

MA 214 - Introduction to Numerical Analysis

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1 Interpolation Theory

1.1 Introduction

- Given finite set of points, reconstructing the original curve is interpolation.
- There will be obviously infinitely many curve.
- Interpolation problem
Given $n+1$ real distinct points: x_0, x_1, \dots, x_n and real numbers: y_0, y_1, \dots, y_n
Find a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x_i) = y_i \quad \text{for } i = 0, 1, \dots, n$$

Such a function is called **interpolant** and points x_i are called **interpolation points**.

We attempt to rebuild original function using polynomial functions. This is called polynomial interpolation and function is polynomial interpolant.

- A polynomial is function of the form

$$p(x) = a_0 + a_1x + \dots + a_nx^n$$

- \mathbb{P}_n is the set of polynomials consisting of all polynomials of degree $\leq n$

1.2 Polynomial Interpolation

Theorem (Joseph-Louis Lagrange Theorem). *Given $n+1$ data points with unique x_i s, then there exists a unique polynomial $p_n \in \mathbb{P}_n$ such that*

$$p(x_i) = y_i \quad \text{for } i = 0, 1, \dots, n$$

Proof. (1) This can be shown by linear algebra. In a n degree polynomial, we substitute the points and get $n+1$ equations in $n+1$ variables (coeff) and all the rows are unique (since x_0, x_1, \dots, x_n are unique), hence in $AX = b$, $|A| \neq 0$. □

Proof. (2) Part 1: Uniqueness : If there is an interpolant, then the interpolant is unique
Let there be 2 interpolants, p_n and q_n and let $r(x) = p(x) - q(x)$,

$$r(x) = 0 \quad \text{for } i = 0, 1, \dots, n$$

This contradicts the fundamental theorem of Algebra. (A polynomial of degree n can have at most n real roots). Therefore

$$\begin{aligned} r(x) &= 0 \quad \forall x \in \mathbb{R} \\ p(x) &= q(x) \quad \forall x \in \mathbb{R} \end{aligned}$$

Part 2: Existence (construction):

Given $n+1$ data points, build $n+1$ Lagrange polynomials

$$L_k^n(x) = \begin{cases} 0 & \text{for } i \neq k \\ 1 & \text{for } i = k \end{cases}$$

$$L_k^n(x) = \frac{(x - x_0) \dots (x - x_{k-1})(x - x_{k+1}) \dots (x - x_n)}{(x_k - x_0) \dots (x_k - x_{k-1})(x_k - x_{k+1}) \dots (x_k - x_n)}$$

$$p(x) = \sum_{k=0}^n y_k L_k^n(x)$$

□

1.3 Closeness between functions

Given two continuous functions $f, g : [a, b] \rightarrow \mathbb{R}$, to evaluate how close the functions are consider the following

$$\max_{x \in [a, b]} |f(x) - g(x)|$$

1.4 Set of continuous Functions

$C[a, b]$ is the set of all continuous functions on $[a, b]$

$C[a, b]$ is a infinite dimensional vector space

$$f, g \in C \implies f + g \in C \text{ and } \lambda f \in C$$

We define norm on $C[a, b]$ as

$$\|f\| = \max_{x \in [a, b]} |f(x)|$$

$C^k[a, b]$ denotes the set of all functions which are continuously k-times differentiable

1.5 Polynomial Approximation and Error

Theorem (Weierstrass approximation Theorem). *Given a function $f \in C[a, b]$ and given $\epsilon > 0$, there exists a polynomial $p(x)$ such that,*

$$\|f(x) - p\| < \epsilon$$

Using Langrange's recipe to approximate

Take $n + 1$ interpolation points in the $[a, b]$ and collect the function values at all the points. We have $n + 1$ data points. Using Lagrange polynomials, find the interpolant

Theorem (Error equation). *Let $f \in C^k[a, b]$, $x_0, x_1, \dots, x_n \in [a, b]$ and $p \in \mathbb{P}_n$ be the interpolant using these points, then for all x , there exists a $\zeta = \zeta(x) \in (a, b)$ such that*

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\zeta) \prod_{k=0}^n (x - x_k)$$

Note: Here ζ is dependent on the x , i.e, for every x you choose, ζ generally changes.

Proof. Consider the function,

$$\psi(t) = (f(t) - p(t)) \prod_{k=0}^n (t - x_k) - (f(x) - p(x)) \prod_{k=0}^n (x - x_k)$$

This $n + 2$ roots ($n+1$ data points and x), applying rolle theorem's gives us that $\psi^{(1)}(t)$ has at least $n+1$ roots. Applying like this repeatedly on its derivatives, we get that $\psi^{(n+1)}$ has at least 1 root in $[a, b]$. Assuming the root to be ζ . We have,

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\zeta) \prod_{k=0}^n (x - x_k)$$

□

Approximating the error:

Taking norm on both the sides of error equation, we have,

$$\max_{x \in [a, b]} |f(x) - p(x)| = \frac{1}{(n+1)!} \|f^{(n+1)}(\zeta)\| \prod_{k=0}^n (x - x_k) \quad (1)$$

$$\max_{x \in [a, b]} |f(x) - p(x)| \leq \frac{1}{(n+1)!} \|f^{(n+1)}\| \max_{x \in [a, b]} \prod_{k=0}^n (x - x_k) \quad (2)$$

Chebyshev interpolation points:

$$x_k = \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{j\pi}{n}\right)$$

These points minimise $\max_{x \in [a, b]} \prod_{k=0}^n (x - x_k)$ and therefore preferred over equally spaced points on real line. These points can be visualised as projections of equally spaced points on the arc of the semicircle with $\frac{a+b}{2}$ as center and $\frac{b-a}{2}$ as radius.

1.6 Some more methods for calculating interpolant

- This is similar to the linear algebra method (given as proof(1) to Joseph-Louis Lagrange Theorem) for finding the interpolant.

Consider the polynomial

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots + a_n(x - x_0)\dots(x - x_{n-1})$$

Find the coefficients a_0, a_1, \dots, a_n by substituting the data points. On substituting x_0 , we get a_0 , again on substituting x_1 and using a_0 , we get a_1 and so on.

- Interpolant $p(x)$ of x_0, x_1, \dots, x_n can be calculated using interpolants $a(x)$ and $b(x)$ of x_1, x_2, \dots, x_n and x_0, x_1, \dots, x_{n-1} respectively as

$$p(x) = \frac{(x - x_0)a(x) - (x - x_n)b(x)}{x_n - x_0}$$

1.7 Divided difference - recursion relation

- **Divided difference** : It is the coefficient of x_n in the interpolant $p \in \mathbb{P}_n$ and denoted by $f[x_0, x_1, \dots, x_n]$.

Using Lagrange polynomials, we have

$$p(x) = \sum_{k=0}^n f(x_k) \prod_{j=0, j \neq k}^n \frac{x - x_j}{x_k - x_j}$$

So the divided difference is

$$f[x_0, x_1, \dots, x_n] = \sum_{k=0}^n f(x_k) \prod_{j=0, j \neq k}^n \frac{1}{x_k - x_j}$$

Theorem (Divided difference recursion theorem).

$$f[x_0, x_1, \dots, x_{m+1}] = \frac{f[x_1, x_2, \dots, x_{m+1}] - f[x_0, x_1, \dots, x_m]}{x_{m+1} - x_0}$$

Proof. Let $p(x)$ be the interpolant for x_0, x_1, \dots, x_m and $q(x)$ be the interpolant for x_1, x_2, \dots, x_{m+1} . Then,

$$L(x) = \frac{(x - x_0)q(x) + (x_{m+1} - x)p(x)}{x_{m+1} - x_0}$$

is an interpolant. Since, interpolant is unique, considering coeff of x_m we have,

$$f[x_0, x_1, \dots, x_m] = \frac{f[x_1, x_2, \dots, x_{m+1}] - f[x_0, x_1, \dots, x_m]}{x_{m+1} - x_0}$$

□

Theorem (Interpolant using divided differences). Suppose x_0, x_1, \dots, x_n be the data points. Then interpolant $p \in \mathbb{P}_n$ is

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0, x_1, \dots, x_n] \prod_{j=0}^{n-1} (x - x_j)$$

Proof. We prove this by induction. Base case $n = 0$ is trivially satisfied. Assume that this is satisfied for p_k ,

$$p_k(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0, x_1, \dots, x_k] \prod_{j=0}^{k-1} (x - x_j)$$

Consider the polynomial $p_{k+1}(x) - p_k(x) \in \mathbb{P}_{k+1}$ which has x_0, x_1, \dots, x_k as roots. Hence,

$$p_{k+1}(x) - p_k(x) = c \prod_{j=0}^k (x - x_j)$$

Comparing leading coefficient on both sides, we have $c = f[x_0, x_1, \dots, x_k]$. Hence,

$$p_{k+1}(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0, x_1, \dots, x_{k+1}] \prod_{j=0}^k (x - x_j)$$

By PMI,

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0, x_1, \dots, x_n] \prod_{j=0}^{n-1} (x - x_j)$$

□

1.8 Weierstrass theorem consequences

- In [Weierstrass approximation Theorem](#), take $\epsilon_n = \frac{1}{n}$. Then weierstrass theorem proves the existence of sequence of polynomials $p^{(1)}, p^{(2)}, \dots$ such that

$$\lim_{n \rightarrow \infty} \|f - p^{(n)}\| = 0$$

- If f is not a polynomial, then

$$\lim_{n \rightarrow \infty} \text{degree of } p(n) = 0$$

1.9 Spline Interpolation

- **Piece wise polynomial:** $\phi \in C[a, b]$ is a piecewise polynomial function, if there exists $a = x_0 < x_1 < \dots < x_n = b$ such that $\phi \in \mathbb{P}_m$ when $x \in [x_i, x_{i+1}]$ for all $i = 0, 1, \dots, n$ and some $m > 0$.
- Piece wise polynomial ϕ need not be polynomial in whole domain.
- Splines interpolation for $f \in C[a, b]$
 - Pick some data points x_0, x_1, \dots, x_n such that $a = x_0 < x_1 < \dots < x_n = b$
 - Fix $m \leq n$
 - Build ϕ in each subinterval $[x_i, x_{i+1}]$ using the following conditions:

$$\phi(x_i) = f_i \quad \text{for } i = 0, 1, \dots, n$$

$$\lim_{h \rightarrow 0+} \phi(x_i - h) = \lim_{h \rightarrow 0+} \phi(x_i + h) \quad \text{for } i = 1, 2, \dots, n-1$$

$$\lim_{h \rightarrow 0+} \frac{d\phi(x_i - h)}{dx} = \lim_{h \rightarrow 0+} \frac{d\phi(x_i + h)}{dx} \quad \text{for } i = 1, 2, \dots, n-1$$

$$\lim_{h \rightarrow 0+} \frac{d^2\phi(x_i - h)}{dx^2} = \lim_{h \rightarrow 0+} \frac{d^2\phi(x_i + h)}{dx^2} \quad \text{for } i = 1, 2, \dots, n-1$$

\vdots

$$\lim_{h \rightarrow 0+} \frac{d^{m-1}\phi(x_i - h)}{dx^{m-1}} = \lim_{h \rightarrow 0+} \frac{d^{m-1}\phi(x_i + h)}{dx^{m-1}} \quad \text{for } i = 1, 2, \dots, n-1$$

- We have $(n+1) + m(n-1) = n(m+1) - (m-1)$ conditions. We need $m-1$ more conditions.

1.10 Linear Splines

Constructing linear splines:

Since the degree is only 1, we can construct the splines using the equation of straight lines between the knots (data points).

$$\begin{aligned} s_0 &= \frac{f_1 - f_0}{x_1 - x_0}x + \frac{x_1 f_0 - x_0 f_1}{x_1 - x_0} && \text{for } x \in [x_0, x_1] \\ s_1 &= \frac{f_2 - f_1}{x_2 - x_1}x + \frac{x_2 f_1 - x_1 f_2}{x_2 - x_1} && \text{for } x \in [x_1, x_2] \\ &\vdots \\ s_{n-1} &= \frac{f_n - f_{n-1}}{x_n - x_{n-1}}x + \frac{x_n f_{n-1} - x_{n-1} f_n}{x_n - x_{n-1}} && \text{for } x \in [x_{n-1}, x_n] \end{aligned}$$

Theorem (Linear Splines error). *Let $f \in C^2[a, b]$ and $s_L(x)$ be the interpolating **linear spline** at $(n+1)$ knots $a = x_0 < x_1 < \dots < x_n = b$ and let h be the maximum subinterval length, then*

$$\|f - s_L\| \leq \frac{h^2}{8} \|f''\|$$

Proof. Consider the interval $[x_i, x_{i+1}]$, then $s_L(x)$ is the interpolating polynomial in this interval. Using the error equation for interpolating polynomials,

$$f(x) - s_L(x) = \frac{1}{2}f''(\zeta)(x - x_i)(x - x_{i+1})$$

Taking absolute value on both the sides,

$$\begin{aligned} |f(x) - s_L(x)| &= \frac{1}{2}|f''(\zeta)|(x - x_i)(x - x_{i+1}) \\ &\leq \frac{1}{2}\|f''\|\frac{h_i^2}{4} \quad \text{where } h_i = \frac{x_{i+1} - x_i}{2} \\ &\leq \frac{h_i^2}{8}\|f''\| \end{aligned}$$

Considering $h = \max(h_i)$, then for $x \in [a, b]$

$$\max_{x \in [a, b]} |f(x) - s_L(x)| \leq \frac{1}{8}h^2\|f''\|$$

□

1.11 Cubic Splines

Constructing cubic splines:

$$s_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$$

Using the conditions given in the section 1.9, we have $4n-2$ equations to for $4n$ coefficients. We choose the other two conditions as

$$s_0''(x_0) = s_{n-1}''(x_n) = 0$$

We have $4n$ variables (coefficients) and $4n$ equations, i.e, we have a $4n \times 4n$ matrix which can be solved to get the coefficients of the spline.

We can simplify this by choosing the form of spline as

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \quad \text{for } i = 0, 1, \dots, n-1$$

We will work only with **equally spaced knots**, i.e, $x_{i+1} - x_i = \text{const}$ for $i = 0, 1, \dots, n-1$. We also define

$$\sigma_i = s''(x_i) \quad \text{for } i = 0, 1, \dots, n$$

After doing a lot of simplification, we get

$$\begin{aligned} a_i &= \frac{\sigma_{i+1} - \sigma_i}{6h} \\ b_i &= \frac{\sigma_i}{2} \\ c_i &= \frac{f_{i+1} - f_i}{h} - \frac{h}{6}(2\sigma_i + \sigma_{i+1}) \\ d_i &= f_i \end{aligned}$$

σ_i s can be obtained by solving these equations

$$\sigma_{i-1} + 4\sigma_i + \sigma_{i+1} = \frac{6}{h^2}(f_{i-1} - 2f_i + f_{i+1})$$

This can be put in matrix form as

$$\begin{bmatrix} 4 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 4 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 4 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 4 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \vdots \\ \sigma_{n-3} \\ \sigma_{n-2} \\ \sigma_{n-1} \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} f_0 - 2f_1 + f_2 \\ f_1 - 2f_2 + f_3 \\ f_2 - 2f_3 + f_4 \\ \vdots \\ f_{n-4} - 2f_{n-3} + f_{n-2} \\ f_{n-3} - 2f_{n-2} + f_{n-1} \\ f_{n-2} - 2f_{n-1} + f_n \end{bmatrix}$$

Theorem (Error - Cubic Splines (equispaced knots)). Let $f \in C^4[a, b]$ and $s(x) \in C^2[a, b]$ be the interpolating **natural cubic spline** at $(n+1)$ **equispaced knots** $a = x_0 < x_1 < \dots < x_n = b$ and let h be the subinterval length ($h = x_{i+1} - x_i$), then

$$\|f - s\| \leq C \|f^{(4)}\| h^4 \quad \text{for some } C > 0$$

Proof. Consider the function g which is defined as $g = f - s_i$ on each subinterval $[x_i, x_{i+1}]$. Then $g(x_i) = 0$ for $i = 0, 1, \dots, n-1$.

We can see that the zero polynomial is the linear spline of $g(x)$ and using the theorem [Linear Splines error](#), we have

$$\begin{aligned} \|g - 0\| &\leq \frac{h^2}{8} \|g''\| \\ \|f - s\| &\leq \frac{h^2}{8} \|f'' - s''\| \end{aligned}$$

Assuming that $\|f'' - s''\| \leq Ch^2 \|f^{(4)}\|$, we have

$$\|f - s\| \leq Ch^4 \|f^{(4)}\| \quad \text{for some } C > 0$$

□

2 Numerical Integration

2.1 Introduction

- Given f be a real valued function, we want to evaluate

$$\int_a^b f(x) dx$$

- If we can find its antiderivative $F(x)$, then we can use the fundamental theorem of calculus and evaluate it. But finding antiderivate is not so straightforward.

$$\int_a^b f(x) dx = F(b) - F(a)$$

2.2 Newton-Cotes Formula

Let $f(x) : [a, b] \rightarrow \mathbb{R}$ and $p(x) \in \mathbb{P}_n$ be the polynomial interpolant using the data points $a = x_1 < x_2 < \dots < x_n = b$, then the definite integral $\int_a^b f(x) dx$ can be approximated as

$$\begin{aligned} \int_a^b f(x) dx &\approx \int_a^b p(x) dx \\ &= \int_a^b \sum_{i=0}^n f(x_i) L_i(x) dx \\ &= \sum_{i=0}^n f(x_i) \int_a^b L_i(x) dx \end{aligned}$$

Let $x_i = a + ih$ for $i = 0, 1, \dots, n$ and $x = a + th$ for $t \in [0, n]$, then we have,

$$\int_a^b L_i(x) dx = \int_a^b \prod_{k=0, k \neq i}^n \frac{x - x_k}{x_i - x_k} dx = \int_0^n \prod_{k=0, k \neq i}^n \frac{t - k}{i - k} h dt = h \int_0^n \varphi_i(t) dt = h w_i$$

$$\text{where } w_i = \int_0^n \varphi_i(t) dt \text{ and } \varphi_i(t) = \prod_{k=0, k \neq i}^n \frac{t - k}{i - k}$$

Then,

$$\int_a^b f(x) dx \approx h \sum_{i=0}^n w_i f(x_i)$$

Note: w_i s are independent of f , end points a and b and h . w_i s are dependent on only dependent on n . **Note:** w_i s are symmetric i.e,

$$w_k = w_{n-k}$$

Trapezium rule ($n = 1$):

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a) + f(b))$$

Simpson's rule ($n = 2$):

$$\int_a^b f(x)dx \approx \frac{h}{3} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right)$$

2.3 Newton-Cotes formula error

We use the [Error equation](#), for interpolation polynomial to calculate the error in Newton-Cotes formula.

$$\begin{aligned} |I_f - I_P| &= \left| \int_a^b f(x) - p(x) dx \right| \\ &\leq \int_a^b |f(x) - p(x)| dx \end{aligned}$$

$$\boxed{|I_f - I_P| \leq \frac{1}{(n+1)!} \|f^{(n+1)}\| \int_a^b \prod_{i=0}^n |x - x_i| dx}$$

For trapezium rule ($n=1$)

$$\begin{aligned} |I_f - I_{P_1}| &\leq \frac{1}{2} \|f''\| \int_a^b |(x-a)(x-b)| dx \\ &= \frac{1}{12} \|f''\| (b-a)^3 \end{aligned}$$

For Simpson's rule ($n=2$)

$$\begin{aligned} |I_f - I_{P_2}| &\leq \frac{1}{6} \|f'''\| \int_a^b |(x-a)(x-\frac{a+b}{2})(x-b)| dx \\ &= \frac{1}{192} \|f'''\| (b-a)^4 \end{aligned}$$

(//todo: stricter bound for Simpsons rule)

2.4 Convergence of the approximation

- The difference $|I_f - I_{P_n}|$ **does not converge** to 0 as we increase n . This is (crudely) because the weights sometimes takes negative values.
- A similar approximation which converges as n increases is Gaussian quadratures.

$$G_n(f) = \sum_{i=0}^n W_i f(x_i)$$

where weights W_i are

$$W_i = \int_a^b (L_i(x))^2 dx = \int_a^b \left[\prod_{k=0, k \neq i}^n \frac{x - x_k}{x_i - x_k} \right]^2 dx$$

Here, the points x_k s are **not** equally spaces.
 x_k s are the roots of Legendre Polynomials

2.5 Composites Rule

Similar to the splines case, we approximate the integral using Newton-Cotes formula in each subinterval.

- **Composite Trapezoidal Rule:** We divide the interval $[a, b]$ into m subintervals and apply trapezoidal rule in each subinterval.

$$\begin{aligned} \int_a^b f(x)dx &\approx \int_a^b C_{P_1} dx \\ &= h \left(\frac{1}{2} f(a) + \frac{1}{2} f(a+h) \right) + h \left(\frac{1}{2} f(a+h) + \frac{1}{2} f(a+2h) \right) + \dots + h \left(\frac{1}{2} f(a+(m-1)h) + \frac{1}{2} f(b) \right) \\ &= h \left(\frac{1}{2} f(a) + f(a+h) + f(a+2h) + \dots + f(a+(m-1)h) + \frac{1}{2} f(b) \right) \end{aligned}$$

- **Composite Simpson's Rule:** We divide the interval $[a, b]$ into $2m$ subintervals and apply simpsons' rule.

$$\begin{aligned}
\int_a^b f(x)dx &\approx \int_a^b C_{p_2} dx \\
&= h\left(\frac{1}{3}f(a) + \frac{4}{3}f(a+h) + \frac{1}{3}f(a+2h)\right) + h\left(\frac{1}{3}f(a+2h) + \frac{4}{3}f(a+3h) + \frac{1}{3}f(a+4h)\right) \\
&\quad + \dots + h\left(\frac{1}{3}f(a+(2m-2)h) + \frac{4}{3}f(a+(2m-1)h) + \frac{1}{3}f(b)\right) \\
&= h\left(\frac{1}{3}f(a) + \frac{4}{3}f(a+h) + \frac{2}{3}f(a+2h) + \frac{4}{3}f(a+3h) + \frac{2}{3}f(a+4h) + \dots\right. \\
&\quad \left.+ \frac{2}{3}f(a+(2m-2)h) + \frac{4}{3}f(a+(2m-1)h) + \frac{1}{3}f(b)\right)
\end{aligned}$$

2.6 Error - Composites Rule

- To get an estimate of the error $|I_f - C_{p_n}|$, we sum the error in each subinterval.
- For composite trapezoid rule,

$$\begin{aligned}
|I_f - C_{p_1}| &\leq \sum_{i=0}^m \frac{1}{12} \|f''\| h^3 \\
&= \frac{1}{12} \|f''\| h^2 (b-a)
\end{aligned}$$

3 Ordinary Differential Equation