Split-step for NLS equation

$$\boxed{i\psi_t + (1 - i\varepsilon a)\nabla^2\psi + \varkappa(1 + i\varepsilon c|\psi|^s)|\psi|^{\frac{4}{d}}\psi = i\phi}$$

$\varkappa = 1$ focusing
$\varkappa = -1$ defocusing

in the code $\varkappa \equiv$ "focus"

in the code $d = 2$ (dimensionality)

in the code $\hat{\phi}_k = f_k \hat{\psi}_k$ where

$f_k = \varepsilon b$ or $f_k = f(k)$

$$\psi_t = \underbrace{i(1 - i\varepsilon a)\nabla^2\psi}_{L} + \underbrace{i\varkappa(1 + i\varepsilon c|\psi|^s)|\psi|^{\frac{4}{d}}\psi}_{NL} + \phi$$

The split-step method alternates linear and nonlinear substeps

$$\psi_t^{(L)} = i(1 - i\varepsilon a)\nabla^2\psi + \phi$$

$$\psi_t^{(N)} = i\varkappa(1 + i\varepsilon c|\psi|^s)|\psi|^{\frac{4}{d}}\psi$$

2nd order splitstep: $\left(\frac{1}{2}L,\ N,\ \frac{1}{2}L\right)$

4th order splitstep:

$$\left(\frac{w}{2}N,\ wL,\ \frac{1-w}{2}N,\ (1-2w)L,\ \frac{1-w}{2}N,\ wL,\ \frac{w}{2}N\right)$$

with $w = \dfrac{1}{2 - 2^{1/3}}$

Split-step : linear substep, advancing to Δt

$$\psi_t = i(1 - i\varepsilon a)\nabla^2\psi + \phi$$

FFT in space

$$\hat{\Phi}_k = f_k \hat{\psi}_k \quad , \text{ in particular } f_k = \varepsilon b$$

$$\frac{d}{dt}\hat{\psi}_k = \underbrace{\left[(\varepsilon a + i)\hat{\nabla}_k + f_k\right]}_{Q_k}\hat{\psi}_k$$

where $\hat{\nabla}_k = -(k_x^2 + k_y^2)\left(\frac{2\pi}{L}\right)^2$

$$\hat{\psi}_k(t) = \hat{\psi}_k(t_0)\exp\left\{Q_k(t - t_0)\right\}$$

Split-step: non-linear substep

Solution is advanced according to ODE solved analytically

$$\boxed{\psi_t = i\varkappa\left(1 + i\varepsilon c|\psi|^s\right)|\psi|^{\frac{4}{d}}\psi}$$

$$\frac{d\psi\psi^*}{dt} = \psi^*\frac{d\psi}{dt} + \psi\frac{d\psi^*}{dt} =$$

$$= \psi^* i\varkappa\left(1+i\varepsilon c|\psi|^s\right)|\psi|^{\frac{4}{d}}\psi + \psi(-i)\varkappa\left(1-i\varepsilon c|\psi|^s\right)|\psi|^{\frac{4}{d}}\psi^* =$$

$$= \varkappa\psi^{\frac{4}{d}+2}\left(i - \varepsilon c|\psi|^s - i - \varepsilon c|\psi|^s\right) = -2\varkappa\varepsilon c|\psi|^{\frac{4}{d}+s+2}$$

$$\frac{d|\psi|^2}{dt} = 2|\psi|\frac{d|\psi|}{dt} = -2\varkappa\varepsilon c|\psi|^{\frac{4}{d}+s+2}$$

$$\boxed{\frac{d|\psi|}{dt} = -\varkappa\varepsilon c|\psi|^{\frac{4}{d}+s+1}}$$

$$|\psi|^{-\frac{4}{d}-s-1}\,d|\psi| = -\varkappa\varepsilon c\,dt$$

$$\frac{1}{-\frac{4}{d}-s}\left(|\psi|^{-\frac{4}{d}-s} - |\psi_0|^{-\frac{4}{d}-s}\right) = -\varkappa\varepsilon c\,(t-t_0)$$

$$|\psi|^{-\frac{4}{d}-s} = |\psi_0|^{-\frac{4}{d}-s} + \varkappa\varepsilon c\left(\frac{4}{d}+s\right)(t-t_0)$$

$$\boxed{|\psi| = |\psi_0|\left[1 + \varkappa\varepsilon c\left(\frac{4}{d}+s\right)|\psi_0|^{\frac{4}{d}+s}(t-t_0)\right]^{-\frac{1}{\frac{4}{d}+s}}}$$

Note: if $\varepsilon c = 0$ then $|\psi| = |\psi_0|$.

$$\psi = |\psi|\, e^{i\alpha(t)}$$

$$\psi_t = \frac{d|\psi|}{dt}\, e^{i\alpha} + i\frac{d\alpha}{dt}|\psi|\, e^{i\alpha} = \left(\frac{1}{|\psi|}\frac{d|\psi|}{dt} + i\frac{d\alpha}{dt}\right)\psi$$

$$i\varkappa\left(1 + i\varepsilon C|\psi|^{s}\right)|\psi|^{\frac{4}{d}}\cancel{\psi} = \left(\frac{1}{|\psi|}(-\varkappa\varepsilon C)|\psi|^{\frac{4}{d}+s+1} + i\frac{d\alpha}{dt}\right)\cancel{\psi}$$

$$i\varkappa|\psi|^{\frac{4}{d}} - \varkappa\varepsilon C|\psi|^{\frac{4}{d}+s} = -\varkappa\varepsilon C|\psi|^{\frac{4}{d}+s} + i\frac{d\alpha}{dt}$$

$$\frac{d\alpha}{dt} = \varkappa|\psi|^{\frac{4}{d}}$$

$$\frac{d\alpha}{dt} = \varkappa|\psi_0|^{\frac{4}{d}}\left[1 + \varkappa\varepsilon C\left(\frac{4}{d}+s\right)|\psi_0|^{\frac{4}{d}+s}(t_0 - t)\right]^{-\frac{\frac{4}{d}}{\frac{4}{d}+s}}$$

$$\alpha = \alpha_0 + \varkappa|\psi_0|^{\frac{4}{d}}\int_{t_0}^{t}\left[1 + \underbrace{\varkappa\varepsilon C\left(\frac{4}{d}+s\right)|\psi_0|^{\frac{4}{d}+s}}_{a}(t - t_0)\right]^{\underbrace{-\frac{\frac{4}{d}}{\frac{4}{d}+s}}_{n}} dt$$

<u>if $\varepsilon C = 0$:</u> $\quad \alpha = \alpha_0 + \varkappa|\psi_0|^{\frac{4}{d}}(t - t_0)$

$$\psi = |\psi_0|\, e^{i\alpha_0 + i\varkappa|\psi_0|^{\frac{4}{d}}(t - t_0)}$$

$$\psi = |\psi_0|\, e^{i\alpha_0}\, e^{i\varkappa|\psi_0|^{\frac{4}{d}}(t - t_0)}$$

$$\boxed{\psi = \psi_0 \exp\left\{i\varkappa|\psi_0|^{\frac{4}{d}}(t - t_0)\right\}}$$

if $\varepsilon c \neq 0$, $s = 0$:

$$\alpha = \alpha_0 + \varkappa |\psi_0|^{\frac{4}{d}} \int_{t_0}^{t} \left[1 + \varkappa \varepsilon c \tfrac{4}{d} |\psi_0|^{\frac{4}{d}} (t - t_0)\right]^{-1} dt$$

$$\int_{t_0}^{t} (1 + a(t - t_0))^{-1} dt = \frac{1}{a} \int (1 + a(t - t_0))^{-1} d(1 + a(t - t_0)) =$$

$$= \frac{1}{a} \ln(1 + a(t - t_0)) \Big|_{t_0}^{t} = \frac{1}{a} \ln(1 + a(t - t_0))$$

$$\alpha = \alpha_0 + \varkappa |\psi_0|^{\frac{4}{d}} \frac{1}{\varkappa \varepsilon c \frac{4}{d} |\psi_0|^{\frac{4}{d}}} \ln\left(1 + \varkappa \varepsilon c \tfrac{4}{d} |\psi_0|^{\frac{4}{d}} (t - t_0)\right)$$

$$\alpha = \alpha_0 + \frac{d}{4\varepsilon c} \ln\left(1 + \varkappa \varepsilon c \tfrac{4}{d} |\psi_0|^{\frac{4}{d}} (t - t_0)\right) \equiv \alpha_0 + \frac{d}{4\varepsilon c} \ln q$$

$$|\psi| = |\psi_0| \left[1 + \varkappa \varepsilon c \tfrac{4}{d} |\psi_0|^{\frac{4}{d}} (t - t_0)\right]^{-\frac{d}{4}} = |\psi_0| q^{-\frac{d}{4}}$$

$$\psi = |\psi| e^{i\alpha} = |\psi_0| q^{-\frac{d}{4}} e^{i\alpha_0 + i\frac{d}{4\varepsilon c} \ln q} = |\psi_0| e^{i\alpha_0} e^{-\frac{d}{4} \ln q} e^{i \frac{d}{4\varepsilon c} \ln q}$$

$$= \psi_0 \exp\left[\frac{d}{4} \left(-1 + i \tfrac{1}{\varepsilon c}\right) \ln q\right]$$

$$\boxed{\psi = \psi_0 \exp\left\{\frac{d}{4} \frac{i - \varepsilon c}{\varepsilon c} \ln\left[1 + \varkappa \varepsilon c \tfrac{4}{d} |\psi_0|^{\frac{4}{d}} (t - t_0)\right]\right\}}$$

$d = 1$: $\quad \psi = \psi_0 \exp\left\{\frac{i - \varepsilon c}{4\varepsilon c} \ln\left[1 + 4\varkappa \varepsilon c |\psi_0|^4 (t - t_0)\right]\right\}$

$d = 2$: $\quad \psi = \psi_0 \exp\left\{\frac{i - \varepsilon c}{2\varepsilon c} \ln\left[1 + 2\varkappa \varepsilon c |\psi_0|^2 (t - t_0)\right]\right\}$

if $\varepsilon c \neq 0,\ s \neq 0$:

$$a \equiv \varkappa \varepsilon c \left(\tfrac{4}{d}+s\right)|\psi_0|^{\frac{4}{d}+s}, \qquad n \equiv -\frac{\frac{4}{d}}{\frac{4}{d}+s}, \qquad n+1 = \frac{s}{\frac{4}{d}+s}$$

$$\int_{t_0}^{t} [1+a(t-t_0)]^n\,dt = \frac{1}{a}\frac{1}{n+1}[1+a(t-t_0)]^{n+1}\Big|_{t_0}^{t} =$$

$$= \frac{1}{a}\frac{1}{n+1}\left([1+a(t-t_0)]^{n+1}-1\right) = \frac{1}{a}\frac{1}{n+1}\left[(1+a\Delta t)^{n+1}-1\right]$$

$$\alpha = \alpha_0 + \varkappa|\psi_0|^{\frac{4}{d}} \frac{1}{\varkappa\varepsilon c\left(\frac{4}{d}+s\right)|\psi_0|^{\frac{4}{d}+s}} \frac{\frac{4}{d}+s}{s}\left[(1+a\Delta t)^{\frac{s}{\frac{4}{d}+s}}-1\right]$$

$$\alpha = \alpha_0 + \frac{1}{\varepsilon c s|\psi_0|^s}\left[(1+a\Delta t)^{\frac{s}{\frac{4}{d}+s}}-1\right] = \alpha_0 + \frac{e^q-1}{\varepsilon c s|\psi_0|^s}$$

$$(1+a\Delta t)^{\frac{s}{\frac{4}{d}+s}} \equiv e^q, \qquad q = \frac{s}{\frac{4}{d}+s}\ln(1+a\Delta t)$$

$$|\psi| = |\psi_0|\,(1+a\Delta t)^{-\frac{1}{\frac{4}{d}+s}} = |\psi_0|\exp\left\{\frac{\frac{4}{d}+s}{s}\,q\left(-\frac{1}{\frac{4}{d}+s}\right)\right\} = |\psi_0|\exp\left\{-\frac{q}{s}\right\}$$

$$\psi = |\psi|e^{i\alpha} = |\psi_0|\exp\left\{-\frac{q}{s}\right\}e^{i\alpha_0}\exp\left\{i\frac{e^q-1}{\varepsilon c s|\psi_0|^s}\right\}$$

$$\boxed{\psi = \psi_0\exp\left\{-\frac{q}{s} + i\frac{e^q-1}{\varepsilon c s|\psi_0|^s}\right\} \quad \text{where } q = \frac{s}{\frac{4}{d}+s}\ln\left[1+\varkappa\varepsilon c\left(\tfrac{4}{d}+s\right)|\psi_0|^{\frac{4}{d}+s}\Delta t\right]}$$

OR $\psi = \psi_0\exp\left\{-\frac{q}{\frac{4}{d}+s} + i\frac{\exp\left(\frac{s}{\frac{4}{d}+s}q\right)-1}{\varepsilon c s|\psi_0|^s}\right\}$, where $q = \ln\left[1+\varkappa\varepsilon c\left(\frac{4}{d}+s\right)|\psi_0|^{\frac{4}{d}+s}\Delta t\right]$

OR $\psi = \psi_0\exp\left\{-\frac{\ln q}{\frac{4}{d}+s} + i\frac{q^{\frac{s}{\frac{4}{d}+s}}-1}{\varepsilon c s|\psi_0|^s}\right\}$, where $q = 1+\varkappa\varepsilon c\left(\frac{4}{d}+s\right)|\psi_0|^{\frac{4}{d}+s}\Delta t$

OR $\psi = \psi_0\, q^{-\frac{1}{\frac{4}{d}+s}}\exp\left\{\frac{i}{\varepsilon c s}|\psi_0|^{-s}\left(q^{\frac{s}{\frac{4}{d}+s}}-1\right)\right\}$ where

$q = 1+\varkappa\varepsilon c\left(\frac{4}{d}+s\right)|\psi_0|^{\frac{4}{d}+s}\Delta t$

Split-step: summary for nonlinear substep, d=2

ODE: $\psi_t = i\varkappa(1 + i\varepsilon c|\psi|^s)|\psi|^2\psi$

$$\Delta t \equiv t - t_0$$

$$a \equiv \varkappa(2+s)\,\varepsilon c\,\Delta t$$

$\varepsilon c = 0$:

$$Q = i\varkappa\Delta t\,|\psi_0|^2$$

$$\psi = \psi_0 \exp(Q)$$

$\varepsilon c \neq 0,\ s = 0$:

$$q = \ln(1 + a|\psi_0|^2)$$

$$Q = -\frac{q}{2} + i\frac{q}{2\varepsilon c}$$

$$\psi = \psi_0 \exp(Q)$$

$\varepsilon c \neq 0,\ s \neq 0$

$$q = \ln\left(1 + a|\psi_0|^{2+s}\right)$$

$$Q = -\frac{q}{s+2} + i\,\frac{\exp\left(\frac{s}{s+2}q\right) - 1}{\varepsilon c s|\psi_0|^s}$$

$$\psi = \psi_0 \exp(Q)$$

Note 1) $x^a = \exp(\ln x^a) = \exp(a\ln x)$

$$|\psi_0|^s = (|\psi_0|^2)^{\frac{s}{2}} = \exp\left(\frac{s}{2}\ln|\psi_0|^2\right)$$

Note 2) the expression for $s \neq 0$ reduces to the expression for $s = 0$ at $s \to 0$.