1\, \sigma_i(X)\right| = \sum_{i=1}^{n} \sigma_i(X)$$

Thus we can express

$$\sum_{i=1}^{n} \sigma_i(X) = \sup_{Q:Q^TQ=I} \underbrace{|tr(QX)|}_{g(Q,X)}$$

For a fixed Q, we have that QX is affine on X.

Since $|\cdot|$ is convex and $tr(\cdot)$ is affine then $|tr(\cdot)|$ is convex $\Rightarrow |tr(QX)|$ is convex

Finally the sup over Q preserves convexity,

so f is convex

Convex Optimization $\boxed{Ex\ 4}$ optionals

1) • Let $\alpha, \beta \geq 0$ and $x, y \in K_{m+}$

$\alpha x + \beta y = (\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n)$

$\alpha \underbrace{x_1}_{\geq x_2} + \beta \underbrace{y_1}_{\geq y_2} \geq \alpha \underbrace{x_2}_{\geq x_3} + \beta \underbrace{y_2}_{\geq y_3} \geq \ldots \geq \alpha x_n + \beta y_n$

Then $\alpha x + \beta y \in K_{m+} \Rightarrow K_{m+}$ is a convex cone

• K_{m+} is closed since it's defined with n (not strict) linear inequalitis. Thus K_{m+} contains its boundary.

• Let $p = (n, n-1, n-2, \ldots, 1)$

$p \in K_{m+}$ and strictly satisfies the inequalities
Then K_{m+} has non-empty interior

• If $x \in K_{m+}$ and $-x \in K_{m+}$ we have

$x_1 \geq x_2, \quad x_2 \geq x_3, \quad \ldots, \quad x_n \geq 0$
$x_1 \leq x_2 \quad x_2 \leq x_3 \quad\quad x_n \leq 0$

Then $x_1 = x_2 = \ldots = x_n = 0$

Finally, K_{m+} is a proper cone

2) $K_{m+}^* = \{ y \,/\, y^T x \geq 0 \ \forall x \in K_{m+} \}$

$$y^T x = \sum_{i=1}^{n} y_i x_i = \overbrace{(x_1 - x_2)}^{\geq 0} y_1 + \overbrace{(x_2 - x_3)}^{\geq 0}(y_1 + y_2) + \ldots + \underbrace{(x_{n-1} - x_n)}_{\geq 0}(y_1 + \ldots + y_{n-1}) + \underbrace{x_n}_{\geq 0}(y_1 + \ldots + y_n)$$

Then we need $y_1 \geq 0, \ y_1 + y_2 \geq 0, \ldots, (y_1 + \ldots + y_n) \geq 0$

$$K_{m+}^* = \{ y \,/\, \sum_{i=1}^{k} y_i \geq 0 \ \forall k = 1, \ldots, n \}$$

Convex Optimization: Optionals

ExS 1) $f(x) = \max_{i=1,\dots,n} x_i$ on $\mathbb{R}^n$

$$f^*(y) = \sup_x \left(y^T x - \max_{i=1,\dots,n} x_i \right) = \sup_x g(x,y)$$

$\text{dom } f^*$ is the set of $y \in \mathbb{R}^n$ for which the sup is finite

- If $y_i < 0$ for some i. Let $x_i = -\alpha$ and $x_j = 0 \ \forall j \neq i$

Then $g(x,y) = \sum_{i=1}^n y_i x_i - \max_i x_i = -\alpha y_i \xrightarrow{\alpha \to +\infty} +\infty$

- If $y_i \geq 0 \ \forall i$

Let $x_i = \alpha \ \forall i$: $g(x,y) = \alpha \sum_{i=1}^n y_i - \alpha = \alpha\left(\sum_{i=1}^n y_i - 1\right)$

$$\left.\begin{array}{ll} \cdot \text{ if } \sum_{i=1}^n y_i > 1 & g(x,y) \xrightarrow{\alpha \to +\infty} +\infty \\ \cdot \text{ if } \sum_{i=1}^n y_i < 1 & g(x,y) \xrightarrow{\alpha \to -\infty} +\infty \end{array}\right\} \text{ then the sup is } +\infty$$

- if $\sum_{i=1}^n y_i = 1 \quad g(x,y) = 0$

$$g(x,y) \leq \sum_{i=1}^n y_i \max_i\{x_i\} - \max_i\{x_i\} \quad \forall x \in \mathbb{R}^n$$

$$= \max_i\{x_i\} \underbrace{\left(\sum_{i=1}^n y_i - 1\right)}_{=0} = 0 \quad \forall x \in \mathbb{R}^n$$

Then, for $y_i \geq 0 \ \forall i$ with $\sum_{i=1}^n y_i = 1$ $\sup_x g(x,y) = 0$

Finally $f^*(y) = \begin{cases} 0 & \text{if } y \geq 0 \text{ and } y^T \mathbb{1} = 1 \\ +\infty & \text{otherwise} \end{cases}$

2) $f(x) = \sum_{i=1}^{r} x_{[i]}$ on $\mathbb{R}^n$

$$f^*(y) = \sup_x \left(\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{r} x_{[i]} \right)$$

- If $y_i > 0$ for some i:

Let $x_i = \alpha$ and $x_j = 0 \ \forall i \neq j$

Then $g(x,y) = \alpha y_i - \alpha = \alpha(y_i - 1)$

If $y_i > 1$ then $g(x,y) \xrightarrow{\alpha \to +\infty} +\infty$ | the sup is $+\infty$ if some $y_i > 0$.

- If $0 \leq y_i \leq 1 \ \forall i$:

Let $x_i = \alpha \ \forall i = 1, \dots, n \Rightarrow g(x,y) = \alpha\left(\sum_{i=1}^{n} y_i - r\right)$

- If $\sum_{i=1}^{n} y_i > r \Rightarrow g(x,y) \xrightarrow{\alpha \to +\infty} +\infty$
- If $\sum_{i=1}^{n} y_i < r \Rightarrow g(x,y) \xrightarrow{\alpha \to -\infty} +\infty$
- If $\sum_{i=1}^{n} y_i = r \Rightarrow g(x,y) = 0$

Now let's find a bound for $g(x,y)$

$$\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{r} x_{[i]} \leq \max_i \{x_i\} \left(\sum_{i=1}^{n} y_i - r \right) = 0 \quad \text{if } \sum_{i=1}^{n} y_i = r$$

Then $f^*(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq 1 \text{ and } y^T \mathbb{1} = r \\ +\infty & \text{otherwise.} \end{cases}$