Phương pháp tính MAT1099

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Chapter 1

Error analysis

Exercise 1

Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

Solution 1

as hey

Chapter 2

Solution approximation

2.1 The Bisection Method

Exercise 1

Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

Solution 1

$f(0) = -1$ and $f(1) \approx 0.459\,697\,694$ have the opposite signs, so there's a root in $[0, 1]$.

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	$-0.170\,475\,781$
2	0.5	1	0.75	$0.134\,336\,535$
3	0.5	0.75	0.625	$-0.020\,393\,704$

So $p_3 = 0.625$.

Exercise 2

Let $f(x) = 3(x+1)(x - \frac{1}{2})(x-1)$. Use the bisection method to find p_3 in the following intervals:

(a) $[-2, 1.5]$

(b) $[-1.5, 2.5]$

Solution 2

(a) $f(-2) = -22.5$ and $f(1.5) = 3.75$ have the opposite signs, so there's a root in $[-2, 1.5]$.

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-2	1.5	-0.25	2.109 375
2	-2	-0.25	-1.125	$-1.294\,921\,875$
3	-1.125	-0.25	-0.6875	1.878 662 109

So $p_3 = -0.6875$.

(b) $f(-1.25) = -2.953\,125$ and $f(2.5) = 31.5$ have the opposite signs, so there's a root in $[-1.25, 2.5]$.

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-1.5	2.5	0.5	0

The solution is found in the first iteration so p_3 doesn't exist.

Exercise 3

Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ in the following intervals:

(a) $[0, 1]$ (b) $[1, 3.2]$ (c) $[3.2, 4]$

Solution 3

(a) $f(0) = -6$ and $f(1) = 2$ have the opposite signs, so there's a root in $[0, 1]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	-0.625
2	0.5	1	0.75	0.984 375
3	0.5	0.75	0.625	0.259 766
4	0.5	0.625	0.5625	$-0.161\,865$
5	0.5625	0.625	0.593 75	0.054 047
6	0.5625	0.593 75	0.578 125	$-0.052\,624$
7	0.578 125	0.593 75	0.585 937 5	0.001 031

So $p \approx 0.5859$.

(b) $f(1) = 2$ and $f(3.2) = -0.112$ have the opposite signs, so there's a root in $[1, 3.2]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{3.2 - 1}{2^n} < 10^{-2} \iff n \geq 8$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	3.2	2.1	1.791
2	2.1	3.2	2.65	0.552 125
3	2.65	3.2	2.925	0.085 828
4	2.925	3.2	3.0625	−0.054 443
5	2.925	3.0625	2.993 75	0.006 328
6	2.993 75	3.0625	3.028 125	−0.026 521
7	2.993 75	3.028 13	3.010 938	−0.010 697
8	2.993 75	3.010 938	3.002 344	−0.002 333

So $p \approx 3.0023$.

(c) $f(3.2) = -0.112$ and $f(4) = 2$ have the opposite signs, so there's a root in $[3.2, 4]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{4 - 3.2}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	3.2	4	3.6	0.336
2	3.2	3.6	3.4	−0.016
3	3.4	3.6	3.5	0.125
4	3.4	3.5	3.45	0.046 125
5	3.4	3.45	3.425	0.013 016
6	3.4	3.425	3.4125	−0.001 998
7	3.4125	3.425	3.418 75	0.005 382

So $p \approx 3.4188$.

Exercise 4

Use the Bisection method to find solutions accurate to within 10^{-2} for $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ for the following intervals:

(a) $[-2, -1]$ (b) $[0, 2]$ (c) $[2, 3]$ (d) $[-1, 0]$

Solution 4

(a) $f(-2) = 12$ and $f(-1) = -1$ have the opposite signs, so there's a root in $[-2, -1]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{-1-(-2)}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-2	-1	-1.5	0.8125
2	-1.5	-1	-1.25	$-0.902\,344$
3	-1.5	-1.25	-1.375	$-0.288\,818$
4	-1.5	-1.375	-1.4375	$0.195\,328$
5	-1.4375	-1.375	$-1.406\,25$	$-0.062\,667$
6	-1.4375	$-1.406\,25$	$-1.421\,875$	$0.062\,263$
7	$-1.421\,875$	$-1.406\,25$	$-1.414\,063$	$-0.001\,208$

So $p \approx -1.4141$.

(b) $f(0) = 4$ and $f(2) = -4$ have the opposite signs, so there's a root in $[0, 2]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{2-0}{2^n} < 10^{-2} \iff n \geq 8$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	2	1	3
2	1	2	1.5	-0.6875
3	1	1.5	1.25	$1.285\,156$
4	1.25	1.5	1.375	$0.312\,744$
5	1.375	1.5	1.4375	$-0.186\,508$
6	1.375	1.4375	$1.406\,25$	$0.063\,676$
7	$1.406\,25$	1.4375	$1.421\,875$	$-0.061\,318$
8	$1.406\,25$	$1.421\,875$	$1.414\,063$	$0.001\,208$

So $p \approx 1.4141$.

(c) $f(2) = -4$ and $f(3) = 7$ have the opposite signs, so there's a root in $[2, 3]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{3-2}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	2	3	2.5	−3.1875
2	2.5	3	2.75	0.347 656
3	2.5	2.75	2.625	−1.757 568
4	2.625	2.75	2.6875	−0.795 639
5	2.6875	2.75	2.718 75	−0.247 466
6	2.718 75	2.75	2.734 375	0.044 125
7	2.718 75	2.734 375	2.726 563	−0.103 151

So $p \approx 2.7266$.

(d) $f(-1) = -1$ and $f(0) = 4$ have the opposite signs, so there's a root in $[-1, 0]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{0-(-1)}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	−1	0	−0.5	1.3125
2	−1	−0.5	−0.75	−0.089 844
3	−0.75	−0.5	−0.625	0.578 369
4	−0.75	−0.625	−0.6875	0.232 681
5	−0.75	−0.6875	−0.718 75	0.068 086
6	−0.75	−0.718 75	−0.734 375	−0.011 768
7	−0.734 375	−0.718 75	−0.726 563	0.027 943

So $p \approx -0.7266$.

Exercise 5

Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems:

(a) $x - 2^{-x} = 0$, $x \in [0, 1]$

(b) $e^x - x^2 + 3x - 2 = 0$, $x \in [0, 1]$

(c) $2x \cos 2x - (x + 1)^2 = 0$, $x \in [-3, -2]$

(d) $x \cos x - 2x^2 + 3x - 1 = 0$, $x \in [0.2, 0.3]$

Solution 5

(a) $f(0) = -1$ and $f(1) = 0.5$ have the opposite signs, so there's a root in $[0, 1]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	−0.207 106 781
2	0.5	1	0.75	0.155 396 442
3	0.5	0.75	0.625	−0.023 419 777
4	0.625	0.75	0.6875	0.066 571 094
5	0.625	0.6875	0.656 25	0.021 724 521
6	0.625	0.656 25	0.640 625	−0.000 810 008
7	0.640 625	0.656 25	0.648 437 5	0.010 466 611
8	0.640 625	0.648 437 5	0.644 531 25	0.004 830 646
9	0.640 625	0.644 531 25	0.642 578 125	0.002 010 906
10	0.640 625	0.642 578 125	0.641 601 562	0.000 600 596
11	0.640 625	0.641 601 562	0.641 113 281	−0.000 104 669
12	0.641 113 281	0.641 601 562	0.641 357 422	0.000 247 972
13	0.641 113 281	0.641 357 422	0.641 235 352	0.000 071 654
14	0.641 113 281	0.641 235 352	0.641 174 316	−0.000 016 507
15	0.641 174 316	0.641 235 352	0.641 204 834	0.000 027 573
16	0.641 174 316	0.641 204 834	0.641 189 575	0.000 005 533
17	0.641 174 316	0.641 189 575	0.641 181 946	−0.000 005 487

So $p \approx -0.641\,182$.

(b) $f(0) = -1$ and $f(1) = e$ have the opposite signs, so there's a root in $[0, 1]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	0.898 721 271
2	0	0.5	0.25	−0.028 474 583
3	0.25	0.5	0.375	0.439 366 415
4	0.25	0.375	0.3125	0.206 681 691
5	0.25	0.3125	0.281 25	0.089 433 196
6	0.25	0.281 25	0.265 625	0.030 564 234
7	0.25	0.265 625	0.257 812 5	0.001 066 368
8	0.25	0.257 812 5	0.253 906 25	−0.013 698 684
9	0.253 906 25	0.257 812 5	0.255 859 375	−0.006 314 807
10	0.255 859 375	0.257 812 5	0.256 835 938	−0.002 623 882
11	0.256 835 938	0.257 812 5	0.257 324 219	−0.000 778 673
12	0.257 324 219	0.257 812 5	0.257 568 359	0.000 143 868
13	0.257 324 219	0.257 568 359	0.257 446 289	−0.000 317 397
14	0.257 446 289	0.257 568 359	0.257 507 324	−0.000 086 763
15	0.257 507 324	0.257 568 359	0.257 537 842	0.000 028 553
16	0.257 507 324	0.257 537 842	0.257 522 583	−0.000 029 105
17	0.257 522 583	0.257 537 842	0.257 530 212	−0.000 000 276

So $p \approx 0.257\,53$.

(c) $f(-3) \approx -9.761\,021\,72$ and $f(-2) \approx 1.614\,574\,483$ have the opposite signs, so there's a root in $[-3, -2]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{-2-(-3)}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	−3	−2	−2.5	−3.668 310 93
2	−2.5	−2	−2.25	−0.613 918 903
3	−2.25	−2	−2.125	0.630 246 832
4	−2.25	−2.125	−2.1875	0.038 075 532
5	−2.25	−2.1875	−2.218 75	−0.280 836 176
6	−2.218 75	−2.1875	−2.203 125	−0.119 556 815
7	−2.203 125	−2.1875	−2.195 312 5	−0.040 278 514

n	a_n	b_n	p_n	$f(p_n)$
8	-2.1953125	-2.1875	-2.19140625	-0.000985195
9	-2.19140625	-2.1875	-2.18945312	0.018574337
10	-2.19140625	-2.18945312	-2.19042969	0.008801851
11	-2.19140625	-2.19042969	-2.19091797	0.003910147
12	-2.19140625	-2.19091797	-2.19116211	0.00146293
13	-2.19140625	-2.19116211	-2.19128418	0.000238981
14	-2.19140625	-2.19128418	-2.19134521	-0.000373078
15	-2.19134521	-2.19128418	-2.1913147	-0.000067041
16	-2.1913147	-2.19128418	-2.19129944	0.000085972

So $p \approx -2.191299$.

(d) $f(0.2) \approx -0.283986684$ and $f(0.3) \approx 0.006600946$ have the opposite signs, so there's a root in $[0.2, 0.3]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{0.3 - 0.2}{2^n} < 10^{-5} \iff n \geq 14$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0.2	0.3	0.25	-0.132771895
2	0.25	0.3	0.275	-0.061583071
3	0.275	0.3	0.2875	-0.027112719
4	0.2875	0.3	0.29375	-0.010160959
5	0.29375	0.3	0.296875	-0.001756232
6	0.296875	0.3	0.2984375	0.002428306
7	0.296875	0.2984375	0.29765625	0.000337524
8	0.296875	0.29765625	0.297265625	-0.000708983
9	0.297265625	0.29765625	0.297460938	-0.000185637
10	0.297460938	0.29765625	0.297558594	0.000075967
11	0.297460938	0.297558594	0.297509766	-0.000054829
12	0.297509766	0.297558594	0.29753418	0.00001057
13	0.297509766	0.29753418	0.297521973	-0.000022129
14	0.297521973	0.29753418	0.297528076	-0.000005779

So $p \approx 0.297528$.

Exercise 6

Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems:

(a) $3x - e^x = 0,\ x \in [1, 2]$

(b) $2x + 3\cos x - e^x = 0,\ x \in [0, 1]$

(c) $x^2 - 4x + 4 - \ln x = 0,\ x \in [1, 2]$

(d) $x + 1 - 2\sin \pi x = 0,\ x \in [0, 0.5]$

Solution 6

(a) $f(1) \approx 0.281\,718\,172$ and $f(2) \approx -1.389\,056\,099$ have the opposite signs, so there's a root in $[1, 2]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \le \frac{2-1}{2^n} < 10^{-5} \iff n \ge 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	0.018 310 93
2	1.5	2	1.75	−0.504 602 676
3	1.5	1.75	1.625	−0.203 419 037
4	1.5	1.625	1.5625	−0.083 233 182
5	1.5	1.5625	1.531 25	−0.030 203 153
6	1.5	1.531 25	1.515 625	−0.005 390 404
7	1.5	1.515 625	1.507 812 5	0.006 598 107
8	1.507 812 5	1.515 625	1.511 718 75	0.000 638 447
9	1.511 718 75	1.515 625	1.513 671 88	−0.002 367 313
10	1.511 718 75	1.513 671 88	1.512 695 31	−0.000 862 268
11	1.511 718 75	1.512 695 31	1.512 207 03	−0.000 111 37
12	1.511 718 75	1.512 207 03	1.511 962 89	0.000 263 674
13	1.511 962 89	1.512 207 03	1.512 084 96	0.000 076 186
14	1.512 084 96	1.512 207 03	1.512 146	−0.000 017 584
15	1.512 084 96	1.512 146	1.512 115 48	0.000 029 303
16	1.512 115 48	1.512 146	1.512 130 74	0.000 005 86
17	1.512 130 74	1.512 146	1.512 138 37	−0.000 005 861

So $p \approx 1.512\,138$.

(b) $f(0) = 2$ and $f(1) \approx 0.902\,625\,089$ have the same sign, so there's no guaranteed root in $[0, 1]$.

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	1.984 026 41
2	0.5	1	0.75	1.578 066 59
3	0.75	1	0.875	1.274 115 28

n	a_n	b_n	p_n	$f(p_n)$
4	0.875	1	0.9375	1.096 825 77
5	0.9375	1	0.968 75	1.001 841 5
6	0.968 75	1	0.984 375	0.952 762 63
7	0.984 375	1	0.992 187 5	0.927 826 236
8	0.992 187 5	1	0.996 093 75	0.915 258 762
9	0.996 093 75	1	0.998 046 875	0.908 950 201
10	0.998 046 875	1	0.999 023 438	0.905 789 714
11	0.999 023 438	1	0.999 511 719	0.904 207 919
12	0.999 511 719	1	0.999 755 859	0.903 416 633
13	0.999 755 859	1	0.999 877 93	0.903 020 894
14	0.999 877 93	1	0.999 938 965	0.902 822 999
15	0.999 938 965	1	0.999 969 482	0.902 724 046
16	0.999 969 482	1	0.999 984 741	0.902 674 568
17	0.999 984 741	1	0.999 992 371	0.902 649 829
18	0.999 992 371	1	0.999 996 185	0.902 637 459
19	0.999 996 185	1	0.999 998 093	0.902 631 274
20	0.999 998 093	1	0.999 999 046	0.902 628 182

As $f(p_20) \approx 0.902\,628\,182 > 0$, the method failed.

(c) $f(1) = 1$ and $f(2) = -0.693\,147\,181$ have the opposite signs, so there's a root in $[1, 2]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{2-1}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	−0.155 465 108
2	1	1.5	1.25	0.339 356 449
3	1.25	1.5	1.375	0.072 171 269
4	1.375	1.5	1.4375	−0.046 499 244
5	1.375	1.4375	1.406 25	0.011 612 476
6	1.406 25	1.4375	1.421 875	−0.017 747 908
7	1.406 25	1.421 875	1.414 062 5	−0.003 144 013
8	1.406 25	1.414 062 5	1.410 156 25	0.004 215 136
9	1.410 156 25	1.414 062 5	1.412 109 38	0.000 530 79
10	1.412 109 38	1.414 062 5	1.413 085 94	−0.001 307 804
11	1.412 109 38	1.413 085 94	1.412 597 66	−0.000 388 805
12	1.412 109 38	1.412 597 66	1.412 353 52	0.000 070 918
13	1.412 353 52	1.412 597 66	1.412 475 59	−0.000 158 962
14	1.412 353 52	1.412 475 59	1.412 414 55	−0.000 044 027

n	a_n	b_n	p_n	$f(p_n)$
15	1.412 353 52	1.412 414 55	1.412 384 03	0.000 013 444
16	1.412 384 03	1.412 414 55	1.412 399 29	−0.000 015 292
17	1.412 384 03	1.412 399 29	1.412 391 66	−0.000 000 924

So $p \approx 1.412\,392$.

(d) $f(0) = 1$ and $f(1) = -0.5$ have the opposite signs, so there's a root in $[0, 0.5]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{0.5 - 0}{2^n} < 10^{-5} \iff n \geq 16$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	0.5	0.25	−0.164 213 562
2	0	0.25	0.125	0.359 633 135
3	0.125	0.25	0.1875	0.076 359 534
4	0.1875	0.25	0.218 75	−0.050 036 568
5	0.1875	0.218 75	0.203 125	0.011 726 391
6	0.203 125	0.218 75	0.210 937 5	−0.019 525 681
7	0.203 125	0.210 937 5	0.207 031 25	−0.003 990 833
8	0.203 125	0.207 031 25	0.205 078 125	0.003 845 166
9	0.205 078 125	0.207 031 25	0.206 054 688	−0.000 078 51
10	0.205 078 125	0.206 054 688	0.205 566 406	0.001 881 912
11	0.205 566 406	0.206 054 688	0.205 810 547	0.000 901 347
12	0.205 810 547	0.206 054 688	0.205 932 617	0.000 411 33
13	0.205 932 617	0.206 054 688	0.205 993 652	0.000 166 388
14	0.205 993 652	0.206 054 688	0.206 024 17	0.000 043 934
15	0.206 024 17	0.206 054 688	0.206 039 429	−0.000 017 289
16	0.206 024 17	0.206 039 429	0.206 031 799	0.000 013 322

So $p \approx 0.206\,032$.

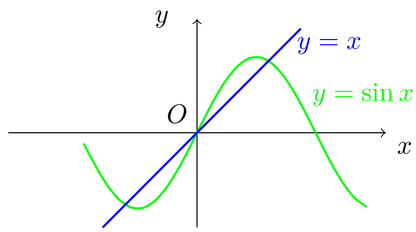
Exercise 7

(a) Sketch the graphs of $y = x$ and $y = 2\sin x$.

(b) Use the Bisection method to find an approximation to within 10^5 to the first positive value of x with $x = 2\sin x$.

Solution 7

(a) Graph of $y = x$ and $y = 2\sin x$ is as follow:



(b) According to the graph, the first positive root p of $f = x - 2\sin x$ is in $[\frac{\pi}{2}, \pi]$.

The number of iteration n needed to approximate p to within 10^{-5} in that interval is:

$$|p_n - p| \leq \frac{\pi - \frac{\pi}{2}}{2^n} < 10^{-5} \iff n \geq 18$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1.570 796 33	3.141 592 65	2.356 194 49	0.941 980 928
2	1.570 796 33	2.356 194 49	1.963 495 41	0.115 736 343
3	1.570 796 33	1.963 495 41	1.767 145 87	−0.194 424 693
4	1.767 145 87	1.963 495 41	1.865 320 64	−0.048 560 033
5	1.865 320 64	1.963 495 41	1.914 408 02	0.031 319 893
6	1.865 320 64	1.914 408 02	1.889 864 33	−0.009 192 031
7	1.889 864 33	1.914 408 02	1.902 136 18	0.010 921 526
8	1.889 864 33	1.902 136 18	1.896 000 25	0.000 829 072
9	1.889 864 33	1.896 000 25	1.892 932 29	−0.004 190 408
10	1.892 932 29	1.896 000 25	1.894 466 27	−0.001 682 899
11	1.894 466 27	1.896 000 25	1.895 233 26	−0.000 427 471
12	1.895 233 26	1.896 000 25	1.895 616 76	0.000 200 661
13	1.895 233 26	1.895 616 76	1.895 425 01	−0.000 113 44
14	1.895 425 01	1.895 616 76	1.895 520 88	0.000 043 602
15	1.895 425 01	1.895 520 88	1.895 472 95	−0.000 034 921
16	1.895 472 95	1.895 520 88	1.895 496 92	0.000 004 34
17	1.895 472 95	1.895 496 92	1.895 484 93	−0.000 015 291
18	1.895 484 93	1.895 496 92	1.895 490 92	−0.000 005 476

So $p \approx 1.895\,491$.

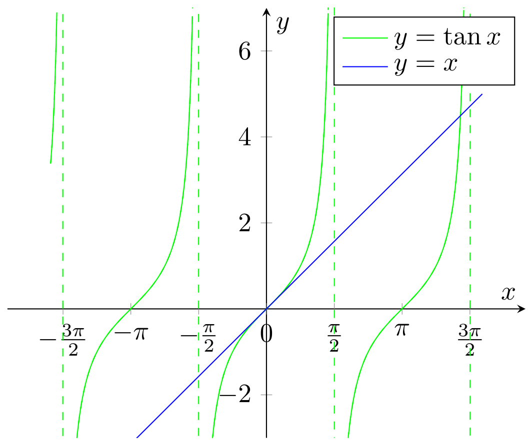
Exercise 8

(a) Sketch the graphs of $y = x$ and $y = \tan x$.

(b) Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $y = \tan x$.

Solution 8

(a) Graph of $y = x$ and $y = \tan x$ is as follow:



(b) According to the graph, the first positive root p of $f = x - \tan x$ is in $[\pi, \dfrac{3\pi}{2}]$.

The number of iteration n needed to approximate p to within 10^{-5} in that interval is:

$$|p_n - p| \leq \frac{\frac{3\pi}{2} - \pi}{2^n} < 10^{-5} \iff n \geq 18$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	3.141 592 65	4.712 388 98	3.926 990 82	2.926 990 82
2	3.926 990 82	4.712 388 98	4.319 689 9	1.905 476 34
3	4.319 689 9	4.712 388 98	4.516 039 44	−0.511 300 053
4	4.319 689 9	4.516 039 44	4.417 864 67	1.121 306 46
5	4.417 864 67	4.516 039 44	4.466 952 05	0.474 728 271
6	4.466 952 05	4.516 039 44	4.491 495 75	0.038 293 523
7	4.491 495 75	4.516 039 44	4.503 767 59	−0.219 861 735
8	4.491 495 75	4.503 767 59	4.497 631 67	−0.086 980 389

n	a_n	b_n	p_n	$f(p_n)$
9	4.491 495 75	4.497 631 67	4.494 563 71	−0.023 432 692
10	4.491 495 75	4.494 563 71	4.493 029 73	0.007 653 323
11	4.493 029 73	4.494 563 71	4.493 796 72	−0.007 833 371
12	4.493 029 73	4.493 796 72	4.493 413 22	−0.000 076 02
13	4.493 029 73	4.493 413 22	4.493 221 48	0.003 792 144
14	4.493 221 48	4.493 413 22	4.493 317 35	0.001 858 936
15	4.493 317 35	4.493 413 22	4.493 365 29	0.000 891 677
16	4.493 365 29	4.493 413 22	4.493 389 25	0.000 407 883
17	4.493 389 25	4.493 413 22	4.493 401 24	0.000 165 946
18	4.493 401 24	4.493 413 22	4.493 407 23	0.000 044 966

So $p \approx 4.493\,407$.

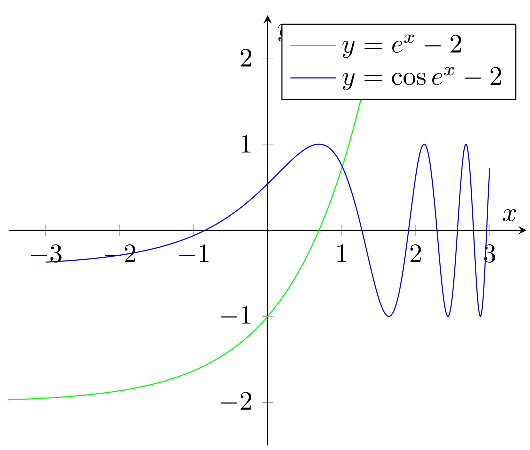
Exercise 9

(a) Sketch the graphs of $y = e^x - 2$ and $y = \cos e^x - 2$.

(b) Use the Bisection method to find an approximation to within 10^{-5} to a value in $[0.5, 1.5]$ with $e^x - 2 = \cos e^x - 2$.

Solution 9

(a) The graphs of the 2 functions are as follow:



(b) Let $f = e^x - 2 - \cos e^x - 2$. $f(0.5) \approx -1.290\,212$ and $f(1.5) \approx 3.271\,74$ have the opposite signs, so there's a root p of f in $[0.5, 1.5]$.

The number of iteration n needed to approximate p to within 10^{-5} in that interval is:

$$|p_n - p| \leq \frac{1.5 - 0.5}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0.5	1.5	1	−0.034 655 726
2	1	1.5	1.25	1.409 976 35
3	1	1.25	1.125	0.609 079 747
4	1	1.125	1.0625	0.266 982 288
5	1	1.0625	1.031 25	0.111 147 764
6	1	1.031 25	1.015 625	0.037 002 875
7	1	1.015 625	1.007 812 5	0.000 864 425
8	1	1.007 812 5	1.003 906 25	−0.016 972 716
9	1.003 906 25	1.007 812 5	1.005 859 38	−0.008 073 44
10	1.005 859 38	1.007 812 5	1.006 835 94	−0.003 609 335
11	1.006 835 94	1.007 812 5	1.007 324 22	−0.001 373 662
12	1.007 324 22	1.007 812 5	1.007 568 36	−0.000 254 92
13	1.007 568 36	1.007 812 5	1.007 690 43	0.000 304 677
14	1.007 568 36	1.007 690 43	1.007 629 39	0.000 024 859
15	1.007 568 36	1.007 629 39	1.007 598 88	−0.000 115 035
16	1.007 598 88	1.007 629 39	1.007 614 14	−0.000 045 089

So $p \approx 1.007\,614$.

Exercise 10

Let $f(x) = (x+2)(x+1)^2 x(x-1)^3(x-2)$. To which zero of f does the Bisection method converge when applied on the following intervals?

(a) $[-1.5, 2.5]$ (b) $[-0.5, 2.4]$ (c) $[-0.5, 3]$ (d) $[-3, -0.5]$

Solution 10

f has 5 zeros: ± 2, ± 1, 0.

(a) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	−1.5	2.5	0.5	0.527 343 75
2	−1.5	0.5	−0.5	−1.582 031 25
3	−0.5	0.5	0	0

So when applied on $[-1.5, 2.5]$, the Bisection method gives 0.

(b) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-0.5	2.4	0.95	0.001 398 666
2	-0.5	0.95	0.225	0.620 709 19

At $n = 2$, the interval shrinks to $[-0.5, 0.95]$. So when applied on $[-0.5, 2.4]$, the Bisection method gives 0.

(c) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-0.5	3	1.25	$-0.241\,012\,573$
2	1.25	3	2.125	15.235 282 5

At $n = 2$, the interval shrinks to $[1.25, 3]$. So when applied on $[-0.5, 3]$, the Bisection method gives 2.

(d) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-3	-0.5	-1.75	$-19.192\,428\,6$
2	-3	-1.75	-2.375	283.204 185

At $n = 2$, the interval shrinks to $[3, -1.75]$. So when applied on $[-3, -0.5]$, the Bisection method gives -2.

Exercise 11

Let $f(x) = (x + 2)(x + 1)x(x - 1)^3(x - 2)$. To which zero of f does the Bisection method converge when applied on the following intervals?

(a) $[-3, 2.5]$

(b) $[-2.5, 3]$

(c) $[-1.75, 1.5]$

(d) $[-1.5, -1.75]$

Solution 11

f has 5 zeros: ± 2, ± 1, 0.

(a) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-3	2.5	-0.25	$-1.441\,955\,57$
2	-0.25	2.5	1.125	$-0.012\,767\,315$
3	1.125	2.5	1.8125	$-1.954\,572\,48$

At $n = 3$, the interval shrinks to $[1.125, 2.5]$. So when applied on $[-3, 2.5]$, the Bisection method gives 2.

(b) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-2.5	3	0.25	0.519 104 004
2	-2.5	0.25	-1.125	3.689 754 01
3	-2.5	-1.125	-1.8125	23.420 173 2

At $n = 3$, the interval shrinks to $[-2.5, -1.125]$. So when applied on $[-2.5, 3]$, the Bisection method gives -2.

(c) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-1.75	1.5	-0.125	$-0.620\,491\,505$
2	-1.75	-0.125	-0.9375	$-1.330\,096\,78$

At $n = 2$, the interval shrinks to $[-1.75, -0.125]$. So when applied on $[-1.75, 1.5]$, the Bisection method gives -1.

(d) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-1.5	1.75	0.125	0.375 359 058
2	0.125	1.75	0.9375	0.001 384 076

At $n = 2$, the interval shrinks to $[0.125, 1.75]$. So when applied on $[-1.5, 1.75]$, the Bisection method gives 1.

Exercise 12

Find an approximation to $\sqrt{3}$ correct to within 10^4 using the Bisection Algorithm.

Solution 12

Let $f(x) = x^2 - 3$. The positive zero of f is $\sqrt{3}$, so by approximating that positive zero, we get an approximation of $\sqrt{3}$.

The positive zero of f clearly is inside $[1, 2]$. Using Bisection, the number of iteration n needed to approximate $\sqrt{3}$ to within 10^{-4} in that interval is:

$$\frac{2-1}{2^n} < 10^{-4} \iff n \geq 14$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	−0.75
2	1.5	2	1.75	0.0625
3	1.5	1.75	1.625	−0.359 375
4	1.625	1.75	1.6875	−0.152 343 75
5	1.6875	1.75	1.718 75	−0.045 898 438
6	1.718 75	1.75	1.734 375	0.008 056 641
7	1.718 75	1.734 375	1.726 562 5	−0.018 981 934
8	1.726 562 5	1.734 375	1.730 468 75	−0.005 477 905
9	1.730 468 75	1.734 375	1.732 421 88	0.001 285 553
10	1.730 468 75	1.732 421 88	1.731 445 31	−0.002 097 13
11	1.731 445 31	1.732 421 88	1.731 933 59	−0.000 406 027
12	1.731 933 59	1.732 421 88	1.732 177 73	0.000 439 703
13	1.731 933 59	1.732 177 73	1.732 055 66	0.000 016 823
14	1.731 933 59	1.732 055 66	1.731 994 63	−0.000 194 605

So $\sqrt{3} \approx 1.731\,99$.

Exercise 13

Find an approximation to $\sqrt[3]{25}$ correct to within 10^4 using the Bisection Algorithm.

Solution 13

Let $f(x) = x^3 - 25$. The zero of f is $\sqrt[3]{25}$, so by approximating that positive zero, we get an approximation of $\sqrt[3]{25}$.

The positive zero of f clearly is inside $[2, 3]$. Using Bisection, the number of iteration n needed to approximate $\sqrt[3]{25}$ to within 10^{-4} in that interval is:

$$\frac{3-2}{2^n} < 10^{-4} \iff n \geq 14$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	2	3	2.5	−9.375
2	2.5	3	2.75	−4.203 125
3	2.75	3	2.875	−1.236 328 12
4	2.875	3	2.9375	0.347 412 109
5	2.875	2.9375	2.906 25	−0.452 972 412
6	2.906 25	2.9375	2.921 875	−0.054 920 197
7	2.921 875	2.9375	2.929 687 5	0.145 709 515
8	2.921 875	2.929 687 5	2.925 781 25	0.045 260 727
9	2.921 875	2.925 781 25	2.923 828 12	−0.004 863 195
10	2.923 828 12	2.925 781 25	2.924 804 69	0.020 190 398
11	2.923 828 12	2.924 804 69	2.924 316 41	0.007 661 51
12	2.923 828 12	2.924 316 41	2.924 072 27	0.001 398 635
13	2.923 828 12	2.924 072 27	2.923 950 2	−0.001 732 411
14	2.923 950 2	2.924 072 27	2.924 011 23	−0.000 166 921

So $\sqrt[3]{25} \approx 2.924\,01$.

Exercise 14

Use Theorem 2.1 (*Định lí 2.2* in the Lectures.pdf of the project) to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-3} to the solution of $x^3 + x4 = 0$ lying in the interval $[1, 4]$. Find an approximation to the root with this degree of accuracy.

Solution 14

Let $f(x) = x^3 + x4$. $f(1) = -2$ and $f(4) = 64$ have the opposite signs, so there's a root p of f in $[1, 4]$.

The number of iteration n needed to approximate p to within 10^{-3} in that interval is:

$$|p_n - p| \leq \frac{4-1}{2^n} < 10^{-3} \iff n \geq 12$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	4	2.5	14.125
2	1	2.5	1.75	3.109 375
3	1	1.75	1.375	−0.025 390 625
4	1.375	1.75	1.5625	1.377 197 27
5	1.375	1.5625	1.468 75	0.637 176 514
6	1.375	1.468 75	1.421 875	0.296 520 233
7	1.375	1.421 875	1.398 437 5	0.133 260 25

n	a_n	b_n	p_n	$f(p_n)$
8	1.375	1.398 437 5	1.386 718 75	0.053 363 502
9	1.375	1.386 718 75	1.380 859 38	0.013 844 214
10	1.375	1.380 859 38	1.377 929 69	−0.005 808 686
11	1.377 929 69	1.380 859 38	1.379 394 53	0.004 008 885
12	1.377 929 69	1.379 394 53	1.378 662 11	−0.000 902 119

So $p \approx 1.3787$.

Exercise 15

Use Theorem 2.1 (*Định lí 2.2* in the Lectures.pdf of the project) to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-4} to the solution of $x^3 - x1 = 0$ lying in the interval $[1, 2]$. Find an approximation to the root with this degree of accuracy.

Solution 15

Let $f(x) = x^3 - x1$. $f(1) = -2$ and $f(4) = 64$ have the opposite signs, so there's a root p of f in $[1, 2]$.

The number of iteration n needed to approximate p to within 10^{-4} in that interval is:

$$|p_n - p| \leq \frac{2-1}{2^n} < 10^{-4} \iff n \geq 14$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	0.875
2	1	1.5	1.25	−0.296 875
3	1.25	1.5	1.375	0.224 609 375
4	1.25	1.375	1.3125	−0.051 513 672
5	1.3125	1.375	1.343 75	0.082 611 084
6	1.3125	1.343 75	1.328 125	0.014 575 958
7	1.3125	1.328 125	1.320 312 5	−0.018 710 613
8	1.320 312 5	1.328 125	1.324 218 75	−0.002 127 945
9	1.324 218 75	1.328 125	1.326 171 88	0.006 208 83
10	1.324 218 75	1.326 171 88	1.325 195 31	0.002 036 651
11	1.324 218 75	1.325 195 31	1.324 707 03	−0.000 046 595
12	1.324 707 03	1.325 195 31	1.324 951 17	0.000 994 791
13	1.324 707 03	1.324 951 17	1.324 829 1	0.000 474 039
14	1.324 707 03	1.324 829 1	1.324 768 07	0.000 213 707

So $p \approx 1.324\,77$.

Exercise 16

Let $f(x) = (x1)^{10}$, $p = 1$, and $p_n = 1 + \frac{1}{n}$. Show that $\left|f(p_n)\right| < 10^{-3}$ whenever $n > 1$ but that $|p - p_n| < 10^{-3}$ requires that $n > 1000$.

Solution 16

For $f(p_n) < 10^{-3}$, it is required that $n > 1$ as:

$$\begin{aligned} & f(p_n) < 10^{-3} \\ \iff & (p_n - 1)^{10} < 10^{-3} \\ \iff & \frac{1}{n^{10}} < 10^{-3} \\ \iff & n > 1 \end{aligned}$$

For $|p - p_n| < 10^{-3}$, it is required that $n > 1000$ as:

$$\begin{aligned} & |p - p_n| < 10^{-3} \\ \iff & \frac{1}{n} < 10^{-3} \\ \iff & n > 1000 \end{aligned}$$

□

Exercise 17

Let $\{p_n\}$ be the sequence defined by $p_n = \sum_{k=1}^{n} \frac{1}{k}$. Show that $\{p_n\}$ diverges even though $\lim_{n\to\infty}(p_n - p_{n-1}) = 0$.

Solution 17

It's clear that the difference of 2 consecutive terms goes to zero:

$$\lim_{n\to\infty}(p_n - p_{n-1}) = \lim_{n\to\infty}\frac{1}{n} = 0$$

However, the sequence diverges as:

$$\begin{aligned} p_n &= \sum_{k=1}^{n}\frac{1}{k} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \\ &> 1 + (\frac{1}{2}) + (\frac{1}{4} + \frac{1}{4}) + \dots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \dots \\ &= \infty \end{aligned}$$

Exercise 18

The function defined by $f(x) = \sin \pi x$ has zeros at every integer. Show that when $1 < a < 0$ and $2 < b < 3$, the Bisection method converges to

(a) 0 if $a + b < 2$ (b) 2 if $a + b > 2$ (c) 1 if $a + b = 2$

Solution 18

Let p be the zero converged by Bisection.
With $-1 < a < 0$ and $2 < b < 3$:

$$\sin \pi a < 0$$
$$\sin \pi b > 0$$
$$1 < a + b < 3$$

(a) If $a + b < 2$, then $0.5 < p_1 = \frac{a+b}{2} < 1$. Then $\sin p_1 > 0$, and the interval shrinks to $[a, p_1]$. 0 is the only zero in that interval, so $p = 0$.

(b) If $a + b > 2$, then $1 < p_1 = \frac{a+b}{2} < 1.5$. Then $\sin p_1 < 0$, and the interval shrinks to $[p_1, b]$. 2 is the only zero in that interval, so $p = 0$.

(c) If $a + b = 2$, then $p_1 = \frac{a+b}{2} = 1$. Then $\sin p_1 = 0$, and a zero $p = 1$ is found.

Exercise 19

A trough of length L has a cross section in the shape of a semicircle with radius r. When filled with water to within a distance h of the top, the volume V of water is:

$$V = L(0.5\pi r^2 - r^2 \arcsin \frac{h}{r} - h\sqrt{r^2 - h^2})$$

Suppose $L = 10\,\mathrm{ft}$, $r = 1\,\mathrm{ft}$, and $V = 12.4\,\mathrm{ft}^3$. Find the depth of water in the trough to within $0.01\,\mathrm{ft}$.

Solution 19

Let d be the depth of the water, so $d = r - h$. Let

$$f(h) = 10(0.5\pi - \arcsin(h) - h\sqrt{1 - h^2}) - 12.4$$

Instead of finding d directly, we find h, also to within $0.01\,\mathrm{ft}$. The number of iteration n needed to approximate h to within 0.01 in $[0, r]$ is:

$$|h - h_n| < \frac{1 - 0}{2^n} < 0.01 \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	−6.258 151 51
2	0	0.5	0.25	−1.639 453 87
3	0	0.25	0.125	0.814 489 029
4	0.125	0.25	0.1875	−0.419 946 724
5	0.125	0.1875	0.156 25	0.195 725 903
6	0.156 25	0.1875	0.171 875	−0.112 536 394
7	0.156 25	0.171 875	0.164 062 5	0.041 493 241

So $h \approx 0.1641$, hence $d = r - h \approx 0.8359$.

Exercise 20

A particle starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate ω such that:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega < 0$$

At the end of t seconds, the position of the object is given by:

$$x(t) = -\frac{g}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{x} - \sin \omega t\right)$$

Suppose the particle has moved 1.7 ft in 1 s. Find, to within 10^5, the rate ω at which θ changes. Assume that $g = 32.17\,\mathrm{ft/s^2}$.

Solution 20

As $\omega < 0$, the plane rotates clockwise. After 1 s, the particle still sticks to the plane, so:

$$\theta(1) < \frac{\pi}{2} \iff -\frac{\pi}{2} < \omega < 0$$

After 1 s, the particle has moved 1.7 ft, so that:

$$x(1) = 1.7 = -\frac{32.17}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin \omega t\right)$$

Let

$$f(\omega) = 3.4\omega^2 + 32.17\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin \omega t\right)$$

The root of the above function in $(-\frac{\pi}{2}, 0)$ will be the solution of the problem.

Applying Bisection on f on $[-\frac{\pi}{2}, 0]$ fails as $f(0) = 0$. We need to expand (arbitrarily even) the searching interval a bit for the method to work, and check the solution later on. Hence, we use the interval $[-\frac{\pi}{2}, 1]$.

The number of iteration n needed to approximate ω to within 10^{-5} is:

$$|\omega - \omega_n| < \frac{1-(-0.5\pi)}{2^n} < 10^{-5} \iff n \geq 18$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	−1.570 796 33	1	−0.285 398 163	0.027 657 569
2	−1.570 796 33	−0.285 398 163	−0.928 097 245	−5.651 487 86
3	−0.928 097 245	−0.285 398 163	−0.606 747 704	−1.143 969 69
4	−0.606 747 704	−0.285 398 163	−0.446 072 934	−0.275 313 029
5	−0.446 072 934	−0.285 398 163	−0.365 735 549	−0.069 822 38
6	−0.365 735 549	−0.285 398 163	−0.325 566 856	−0.009 667 545
7	−0.325 566 856	−0.285 398 163	−0.305 482 51	0.011 587 981
8	−0.325 566 856	−0.305 482 51	−0.315 524 683	0.001 641 051
9	−0.325 566 856	−0.315 524 683	−0.320 545 769	−0.003 838 965
10	−0.320 545 769	−0.315 524 683	−0.318 035 226	−0.001 055 895
11	−0.318 035 226	−0.315 524 683	−0.316 779 954	0.000 303 28
12	−0.318 035 226	−0.316 779 954	−0.317 407 59	−0.000 373 625
13	−0.317 407 59	−0.316 779 954	−0.317 093 772	−0.000 034 503
14	−0.317 093 772	−0.316 779 954	−0.316 936 863	0.000 134 556
15	−0.317 093 772	−0.316 936 863	−0.317 015 318	0.000 050 068
16	−0.317 093 772	−0.317 015 318	−0.317 054 545	0.000 007 793
17	−0.317 093 772	−0.317 054 545	−0.317 074 159	−0.000 013 352
18	−0.317 074 159	−0.317 054 545	−0.317 064 352	−0.000 002 779

As $-0.317\,064 \in (-\frac{\pi}{2}, 0)$, it is a valid approximation of ω. We conclude that $\omega \approx -0.317\,064$.

2.2 Fixed-Point Iteration

Exercise 1

Use algebraic manipulation to show that each of the following functions has a fixed-point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

a) $g_1(x) = (3 + x - 2x^2)^{1/4}$

b) $g_2(x) = \left(\dfrac{x + 3 - x^4}{2}\right)^{1/2}$

c) $g_3(x) = \left(\dfrac{x+3}{x^2+2}\right)^{1/2}$

d) $g_4(x) = \dfrac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$

Solution 1

a) For $x = p$:

$$g_1(p) = (3 + p - 2p^2)^{\frac{1}{4}} = (p^4 - f(p))^{1/4} = |p|$$

So p is a fixed-point of g_1.

b) For $x = p$:

$$\begin{aligned} g_2(p) &= \left(\frac{p + 3 - p^4}{2}\right)^{1/2} \\ &= \left(\frac{2p^2}{2}\right)^{\frac{1}{2}} \\ &= |p| \end{aligned}$$

So p is a fixed-point of g_2.

c) For $x = p$:

$$\begin{aligned} g_3(p) &= \left(\frac{p + 3}{p^2 + 2}\right)^{1/2} \\ &= \left(\frac{p^4 + 2p^2}{p^2 + 2}\right)^{1/2} \\ &= |p| \end{aligned}$$

So p is a fixed-point of g_3.

d) For $x = p$:

$$\begin{aligned} g_4(p) &= \frac{3p^4 + 2p^2 + 3}{4p^3 + 4p - 1} \\ &= \frac{4p^4 - (3 + p - 2p^2) + 2p^2 + 3}{4p^3 + 4p - 1} \\ &= \frac{4p^4 + 4p^2 - p}{4p^3 + 4p - 1} \\ &= p \end{aligned}$$

So p is a fixed-point of g_4.

Exercise 2

a) Perform four iterations, if possible, on each of the functions g defined in Exercise 1. Let $p_0 = 1$ and $p_{n+1} = g(p_n)$, for $n = 0, 1, 2, 3$.

b) Which function do you think gives the best approximation to the solution?

Solution 2

a) Applying fixed-point method on the four functions g generates the following table:

n	p_n by g_1	p_n by g_2	p_n by g_3	p_n by g_4
0	1	1	1	1
1	1.189 207 115	1.224 744 871	1.154 700 538	1.142 857 143
2	1.080 057 753	0.993 666 159	1.116 427 41	1.124 481 69
3	1.149 671 431	1.228 568 645	1.126 052 233	1.124 123 164
4	1.107 820 053	0.987 506 429	1.123 638 885	1.124 123 03

b) g_4 gives the best approximation as it generates the smallest difference between p_3 and p_4: $|p_4 - p_3| = -134 \times 10^{-7}$.

Exercise 3

The following four methods are proposed to compute $21^{1/3}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

a) $p_n = \dfrac{20p_{n-1} + 21/p_{n-1}^2}{21}$

b) $p_n = p_{n-1} - \dfrac{p_{n-1}^3 - 21}{3p_{n-1}^2}$

c) $p_n = p_{n-1} - \dfrac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21}$

d) $p_n = \left(\dfrac{21}{p_{n-1}}\right)^{1/2}$

Solution 3

Applying fixed-point method on the four sequences generate the following table:

n	a)	b)	c)	d)
0	1	1	1	1
1	1.952 380 952	7.666 666 667	0	4.582 575 695
2	2.121 754 174	5.230 203 739	0	2.140 695 143
3	2.242 849 692	3.742 696 919		3.132 075 595
4	2.334 839 673	2.994 853 568		2.589 366 527
5	2.401 093 38	2.777 022 226		2.847 822 274

n	a)	b)	c)	d)
6	2.465 059 288	2.759 041 866		2.715 521 253
7	2.512 243 463	2.758 924 181		2.780 885 095
8	2.551 057 096	2.758 924 176		2.748 008 838
9	2.583 237 767	2.758 924 176		2.764 398 093
10	2.610 081 445			2.756 191 284
11	2.632 580 301			2.760 291 639
12	2.651 509 504			2.758 240 699
13	2.667 484 488			2.759 265 978
14	2.681 000 202			2.758 753 291
15	2.692 458 887			2.759 009 623
16	2.702 190 249			2.758 881 454
17	2.710 466 453			2.758 945 538
18	2.717 513 483			2.758 913 496
19	2.723 519 902			2.758 929 517

Apparently, the speed of convergence is ranked in descending order as follow: b), d), a). c) does not converge.

Exercise 4

The following four methods are proposed to compute $7^{1/5}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

a) $p_n = p_{n-1} - \left(1 + \frac{7-p_{n-1}^5}{p_{n-1}^2}\right)^3$

b) $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{p_{n-1}^2}$

c) $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{5p_{n-1}^4}$

d) $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{12}$

Solution 4

Applying fixed-point method on the four sequences generate the following table:

n	a)	b)	c)	d)
0	1	1	2.2	1
1	343	7	1.819 763 677	1.5
2	-2.25×10^{25}	-335.857	1.583 474 83	1.450 520 833
3		37 884 356	1.489 460 974	1.498 749 661
4			1.476 022 436	1.451 903 535
5			1.475 773 246	1.497 577 067
6			1.475 773 162	1.453 192 29
7			1.475 773 162	1.496 475 364
9				1.454 396 119

n	a)	b)	c)	d)
8				1.495 438 587
10				1.455 522 81
11				1.494 461 513
12				1.456 579 138
13				1.493 539 533
14				1.457 571 031
15				1.492 668 56
16				1.458 803 715
17				1.491 844 948
18				1.459 381 814
19				1.491 065 425

Apparently, the speed of convergence is ranked in descending order as follow: c), d). a) and b) do not converge.

Exercise 5

Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^4 - 3x^2 - 3 = 0$ on $[1, 2]$. Use $p_0 = 1$.

Solution 5

Let $f(x) = x^4 - 3x^2 - 3$. Let p be the root of f in $[1, 2]$. We need to find a function g for which $p = g(p)$ to perform the fixed-point method.

Extract p to RHS gives:

$$p^4 = 3p^2 + 3 \iff |p| = (3p^2 + 3)^{1/4}$$

Then g is chosen as:

$$g(x) = (3x^2 + 3)^{1/4}$$

Applying fixed-point method on g generate the following table:

n	p_n	n	p_n
0	1	4	1.922 847 844
1	1.565 084 58	5	1.937 507 54
2	1.793 572 879	6	1.943 316 93
3	1.885 943 743		

We can try the other obvious option

$$g(x) = \left(\frac{x^4 - 3}{3}\right)^{0.5}$$

which fails on the first iteration. A reasonable explanation for the choice of g is that we need $|g'|$ to be as small as possible. On $[1,2]$, the $O(x^{0.5})$ of the first choice clearly has an advantage over $O(x^2)$ of the second choice of g.

We conclude that $p \approx 1.943$.

Exercise 6

Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on $[1,2]$. Use $p_0 = 1$.

Solution 6

Let $f(x) = x^3 - x - 1 = 0$. Let p be the root of f in $[1,2]$. We need to find a function g for which $p = g(p)$ to perform the fixed-point method.

Extract p to RHS gives:

$$p^3 = p + 1 \iff p = (p+1)^{1/3}$$

Then g is chosen as:

$$g(x) = (p+1)^{1/3}$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n
0	1	3	1.322 353 819
1	1.259 921 05	4	1.324 268 745
2	1.312 293 837		

We conclude that $p \approx 1.324$.

Exercise 7

Use Theorem 2.3 (Định lý 2.3 in the accompanying Lectures.pdf) to show that $g(x) = \pi + 0.5 \sin 0.5x$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2}. Use Corollary 2.5 (Hệ quả 2.1) to estimate the number of iterations required to achieve 10^{-2} accuracy, and compare this theoretical estimate to the number actually needed.

Solution 7

From the formula of g:

$$\begin{aligned} g(x) &= \pi + 0.5 \sin 0.5x \\ \Rightarrow g(x) &\in [\pi - 0.5, \pi + 0.5]\ \forall x \end{aligned}$$

Consider the interval $I = [\pi - 0.5, \pi + 0.5] \in [0, 2\pi]$. From the above equations, we know that:

- $g \in CI$
- $g(x) \in I \, \forall x \in I$

According to Theorem 2.3, there exists a fixed point of g on I.

Differentiating g gives:

$$g'(x) = -0.25 \cos 0.5x \Rightarrow \left|g'(x)\right| \leq k = 0.25 < 1 \, \forall x$$

Again, according to Theorem 2.3, there exists one and only one fixed point of g on I.

Applying fixed-point method on g, with $p_0 = \pi$, generates the following table:

n	p_n	n	p_n
0	3.141 592 654	2	3.626 048 864
1	3.641 592 654	3	3.626 995 622

Using corollary 2.5, the number of iterations n required to achieve 10^{-2} accuracy is

$$|p_n - p| \leq k^n 0.5 < 10^{-2} \iff n \geq 3$$

which is in line with the number of iteration actually performed.

Exercise 8

Use Theorem 2.3 (Định lý 2.3 in the accompanying Lectures.pdf) to show that $g(x) = 2^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-4}. Use Corollary 2.5 (Hệ quả 2.1) to estimate the number of iterations required to achieve 10^{-4} accuracy, and compare this theoretical estimate to the number actually needed.

Solution 8

From the formula of g:

$$\begin{aligned} g(x) &= 2^{-x} \\ \Rightarrow g'(x) &= -2^{-x} \ln 2 \end{aligned}$$

It is clear that $g \in C^1 R$.

Consider the interval $I = [\frac{1}{3}, 1]$, $I_{open} = (\frac{1}{3}, 1)$:

$$
\begin{aligned}
& g'(x) < 0 \forall x \in I \\
& \Rightarrow 1 > g(\frac{1}{3}) = 2^{-1/3} \geq g(x) \geq g(1) = 2^{-1} > \frac{1}{3} \\
& \Rightarrow g(x) \in I \, \forall x \in I
\end{aligned}
$$

So far, we know that:

- $g \in CI$ ($g \in CR$ even)
- $g(x) \in I \, \forall x \in I$

According to Theorem 2.3, there exists a fixed point of g on I.
Consider g':

$$
\begin{aligned}
-1 < -\ln 2 \leq g'(x) \leq -\frac{1}{3} \ln 2 < 0 \, \forall x \in I \\
\Rightarrow \left| g'(x) \right| \leq k = \ln 2 < 1 \, \forall x \in I
\end{aligned}
$$

Again, according to Theorem 2.3, there exists one and only one fixed point of g on I.
Applying fixed-point method on g, with $p_0 = \frac{2}{3}$, generates the following table:

n	p_n	n	p_n
0	0.666 666 667	5	0.640 746 653
1	0.629 960 525	6	0.641 380 922
2	0.646 194 096	7	0.641 099 006
3	0.638 963 711	8	0.641 224 295
4	0.642 174 057	9	0.641 168 611

Using Corollary 2.5, the number of iterations n required to achieve 10^{-4} accuracy is

$$
|p_n - p| \leq k^n \frac{1}{3} < 10^{-4} \iff n \geq 23
$$

which is quit a bit higher than the number of iteration actually performed.

Exercise 9

Use a fixed-point iteration method to find an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 12 of Section 2.1.

Solution 9

Let $f(x) = x^2 - 3$, $p > 0$ is a zero of f. Then $p = \sqrt{3}$, and an approximation of p is an approximation of $\sqrt{3}$.

Consider $g(x) = \frac{3}{x}$. It is clear that this is a bad choice, as applying g on any p_0 generates a sequence that jumps between p_0 and $\frac{3}{p_0}$.

From the textbook examples, we choose $g(x) = x - \frac{x^2-3}{x^2}$. Applying fixed-point method on g with $p_0 = 1.5$ generates the following table:

n	p_n	n	p_n
0	1.5	4	1.731 898 58
1	1.833 333 33	5	1.732 074 38
2	1.725 895 32	6	1.732 047 16
3	1.733 041 14		

We conclude that $\sqrt{3} \approx 1.732\,05$. In exercise 12 of section 2.1, 14 iteration is needed, much higher than that of this method.

Exercise 10

Use a fixed-point iteration method to find an approximation to $\sqrt[3]{25}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 13 of Section 2.1.

Solution 10

Let $f(x) = x^3 - 25$, $p > 0$ is a zero of f. Then $p = \sqrt[3]{25}$, and an approximation of p is an approximation of $\sqrt[3]{25}$.

We choose $g(x) = x - \frac{x^3-25}{x^3}$. Applying fixed-point method on g with $p_0 = 2.5$ generates the following table:

n	p_n	n	p_n
0	2.5	3	2.923 783 69
1	3.1	4	2.924 023 86
2	2.939 179 62	5	2.924 017 58

We conclude that $\sqrt[3]{25} \approx 2.924\,02$. In exercise 13 of section 2.1, 14 iteration is needed, much higher than that of this method.

Exercise 11

For each of the following equations, determine an interval $[a, b]$ on which fixed-point iteration converges. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and perform the calculations.

a) $x = \dfrac{2 - e^x + x^2}{3}$

b) $x = \dfrac{5}{x^2} + 2$

c) $x = (e^x/3)^{1/2}$

d) $x = 5^{-x}$

e) $x = 6^{-x}$

f) $x = 0.5(\sin x + \cos x)$

Solution 11

a) Let

$$
\begin{aligned}
& g(x) = \frac{2 - e^x + x^2}{3} \\
\Rightarrow \qquad & g'(x) = \frac{2x - e^x}{3} \\
\Rightarrow \qquad & g''(x) = \frac{2 - e^x}{3}
\end{aligned}
$$

It is clear that g is continuous in $\mathbb{R}$.

Consider g'':

- $g''(x) > 0 \iff x < \ln 2$
- $g''(x) = 0 \iff x = \ln 2$
- $g''(x) < 0 \iff x > \ln 2$

So, $\max g'(x) = g'(\ln 2) = \dfrac{\ln 4 - 2}{3} < 0$. So g is monotonically decreasing in $\mathbb{R}$.

Consider the interval $I = [0, 1]$:

$$
\begin{aligned}
1 > g(0) = \frac{1}{3} > & g(x) > g(1) = \frac{3 - e}{3} > 0 \,\forall x \in I \\
\Rightarrow & g(x) \in I \,\forall x \in I
\end{aligned}
$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 0.5$ generates the following table:

n	p_n	n	p_n
0	0.5	5	0.257 265 636
1	0.200 426 243	6	0.257 598 985
2	0.272 749 065	7	0.257 512 455
3	0.253 607 157	8	0.257 534 914
4	0.258 550 376	9	0.257 529 084

We conclude that the fixed point $p \approx 0.257\,529$.

b) Let

$$g = \frac{5}{x^2} + 2$$

Consider the interval $I = [2.5, 3]$. $0 \notin I$, so g is continuous in I.

x^2 is monotonically increasing in I, so g is monotonically decreasing in I. So that:

$$3 > g(2.5) = 2.8 > g(x) > g(3) = {}^{23}/_{9} > 2.5\,\forall x \in I$$
$$\Rightarrow g(x) \in I\,\forall x \in I$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 2.75$ generates the following table:

n	p_n	n	p_n	n	p_n
0	2.75	6	2.691 710 92	12	2.690 666 91
1	2.661 157 02	7	2.690 101 82	13	2.690 637 46
2	2.706 039 5	8	2.690 927 64	14	2.690 652 58
3	2.682 812 93	9	2.690 503 63	15	2.690 644 82
4	2.694 687 08	10	2.690 721 29		
5	2.688 578 29	11	2.690 609 54		

We conclude that the fixed point $p \approx 2.690\,645$.

c) Let

$$g(x) = \left(\frac{e^x}{3}\right)^{1/2}$$

It is clear that g is continuous in $\mathbb{R}$.

g is monotonically increasing in $\mathbb{R}$. Consider the interval $I = [0, 1]$:

$$0 < g(0) = \frac{1}{\sqrt{3}} < g(x) < g(1) = \sqrt{\frac{e}{3}} < 1$$
$$\Rightarrow g(x) \in I\,\forall x \in I$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 0.5$ generates the following table:

n	p_n	n	p_n	n	p_n
0	0.5	5	0.903 281 143	10	0.909 876 791
1	0.741 332 42	6	0.906 952 163	11	0.909 948 068
2	0.836 407 007	7	0.908 618 411	12	0.909 980 498
3	0.877 127 74	8	0.909 375 718	13	0.909 995 254
4	0.895 169 428	9	0.909 720 122	14	0.910 001 967

We conclude that the fixed point $p \approx 0.910\,002$.

d) Let $g(x) = 5^{-x}$. It is clear that g is continuous in $\mathbb{R}$.

5^x is monotonically increasing in $\mathbb{R}$, so g is monotonically decreasing in $\mathbb{R}$.

Consider the interval $I = [0, 1]$:

$$\begin{aligned} 0 < g(1) = 0.2 <& g(x) < g(0) = 1 \\ \Rightarrow& g(x) \in I \,\forall x \in I \end{aligned}$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 0.5$ generates the following table:

n	p_n	n	p_n	n	p_n
0	0.5	11	0.468 245 559	22	0.469 685 261
1	0.447 213 595	12	0.470 663 369	23	0.469 574 052
2	0.486 867 866	13	0.468 835 429	24	0.469 658 106
3	0.456 766 207	14	0.470 216 753	25	0.469 594 575
4	0.479 439 843	15	0.469 172 549	26	0.469 642 593
5	0.462 259 591	16	0.469 961 695	27	0.469 606 3
6	0.475 219 673	17	0.469 365 184	28	0.469 633 731
7	0.465 409 992	18	0.469 816 013	29	0.469 612 998
8	0.472 816 23	19	0.469 475 247	30	0.469 628 669
9	0.467 213 774	20	0.469 732 798	31	0.469 616 824
10	0.471 445 6	21	0.469 538 128	32	0.469 625 777

We conclude that the fixed point $p \approx 0.469\,626$.

e) Let $g(x) = 6^{-x}$. It is clear that g is continuous in $\mathbb{R}$.

6^x is monotonically increasing in $\mathbb{R}$, so g is monotonically decreasing in $\mathbb{R}$.

Consider the interval $I = [0, 1]$:

$$\begin{aligned} 0 < g(1) = \frac{1}{6} <& g(x) < g(0) = 1 \\ \Rightarrow& g(x) \in I \,\forall x \in I \end{aligned}$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 0.5$ generates the following table:

n	p_n	n	p_n	n	p_n
0	0.5	15	0.446 190 464	30	0.448 132 603
1	0.408 248 29	16	0.449 568 975	31	0.448 007 263
2	0.481 194 974	17	0.446 855 739	32	0.448 107 887
3	0.422 238 208	18	0.449 033 402	33	0.448 027 103
4	0.469 282 988	19	0.447 284 756	34	0.448 091 958
5	0.431 347 074	20	0.448 688 365	35	0.448 039 891
6	0.461 686 032	21	0.447 561 363	36	0.448 081 691
7	0.437 258 678	22	0.448 466 044	37	0.448 048 133
8	0.456 821 582	23	0.447 739 682	38	0.448 075 074
9	0.441 086 448	24	0.448 322 78	39	0.448 053 445
10	0.453 699 216	25	0.447 854 63	40	0.448 070 809
11	0.443 561 035	26	0.448 230 453	41	0.448 056 869
12	0.451 692 029	27	0.447 928 723	42	0.448 068 06
13	0.445 159 128	28	0.448 170 951	43	0.448 059 076
14	0.450 400 504	29	0.447 976 481		

We conclude that the fixed point $p \approx 0.448\,059$.

f) Let $g(x) = 0.5(\sin x + \cos x)$. It is clear that g is continuous in $\mathbb{R}$.

Manipulating g gives:

$$\begin{aligned}
\sin x + \cos x &= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) \\
&= \sqrt{2}\left(\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x\right) \\
&= \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) \\
\Rightarrow g(x) &= 0.5(\sin x + \cos x) \\
&= \frac{1}{\sqrt{2}}\sin\left(x + \frac{\pi}{4}\right)
\end{aligned}$$

Consider the interval $I = [0, \frac{\pi}{4}]$. $sinx$ is monotonically increasing in $[0, \frac{\pi}{2}]$, so $\sin x + \frac{\pi}{4}$ also is monotonically increasing in I. It follows that:

$$\begin{aligned}
0 < g(0) = 0.5 < g(x) <&g(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} < \frac{\pi}{4} \\
\Rightarrow&g(x) \in I\,\forall x \in I
\end{aligned}$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = \frac{\pi}{8}$ generates the following table:

n	p_n	n	p_n
0	0.392 699 082	4	0.704 799 153
1	0.653 281 482	5	0.704 811 271
2	0.700 944 543	6	0.704 811 96
3	0.704 586 59		

We conclude that the fixed point $p \approx 0.704\,812$.

Exercise 12

For each of the following equations, use the given interval or determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and perform the calculations.

a) $2 + \sin x - x = 0$ on $[2, 3]$

b) $x^3 - 3x - 5 = 0$ on $[2, 3]$

c) $3x^2 - e^x = 0$

d) $x - \cos x = 0$

Solution 12

a) Let $I = [2, 3]$ and

$$\begin{aligned} g(x) &= \sin x + 2 \\ \Rightarrow g'(x) &= \cos x \end{aligned}$$

A fixed point p of g is also a root of the problem.

Consider g. It is clear that g is continuous on $\mathbb{R}$. $\sin x$ is monotonically decreasing in I, so that:

$$2 < g(3) = \sin 3 + 2 < g(x) < g(2) = \sin 2 + 2 < 3$$

Consider g'. $\cos x$ is monotonically decreasing in I, so that:

$$\begin{aligned} \cos 3 \leq g'(x) \leq \cos 2 < 0 \, \forall x \in I \\ \Rightarrow \left| g'(x) \right| \leq k = -\cos 3 < 1 \end{aligned}$$

Therefore, all the conditions in Corollary 2.5 hold. Using Corollary 2.5, with $p_0 = 2.5$, the number of iteration n required to obtain approximations accurate to within 10^{-5} is:

$$|p_n - p| \leq k^n 0.5 < 10^{-5} \iff n \geq 1076$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n	n	p_n
0	2.5	18	2.552 225 43	36	2.554 123 46
1	2.598 472 14	19	2.555 835 11	37	2.554 256 29
2	2.516 809 97	20	2.552 830 8	38	2.554 145 73
3	2.584 921 02	21	2.555 331 77	39	2.554 237 76
4	2.528 363 28	22	2.553 250 15	40	2.554 161 15
5	2.575 511 41	23	2.554 982 97	41	2.554 224 92
6	2.536 328 7	24	2.553 540 68	42	2.554 171 84
7	2.568 979 15	25	2.554 741 28	43	2.554 216 02
8	2.541 830 51	26	2.553 741 95	44	2.554 179 25
9	2.564 446 15	27	2.554 573 8	45	2.554 209 86
10	2.545 634 87	28	2.553 881 4	46	2.554 184 38
11	2.561 301 68	29	2.554 457 76	47	2.554 205 59
12	2.548 267 3	30	2.553 978 01	48	2.554 187 93
13	2.559 121 11	31	2.554 377 35	49	2.554 202 63
14	2.550 089 61	32	2.554 044 95	50	2.554 190 4
15	2.557 609 33	33	2.554 321 64	51	2.554 200 58
16	2.551 351 48	34	2.554 091 33	52	2.554 192 1
17	2.556 561 41	35	2.554 283 04		

So one solution of the problem is $p \approx 2.554\,192$.

b) Let $I = [2, 3]$ and

$$g(x) = \sqrt[3]{2x+5}$$
$$\Rightarrow g'(x) = \frac{2}{3}(2x+5)^{-2/3}$$

A fixed point p of g is also a solution of the problem.

Consider g. It is clear that g is continuous and monotonically increasing on $\mathbb{R}$, so that:

$$2 < g(2) = \sqrt[3]{9} < g(x) < g(3) = \sqrt[3]{11} < 3$$
$$\Rightarrow g(x) \in I \,\forall x \in I$$

Consider g'. Since $-2/3 < 0$ and $I > 0$, $g'(x)$ is monotonically decreasing in I, so that:

$$g'(2) = \frac{2}{9\sqrt[3]{3}} \geq g'(x) \geq g'(3) = \frac{2}{3\sqrt[3]{121}}$$
$$\Rightarrow \left|g'(x)\right| \leq k = \frac{2}{9\sqrt[3]{3}} < 1$$

Therefore, all the conditions in Corollary 2.5 hold. Using Corollary 2.5, with $p_0 = 2.5$, the number of iteration n required to obtain approximations accurate to within 10^{-5} is:

$$|p_n - p| \leq k^n 0.5 < 10^{-5} \iff n \geq 6$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n
0	2.5	4	2.094 760 55
1	2.154 434 69	5	2.094 583 25
2	2.103 612 03	6	2.094 556 31
3	2.095 927 41	7	2.094 552 22

So one solution of the problem is $p \approx 2.094\,552$.

c) Let $I = [3, 4]$ and

$$g(x) = \ln 3x^2 = 2\ln x + \ln 3$$
$$\Rightarrow g'(x) = \frac{2}{x}$$

A fixed point p of g is also a solution of the problem.

Consider g. It is clear that g is continuous and monotonically increasing on I, so that:

$$3 < g(3) = \ln 27 < g(x) < g(4) = \ln 48 < 4$$
$$\Rightarrow g(x) \in I\, \forall x \in I$$

Consider g'. Since $I > 0$, $g'(x)$ is monotonically decreasing in I, so that:

$$g'(3) = \frac{2}{3} \geq g'(x) \geq g'(4) = \frac{1}{2}$$
$$\Rightarrow \left|g'(x)\right| \leq k = \frac{2}{3} < 1$$

Therefore, all the conditions in Corollary 2.5 hold. Using Corollary 2.5, with $p_0 = 3.5$, the number of iteration n required to obtain approximations accurate to within 10^{-5} is:

$$|p_n - p| \leq k^n 0.5 < 10^{-5} \iff n \geq 27$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n	n	p_n
0	3.5	6	3.727 177 12	12	3.732 939 23
1	3.604 138 23	7	3.729 914 58	13	3.733 004 13
2	3.662 777 67	8	3.731 382 95	14	3.733 038 9
3	3.695 055 86	9	3.732 170 15	15	3.733 057 53
4	3.712 603 63	10	3.732 592 04	16	3.733 067 51
5	3.722 079 13	11	3.732 818 1		

So one solution of the problem is $p \approx 3.733\,068$.

d) Let $I = [0, 1]$ and

$$g(x) = \cos x$$
$$\Rightarrow g'(x) = -\sin x$$

A fixed point p of g is also a solution of the problem.

Consider g. It is clear that g is continuous and monotonically decreasing on I, so that:

$$1 = g(0) \geq g(x) \geq g(1) = \cos 1 > 0$$
$$\Rightarrow g(x) \in I \,\forall x \in I$$

Consider g'. Since $I > 0$, $g'(x)$ is monotonically decreasing in I, so that:

$$g'(0) = 0 \geq g'(x) \geq g'(1) = -\sin 1$$
$$\Rightarrow \left|g'(x)\right| \leq k = \sin 1 < 1$$

Therefore, all the conditions in Corollary 2.5 hold. Using Corollary 2.5, with $p_0 = 0.5$, the number of iteration n required to obtain approximations accurate to within 10^{-5} is:

$$|p_n - p| \leq k^n 0.5 < 10^{-5} \iff n \geq 63$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n	n	p_n
0	0.5	10	0.735 006 309	20	0.739 006 78
1	0.877 582 562	11	0.741 826 523	21	0.739 137 911
2	0.639 012 494	12	0.737 235 725	22	0.739 049 581
3	0.802 685 101	13	0.740 329 652	23	0.739 109 081
4	0.694 778 027	14	0.738 246 238	24	0.739 069 001
5	0.768 195 831	15	0.739 649 963	25	0.739 096
6	0.719 165 446	16	0.738 704 539	26	0.739 077 813
7	0.752 355 759	17	0.739 341 452	27	0.739 090 064
8	0.730 081 063	18	0.738 912 449	28	0.739 081 812
9	0.745 120 341	19	0.739 201 444		

So one root of the problem is $p \approx 0.739\,082$.

Exercise 13

Find all the zeros of $f(x) = x^2 + 10\cos x$ by using the fixed-point iteration method for an appropriate iteration function g. Find the zeros accurate to within 10^{-4}.

Solution 13

Consider $f = 0$. Since $x^2 \geq 0$, $\cos x$ must be negative for the equation to hold, so that:

$$x \in I_k = [\frac{\pi}{2} + k2\pi, \frac{3\pi}{2} + k2\pi]\, \forall k \in \mathbb{N} \tag{1}$$

Also, since $10\cos x \in [-10, 0]$:

$$x \in [-\sqrt{10}, \sqrt{10}] \tag{2}$$

Combining (1) and (2) gives:

$$x \in I = I_a \cup I_b \text{ where } I_a = [-\sqrt{10}, -\frac{\pi}{2}] \text{ and } I_b = [\frac{\pi}{2}, \sqrt{10}]$$

As x^2 and $\cos x$ take Oy as a symmetry axis, each zero z_b of f in I_b results in another zero $z_a = -z_b$ in I_a. Hence, from now on, we just need to examine on I_b.

Differentiating f gives:

$$f'(x) = 2x - 10\sin x$$

x is monotonically increasing on I_b, $\sin x$ is monotonically decreasing on I_b. It follows that f' is monotonically increasing on I_b, which means:

$$f'(\frac{\pi}{2}) = \pi - 10 \leq f'(x) \leq f'(\sqrt{10}) = 2\sqrt{10} - 10\sin\sqrt{10}$$

Combining with the fact that f' is continuous on I_b, according to Intermediate Value Theorem, f' has one zero in I_b. It follows that f has at most two zeros in I_b.

Let

$$g(x) = x - \frac{-10\cos x}{x^2} + 1 = x + \frac{10\cos x}{x^2} + 1$$

A fixed point of g is also a zero of f. *Try* applying fixed-point method on g with several p_0, we found two fixed points:

- $p_0 = \frac{\pi}{2}$ generates the following table:

n	p_n	n	p_n	n	p_n
0	1.570 796 33	4	1.953 548 67	8	1.968 593 28
1	2.570 796 33	5	1.974 930 8	9	1.968 974 39
2	2.297 575 29	6	1.966 757 33	10	1.968 836 22
3	2.038 843 43	7	1.969 648 71	11	1.968 886 24

- $p_0 = -\sqrt{10}$ generates the following table:

n	p_n
0	−3.162 277 66
1	−3.162 063 73
2	−3.161 989 49

The second fixed point is interesting. It is indeed a fixed point of g, a zero of f, but it belongs to I_a. Due to the symmetry property, we conclude that f has 4 zeros: $\pm 1.968\,89$ and $\pm 3.161\,99$.

Exercise 14

Use a fixed-point iteration method to determine a solution accurate to within 10^{-4} for $x = \tan x$, for $x \in [4, 5]$.

Solution 14

Let

$$g(x) = x - \sqrt[3]{\frac{\tan x}{x}} + 1$$

A fixed point p of g is also a solution of the problem. Applying fixed-point method on g generates the following table:

n	p_n	n	p_n	n	p_n
0	4	4	4.495 344 11	8	4.493 529 55
1	4.338 504 07	5	4.492 429 47	9	4.493 349 61
2	4.500 975 94	6	4.493 893 01	10	4.493 439 23
3	4.489 378 73	7	4.493 167 7		

So $p \approx 4.493\,44$ is a solution of the problem in $[4, 5]$.

Exercise 15

Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $2 \sin \pi x + x = 0$ on $[1, 2]$. Use $p_0 = 1$.

Solution 15

Consider f:

$$\begin{aligned} & f(x) = 0 \\ \iff \quad & 2 \sin \pi x = -x \\ \iff \quad & \pi x = \arcsin -0.5x + k2\pi \ (k \in \mathbb{N}) \\ \iff \quad & x = \frac{\arcsin -0.5x}{\pi} + 2k \end{aligned}$$

Let

$$g(x) = \frac{\arcsin -0.5x}{\pi} + 2$$

arcsin is chosen as it "behaves" nicer than normal sin. Since arcsin returns values in principal branch $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we need to use $k = 1$ to shift the value to cover $[1, 2]$.

A fixed point p of g is also a solution of the problem. Applying fixed-point method on g generates the following table:

n	p_n	n	p_n
0	1	3	1.696 498
1	1.833 333 33	4	1.677 657 06
2	1.630 869 25	5	1.683 240 99

So $p \approx 1.683$ is a solution of the problem in $[1, 2]$.

Exercise 16

Let A be a given positive constant and $g(x) = 2x - Ax^2$.

a) Show that if fixed-point iteration converges to a nonzero limit, then the limit is $p = {}^1/_A$, so the inverse of a number can be found using only multiplications and subtractions.

b) Find an interval about ${}^1/_A$ for which fixed-point iteration converges, provided p_0 is in that interval.

Solution 16

a) If fixed-point iteration converges to a nonzero limit p, then:

$$\begin{aligned} p &= \lim_{n\to\infty} p_n \\ &= \lim_{n\to\infty} g(p_{n-1}) \\ &= \lim_{n\to\infty} \left(2p_{n-1} - Ap_{n-1}^2\right) \\ &= 2p - Ap^2 \\ \iff p = Ap^2 &\iff p = \frac{1}{A} \end{aligned}$$

b) We try to find $\delta > 0$ such that fixed-point method converges on $I = [{}^1/_A - \delta, {}^1/_A + \delta]$ using Fixed Point Theorem.

The condition that g is continuous on I is satisfied with any δ.

Consider g:

$$g(x) = -Ax^2 + 2x = -A\left(x - \frac{1}{A}\right)^2 + \frac{1}{A}$$

So $x = \frac{1}{A}$ is the axis of symmetry for g.

Differentiating g gives:

$$g'(x) = 2 - 2Ax$$

It follows that:

- $g'(x) < 0 \iff x > \frac{1}{A}$
- $g'(x) = 0 \iff x = \frac{1}{A}$
- $g'(x) > 0 \iff x < \frac{1}{A}$

Combining with the fact that $x = \frac{1}{A}$ is the symmetry axis of g gives:

$$\begin{gathered}
g\left(\frac{1}{A}+\delta\right) = g\left(\frac{1}{A}-\delta\right) = g\left(\frac{1}{A}\pm\delta\right) \leq g(x) \leq g\left(\frac{1}{A}\right)\ \forall x \in I \\
\iff \frac{2}{A} - A\delta^2 \leq g(x) \leq \frac{1}{A}
\end{gathered}$$

Then, to satisfy the condition that $g(x) \in I\, \forall x \in I$, δ must satisfy the following:

$$\begin{aligned}
& & \frac{2}{A} - A\delta^2 \geq \frac{1}{A} - \delta \\
&\iff & (A\delta)^2 - A\delta - 1 \leq 0 \\
&\iff & 0 < \delta \leq \frac{1+\sqrt{5}}{2A}\ (\text{as } \delta > 0) \qquad (1)
\end{aligned}$$

Consider g'. g' is monotonically decreasing on $\mathbb{R}$, so:

$$\begin{gathered}
g'\left(\frac{1}{A}-\delta\right) = 2A\delta \geq g'(x) \geq g'\left(\frac{1}{A}-\delta\right) = -2A\delta \\
\iff \left|g'(x)\right| \leq 2A\delta \text{ (equal sign only at either end)} \qquad (2)
\end{gathered}$$

Then, to satisfy the condition that $\left|g'(x)\right| < 1\, \forall x \in I_{open} = (1/A - \delta, 1/A + \delta)$, δ must satisfy the following:

$$2A\delta \leq 1 \iff \delta \leq \frac{1}{2A}$$

From (1) and (2):

$$0 < \delta < \frac{1}{2A}$$

As all the conditions needed for Fixed Point Theorem hold, we conclude that for any $\delta \in (0, \frac{1}{2A}]$, applying fixed-point method on g with $p_0 \in I$ converges to the fixed point.

Exercise 17

Find a function g defined on $[0, 1]$ that satisfies none of the hypotheses of Theorem 2.3 but still has a unique fixed point on $[0, 1]$.

Solution 17

Let $I = [0, 1]$, $g = \dfrac{1}{x + 0.5}$.

Consider g. g is defined on $\mathbb{R} \setminus \{-0.5\}$, so it is defined on I.

$g(x) > 1\,\forall x \in [-0.5, 0.5]$, so the condition that $g(x) \in I\,\forall x \in I$ does not hold.

Differentiating g gives:

$$g'(x) = -\frac{1}{(x + 0.5)^2} < -1 \iff x \in (-1.5, 0.5) \setminus \{-0.5\}$$

So the condition that $\left|g'(x)\right| < 1\,\forall x \in I$ does not hold.

Yet, g has a fixed point at $x = \dfrac{\sqrt{17} - 1}{4}$.

Exercise 18

a) Show that Theorem 2.2 is true if the inequality $\left|g'(x)\right| \leq k$ is replaced by $g'(x) \leq k$, for all $x \in (a, b)$. [Hint: Only uniqueness is in question.]

b) Show that Theorem 2.3 may not hold if inequality $\left|g'(x)\right| \leq k$ is replaced by $g'(x) \leq k$.

Solution 18

a) Where the fuck is Theorem 2.2 in the fucking book?

b) In the proof of Theorem 2.3, if $\left|g'(x) \leq k\right|$ is replaced with $g'(x) \leq k$, then there is a chance that $g'(\xi) = -1$. In that case, the assumption is no longer a contradiction, therefore the proof is invalid, and the theorem doesn't hold.

Exercise 19

a) Use Theorem 2.4 (Định lí 2.5 in the accompanying Lectures.pdf) to show that the sequence defined by:

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \text{ for } n \leq 1$$

converges to $\sqrt{2}$ whenever $x_0 > \sqrt{2}$.

b) Use the fact that $0 < (x_0 - \sqrt{2})^2$ whenever $x_0 \neq \sqrt{2}$ to show that if $0 < x_0 < \sqrt{2}$, then $x_1 > \sqrt{2}$.

c) Use the above results to show that the sequence in (a) converges to $\sqrt{2}$ whenever $x_0 > 0$.

Solution 19

a) Let g be the function that generates the sequence $\{x_n\}$:

$$g(x) = \frac{x}{2} + \frac{1}{x} = \frac{x^2+2}{2x}$$
$$\Rightarrow g'(x) = \frac{1}{2} - \frac{1}{x^2} = \frac{x^2-2}{2x^2}$$

Consider $I = [\sqrt{2}, b]$, for any $b > \sqrt{2}$. It is clear that g and g' exists on I. Since $g'(x) \leq 0 \,\forall x \in I$, g is monotonically increasing on I.

Consider g'. x^2 is strictly increasing on I, so g' is strictly decreasing on I, therefore:

$$\frac{1}{2} > g'(x) \leq g'(\sqrt{2}) = 0 \,\forall x \in I$$
$$\Rightarrow \left|g'(x)\right| < 1 \,\forall x \in I$$

Let

$$f(x) = g(x) - x = \frac{1}{x} - \frac{x}{2}$$

$1/x$ is strictly decreasing on I, and so is $-x$. Therefore, f is strictly decreasing on I, so:

$$f(\sqrt{2}) = 0 \leq f(x) \,\forall x \in I$$

In other words, $g(x) \leq x \,\forall x \in I$. It means that for any b, $g(b) < b$. Combining with the fact that $g(\sqrt{2}) = \sqrt{2}$, it is guaranteed that:

$$g(x) \in I \,\forall x \in I$$

All the conditions of Theorem 2.4 hold, so we can apply it here: for any $x_0 \in I$, applying fixed-point method on g converges to the unique fixed point in I, using any $x_0 \in I$.

Trivially, $\sqrt{2}$ is a fixed point of g, therefore it must be the unique fixed point on I.

We can conclude that for any $x_0 > \sqrt{2}$, the sequence converges to $\sqrt{2}$.

b) When $0 < x < \sqrt{2}$, $g'(x) < 0$, which means g is monotonically decreasing. Applying this on $0 < x_0 < \sqrt{2}$ gives:

$$x_1 = g(x_0) > g(\sqrt{2}) = \sqrt{2}$$

c) We have:

- If $x_0 > \sqrt{2}$: proven.
- If $x_0 = \sqrt{2}$: it is exactly the fixed point.
- If $0 < x_0 < \sqrt{2}$: $x_1 = g(x_0) > \sqrt{2}$, then from x_1 onwards, the sequence converges to $\sqrt{2}$, as proven with the case $x_0 > \sqrt{2}$.

Therefore, we can conclude that the sequence converges to $\sqrt{2}$ whenever $x_0 > 0$.

Exercise 20

a) Show that if A is any positive number, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \text{ for } n \geq 1$$

converges to $\sqrt{A}$ whenever $x_0 > 0$.

b) What happens if $x_0 < 0$?

Solution 20

a) Let

$$\begin{aligned} g(x) &= \frac{x}{2} + \frac{A}{2x} = \frac{x^2 + A}{2x} \\ \Rightarrow g'(x) &= \frac{1}{2} - \frac{A}{2x^2} = \frac{x^2 - A}{2x^2} \end{aligned}$$

Trivially, we can find out that $\sqrt{A}$ is a fixed point of g.

Let

$$\begin{aligned} f(x) = g(x) - x &= \frac{A}{2x} - \frac{x}{2} = \frac{A - x^2}{2x} \\ \Rightarrow f'(x) &= -\frac{A}{2x^2} - \frac{1}{2} = -\frac{x^2 + A}{2x^2} \end{aligned}$$

Since $f'(x) < 0 \, \forall x \neq 0$, $f(x)$ is monotonically increasing when $x > 0$.

Consider the sign of g':

- $g'(x) < 0 \iff |x| < \sqrt{A}$
- $g'(x) = 0 \iff |x| = \sqrt{A}$

- $g'(x) > 0 \iff |x| > \sqrt{A}$

If $x > \sqrt{A}$, then:

- $g' > 0$, which means g is monotonically increasing. It follows that:

$$g(x) > g(\sqrt{A}) = \sqrt{A}$$

- $f(x) < f(\sqrt{A}) = 0$, which means $g(x) < x$, making $\{x_n\}$ a decreasing sequence.

From both of the above, we know that $\{x_n\}$ is a lower-bounded decreasing sequence, and therefore must converge:

$$\begin{aligned} x &= \lim_{n\to\infty} x_n \\ &= \lim_{n\to\infty} g(x_{n-1}) \\ &= \lim_{n\to\infty} \frac{x_{n-1}}{2} + \frac{A}{2x_{n-1}} \\ &= \frac{x}{2} + \frac{A}{2x} \\ \iff x &= \sqrt{A} \end{aligned}$$

So, for all $x_0 > \sqrt{A}$, the sequence converges to $\sqrt{A}$.

If $x = \sqrt{A}$, then $g(x) = x = \sqrt{A}$. Hence $x_n = \sqrt{A}\, \forall n \geq 0$. So, for $x_0 = \sqrt{A}$, the sequence converges to $\sqrt{A}$.

If $0 < x < \sqrt{A}$, then $g' < 0$, which means g is monotonically decreasing. It follows that:

$$g(x) > g(\sqrt{A}) = \sqrt{A}$$

So, for $0 < x_0 < \sqrt{A}$, $x_1 = g(x_0) > \sqrt{A}$, then from x_1 onwards, the sequence converges to $\sqrt{A}$, as proven with the case $x_0 > \sqrt{A}$.

We can conclude that the sequence $\{x_n\}$ converges to $\sqrt{2}$ whenever $x_0 > 0$.

b) If $x_0 < 0$, then similar to the above proof, we conclude that the sequence converges to $-\sqrt{A}$.

Exercise 21

Replace the assumption in Theorem 2.4 that "a positive number $k < 1$ exists with $\left|g(x)\right| \leq k$" with "g satisfies a Lipschitz condition on the interval $[a, b]$ with Lipschitz constant $L < 1$" (See Exercise 27, Section 1.1.) Show that the conclusions of this theorem are still valid.

Solution 21

g satisfies a Lipschitz condition on the interval $[a, b]$ with Lipschitz constant $L < 1$ means that:

$$\frac{g(x_1) - g(x_2)}{x_1 - x_2} \leq L \, \forall x_1, x_2 \in [a, b] \tag{*}$$

In the proof of Theorem 2.4, we see that:

$$|p - p_n| = \left|g(p) - g(p_{n-1})\right|$$

From the previous section of the proof, we already proved that p and p_{n-1} is in $[a, b]$. Applying (*) with $x_1 = p$, $x_2 = p_{n-1}$ gives:

$$|p - p_n| = \left|g(p) - g(p_{n-1})\right| \leq L|p - p_{n-1}|$$

Then the proof proceeds normally, replacing k with L.

Exercise 22

Suppose that g is continuously differentiable on some interval (c, d) that contains the fixed point p of g. Show that if $\left|g'(p)\right| < 1$, then there exists a $\delta > 0$ such that if $|p_0 - p| \leq \delta$, then the fixed-point iteration converges.

Solution 22

Since p is a fixed point in (c, d) of g, $g(p) = p$.

Since g' is continuous at p, according to the definition of continuity and limit, for every $\varepsilon > 0$, there exist $\delta > 0$ such that:

$$\begin{aligned} &\left|g'(x) - g'(p)\right| < \varepsilon \, \forall x \in D = [p - \delta, p + \delta] \\ \iff \; & g'(x) \in E = [g'(p) - \varepsilon, g'(p) + \varepsilon] \, \forall x \in D \end{aligned}$$

We can always choose a ε such that $E \subset (-1, 1)$. Then the proof proceeds normally, replacing $[a, b]$ with E.

Exercise 23

An object falling vertically through the air is subjected to viscous resistance as well as to the force of gravity. Assume that an object with mass m is dropped from a height s_0 and that the height of the object after t seconds is:

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}(1 - e^{-kt/m})$$

where $g = 32.17\,\text{ft/s}^2$ and k represents the coefficient of air resistance in lb/s. Suppose $s_0 = 300\,\text{ft}$, $m = 0.25\,\text{lb}$, and $k = 0.1\,\text{lb/s}$. Find, to within $0.01\,\text{s}$, the time it takes this quarter-pounder to hit the ground.

Solution 23

Replacing symbols in $s(t)$ with number gives:

$$s(t) = 501.0625 - 80.425t - 201.0625e^{-0.4t}$$

Let

$$g(t) = \frac{1}{80.425}(501.0625 - 201.0625e^{-0.4t})$$

A fixed point p of g is also a root of $s(t) = 0$, which is the time it takes the quarter-pounder to hit the ground.

Applying fixed-point method on g with $p_0 = 3$ generates the following table:

n	p_n	n	p_n
0	3	3	5.998 865 94
1	5.477 197 87	4	6.003 285 61
2	5.950 637 4		

We conclude that it takes approximately 6.003 s for the quarter-pounder to hit the ground.

Exercise 24

Let $g \in C^1[a, b]$ and p be in (a, b) with $g(p) = p$ and $\left|g'(p)\right| > 1$. Show that there exists a $\delta > 0$ such that if $0 < |p_0 - p| < \delta$, then $|p_0 - p| < |p_1 - p|$. Thus, no matter how close the initial approximation p_0 is to p, the next iterate p_1 is farther away, so the fixed-point iteration does not converge if $p_0 \neq p$.

Solution 24

This problem is similar to Exercise 22.

Since g' is continuous at p, according to the definition of continuity and limit, for every $\varepsilon > 0$, there exist $\delta > 0$ such that:

$$\begin{aligned} &\left|g'(x) - g'(p)\right| < \varepsilon \,\forall x \in D = [p - \delta, p + \delta] \\ \iff &\; g'(x) \in E = [g'(p) - \varepsilon, g'(p) + \varepsilon] \,\forall x \in D \end{aligned}$$

We can always choose a ε such that $E \subset (1, \infty)$.

If $p_0 \in D$, then according to Mean Value Theorem, there exist a $\xi \in D$ such that:

$$|p_1 - p| = \left|g(p_0) - g(p)\right| = \left|g'(\xi)\right||p_0 - p| > |p_0 - p|$$