Image Denoising - Homework 2

Ex 5.1 u_0 : initial image of shape $n \times n$
u_1: lower scale image of shape $K \times K$

Then $u_1 = IDCT_K^{iso}(ZP_K(DCT_n^{iso}(u_0)))$
$= DCT_K^{iso\,T}(ZP_K(DCT_n^{iso}(u_0)))$

$Var(Ax) = A\,Var(x)A^T$

$$\sigma'^2 Id = Var(u_1) = Var((\overbrace{DCT_K^T\, ZP_K\, DCT_n}^{A})u_0)$$
$$= DCT_K^T\, ZP_K\, DCT_n \cdot \sigma^2 I\, DCT_n^T\, ZP_K^T\, DCT_K$$

Note on ZP_K:

Let $A = \begin{pmatrix} a & b & 1 \\ c & d & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (small example)

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & \cdots & & & \\ 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 1 & 0 & \cdots & \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & & & & & & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & & & & & 0 \end{pmatrix}}_{ZP_K \text{ for the "flattened" matrix}} \begin{pmatrix} a \\ b \\ 1 \\ c \\ d \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\Rightarrow ZP_K\, ZP_K^T = Id$

Since $DCT^T DCT = Id$ and $ZP_K^T ZP_K = Id$
we are getting $\sigma'^2 Id = \sigma^2 Id$ instead of
the intuitive result due to the averaging
(We discussed it at today's class
but weren't able to spot the error)

- If we assume that the DCT is actually averaging on the spatial domain, the result is easy to show. If we go from $N \times N$ to $K \times K$ by averaging the law of variance ensurse the noise is reduced. If we go from N^2 to K^2 by averaging, we need to average $\frac{N^2}{K^2}$ pixels $\sigma'^2 = Var\left(\frac{\sum_{i=1}^{N^2/K^2} X_i}{\frac{N^2}{K^2}}\right) = \frac{1}{(\frac{N}{K})^2}\cancel{(\frac{N}{K})}\sigma^2$

$$\Rightarrow \boxed{\sigma' = \frac{\sigma}{\sqrt{\frac{N^2}{K}}}}$$

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Ex 6.1 Let $n[i]$ be a white noise sequence
$a[i] \geq 0$ with support $[-m, m]$

a) $a * n[i] = \sum_j n(i-j)\, a(j)$

$Var(x) = \mathbb{E}(x^2) - \mathbb{E}(x)^2$

$$\mathbb{E}[a * n[i]] = \sum_j \mathbb{E}[n(i-j)\, a(j)] = \sum_j a(j)\, \underbrace{\mathbb{E}[n(i-j)]}_{=0} = 0$$

$$\mathbb{E}\Big[\big(\sum_j n(i-j)\, a(j)\big)^2\Big] = \mathbb{E}\Big[\sum_j \sum_k n(i-j)\, n(i-k)\, a(j)\, a(k)\Big]$$

$$= \sum_j \sum_k \underbrace{\mathbb{E}[n(i-j)\, n(i-k)]}_{= \begin{cases} \sigma^2 & \text{if } j = k \\ 0 & \text{o.w} \end{cases}} . a(i)\, a(k)$$

$$= \sum_j \sigma^2 a(j)^2 = \sigma^2 \sum_{j=-m}^{m} a(j)^2$$

$$\Rightarrow \boxed{Var(a * n) = \sigma^2 \sum_{j=-m}^{m} a(j)^2}$$

b) Let $\sum_{j=-m}^{m} a(j) = 1$

Since $a(j) \geq 0 \ \forall j$

$\Rightarrow a(j) \leq 1 \ \forall j$

$$1 = \sum_{j=-m}^{m} a(j) \geq \sum_{j=-m}^{m} a^2(j) \Leftrightarrow \sigma^2 \geq \sigma^2 \sum_{j=-m}^{m} a(j)^2$$

c) $\min \sum_{j=-m}^{m} a(j)^2$ st $\sum_{j=-m}^{m} a(j) = 1$

$$L = \sum_{j=-m}^{m} a(j)^2 + \mu\Big(\sum_{j=-m}^{m} a(j) - 1\Big) ; \quad \frac{\partial L}{\partial a(i)} = 2a_i + \mu = 0$$

$$\Rightarrow a_i = -\frac{\mu}{2} \Rightarrow 1 = \sum_{j=-m}^{m} -\frac{\mu}{2} \Rightarrow \mu = \frac{-2}{2m+1}$$

$$\Rightarrow \boxed{a(i) = \frac{1}{2m+1} \geq 0}$$

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