0.1 Fixed-Point Iteration

Exercise 0.1.1

Use algebraic manipulation to show that each of the following functions has a fixed-point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

a) $g_1(x) = (3 + x - 2x^2)^{1/4}$

b) $g_2(x) = \left(\dfrac{x + 3 - x^4}{2}\right)^{1/2}$

c) $g_3(x) = \left(\dfrac{x + 3}{x^2 + 2}\right)^{1/2}$

d) $g_4(x) = \dfrac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$

Solution 0.1.1

a) For $x = p$:

$$g_1(p) = (3 + p - 2p^2)^{\frac{1}{4}} = (p^4 - f(p))^{1/4} = |p|$$

So p is a fixed-point of g_1.

b) For $x = p$:

$$\begin{aligned} g_2(p) &= \left(\frac{p + 3 - p^4}{2}\right)^{1/2} \\ &= \left(\frac{2p^2}{2}\right)^{\frac{1}{2}} \\ &= |p| \end{aligned}$$

So p is a fixed-point of g_2.

c) For $x = p$:

$$\begin{aligned} g_3(p) &= \left(\frac{p + 3}{p^2 + 2}\right)^{1/2} \\ &= \left(\frac{p^4 + 2p^2}{p^2 + 2}\right)^{1/2} \\ &= |p| \end{aligned}$$

So p is a fixed-point of g_3.

d) For $x = p$:

$$\begin{aligned} g_4(p) &= \frac{3p^4 + 2p^2 + 3}{4p^3 + 4p - 1} \\ &= \frac{4p^4 - (3 + p - 2p^2) + 2p^2 + 3}{4p^3 + 4p - 1} \\ &= \frac{4p^4 + 4p^2 - p}{4p^3 + 4p - 1} \\ &= p \end{aligned}$$

So p is a fixed-point of g_4.

Exercise 0.1.2

a) Perform four iterations, if possible, on each of the functions g defined in Exercise 1. Let $p_0 = 1$ and $p_{n+1} = g(p_n)$, for $n = 0, 1, 2, 3$.

b) Which function do you think gives the best approximation to the solution?

Solution 0.1.2

a) Applying fixed-point method on the four functions g generates the following table:

n	p_n by g_1	p_n by g_2	p_n by g_3	p_n by g_4
0	1	1	1	1
1	1.189 207 115	1.224 744 871	1.154 700 538	1.142 857 143
2	1.080 057 753	0.993 666 159	1.116 427 41	1.124 481 69
3	1.149 671 431	1.228 568 645	1.126 052 233	1.124 123 164
4	1.107 820 053	0.987 506 429	1.123 638 885	1.124 123 03

b) g_4 gives the best approximation as it generates the smallest difference between p_3 and p_4: $|p_4 - p_3| = -134 \times 10^{-7}$.

Exercise 0.1.3

The following four methods are proposed to compute $21^{1/3}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

a) $p_n = \dfrac{20p_{n-1} + 21/p_{n-1}^2}{21}$

b) $p_n = p_{n-1} - \dfrac{p_{n-1}^3 - 21}{3p_{n-1}^2}$

c) $p_n = p_{n-1} - \dfrac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21}$

d) $p_n = \left(\dfrac{21}{p_{n-1}}\right)^{1/2}$

Solution 0.1.3

Applying fixed-point method on the four sequences generate the following table:

n	a)	b)	c)	d)
0	1	1	1	1
1	1.952 380 952	7.666 666 667	0	4.582 575 695
2	2.121 754 174	5.230 203 739	0	2.140 695 143
3	2.242 849 692	3.742 696 919		3.132 075 595
4	2.334 839 673	2.994 853 568		2.589 366 527
5	2.401 093 38	2.777 022 226		2.847 822 274
6	2.465 059 288	2.759 041 866		2.715 521 253
7	2.512 243 463	2.758 924 181		2.780 885 095
8	2.551 057 096	2.758 924 176		2.748 008 838
9	2.583 237 767	2.758 924 176		2.764 398 093
10	2.610 081 445			2.756 191 284
11	2.632 580 301			2.760 291 639
12	2.651 509 504			2.758 240 699
13	2.667 484 488			2.759 265 978
14	2.681 000 202			2.758 753 291
15	2.692 458 887			2.759 009 623
16	2.702 190 249			2.758 881 454
17	2.710 466 453			2.758 945 538
18	2.717 513 483			2.758 913 496
19	2.723 519 902			2.758 929 517

Apparently, the speed of convergence is ranked in descending order as follow: b), d), a). c) does not converge.

Exercise 0.1.4

The following four methods are proposed to compute $7^{1/5}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

a) $p_n = p_{n-1} - \left(1 + \frac{7-p_{n-1}^5}{p_{n-1}^2}\right)^3$

b) $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{p_{n-1}^2}$

c) $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{5p_{n-1}^4}$

d) $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{12}$

Solution 0.1.4

Applying fixed-point method on the four sequences generate the following table:

n	a)	b)	c)	d)
0	1	1	2.2	1
1	343	7	1.819 763 677	1.5
2	-2.25×10^{25}	-335.857	1.583 474 83	1.450 520 833
3		37 884 356	1.489 460 974	1.498 749 661
4			1.476 022 436	1.451 903 535
5			1.475 773 246	1.497 577 067
6			1.475 773 162	1.453 192 29
7			1.475 773 162	1.496 475 364
9				1.454 396 119
8				1.495 438 587
10				1.455 522 81
11				1.494 461 513
12				1.456 579 138
13				1.493 539 533
14				1.457 571 031
15				1.492 668 56
16				1.458 803 715
17				1.491 844 948
18				1.459 381 814
19				1.491 065 425

Apparently, the speed of convergence is ranked in descending order as follow: c), d). a) and b) do not converge.

Exercise 0.1.5

Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^4 - 3x^2 - 3 = 0$ on $[1, 2]$. Use $p_0 = 1$.

Solution 0.1.5

Let $f(x) = x^4 - 3x^2 - 3$. Let p be the root of f in $[1, 2]$. We need to find a function g for which $p = g(p)$ to perform the fixed-point method.

Extract p to RHS gives:

$$p^4 = 3p^2 + 3 \iff |p| = (3p^2 + 3)^{1/4}$$

Then g is chosen as:

$$g(x) = (3x^2 + 3)^{1/4}$$

Applying fixed-point method on g generate the following table:

n	p_n	n	p_n
0	1	4	1.922 847 844
1	1.565 084 58	5	1.937 507 54
2	1.793 572 879	6	1.943 316 93
3	1.885 943 743		

We can try the other obvious option

$$g(x) = \left(\frac{x^4 - 3}{3}\right)^{0.5}$$

which fails on the first iteration. A reasonable explanation for the choice of g is that we need $|g'|$ to be as small as possible. On $[1, 2]$, the $O(x^{0.5})$ of the first choice clearly has an advantage over $O(x^2)$ of the second choice of g.

We conclude that $p \approx 1.943$.

Exercise 0.1.6

Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on $[1, 2]$. Use $p_0 = 1$.

Solution 0.1.6

Let $f(x) = x^3 - x - 1 = 0$. Let p be the root of f in $[1, 2]$. We need to find a function g for which $p = g(p)$ to perform the fixed-point method.

Extract p to RHS gives:

$$p^3 = p + 1 \iff p = (p + 1)^{1/3}$$

Then g is chosen as:

$$g(x) = (p + 1)^{1/3}$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n
0	1	3	1.322 353 819
1	1.259 921 05	4	1.324 268 745
2	1.312 293 837		

We conclude that $p \approx 1.324$.

Exercise 0.1.7

Use Theorem 2.3 (Định lý 2.3 in the accompanying Lectures.pdf) to show that $g(x) = \pi + 0.5 \sin 0.5x$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2}. Use Corollary 2.5 (Hệ quả 2.1) to estimate the number of iterations required to achieve 10^{-2} accuracy, and compare this theoretical estimate to the number actually needed.

Solution 0.1.7

From the formula of g:

$$\begin{aligned} g(x) &= \pi + 0.5 \sin 0.5x \\ \Rightarrow g(x) &\in [\pi - 0.5, \pi + 0.5] \,\forall x \end{aligned}$$

Consider the interval $I = [\pi - 0.5, \pi + 0.5] \in [0, 2\pi]$. From the above equations, we know that:

- $g \in CI$
- $g(x) \in I \,\forall x \in I$

According to Theorem 2.3, there exists a fixed point of g on I.

Differentiating g gives:

$$g'(x) = -0.25 \cos 0.5x \Rightarrow \left|g'(x)\right| \leq k = 0.25 < 1 \,\forall x$$

Again, according to Theorem 2.3, there exists one and only one fixed point of g on I.

Applying fixed-point method on g, with $p_0 = \pi$, generates the following table:

n	p_n	n	p_n
0	3.141 592 654	2	3.626 048 864
1	3.641 592 654	3	3.626 995 622

Using corollary 2.5, the number of iterations n required to achieve 10^{-2} accuracy is

$$|p_n - p| \leq k^n 0.5 < 10^{-2} \iff n \geq 3$$

which is in line with the number of iteration actually performed.

Exercise 0.1.8

Use Theorem 2.3 (Định lý 2.3 in the accompanying Lectures.pdf) to show that $g(x) = 2^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-4}. Use Corollary 2.5 (Hệ quả 2.1) to estimate the number of iterations required to achieve 10^{-4} accuracy, and compare this theoretical estimate to the number actually needed.

Solution 0.1.8

From the formula of g:

$$g(x) = 2^{-x}$$
$$\Rightarrow g'(x) = -2^{-x}\ln 2$$

It is clear that $g \in C^1R$.
Consider the interval $I = [\frac{1}{3}, 1]$, $I_{open} = (\frac{1}{3}, 1)$:

$$\begin{aligned}
&g'(x) < 0 \forall x \in I \\
&\Rightarrow 1 > g(\frac{1}{3}) = 2^{-1/3} \geq g(x) \geq g(1) = 2^{-1} > \frac{1}{3} \\
&\Rightarrow g(x) \in I \,\forall x \in I
\end{aligned}$$

So far, we know that:

- $g \in CI$ ($g \in CR$ even)
- $g(x) \in I \,\forall x \in I$

According to Theorem 2.3, there exists a fixed point of g on I.
Consider g':

$$-1 < -\ln 2 \leq g'(x) \leq -\frac{1}{3}\ln 2 < 0 \,\forall x \in I$$
$$\Rightarrow \left|g'(x)\right| \leq k = \ln 2 < 1 \,\forall x \in I$$

Again, according to Theorem 2.3, there exists one and only one fixed point of g on I.
Applying fixed-point method on g, with $p_0 = \frac{2}{3}$, generates the following table:

n	p_n	n	p_n
0	0.666 666 667	5	0.640 746 653
1	0.629 960 525	6	0.641 380 922
2	0.646 194 096	7	0.641 099 006
3	0.638 963 711	8	0.641 224 295
4	0.642 174 057	9	0.641 168 611

Using Corollary 2.5, the number of iterations n required to achieve 10^{-4} accuracy is

$$|p_n - p| \leq k^n \frac{1}{3} < 10^{-4} \iff n \geq 23$$

which is quit a bit higher than the number of iteration actually performed.

Exercise 0.1.9

Use a fixed-point iteration method to find an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 12 of Section 2.1.

Solution 0.1.9

Let $f(x) = x^2 - 3$, $p > 0$ is a zero of f. Then $p = \sqrt{3}$, and an approximation of p is an approximation of $\sqrt{3}$.

Consider $g(x) = \frac{3}{x}$. It is clear that this is a bad choice, as applying g on any p_0 generates a sequence that jumps between p_0 and $\frac{3}{p_0}$.

From the textbook examples, we choose $g(x) = x - \frac{x^2-3}{x^2}$. Applying fixed-point method on g with $p_0 = 1.5$ generates the following table:

n	p_n	n	p_n
0	1.5	4	1.731 898 58
1	1.833 333 33	5	1.732 074 38
2	1.725 895 32	6	1.732 047 16
3	1.733 041 14		

We conclude that $\sqrt{3} \approx 1.732\,05$. In exercise 12 of section 2.1, 14 iteration is needed, much higher than that of this method.

Exercise 0.1.10

Use a fixed-point iteration method to find an approximation to $\sqrt[3]{25}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 13 of Section 2.1.

Solution 0.1.10

Let $f(x) = x^3 - 25$, $p > 0$ is a zero of f. Then $p = \sqrt[3]{25}$, and an approximation of p is an approximation of $\sqrt[3]{25}$.

We choose $g(x) = x - \frac{x^3-25}{x^3}$. Applying fixed-point method on g with $p_0 = 2.5$ generates the following table:

n	p_n	n	p_n
0	2.5	3	2.923 783 69
1	3.1	4	2.924 023 86
2	2.939 179 62	5	2.924 017 58

We conclude that $\sqrt[3]{25} \approx 2.924\,02$. In exercise 13 of section 2.1, 14 iteration is needed, much higher than that of this method.

Exercise 0.1.11

For each of the following equations, determine an interval $[a, b]$ on which fixed-point iteration converges. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and perform the calculations.

a) $x = \dfrac{2 - e^x + x^2}{3}$

b) $x = \dfrac{5}{x^2} + 2$

c) $x = \left({}^{e^x}\!/_3\right)^{1/2}$

d) $x = 5^{-x}$

e) $x = 6^{-x}$

f) $x = 0.5(\sin x + \cos x)$

Solution 0.1.11

a) Let

$$\begin{aligned} && g(x) &= \frac{2 - e^x + x^2}{3} \\ \Rightarrow && g'(x) &= \frac{2x - e^x}{3} \\ \Rightarrow && g''(x) &= \frac{2 - e^x}{3} \end{aligned}$$

It is clear that g is continuous in $\mathbb{R}$.

Consider g'':

- $g''(x) > 0 \iff x < \ln 2$
- $g''(x) = 0 \iff x = \ln 2$
- $g''(x) < 0 \iff x > \ln 2$

So, $\max g'(x) = g'(\ln 2) = \dfrac{\ln 4 - 2}{3} < 0$. So g is monotonically decreasing in $\mathbb{R}$.

Consider the interval $I = [0, 1]$:

$$1 > g(0) = \frac{1}{3} > g(x) > g(1) = \frac{3-e}{3} > 0 \, \forall x \in I$$
$$\Rightarrow g(x) \in I \, \forall x \in I$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 0.5$ generates the following table:

n	p_n	n	p_n
0	0.5	5	0.257 265 636
1	0.200 426 243	6	0.257 598 985
2	0.272 749 065	7	0.257 512 455
3	0.253 607 157	8	0.257 534 914
4	0.258 550 376	9	0.257 529 084

We conclude that the fixed point $p \approx 0.257\,529$.

b) Let

$$g = \frac{5}{x^2} + 2$$

Consider the interval $I = [2.5, 3]$. $0 \notin I$, so g is continuous in I.

x^2 is monotonically increasing in I, so g is monotonically decreasing in I. So that:

$$3 > g(2.5) = 2.8 > g(x) > g(3) = {}^{23}/_{9} > 2.5 \, \forall x \in I$$
$$\Rightarrow g(x) \in I \, \forall x \in I$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 2.75$ generates the following table:

n	p_n	n	p_n	n	p_n
0	2.75	6	2.691 710 92	12	2.690 666 91
1	2.661 157 02	7	2.690 101 82	13	2.690 637 46
2	2.706 039 5	8	2.690 927 64	14	2.690 652 58
3	2.682 812 93	9	2.690 503 63	15	2.690 644 82
4	2.694 687 08	10	2.690 721 29		
5	2.688 578 29	11	2.690 609 54		

We conclude that the fixed point $p \approx 2.690\,645$.

c) Let

$$g(x) = \left(\frac{e^x}{3}\right)^{1/2}$$

It is clear that g is continuous in $\mathbb{R}$.

g is monotonically increasing in $\mathbb{R}$. Consider the interval $I = [0, 1]$:

$$0 < g(0) = \frac{1}{\sqrt{3}} < g(x) < g(1) = \sqrt{\frac{e}{3}} < 1$$
$$\Rightarrow g(x) \in I \; \forall x \in I$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 0.5$ generates the following table:

n	p_n	n	p_n	n	p_n
0	0.5	5	0.903 281 143	10	0.909 876 791
1	0.741 332 42	6	0.906 952 163	11	0.909 948 068
2	0.836 407 007	7	0.908 618 411	12	0.909 980 498
3	0.877 127 74	8	0.909 375 718	13	0.909 995 254
4	0.895 169 428	9	0.909 720 122	14	0.910 001 967

We conclude that the fixed point $p \approx 0.910\,002$.

d) Let $g(x) = 5^{-x}$. It is clear that g is continuous in $\mathbb{R}$.

5^x is monotonically increasing in $\mathbb{R}$, so g is monotonically decreasing in $\mathbb{R}$.

Consider the interval $I = [0, 1]$:

$$0 < g(1) = 0.2 < g(x) < g(0) = 1$$
$$\Rightarrow g(x) \in I \; \forall x \in I$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 0.5$ generates the following table:

n	p_n	n	p_n	n	p_n
0	0.5	11	0.468 245 559	22	0.469 685 261
1	0.447 213 595	12	0.470 663 369	23	0.469 574 052
2	0.486 867 866	13	0.468 835 429	24	0.469 658 106
3	0.456 766 207	14	0.470 216 753	25	0.469 594 575
4	0.479 439 843	15	0.469 172 549	26	0.469 642 593
5	0.462 259 591	16	0.469 961 695	27	0.469 606 3

n	p_n	n	p_n	n	p_n
6	0.475 219 673	17	0.469 365 184	28	0.469 633 731
7	0.465 409 992	18	0.469 816 013	29	0.469 612 998
8	0.472 816 23	19	0.469 475 247	30	0.469 628 669
9	0.467 213 774	20	0.469 732 798	31	0.469 616 824
10	0.471 445 6	21	0.469 538 128	32	0.469 625 777

We conclude that the fixed point $p \approx 0.469\,626$.

e) Let $g(x) = 6^{-x}$. It is clear that g is continuous in $\mathbb{R}$.

6^x is monotonically increasing in $\mathbb{R}$, so g is monotonically decreasing in $\mathbb{R}$.

Consider the interval $I = [0, 1]$:

$$\begin{aligned} 0 < g(1) = \frac{1}{6} &< g(x) < g(0) = 1 \\ &\Rightarrow g(x) \in I \, \forall x \in I \end{aligned}$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 0.5$ generates the following table:

n	p_n	n	p_n	n	p_n
0	0.5	15	0.446 190 464	30	0.448 132 603
1	0.408 248 29	16	0.449 568 975	31	0.448 007 263
2	0.481 194 974	17	0.446 855 739	32	0.448 107 887
3	0.422 238 208	18	0.449 033 402	33	0.448 027 103
4	0.469 282 988	19	0.447 284 756	34	0.448 091 958
5	0.431 347 074	20	0.448 688 365	35	0.448 039 891
6	0.461 686 032	21	0.447 561 363	36	0.448 081 691
7	0.437 258 678	22	0.448 466 044	37	0.448 048 133
8	0.456 821 582	23	0.447 739 682	38	0.448 075 074
9	0.441 086 448	24	0.448 322 78	39	0.448 053 445
10	0.453 699 216	25	0.447 854 63	40	0.448 070 809
11	0.443 561 035	26	0.448 230 453	41	0.448 056 869
12	0.451 692 029	27	0.447 928 723	42	0.448 068 06
13	0.445 159 128	28	0.448 170 951	43	0.448 059 076
14	0.450 400 504	29	0.447 976 481		

We conclude that the fixed point $p \approx 0.448\,059$.

f) Let $g(x) = 0.5(\sin x + \cos x)$. It is clear that g is continuous in $\mathbb{R}$.

Manipulating g gives:

$$\sin x + \cos x = \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x\right)$$
$$= \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$$
$$\Rightarrow g(x) = 0.5(\sin x + \cos x)$$
$$= \frac{1}{\sqrt{2}}\sin\left(x + \frac{\pi}{4}\right)$$

Consider the interval $I = [0, \frac{\pi}{4}]$. $sinx$ is monotonically increasing in $[0, \frac{\pi}{2}]$, so $\sin x + \frac{\pi}{4}$ also is monotonically increasing in I. It follows that:

$$0 < g(0) = 0.5 < g(x) < g(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} < \frac{\pi}{4}$$
$$\Rightarrow g(x) \in I\, \forall x \in I$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = \frac{\pi}{8}$ generates the following table:

n	p_n	n	p_n
0	0.392 699 082	4	0.704 799 153
1	0.653 281 482	5	0.704 811 271
2	0.700 944 543	6	0.704 811 96
3	0.704 586 59		

We conclude that the fixed point $p \approx 0.704\,812$.

Exercise 0.1.12

For each of the following equations, use the given interval or determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and perform the calculations.

a) $2 + \sin x - x = 0$ on $[2, 3]$

b) $x^3 - 3x - 5 = 0$ on $[2, 3]$

c) $3x^2 - e^x = 0$

d) $x - \cos x = 0$

Solution 0.1.12

a) Let $I = [2, 3]$ and

$$g(x) = \sin x + 2$$
$$\Rightarrow g'(x) = \cos x$$

A fixed point p of g is also a root of the problem.

Consider g. It is clear that g is continuous on $\mathbb{R}$. $\sin x$ is monotonically decreasing in I, so that:

$$2 < g(3) = \sin 3 + 2 < g(x) < g(2) = \sin 2 + 2 < 3$$

Consider g'. $\cos x$ is monotonically decreasing in I, so that:

$$\cos 3 \leq g'(x) \leq \cos 2 < 0 \,\forall x \in I$$
$$\Rightarrow \left|g'(x)\right| \leq k = -\cos 3 < 1$$

Therefore, all the conditions in Corollary 2.5 hold. Using Corollary 2.5, with $p_0 = 2.5$, the number of iteration n required to obtain approximations accurate to within 10^{-5} is:

$$|p_n - p| \leq k^n 0.5 < 10^{-5} \iff n \geq 1076$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n	n	p_n
0	2.5	18	2.552 225 43	36	2.554 123 46
1	2.598 472 14	19	2.555 835 11	37	2.554 256 29
2	2.516 809 97	20	2.552 830 8	38	2.554 145 73
3	2.584 921 02	21	2.555 331 77	39	2.554 237 76
4	2.528 363 28	22	2.553 250 15	40	2.554 161 15
5	2.575 511 41	23	2.554 982 97	41	2.554 224 92
6	2.536 328 7	24	2.553 540 68	42	2.554 171 84
7	2.568 979 15	25	2.554 741 28	43	2.554 216 02
8	2.541 830 51	26	2.553 741 95	44	2.554 179 25
9	2.564 446 15	27	2.554 573 8	45	2.554 209 86
10	2.545 634 87	28	2.553 881 4	46	2.554 184 38
11	2.561 301 68	29	2.554 457 76	47	2.554 205 59
12	2.548 267 3	30	2.553 978 01	48	2.554 187 93
13	2.559 121 11	31	2.554 377 35	49	2.554 202 63
14	2.550 089 61	32	2.554 044 95	50	2.554 190 4
15	2.557 609 33	33	2.554 321 64	51	2.554 200 58
16	2.551 351 48	34	2.554 091 33	52	2.554 192 1
17	2.556 561 41	35	2.554 283 04		

So one solution of the problem is $p \approx 2.554\,192$.

b) Let $I = [2, 3]$ and

$$g(x) = \sqrt[3]{2x + 5}$$

$$\Rightarrow g'(x) = \frac{2}{3}(2x+5)^{-2/3}$$

A fixed point p of g is also a solution of the problem.

Consider g. It is clear that g is continuous and monotonically increasing on $\mathbb{R}$, so that:

$$2 < g(2) = \sqrt[3]{9} < g(x) < g(3) = \sqrt[3]{11} < 3$$
$$\Rightarrow g(x) \in I \; \forall x \in I$$

Consider g'. Since $-2/3 < 0$ and $I > 0$, $g'(x)$ is monotonically decreasing in I, so that:

$$g'(2) = \frac{2}{9\sqrt[3]{3}} \geq g'(x) \geq g'(3) = \frac{2}{3\sqrt[3]{121}}$$
$$\Rightarrow \left|g'(x)\right| \leq k = \frac{2}{9\sqrt[3]{3}} < 1$$

Therefore, all the conditions in Corollary 2.5 hold. Using Corollary 2.5, with $p_0 = 2.5$, the number of iteration n required to obtain approximations accurate to within 10^{-5} is:

$$|p_n - p| \leq k^n 0.5 < 10^{-5} \iff n \geq 6$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n
0	2.5	4	2.094 760 55
1	2.154 434 69	5	2.094 583 25
2	2.103 612 03	6	2.094 556 31
3	2.095 927 41	7	2.094 552 22

So one solution of the problem is $p \approx 2.094\,552$.

c) Let $I = [3, 4]$ and

$$g(x) = \ln 3x^2 = 2\ln x + \ln 3$$
$$\Rightarrow g'(x) = \frac{2}{x}$$

A fixed point p of g is also a solution of the problem.

Consider g. It is clear that g is continuous and monotonically increasing on I, so that:

$$
\begin{aligned}
3 < g(3) = \ln 27 < &g(x) < g(4) = \ln 48 < 4 \\
\Rightarrow &g(x) \in I \, \forall x \in I
\end{aligned}
$$

Consider g'. Since $I > 0$, $g'(x)$ is monotonically decreasing in I, so that:

$$
\begin{aligned}
g'(3) = \frac{2}{3} \geq g'(x) \geq g'(4) = \frac{1}{2} \\
\Rightarrow \left|g'(x)\right| \leq k = \frac{2}{3} < 1
\end{aligned}
$$

Therefore, all the conditions in Corollary 2.5 hold. Using Corollary 2.5, with $p_0 = 3.5$, the number of iteration n required to obtain approximations accurate to within 10^{-5} is:

$$
|p_n - p| \leq k^n 0.5 < 10^{-5} \iff n \geq 27
$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n	n	p_n
0	3.5	6	3.727 177 12	12	3.732 939 23
1	3.604 138 23	7	3.729 914 58	13	3.733 004 13
2	3.662 777 67	8	3.731 382 95	14	3.733 038 9
3	3.695 055 86	9	3.732 170 15	15	3.733 057 53
4	3.712 603 63	10	3.732 592 04	16	3.733 067 51
5	3.722 079 13	11	3.732 818 1		

So one solution of the problem is $p \approx 3.733\,068$.

d) Let $I = [0, 1]$ and

$$
\begin{aligned}
g(x) &= \cos x \\
\Rightarrow g'(x) &= -\sin x
\end{aligned}
$$

A fixed point p of g is also a solution of the problem.

Consider g. It is clear that g is continuous and monotonically decreasing on I, so that:

$$
\begin{aligned}
1 = g(0) \geq &g(x) \geq g(1) = \cos 1 > 0 \\
\Rightarrow &g(x) \in I \, \forall x \in I
\end{aligned}
$$

Consider g'. Since $I > 0$, $g'(x)$ is monotonically decreasing in I, so that:

$$
g'(0) = 0 \geq g'(x) \geq g'(1) = -\sin 1
$$

$$\Rightarrow \left|g'(x)\right| \leq k = \sin 1 < 1$$

Therefore, all the conditions in Corollary 2.5 hold. Using Corollary 2.5, with $p_0 = 0.5$, the number of iteration n required to obtain approximations accurate to within 10^{-5} is:

$$|p_n - p| \leq k^n 0.5 < 10^{-5} \iff n \geq 63$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n	n	p_n
0	0.5	10	0.735 006 309	20	0.739 006 78
1	0.877 582 562	11	0.741 826 523	21	0.739 137 911
2	0.639 012 494	12	0.737 235 725	22	0.739 049 581
3	0.802 685 101	13	0.740 329 652	23	0.739 109 081
4	0.694 778 027	14	0.738 246 238	24	0.739 069 001
5	0.768 195 831	15	0.739 649 963	25	0.739 096
6	0.719 165 446	16	0.738 704 539	26	0.739 077 813
7	0.752 355 759	17	0.739 341 452	27	0.739 090 064
8	0.730 081 063	18	0.738 912 449	28	0.739 081 812
9	0.745 120 341	19	0.739 201 444		

So one root of the problem is $p \approx 0.739\,082$.

Exercise 0.1.13

Find all the zeros of $f(x) = x^2 + 10\cos x$ by using the fixed-point iteration method for an appropriate iteration function g. Find the zeros accurate to within 10^{-4}.

Solution 0.1.13

Consider $f = 0$. Since $x^2 \geq 0$, $\cos x$ must be negative for the equation to hold, so that:

$$x \in I_k = [\frac{\pi}{2} + k2\pi, \frac{3\pi}{2} + k2\pi]\,\forall k \in \mathbb{N} \tag{1}$$

Also, since $10\cos x \in [-10, 0]$:

$$x \in [-\sqrt{10}, \sqrt{10}] \tag{2}$$

Combining (1) and (2) gives:

$$x \in I = I_a \cup I_b \text{ where } I_a = [-\sqrt{10}, -\frac{\pi}{2}] \text{ and } I_b = [\frac{\pi}{2}, \sqrt{10}]$$

As x^2 and $\cos x$ take Oy as a symmetry axis, each zero z_b of f in I_b results in another zero $z_a = -z_b$ in I_a. Hence, from now on, we just need to examine on I_b.

Differentiating f gives:

$$f'(x) = 2x - 10\sin x$$

x is monotonically increasing on I_b, $\sin x$ is monotonically decreasing on I_b. It follows that f' is monotonically increasing on I_b, which means:

$$f'(\frac{\pi}{2}) = \pi - 10 \leq f'(x) \leq f'(\sqrt{10}) = 2\sqrt{10} - 10\sin\sqrt{10}$$

Combining with the fact that f' is continuous on I_b, according to Intermediate Value Theorem, f' has one zero in I_b. It follows that f has at most two zeros in I_b.

Let

$$g(x) = x - \frac{-10\cos x}{x^2} + 1 = x + \frac{10\cos x}{x^2} + 1$$

A fixed point of g is also a zero of f. *Try* applying fixed-point method on g with several p_0, we found two fixed points:

- $p_0 = \frac{\pi}{2}$ generates the following table:

n	p_n	n	p_n	n	p_n
0	1.570 796 33	4	1.953 548 67	8	1.968 593 28
1	2.570 796 33	5	1.974 930 8	9	1.968 974 39
2	2.297 575 29	6	1.966 757 33	10	1.968 836 22
3	2.038 843 43	7	1.969 648 71	11	1.968 886 24

- $p_0 = -\sqrt{10}$ generates the following table:

n	p_n
0	−3.162 277 66
1	−3.162 063 73
2	−3.161 989 49

The second fixed point is interesting. It is indeed a fixed point of g, a zero of f, but it belongs to I_a. Due to the symmetry property, we conclude that f has 4 zeros: $\pm 1.968\,89$ and $\pm 3.161\,99$.

Exercise 0.1.14

Use a fixed-point iteration method to determine a solution accurate to within 10^{-4} for $x = \tan x$, for $x \in [4, 5]$.

Solution 0.1.14

Let

$$g(x) = x - \sqrt[3]{\frac{\tan x}{x}} + 1$$

A fixed point p of g is also a solution of the problem. Applying fixed-point method on g generates the following table:

n	p_n	n	p_n	n	p_n
0	4	4	4.495 344 11	8	4.493 529 55
1	4.338 504 07	5	4.492 429 47	9	4.493 349 61
2	4.500 975 94	6	4.493 893 01	10	4.493 439 23
3	4.489 378 73	7	4.493 167 7		

So $p \approx 4.493\,44$ is a solution of the problem in $[4, 5]$.

Exercise 0.1.15

Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $2 \sin \pi x + x = 0$ on $[1, 2]$. Use $p_0 = 1$.

Solution 0.1.15

Consider f:

$$\begin{aligned} & f(x) = 0 \\ \iff \quad & 2 \sin \pi x = -x \\ \iff \quad & \pi x = \arcsin -0.5x + k2\pi \ (k \in \mathbb{N}) \\ \iff \quad & x = \frac{\arcsin -0.5x}{\pi} + 2k \end{aligned}$$

Let

$$g(x) = \frac{\arcsin -0.5x}{\pi} + 2$$

arcsin is chosen as it "behaves" nicer than normal sin. Since arcsin returns values in principal branch $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we need to use $k = 1$ to shift the value to cover $[1, 2]$.

A fixed point p of g is also a solution of the problem. Applying fixed-point method on g generates the following table:

n	p_n	n	p_n
0	1	3	1.696 498
1	1.833 333 33	4	1.677 657 06
2	1.630 869 25	5	1.683 240 99

So $p \approx 1.683$ is a solution of the problem in $[1, 2]$.

Exercise 0.1.16

Let A be a given positive constant and $g(x) = 2x - Ax^2$.

a) Show that if fixed-point iteration converges to a nonzero limit, then the limit is $p = {}^1/_A$, so the inverse of a number can be found using only multiplications and subtractions.

b) Find an interval about ${}^1/_A$ for which fixed-point iteration converges, provided p_0 is in that interval.

Solution 0.1.16

a) If fixed-point iteration converges to a nonzero limit p, then:

$$\begin{aligned} p &= \lim_{n\to\infty} p_n \\ &= \lim_{n\to\infty} g(p_{n-1}) \\ &= \lim_{n\to\infty} \left(2p_{n-1} - Ap_{n-1}^2\right) \\ &= 2p - Ap^2 \\ \iff p &= Ap^2 \iff p = \frac{1}{A} \end{aligned}$$

b) We try to find $\delta > 0$ such that fixed-point method converges on $I = [{}^1/_A - \delta, {}^1/_A + \delta]$ using Fixed Point Theorem.

The condition that g is continuous on I is satisfied with any δ.

Consider g:

$$g(x) = -Ax^2 + 2x = -A\left(x - \frac{1}{A}\right)^2 + \frac{1}{A}$$

So $x = \frac{1}{A}$ is the axis of symmetry for g.

Differentiating g gives:

$$g'(x) = 2 - 2Ax$$

It follows that:

- $g'(x) < 0 \iff x > \frac{1}{A}$
- $g'(x) = 0 \iff x = \frac{1}{A}$
- $g'(x) > 0 \iff x < \frac{1}{A}$

Combining with the fact that $x = \frac{1}{A}$ is the symmetry axis of g gives:

$$g\left(\frac{1}{A}+\delta\right) = g\left(\frac{1}{A}-\delta\right) = g\left(\frac{1}{A}\pm\delta\right) \leq g(x) \leq g\left(\frac{1}{A}\right) \; \forall x \in I$$
$$\iff \frac{2}{A} - A\delta^2 \leq g(x) \leq \frac{1}{A}$$

Then, to satisfy the condition that $g(x) \in I \, \forall x \in I$, δ must satisfy the following:

$$\begin{aligned} & \frac{2}{A} - A\delta^2 \geq \frac{1}{A} - \delta \\ \iff \quad & (A\delta)^2 - A\delta - 1 \leq 0 \\ \iff \quad & 0 < \delta \leq \frac{1+\sqrt{5}}{2A} \text{ (as } \delta > 0\text{)} \end{aligned} \tag{1}$$

Consider g'. g' is monotonically decreasing on $\mathbb{R}$, so:

$$g'\left(\frac{1}{A}-\delta\right) = 2A\delta \geq g'(x) \geq g'\left(\frac{1}{A}-\delta\right) = -2A\delta$$
$$\iff \left|g'(x)\right| \leq 2A\delta \text{ (equal sign only at either end)} \tag{2}$$

Then, to satisfy the condition that $\left|g'(x)\right| < 1 \, \forall x \in I_{open} = (1/A - \delta, 1/A + \delta)$, δ must satisfy the following:

$$2A\delta \leq 1 \iff \delta \leq \frac{1}{2A}$$

From (1) and (2):

$$0 < \delta < \frac{1}{2A}$$

As all the conditions needed for Fixed Point Theorem hold, we conclude that for any $\delta \in (0, \frac{1}{2A}]$, applying fixed-point method on g with $p_0 \in I$ converges to the fixed point.

Exercise 0.1.17

Find a function g defined on $[0, 1]$ that satisfies none of the hypotheses of Theorem 2.3 but still has a unique fixed point on $[0, 1]$.

Solution 0.1.17

Let $I = [0, 1]$, $g = \dfrac{1}{x + 0.5}$.

Consider g. g is defined on $\mathbb{R} \setminus \{-0.5\}$, so it is defined on I.

$g(x) > 1\, \forall x \in [-0.5, 0.5]$, so the condition that $g(x) \in I\, \forall x \in I$ does not hold.

Differentiating g gives:

$$g'(x) = -\frac{1}{(x + 0.5)^2} < -1 \iff x \in (-1.5, 0.5) \setminus \{-0.5\}$$

So the condition that $\left|g'(x)\right| < 1\, \forall x \in I$ does not hold.

Yet, g has a fixed point at $x = \dfrac{\sqrt{17} - 1}{4}$.

Exercise 0.1.18

a) Show that Theorem 2.2 is true if the inequality $\left|g'(x)\right| \leq k$ is replaced by $g'(x) \leq k$, for all $x \in (a, b)$. [Hint: Only uniqueness is in question.]

b) Show that Theorem 2.3 may not hold if inequality $\left|g'(x)\right| \leq k$ is replaced by $g'(x) \leq k$.

Solution 0.1.18

a) Where the fuck is Theorem 2.2 in the fucking book?

b) In the proof of Theorem 2.3, if $\left|g'(x) \leq k\right|$ is replaced with $g'(x) \leq k$, then there is a chance that $g'(\xi) = -1$. In that case, the assumption is no longer a contradiction, therefore the proof is invalid, and the theorem doesn't hold.

Exercise 0.1.19

a) Use Theorem 2.4 (Định lí 2.5 in the accompanying Lectures.pdf) to show that the sequence defined by:

$$x_n = \frac{1}{2} x_{n-1} + \frac{1}{x_{n-1}}, \text{ for } n \leq 1$$

converges to $\sqrt{2}$ whenever $x_0 > \sqrt{2}$.

b) Use the fact that $0 < (x_0 - \sqrt{2})^2$ whenever $x_0 \neq \sqrt{2}$ to show that if $0 < x_0 < \sqrt{2}$, then $x_1 > \sqrt{2}$.

c) Use the above results to show that the sequence in (a) converges to $\sqrt{2}$ whenever $x_0 > 0$.

Solution 0.1.19

a) Let g be the function that generates the sequence $\{x_n\}$:

$$g(x) = \frac{x}{2} + \frac{1}{x} = \frac{x^2+2}{2x}$$
$$\Rightarrow g'(x) = \frac{1}{2} - \frac{1}{x^2} = \frac{x^2-2}{2x^2}$$

Consider $I = [\sqrt{2}, b]$, for any $b > \sqrt{2}$. It is clear that g and g' exists on I. Since $g'(x) \leq 0\,\forall x \in I$, g is monotonically increasing on I.

Consider g'. x^2 is strictly increasing on I, so g' is strictly decreasing on I, therefore:

$$\frac{1}{2} > g'(x) \leq g'(\sqrt{2}) = 0\,\forall x \in I$$
$$\Rightarrow \left|g'(x)\right| < 1\,\forall x \in I$$

Let

$$f(x) = g(x) - x = \frac{1}{x} - \frac{x}{2}$$

$1/x$ is strictly decreasing on I, and so is $-x$. Therefore, f is strictly decreasing on I, so:

$$f(\sqrt{2}) = 0 \leq f(x)\,\forall x \in I$$

In other words, $g(x) \leq x\,\forall x \in I$. It means that for any b, $g(b) < b$. Combining with the fact that $g(\sqrt{2}) = \sqrt{2}$, it is guaranteed that:

$$g(x) \in I\,\forall x \in I$$

All the conditions of Theorem 2.4 hold, so we can apply it here: for any $x_0 \in I$, applying fixed-point method on g converges to the unique fixed point in I, using any $x_0 \in I$.

Trivially, $\sqrt{2}$ is a fixed point of g, therefore it must be the unique fixed point on I.

We can conclude that for any $x_0 > \sqrt{2}$, the sequence converges to $\sqrt{2}$.

b) When $0 < x < \sqrt{2}$, $g'(x) < 0$, which means g is monotonically decreasing. Applying this on $0 < x_0 < \sqrt{2}$ gives:

$$x_1 = g(x_0) > g(\sqrt{2}) = \sqrt{2}$$

c) We have:

- If $x_0 > \sqrt{2}$: proven.
- If $x_0 = \sqrt{2}$: it is exactly the fixed point.
- If $0 < x_0 < \sqrt{2}$: $x_1 = g(x_0) > \sqrt{2}$, then from x_1 onwards, the sequence converges to $\sqrt{2}$, as proven with the case $x_0 > \sqrt{2}$.

Therefore, we can conclude that the sequence converges to $\sqrt{2}$ whenever $x_0 > 0$.

Exercise 0.1.20

a) Show that if A is any positive number, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \text{ for } n \geq 1$$

converges to $\sqrt{A}$ whenever $x_0 > 0$.

b) What happens if $x_0 < 0$?

Solution 0.1.20

a) Let

$$\begin{aligned} g(x) &= \frac{x}{2} + \frac{A}{2x} = \frac{x^2 + A}{2x} \\ \Rightarrow g'(x) &= \frac{1}{2} - \frac{A}{2x^2} = \frac{x^2 - A}{2x^2} \end{aligned}$$

Trivially, we can find out that $\sqrt{A}$ is a fixed point of g.

Let

$$\begin{aligned} f(x) = g(x) - x &= \frac{A}{2x} - \frac{x}{2} = \frac{A - x^2}{2x} \\ \Rightarrow f'(x) &= -\frac{A}{2x^2} - \frac{1}{2} = -\frac{x^2 + A}{2x^2} \end{aligned}$$

Since $f'(x) < 0 \, \forall x \neq 0$, $f(x)$ is monotonically increasing when $x > 0$.

Consider the sign of g':

- $g'(x) < 0 \iff |x| < \sqrt{A}$
- $g'(x) = 0 \iff |x| = \sqrt{A}$
- $g'(x) > 0 \iff |x| > \sqrt{A}$

If $x > \sqrt{A}$, then:

- $g' > 0$, which means g is monotonically increasing. It follows that:

$$g(x) > g(\sqrt{A}) = \sqrt{A}$$

- $f(x) < f(\sqrt{A}) = 0$, which means $g(x) < x$, making $\{x_n\}$ a decreasing sequence.

From both of the above, we know that $\{x_n\}$ is a lower-bounded decreasing sequence, and therefore must converge:

$$\begin{aligned} x &= \lim_{n\to\infty} x_n \\ &= \lim_{n\to\infty} g(x_{n-1}) \\ &= \lim_{n\to\infty} \frac{x_{n-1}}{2} + \frac{A}{2x_{n-1}} \\ &= \frac{x}{2} + \frac{A}{2x} \\ \iff x &= \sqrt{A} \end{aligned}$$

So, for all $x_0 > \sqrt{A}$, the sequence converges to $\sqrt{A}$.

If $x = \sqrt{A}$, then $g(x) = x = \sqrt{A}$. Hence $x_n = \sqrt{A}\,\forall n \geq 0$. So, for $x_0 = \sqrt{A}$, the sequence converges to $\sqrt{A}$.

If $0 < x < \sqrt{A}$, then $g' < 0$, which means g is monotonically decreasing. It follows that:

$$g(x) > g(\sqrt{A}) = \sqrt{A}$$

So, for $0 < x_0 < \sqrt{A}$, $x_1 = g(x_0) > \sqrt{A}$, then from x_1 onwards, the sequence converges to $\sqrt{A}$, as proven with the case $x_0 > \sqrt{A}$.

We can conclude that the sequence $\{x_n\}$ converges to $\sqrt{2}$ whenever $x_0 > 0$.

b) If $x_0 < 0$, then similar to the above proof, we conclude that the sequence converges to $-\sqrt{A}$.

Exercise 0.1.21

Replace the assumption in Theorem 2.4 that “a positive number $k < 1$ exists with $\left|g(x)\right| \leq k$” with “g satisfies a Lipschitz condition on the interval $[a, b]$ with Lipschitz constant $L < 1$” (See Exercise 27, Section 1.1.) Show that the conclusions of this theorem are still valid.

Solution 0.1.21

g satisfies a Lipschitz condition on the interval $[a, b]$ with Lipschitz constant $L < 1$ means that:

$$\frac{g(x_1) - g(x_2)}{x_1 - x_2} \leq L \,\forall x_1, x_2 \in [a, b] \tag{*}$$

In the proof of Theorem 2.4, we see that:

$$|p - p_n| = \left|g(p) - g(p_{n-1})\right|$$

From the previous section of the proof, we already proved that p and p_{n-1} is in $[a, b]$. Applying (*) with $x_1 = p$, $x_2 = p_{n-1}$ gives:

$$|p - p_n| = \left|g(p) - g(p_{n-1})\right| \leq L|p - p_{n-1}|$$

Then the proof proceeds normally, replacing k with L.

Exercise 0.1.22

Suppose that g is continuously differentiable on some interval (c, d) that contains the fixed point p of g. Show that if $\left|g'(p)\right| < 1$, then there exists a $\delta > 0$ such that if $|p_0 - p| \leq \delta$, then the fixed-point iteration converges.

Solution 0.1.22

Since p is a fixed point in (c, d) of g, $g(p) = p$.

Since g' is continuous at p, according to the definition of continuity and limit, for every $\varepsilon > 0$, there exist $\delta > 0$ such that:

$$\begin{aligned} &\left|g'(x) - g'(p)\right| < \varepsilon \,\forall x \in D = [p - \delta, p + \delta] \\ \iff\ & g'(x) \in E = [g'(p) - \varepsilon, g'(p) + \varepsilon] \,\forall x \in D \end{aligned}$$

We can always choose a ε such that $E \subset (-1, 1)$. Then the proof proceeds normally, replacing $[a, b]$ with E.

Exercise 0.1.23

An object falling vertically through the air is subjected to viscous resistance as well as to the force of gravity. Assume that an object with mass m is dropped from a height s_0 and that the height of the object after t seconds is:

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}(1 - e^{-kt/m})$$

where $g = 32.17\,\text{ft/s}^2$ and k represents the coefficient of air resistance in lb/s. Suppose $s_0 = 300\,\text{ft}$, $m = 0.25\,\text{lb}$, and $k = 0.1\,\text{lb/s}$. Find, to within $0.01\,\text{s}$, the time it takes this quarter-pounder to hit the ground.

Solution 0.1.23

Replacing symbols in $s(t)$ with number gives:

$$s(t) = 501.0625 - 80.425t - 201.0625e^{-0.4t}$$

Let

$$g(t) = \frac{1}{80.425}(501.0625 - 201.0625e^{-0.4t})$$

A fixed point p of g is also a root of $s(t) = 0$, which is the time it takes the quarter-pounder to hit the ground.

Applying fixed-point method on g with $p_0 = 3$ generates the following table:

n	p_n	n	p_n
0	3	3	5.998 865 94
1	5.477 197 87	4	6.003 285 61
2	5.950 637 4		

We conclude that it takes approximately 6.003 s for the quarter-pounder to hit the ground.

Exercise 0.1.24

Let $g \in C^1[a,b]$ and p be in (a,b) with $g(p) = p$ and $\left|g'(p)\right| > 1$. Show that there exists a $\delta > 0$ such that if $0 < |p_0 - p| < \delta$, then $|p_0 - p| < |p_1 - p|$. Thus, no matter how close the initial approximation p_0 is to p, the next iterate p_1 is farther away, so the fixed-point iteration does not converge if $p_0 \neq p$.

Solution 0.1.24

This problem is similar to Exercise 22.

Since g' is continuous at p, according to the definition of continuity and limit, for every $\varepsilon > 0$, there exist $\delta > 0$ such that:

$$\begin{aligned} &\left|g'(x) - g'(p)\right| < \varepsilon \, \forall x \in D = [p - \delta, p + \delta] \\ \iff &g'(x) \in E = [g'(p) - \varepsilon, g'(p) + \varepsilon] \, \forall x \in D \end{aligned}$$

We can always choose a ε such that $E \subset (1, \infty)$.

If $p_0 \in D$, then according to Mean Value Theorem, there exist a $\xi \in D$ such that:

$$|p_1 - p| = \left|g(p_0) - g(p)\right| = \left|g'(\xi)\right||p_0 - p| > |p_0 - p|$$