ME 202 - Strength of Materials

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1 Review

1.1 Strain Energy

$$u = \frac{1}{2}E\epsilon^2 = \frac{1}{2}\sigma\epsilon = \frac{1}{2}\frac{\sigma^2}{E}$$

1.2 Poison's Ratio

$\epsilon_{axial} = \frac{\delta}{L}$, δ is the change in longitudinal length
$\epsilon_{radial} = \frac{\delta'}{r}$, δ' is the change in radius

Poisson's ratio -

$$\nu = -\frac{\epsilon_{radial}}{\epsilon_{axial}}$$

$$0 \leq \nu \leq 0.5$$

Shear stress and strain

$$\tau = G\gamma$$

Relation between G, E and ν -

$$G = \frac{E}{2(1+\nu)}$$

For a axial load,

$$d\delta = \frac{N(x)dx}{A(x)E(x)}$$

1.3 Stresses in a Plane and Mohr's Circle

Using physical equations or using cauchy's stress tensor with rotation matrix, we get

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}cos2\theta + \tau_{xy}sin2\theta \tag{1}$$

$$\tau_{xy'} = -\frac{\sigma_x - \sigma_y}{2}sin2\theta + \tau_{xy}cos2\theta \tag{2}$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}cos2\theta - \tau_{xy}sin2\theta \tag{3}$$

The above equations can be viewed as parametric equations to a circle.
On squaring and adding, we get Mohr's circle

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

Center = $(\frac{\sigma_x+\sigma_y}{2}, 0)$ and Radius = $\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$

- This is plotted by taking $\sigma_{x'}$ on x axis and $\tau xy'$ on **negative** y-axis and angle 2θ anticlockwise as positive.

- τxy is taken as positive if it tends to rotate in the anticlockwise direction, negative otherwise.
- σ_x is taken as positive if it is tensile and negative for compressive.
- A rotation of angle θ in the plane corresponds to a rotation of 2θ on the Mohr's circle.

1.4 Hooke's law for plane stresses

For a point (isotropic material) under planar stresses ($\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$), we have

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$
$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\sigma_x = \frac{E}{1-\nu^2}(1 + \nu\epsilon_x)$$
$$\sigma_y = \frac{E}{1-\nu^2}(1 + \nu\epsilon_y)$$
$$\tau_{xy} = G\gamma_{xy}$$

1.5 Volume change and Strain-energy density

Let a cuboid of sides a,b and c be under stresses.

$$V_0 = abc$$

$$\begin{aligned} V &= (a + a\epsilon_x)(b + b\epsilon_y)(c + c\epsilon_z) \\ &= abc(1 + \epsilon_x)(1 + \epsilon_y)(1 + \epsilon_z) \\ &= V_0(1 + \epsilon_x + \epsilon_y + \epsilon_z) && \text{ignoring the } \epsilon_x\epsilon_y \text{ terms} \end{aligned}$$

Unit volume change e, also known as **dialatation**

$$e = \frac{\Delta V}{V_0} = (1 + \epsilon_x)(1 + \epsilon_y)(1 + \epsilon_z)$$

Strain-energy density in plane stress,

$$u = \frac{1}{2}(\sigma_x\epsilon_x + \sigma_y\sigma_y + \tau_{xy}\gamma_{xy})$$
$$u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y + \frac{\tau_{xy}^2}{2G})$$
$$u = \frac{E}{2(1-\nu^2)}(\epsilon_x^2 + \epsilon_y^2 + 2\nu\epsilon_x\epsilon_y) + \frac{G\gamma_{xy}^2}{2}$$

1.6 Hooke's law for triaxial stresses

($\sigma_z = \tau_{xz} = \tau_{yz} = 0$)

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z)$$
$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_x + \nu(\epsilon_y + \epsilon_z)]$$

1.7 Strain Energy in Torsion

$$d\phi = \frac{Tdx}{GJ}$$

$$\begin{aligned} \text{Energy stored due to torsion, } U &= \int_0^L \frac{1}{2}Td\phi \\ &= \int_0^L \frac{T^2dx}{2GJ} \end{aligned}$$

2 Torsion

- **Angle of twist** : It is the angle by which one end of the rod is displaced wrt the other end of the rod under the effect of some torsion (twisting).

- Consider a small cylidrical section which has length dx, one end has been displaced wrt another by an angle of $d\phi$. Consider a line ab on the circumference along the length of the cylinder which has be changed to ab'. Then shear strain

$$\gamma_{max} = \frac{bb'}{ab} \qquad \text{(bb' can be assumed to be a straight line)}$$

$$\gamma_{max} = \frac{rd\phi}{dx}$$

- Rate of twist or angle of twist per unit length

$$\theta = \frac{\mathrm{d}\phi}{\mathrm{d}x}$$

$$\gamma_{max} = r\theta$$

If θ is constant, then

$$\gamma_{max} = \frac{r\phi}{L}$$

This is called as γ_{max} because we are measuring the shear strain at the outer end i.e, with maximum radius and hence, maximum shear strain.

$$\boxed{\gamma = \rho\theta = \rho\frac{\mathrm{d}\phi}{\mathrm{d}x}}$$

where ρ is the perpendicular distance of the point from the axis (radius) we are considering.

$$\boxed{\gamma = \frac{\rho}{r}\gamma_{max}}$$

2.1 Hooke's Law

Hooke' law for shear stress and shear strain

$$\tau = G\gamma = G\frac{\rho}{r}\gamma_{max} = \frac{\rho}{r}\tau_{max}$$

$$\tau_{max} = G\gamma_{max}$$

2.2 Torsion Formula

- Polar moment of inertia (this is integral over area - double integral)-

$$\boxed{I_P = \int_A \rho^2 dA}$$

where ρ is the distance at which area element dA is located.

- For a circle of radius r,

$$I_P = \int_A \rho^2 dA = \int_{\theta=0}^{2\pi}\int_{\rho=0}^{r} \rho^2(\rho d\theta d\rho) = \frac{\pi r^4}{2}$$

$$\boxed{I_{Pcircle} = \frac{\pi r^4}{2}}$$

- Consider a cross-section of any shape, we are trying to sum all the small torques and equate it to the torque applied on this

$$\begin{aligned}
T = \int_A dM &= \int_A \rho\tau dA \\
&= \int_A \frac{\rho^2}{r}\tau_{max} dA \\
&= \frac{\tau_{max}}{r}\int_A \rho^2 dA \\
T &= \frac{\tau_{max}}{r} I_P
\end{aligned}$$

Or,

$$\tau_{max} = \frac{Tr}{I_P}$$

- The shear stress at distance ρ from the center of the bar with polar moment of inertia I_P is

$$\boxed{\tau = \frac{T\rho}{I_P}}$$

So, I_P represents the resistance to change in twist angle (or shear stress) by virtue of its cross section.
Note: Here, the I_P is constant for any distance ρ.
For a rod of **circular cross section** of radius r,

$$\tau_{max} = \frac{2T}{\pi r^3} = \frac{16T}{\pi d^3}$$

- Rate of twist

$$\theta = \frac{T}{GI_P}$$

Hence, GI_P is also known as **Torsional Rigidity**

- For a bar in pure torsion ($\theta = const$), the total angle of twist

$$\boxed{\phi = \frac{TL}{GI_P}}$$

The quantity $\frac{GI_P}{L}$ is also known as **torsional stiffness** of the bar.

- For a thin circular tube, $I_P = 2\pi r^3 t$

2.3 Non-uniform Torsion

For a bar with non-uniform cross-section, tension :

$$\boxed{\phi = \int_0^L \frac{T(x)dx}{GI_P(x)}}$$

2.4 Torsion Formula for non-prismatic bars

- For elliptical cross section, maximum shear stress

$$\tau_{max} = \frac{2T}{\pi ab^2}$$
$$\phi = \frac{TL}{GJ_e}$$
$$J_e = \frac{\pi a^3 b^3}{a^2 + b^2}$$

- For triangular cross section

$$\tau_{max} = \frac{T\frac{h}{2}}{J_t}$$
$$\phi = \frac{TL}{GJ_t}$$
$$J_e = \frac{h_t}{15\sqrt{3}}$$

- For rectangular cross section

$$\tau_{max} = \frac{T}{k_1 bt^2}$$
$$\phi = \frac{TL}{(k_2 bt^3)G} = \frac{TL}{J_r G}$$
$$J_r = k_2 bt^3$$

k_1 and k_2 are emphirically determined and are dependent on $\frac{b}{t}$

- Thin walled open cross sections We treat flange and web as seperate rectangles.

$$J = J_w + 2J_f$$

$$J_f = k_2 b_f t_f^3$$
$$J_w = k_2(b_w - 2t_f)t_w^3$$
$$\tau_{max} = \frac{2T(\frac{t}{2})}{J}$$
$$\phi = \frac{TL}{GJ}$$

2.5 Thin walled tubes

Consider a small element of length of dx, then shear stresses are τ_a,τ_b, τ_c and τ_d. Then the shear stresses on opposite wall should be equal (to satisfy Newton's second law). Hence,

$$\begin{aligned} F_a &= F_c \\ \tau_a t_a dx &= \tau_c t_c dx \\ \tau_a t_a &= \tau_c t_c \end{aligned}$$

Shear flow f,

$$\boxed{f = \tau t = const}$$

Deriving torsion formula for the thin walled tubes-
(L_m is the total circumferencial length, A_m is the area enclosed by the median line, r is the distance of the element to the center)

$$\begin{aligned} T &= \int \tau dAr \\ &= \int_0^{L_m} frds \\ &= 2fA_m \end{aligned}$$

$$\boxed{\tau = \frac{T}{2tA_m}}$$

Shear energy density of a solid under pure shear stress is $\frac{\tau^2}{2G}$.

$$\begin{aligned} \text{Total shear energy } U &= \int\int \frac{\tau^2}{2G} tdxds \\ &= \int\int \frac{f^2}{2Gt} dxds \\ &= \frac{f^2}{2G} \int_0^L dx \int_0^{L_m} \frac{ds}{t} \\ &= \frac{T^2 L}{2GA_m^2} \oint \frac{ds}{t} \end{aligned}$$

Also, $U = \frac{T^2 L}{2GJ}$. Hence,

$$\boxed{J = \frac{4A_m^2}{\oint \frac{dS}{t}}}$$

For tube with **constant thickness**,

$$\boxed{J = \frac{4A_m^2 t}{L_m}}$$

For circular tube,

$$J = 2\pi r^3 t$$

For rectangular tube,

$$J = \frac{2b^2 h^2 t_1 t_2}{bt_1 + ht_2}$$

Angle of twist,

$$\begin{aligned} \phi &= \frac{TL}{GJ} \\ &= \frac{TL}{4GA_m^2} \oint \frac{ds}{t} \end{aligned}$$

$$\boxed{\phi = \frac{TL}{4GA_m^2} \int_0^{L_m} \frac{ds}{t}}$$

2.6 Torsional stress concentration

- If there is a sudden discontinuity in the cross section such as holes, , then local torsional stresses will develop and usual torsional formula cannot be applied.

- The maximum stress found near the discontinuity can be found using

$$\tau_{max} = K\frac{Tc}{J}$$

where K is the torsional stress concentration factor, r is the fillet radius and c is the radius of the smaller section at the shaft.

- K values for various values of $\frac{r}{d}$ and $\frac{D}{d}$ are empirically (or using advaced theory) are determined and tabulated, where r is the fillet radius, d is the smaller diameter and D is the larger diameter

- Ref: Torsion Stress Concentrations in Circular Shafts

2.7 Inelastic Torsion

- If the applied torque is more than a certain threshold (yielding torque), the hooke's law ($\tau = G\gamma$) is no longer valid. But the strain still varies linearly across the cross section i.e, $\gamma = r\frac{\mathrm{d}\phi}{\mathrm{d}x}$ since in the derivation of this(2), we only used basic geometry.

- This is valid for inelastic torsion

$$T = \int \rho\tau dA = 2\pi\int_0^r \tau\rho^2 d\rho$$

2.8 Elasto plastic Torque

- Consider a bar with circular cross section made up of elastic plastic material with torque applied along its axis.

- If we keep increasing torque, at yielding torque (T_Y), the outer boundary of the bar yields while the rest of the circular cross section is still in the linear region. As we keep increasing torque, the material yield further toward the center.

- At limiting torque T_P, the bar completely yields.

- The shear stress in the yielded part of the bar is constant and equal to the yielding shear stress τ_Y.

- We can calculate T in terms of τ at any point after yielding. (Here, ρ_Y is the radius of the material which has not yielded)

$$\begin{aligned}
T &= \int \rho\tau dA \\
&= 2\pi\int_0^r \tau\rho^2 d\rho \\
&= 2\pi\left(\int_0^{\rho_Y} (\frac{\tau_Y}{\rho_Y}\rho)\rho^2 d\rho + \int_{\rho_Y}^r \tau_Y\rho^2 d\rho\right) \\
&= \frac{\pi\tau_Y}{6}(4r^3 - \rho_Y^3)
\end{aligned}$$

$$\boxed{T = \frac{\pi\tau_Y}{6}(4r^3 - \rho_Y^3)}$$

- After the material has completed yielded $\rho_Y = r$, substituting in above equation

$$T_P = \frac{2\pi}{3}\tau_Y r^3 = \frac{4}{3}T_Y$$

$$\boxed{T_P = \frac{4}{3}T_Y}$$

- Ref: Torsion Loading

2.9 Power due to torque

- Power due to applied torque,

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{Td\theta}{dt} = T\omega$$

3 Bending

3.1 Sign Convention

Sign convention that prof is following in this course: On rightward end, anticlockwise moment and downward shear force is taken as positive

3.2 Relation between V, M and q

Consider a rod under stress. Let the distributed load on it be q(x) (upwards is +ve), shear force V(x) and bending moment M(x). Consider a small element of length dx at a distance x.

Shear force

$$\sum F_y = 0$$
$$V + qdx - (V + dV) = 0$$
$$\boxed{\frac{\mathrm{d}V}{\mathrm{d}x} = -q}$$

Bending Moment

$$\sum M = 0$$
$$(M + dM) - M + qdx(\frac{dx}{2}) - Vdx = 0$$
$$\frac{\mathrm{d}M}{\mathrm{d}x} = V \text{ (since dx dx term = 0)}$$

$$\boxed{\frac{\mathrm{d}M}{\mathrm{d}x} = V}$$

3.3 Pure bending

Ref: Youtube video
Let a beam be under bending moment having a radius of curvature ρ. Then curvature κ is defined as
$$\kappa = \frac{1}{\rho} = \frac{\mathrm{d}\theta}{\mathrm{d}s}$$
(since $\rho = \frac{\mathrm{d}s}{\mathrm{d}\theta}$)

Curved upward is the positive curvature.

For a small length element dx from distance of length y (upwards) from the neutral axis and $d\theta$ be the angle it subtends at the center of curva (take x axis along the neutral axis). Then the strain in that element is

$$\begin{aligned}
\sigma_x &= E\epsilon_x \qquad \text{(Assuming Hooke's law)} \\
&= E\frac{(\rho - y)d\theta - dx}{dx} \\
&= E\frac{(dx - yd\theta) - dx}{dx} \\
&= -\frac{Ey}{\rho} \\
&= -E\kappa y
\end{aligned}$$

$$\boxed{\sigma_x = -\frac{Ey}{\rho}} \quad \text{(Assuming Hooke's law)}$$

Since, the beam is in equilibrium, (in the following calculation, the area (vector) is along x direction - since we want to sum σ_x)

$$\begin{aligned}
&\int_A \sigma_x dA = 0 \\
\implies &\int_A -\frac{Ey}{\rho} dA = 0 \\
\implies &\int_A y dA = 0
\end{aligned}$$

This implies that the neutral axis should pass through the centroid. (or $\int_A ydA = 0$)

3.4 Flexure Formula

Continuing the above case, the sum of all the moments is equal to the bending moment.

$$\begin{aligned} M &= -\int_A \sigma_x y dA \qquad \text{(-ve sign because +ve } \sigma \text{ produces a negative moment)} \\ &= -\int_A \frac{-Ey}{\rho} y dA \\ &= \frac{E}{\rho} I \end{aligned}$$

where $I = \int_A y^2 dA$ is the second moment of area.

$$\boxed{\kappa = \frac{1}{\rho} = \frac{M}{EI}}$$

EI is known as the **Flexure Rigidity**

$$\boxed{\sigma_x = \frac{-My}{I}}$$

Maximum Bending Moment: It occurs at maximum distance (y) from the neutral axis.

Let c_1 and c_2 be maximum distances (along y) from the neutral axis in either directions, then **section modulus** is

$$S_1 = \frac{I}{c_1}$$

Maximum longitudinal (along x) is

$$\boxed{(\sigma_x)_{max} = \frac{M_{max} c}{I}}$$

4 Non Uniform Bending

4.1 Transverse Shear Stress

Consider a small element (along lenght) of a rod under stress. Let the bending on left end be M and right be M+dM and shear force V and $V + dV$. Consider the top portion of the element upto y distance from the neutral axis (top to y). Let σ_1 and σ_2 be the variation of logitudinal stress.Then,

$$\begin{aligned} \tau dA(\text{along x-z plane}) &= \int (\sigma_2 - \sigma_1) dA(\text{along y-z plane}) \\ &= \int (\frac{(M + dM)y}{I} - \frac{My}{I} dA) \end{aligned}$$

Assuming that transverse stress is uniformly distributed along the width

$$\begin{aligned} \tau b dx &= \frac{dM}{I} \int y dA \\ \tau &= \frac{dM}{dx} \frac{1}{Ib} \int y dA \end{aligned}$$

$$\boxed{\tau = \frac{vQ}{Ib}}$$

where, $Q = \int y dA$, area varies from the top portion to the untill plane we want to compute τ. This is called **shear formula**.

For rectangualar cross sections, Consider a portion above the height y_1 above the neutral axis,

$$Q = \int_{y_1}^{\frac{h}{2}} y b dy = \frac{b}{2} \left(\frac{h^2}{4} - y^2 \right)$$

$$\tau = \frac{VQ}{Ib} = \frac{Vb}{2I} \left(\frac{h^2}{4} - y^2 \right)$$

At $y = 0$,

$$\tau_{max} = \frac{3}{2} \frac{V}{A}$$

(//todo:limitations)
For circlular cross sections at the center,

$$Q = \frac{\pi r^2}{2}\frac{4r}{3\pi} = \frac{2r^3}{3}$$

$$\tau_{max} = \frac{VQ}{Ib} = \frac{V(\frac{2}{3}r^3)}{(\frac{\pi r^4}{4})(2r)} = \frac{4V}{3A}$$

For hollow cyclinder,

$$\tau_{max} = \frac{V(\frac{2}{3}(r_2^3 - r_1^3))}{(\frac{\pi}{4}(r_2^4 - r_1^4))(2(r_2 - r_1))} = \frac{3V}{4A}\left(\frac{r_2^2 + r_1 r_2 + r_1^2}{r_2^2 + r_1^2}\right)$$