An Introduction to Polar Codes[1]

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[1] Reading Project with Prahladh H, Vinod P and Sasank M

The basic problem

- Goal : Recover the correct codeword from a noisy received codeword
- Errors?

Communication Channel

Abstraction of a physical transmission medium

- Input alphabet
- Output alphabet
- Conditional probability distribution

Notion of *capacity*

$C = \text{Max}_X I(X; Y)$

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Coding Scheme

- Encoding
- Decoding

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Shannon's random coding approach

Given a noisy channel with channel capacity C and information transmitted at a rate R, then if R < C there exist codes that allow the probability of error at the receiver to be made arbitrarily small.

The converse is also true. For R > C, an arbitrarily small probability of error is not achievable.

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Extremal Channels

Among all channels, there are two classes for which it is easy to communicate optimally:

- The perfect channels: the output Y determines the input X.
- The useless channels: the output Y is independent of the input X.

Arikan's polar coding is a technique to convert any binary-input channel to a mixture of binary-input extremal channels.

- The technique is information lossless, and of low complexity*.

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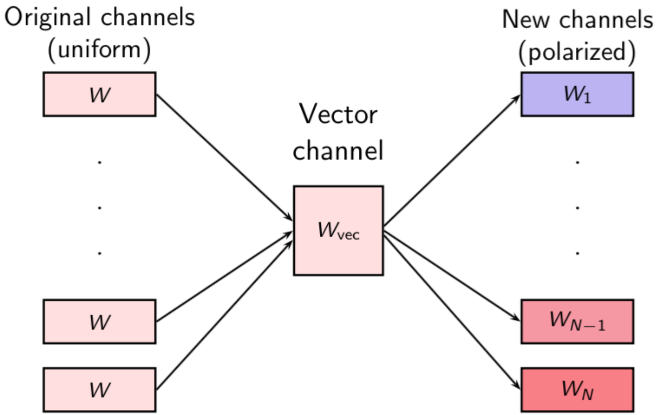
Polarization construction

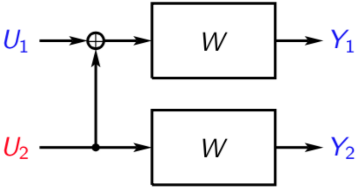
The method: aggregate and redistribute capacity



Building block

Given two copies of a binary input channel $W\colon \mathbb{F}_2 \to \mathcal{Y}$

- Set
$$X_1 = U_1 + U_2$$
$$X_2 = U_2$$
with U_1, U_2 i.i.d., uniform on $\mathbb{F}_2$.

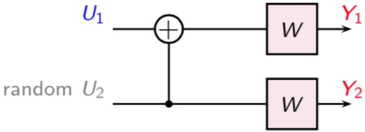


- This induces two synthetic channels $W^-\colon \mathbb{F}_2 \to \mathcal{Y}^2$ and $W^+\colon \mathbb{F}_2 \to \mathcal{Y}^2 \times \mathbb{F}_2$.

Channel Splitting

The first bit-channel W_1

$$W_1 : U_1 \to (Y_1, Y_2)$$

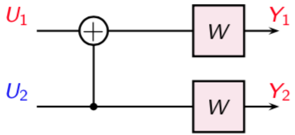


$$C(W_1) = I(U_1; Y_1, Y_2)$$

Channel Splitting

The second bit-channel W_2

$$W_2 : U_2 \rightarrow (Y_1, Y_2, U_1)$$

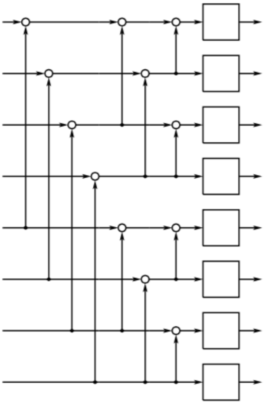


$$C(W_2) = I(U_2; Y_1, Y_2, U_1)$$

Construction ctd.

What we can do once, we can do many times. Given $W : \mathbb{F}_2 \to \mathcal{Y}$,

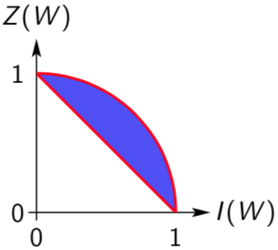
- Duplicate W and obtain W^- and W^+.
- Duplicate W^- (and W^+),
- and obtain W^{--} and W^{-+} (and W^{+-} and W^{++}).
- Duplicate W^{--} (and W^{-+}, W^{+-}, W^{++}) and obtain W^{---} and W^{--+} (and W^{-+-}, W^{-++}, W^{+--}, W^{+-+}, W^{++-}, W^{+++}).



Polarization

Properties of $Z(W)$:

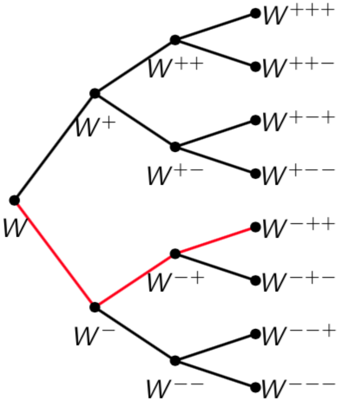
- $Z(W) \in [0, 1]$.
- $Z(W) \approx 0$ iff $I(W) \approx 1$.
- $Z(W) \approx 1$ iff $I(W) \approx 0$.
- $Z(W^+) = Z(W)^2$.
- $Z(W^-) \leq 2Z(W)$.



Since $Z(W)$ upper bounds on probability of error for uncoded transmission over W, we can choose the good indices on the basis of $Z(W)$. The sum of the Z's of the chosen channels will upper bound the block error probability. This suggests studying the polarization speed of Z.

Polarization speed

- Organize the synthetic channels as a tree.
- On a typical path the Z values we encouter are squared half the time.
- On a fraction $I(W)$ of the paths Z converges to zero.
- On a 'typical' such path, Z approaches zero like $\exp(-2^{n/2})$.



Summary

- Polar codes and I(W) achieving codes.
- Encoding complexity is N log N.
- With successive decoding, the decoding complexity is N log N
- Probability of error decays as $2^{-\sqrt{N}}$.
- Moreover, one can identify almost all the ‘good’ channels with $N^2 logN$ complexity.

Future work

Do Reed Muller codes achieve capacity for binary symmetric channels? Is the decoding efficient?

Further Reading

Erdal Arikan
Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels.
IEEE Transactions on Theory, 2008.

C. Shannon
A Mathematical Theory of Communication.
1949.

Thank You.