

CS 213 - Data Structures and Algorithms

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1 Recursion

1.1 Design principle

1. Reduce the problem of size n to a size $n-1$
2. Figure out the base case (usually $n=0$ or 1)
3. Terminate recursion at the base case. Make sure that every n reaches the base case.

1.2 Parameterization

- It consumes extra memory if pass a new array (part of old array copied into this) for recursion. Instead we should pass the indices as the parameters.
- In creating recursive methods, it is often useful to define additional functions(or methods) to facilitate recursion.

1.3 Tail Recursion

- Recursion has to maintain function calls on stack that makes it more expensive than iterative methods.
- To reduce this extra usage of resources, we can write an algorithm in a tail recursive way i.e, the function should directly return the function call (without any other operations on the result returned by the function call).

2 Algorithm analysis

We primarily compare running time in this course. To analyse algorithms, runtime should be machine independent for which we use 'RAM' model of computation.

2.1 RAM model

- Generic single processor model
- Computer supports simple constant time instructions
 - Arithmetic ($+$, $-$, \times , $/$, $floor$, ..)
 - Data movement (load, store, copy)
 - Control (branch, function call)
- We assume that the cost (runtime) of all simple instructions is 1
- Sequential execution - No concurrent execution
- Flat memory model and accessing a memory costs 1 unit.

2.2 Asymptotic Notation

- Θ notation: Given functions g , we define

$$\Theta(g(n)) = \{f(n) : \exists \text{ positive constants } c_1, c_2, n_0 \text{ such that } c_1g(n) \leq f(n) \leq c_2g(n), \forall n \geq n_0\}$$

$g(n)$ is an asymptotic tight bound for $f(n)$.

- O notation: Given functions g , we define

$$O(g(n)) = \{f(n) : \exists \text{ positive constants } c, n_0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}$$

$g(n)$ is an asymptotic upper bound for $f(n)$.

- Ω notation: Given functions g , we define

$$\Omega(g(n)) = \{f(n) : \exists \text{ positive constants } c, n_0 \text{ such that } cg(n) \leq f(n), \forall n \geq n_0\}$$

$g(n)$ is a asymptotic lower bound for $f(n)$.

3 Divide and Conquer

- Divide and conquer algorithm is recursive in nature. Basically, we divide a problem into smaller problems (untill we hit the base case) and combine them to get the solution for larger problem.
- For the time complexity of a divide and conquer algorithm, we can write a recurrence relation

$$T(n) = aT\left(\frac{n}{b}\right) + C(n) + D(n)$$

where a = no of subproblems a problem is divided, b = factor by which n is reduced, $C(n)$ and $D(n)$ are the time taken in combine and divide step.

4 Recurrence Relation

The recurrence relations of type $T(n) = aT\left(\frac{n}{b}\right) + f(n)$ can be solved using the following methods:

1. **Plug and chuck** : Keep on substituting in the value of $T\left(\frac{n}{b}\right)$ in terms of smaller n untill we see a pattern. Pattern can be used to easily get the solution
2. **Substitution method** : (This is more like check for your solution) In this method, we first guess the solution and prove that using induction, i.e, substitute the solution for $T\left(\frac{n}{b}\right)$ and check whether it you can arrive at the proposed relation for $T(n)$
3. **Recursion-tree method**: We draw the recursion tree for this method. In the recursion tree, leaves contain the cost for the base case and each node contains the cost for the merge and divide process. We sum all of them to get a solution for the recurrence relation.
4. **Using master theorem**: Some of the recurrence relations encountered while working with algorithms can be solved using -

Theorem (The master theorem). *Let $a \geq 1$, $b > 1$ and $f(n)$ be a function, then let $T(n)$ be defined on non-negative integers, then the recurrence relation*

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

is bound assymptotically in three cases. (If possible, let $f(n) = \Theta(n^d)$)

$$\begin{aligned} T(n) &= \sum_{k=0}^{\log_b n} a^k f\left(\frac{n}{b^k}\right) \\ &= \sum_{k=0}^{\log_b n} a^k \left(\frac{n}{b^k}\right)^d \\ &= \begin{cases} O(n^d) & \text{if } a < b^d \text{ or } f(n) = \Omega(n^{\log_b a + \epsilon}) \\ O(n^d \log n) & \text{if } a = b^d \text{ or } f(n) = \Theta(n^{\log_b a}) \\ O(n^{\log_b a}) & \text{if } a > b^d \text{ or } f(n) = O(n^{\log_b a - \epsilon}) \end{cases} \end{aligned}$$

- In the first case ($a < b^d$ or $f(n) = \Omega(n^{\log_b a + \epsilon})$), the work done at the first node dominates the time taken. Hence, total time taken = time taken at first node. Also this means that as the depth increases, the work done at each level decreases.
- In the second case ($a = b^d$ or $f(n) = \Theta(n^{\log_b a})$), the work done at all levels is equal. Hence, the total time taken = time at initial node \times no of levels.
- In the third case ($a > b^d$ or $f(n) = O(n^{\log_b a - \epsilon})$), the work done at the deepest level dominates the time taken. Hence, total time taken = time taken at the lowest level

5. Solving Recurrence relation of type:

$$a_n T_n = b_n T_{n-1} + c_n$$

Our goal here is make the above equation in the form $T_n - T_{n-1} = c_n$, so that we can solve this by telescopic summation

- (a) We multiply both the sides by a factor σ_n which has the property $\sigma_n b_n = \sigma_{n-1} a_{n-1}$, so
- $$\frac{\sigma_n}{\sigma_{n-1}} = \frac{a_{n-1}}{b_n}$$

$$\sigma_n a_n T_n = \sigma_n b_n T_{n-1} + \sigma_n c_n$$

$$\sigma_n a_n T_n - \sigma_{n-1} b_{n-1} T_{n-1} = \sigma_n c_n$$

- (b) Now we sum all the terms to get (telescopic sum),

$$\sigma_n a_n T_n = \sigma_0 a_0 T_0 + \sum_{k=1}^n c_k \sigma_k$$

$$T_n = \frac{a_0 \sigma_0 T_0 + \sum_{k=1}^n c_k \sigma_k}{\sigma_n a_n}$$

6. Solving by generating functions: We will use this technique to get a closed form solution for fibonacci series.

- (a) Write a single recurrence equation for g_n .
In the case of Fibannaci series, it is

$$g_n = g_{n-1} + g_{n-2} + \delta_1$$

where δ_1 is the dirac delta function. This is used because we can assume $g_n = 0$ for all non positive numbers and still satisfy this for $n = 1$.

- (b) Multiply both sides by z^n and z transform LHS and RHS.
For fibanacci series,

$$G(z) = zG(z) + z^2G(z) + z$$

where $G(z)$ is the z transform of $g(z)$

- (c) Solve for $G(z)$ in the z domain
For fibanacci series

$$G(z) = \frac{z}{1 - z - z^2}$$

- (d) Inverse transform $G(z)$ to get $g(z)$

$$g_n = \frac{1}{\sqrt{5}}(\phi^n - \bar{\phi}^n)$$

5 Amortized Complexity

(//todo: aggregate and potential(work required))