0.1 Gauss elimination

Exercise 1

For each of the following linear systems, obtain a solution by graphical methods, if possible. Explain the results from a geometrical standpoint.

a)
$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 - x_2 &= 0 \end{aligned}$$

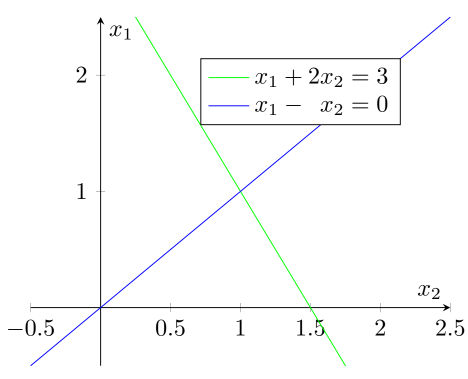
b)
$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 + 4x_2 &= 6 \end{aligned}$$

c)
$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 2x_1 + 4x_2 &= 0 \end{aligned}$$

d)
$$\begin{aligned} 2x_1 + 2x_2 &= -1 \\ 4x_1 + 2x_2 &= -2 \\ x_1 - 3x_2 &= 5 \end{aligned}$$

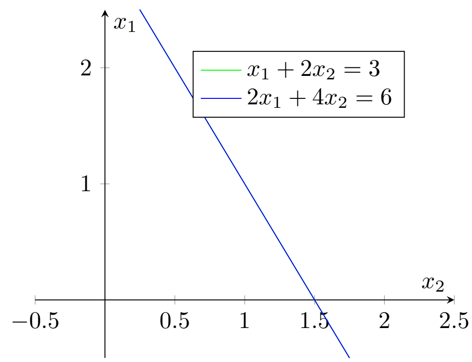
Solution 1

a) The graphs of the equations are as follow:



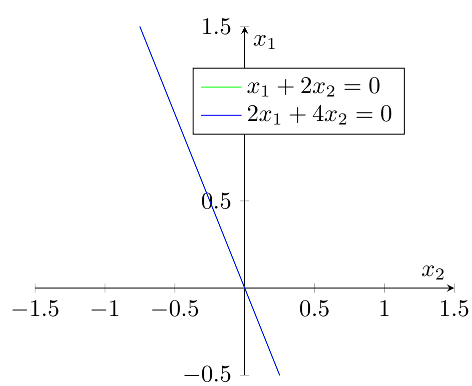
The solution is $x_1 = 1$, $x_2 = 1$ as the lines intersect at $(1, 1)$.

b) The graphs of the equations are as follow:



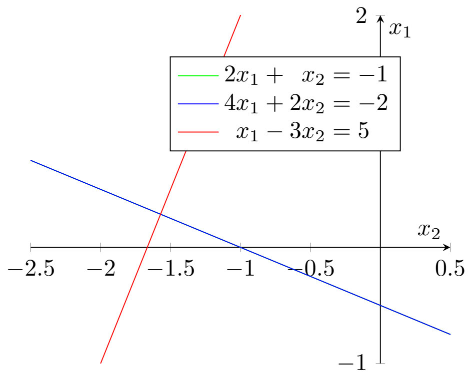
The system of equation has an infinite number of solutions, as the line coincide.

c) The graphs of the equations are as follow:



The system of equation has an infinite number of solutions, as the lines coincide.

d) The graphs of the equations are as follow:



The solution is $x_1 = -\frac{11}{7}$, $x_2 = \frac{2}{7}$ as the lines intersect at $(\frac{2}{7}, -\frac{11}{7})$.

Exercise 2

For each of the following linear systems, obtain a solution by graphical methods, if possible. Explain the results from a geometrical standpoint.

Solution 2

a)

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}$$

b)

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ -2x_1 - 4x_2 &= 6 \end{aligned}$$

c)

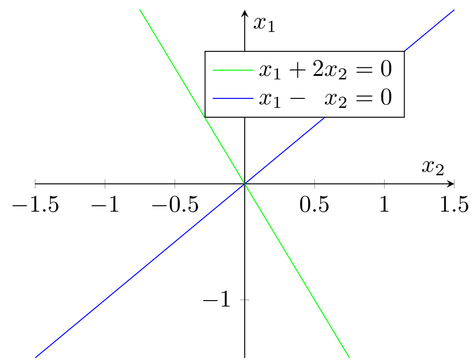
$$\begin{aligned} 2x_1 + x_2 &= -1 \\ x_1 + x_2 &= 2 \\ x_1 - 3x_2 &= 5 \end{aligned}$$

d)

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 - x_3 &= -1 \end{aligned}$$

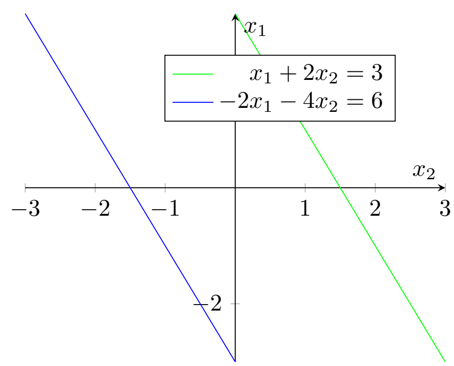
Solution 2

a) The graphs of the equations are as follow:



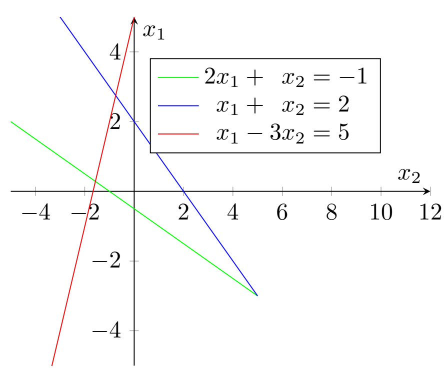
The solution is $x_1 = 0$, $x_2 = 0$ as the lines intersect at $(0, 0)$.

b) The graphs of the equations are as follow:



The system of equation has no solution, as the lines are parallel to each other.

c) The graphs of the equations are as follow:



The system of equation has no solution, as the lines do not intersect.

Exercise 3

Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations.

a)

$$\begin{aligned} 4x_1 - x_2 + x_3 &= 8 \\ 2x_1 + 5x_2 - 2x_3 &= 3 \\ x_1 + 2x_2 - 4x_3 &= 11 \end{aligned}$$

b)

$$\begin{aligned} 4x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 1x_3 &= -5 \\ x_1 + x_2 - 3x_3 &= -9 \end{aligned}$$

Solution 3

a) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{rrr:r} 4 & -1 & 1 & 8 \\ 2 & 5 & 2 & 3 \\ 1 & 2 & 4 & 11 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 0.5E_1;\ E_3 \coloneqq E_3 - 0.25E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{lll:r} 4 & -1 & 1 & 8 \\ 0 & 5.5 & 1.5 & -1 \\ 0 & 2.25 & 3.75 & 9 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - \frac{9}{22}E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{lll:l} 4 & -1 & 1 & 8 \\ 0 & 5.5 & 1.5 & -1 \\ 0 & 0 & 3.136\,36 & 9.409\,09 \end{array}\right)$$

The solution is $x_3 \approx 3$, $x_2 \approx -1$, $x_1 \approx 1$.

b) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{rrr:r} 4 & 1 & 2 & 9 \\ 2 & 4 & -1 & -5 \\ 1 & 1 & -3 & -9 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 0.5E_1;\ E_3 \coloneqq E_3 - 0.25E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{ccc:c} 4 & 1 & 2 & 9 \\ 0 & 3.5 & -2 & -9.5 \\ 0 & 0.75 & -3.5 & -11.25 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - \frac{3}{14} E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{ccc:c} 4 & 1 & 2 & 9 \\ 0 & 3.5 & -2 & -9.5 \\ 0 & 0 & -3.071\,43 & -9.214\,29 \end{array}\right)$$

The solution is $x_3 \approx 3$, $x_2 \approx -1$, $x_1 \approx 1$.

Exercise 4

Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations.

a)

$$\begin{aligned} -1x_1 + 4\,x_2 + x_3 &= 8 \\ \frac{5}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 11 \end{aligned}$$

b)

$$\begin{aligned} 4x_1 + 2\,x_2 - x_3 &= -5 \\ \frac{1}{9}x_1 + \frac{1}{9}x_2 - \frac{1}{3}x_3 &= -1 \\ 1x_1 + 4\,x_2 + 2x_3 &= 9 \end{aligned}$$

Solution 4

a) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{ccc:c} -1 & 4 & 1 & 8 \\ 1.666\,67 & 0.666\,67 & 0.666\,67 & 1 \\ 2 & 1 & 4 & 11 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - (-1.6667)E_1;\; E_3 \coloneqq E_3 - (-2)E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{ccc:c} -1 & 4 & 1 & 8 \\ 0 & 7.333\,33 & 2.333\,33 & 14.333\,33 \\ 0 & 9 & 6 & 27 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - 1.227\,27 E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{lll:l} -1 & 4 & 1 & 8 \\ 0 & 7.333\,33 & 2.333\,33 & 14.333\,33 \\ 0 & 0 & 3.136\,36 & 9.409\,09 \end{array}\right)$$

The solution is $x_3 \approx 3$, $x_2 \approx 1$, $x_1 \approx -1$.

b) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{lll:r} 4 & 2 & -1 & -5 \\ 0.111\,11 & 0.111\,11 & -0.333\,33 & -1 \\ 1 & 4 & 2 & 9 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 0.027\,78E_1;\; E_3 \coloneqq E_3 - 0.25E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{lll:l} 4 & 2 & -1 & -5 \\ 0 & 0.055\,56 & -0.305\,55 & -0.861\,11 \\ 0 & 3.5 & 2.25 & 10.25 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - 63.000\,63E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{lll:l} 4 & 2 & -1 & -5 \\ 0 & 0.055\,56 & -0.305\,55 & -0.861\,11 \\ 0 & 0 & 21.5 & 64.500\,63 \end{array}\right)$$

The solution is $x_3 \approx 3$, $x_2 \approx 1$, $x_1 \approx -1$.

Exercise 5

Use the Gaussian Elimination Algorithm to solve the following linear systems, if possible, and determine whether row interchanges are necessary:

a)

$$\begin{aligned} x_1 - 1x_2 + 3x_3 &= 2 \\ 3x_1 - 3x_2 + 1x_3 &= -1 \\ x_1 + 1x_2 - &= 3 \end{aligned}$$

b)

$$\begin{aligned} 2x_1 - 1.5x_2 + 3x_3 &= 1 \\ -1x_1 + 2x_3 &= 3 \\ 4x_1 - 4.5x_2 + 5x_3 &= 1 \end{aligned}$$

c)

$$\begin{aligned}
2x_1 \qquad\qquad\qquad\qquad &= 3\\
x_1 + 1.5x_2 \qquad\qquad\quad &= 4.5\\
-3x_2 + 0.5x_3 \qquad &= -6.6\\
2x_1 - 2\ \ x_2 + \quad x_3 + x_4 &= 0.8
\end{aligned}$$

d)

$$\begin{aligned}
x_1 + x_2 \qquad\quad + \ \ x_4 &= 2\\
2x_1 + x_2 - \ \ x_3 + \ \ x_4 &= 1\\
4 \quad - x_2 - 2x_3 + 2 \quad &= 0\\
3x_1 - x_2 - \ \ x_3 + 2x_4 &= -3
\end{aligned}$$

Solution 5

a) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{rrr:r} 1 & -1 & 3 & 2\\ 3 & -3 & 1 & -1\\ 1 & 1 & 0 & 3 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 3E_1;\ E_3 \coloneqq E_3 - 1E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{rrr:r} 1 & -1 & 3 & 2\\ 0 & 0 & -8 & -7\\ 0 & 2 & -3 & 1 \end{array}\right)$$

As $a_{22}^{(2)} = 0$, we have to swap row 2 and 3. Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - 63.000\,63E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{rrr:r} 1 & -1 & 3 & 2\\ 0 & 2 & -3 & 1\\ 0 & 0 & -8 & -7 \end{array}\right)$$

The solution is $x_3 = 0.875$, $x_2 = 1.8125$, $x_1 = 1.1875$.

b) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{rrr:r} 2 & -1.5 & 3 & 1\\ -1 & 0 & 2 & 3\\ 4 & -4.5 & 5 & 1 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - (-0.5)E_1;\ E_3 \coloneqq E_3 - 2E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{rrr:r} 2 & -1.5 & 3 & 1\\ 0 & -0.75 & 3.5 & 3.5\\ 0 & -1.5 & -1 & -1 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - 2E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{cccc} 2 & -1.5 & 3 & \vdots & 1 \\ 0 & -0.75 & 3.5 & \vdots & 3.5 \\ 0 & 0 & -8 & \vdots & -8 \end{array}\right)$$

The solution is $x_3 = 1$, $x_2 = 0$, $x_1 = -1$.

c) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{ccccc} 2 & 0 & 0 & 0 & \vdots & 3 \\ 1 & 1.5 & 0 & 0 & \vdots & 4.5 \\ 0 & -3 & 0.5 & 0 & \vdots & -6.6 \\ 2 & -2 & 1 & 1 & \vdots & 0.8 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 0.5E_1;\ E_3 \coloneqq E_3 - 0E_1;\ E_4 \coloneqq E_4 - 1E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{ccccc} 2 & 0 & 0 & 0 & \vdots & 3 \\ 0 & 1.5 & 0 & 0 & \vdots & 3 \\ 0 & -3 & 0.5 & 0 & \vdots & -6.6 \\ 0 & -2 & 1 & 1 & \vdots & -2.2 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - (-2)E_2;\ E_4 \coloneqq E_4 - (-1.333\,33)E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{ccccc} 2 & 0 & 0 & 0 & \vdots & 3 \\ 0 & 1.5 & 0 & 0 & \vdots & 3 \\ 0 & 0 & 0.5 & 0 & \vdots & -0.6 \\ 0 & 0 & 1 & 1 & \vdots & 1.8 \end{array}\right)$$

Eliminating x_3 by these transformation

$$E_4 \coloneqq E_4 - 2E_3$$

gives:

$$\tilde{\boldsymbol{A}}^{(4)} = \left(\begin{array}{ccccc} 2 & 0 & 0 & 0 & \vdots & 3 \\ 0 & 1.5 & 0 & 0 & \vdots & 3 \\ 0 & 0 & 0.5 & 0 & \vdots & -0.6 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{array}\right)$$

The solution is $x_4 = 3$, $x_3 = -1.2$, $x_2 = 2$, $x_1 = 1.5$.

d) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{rrrr|r} 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & -1 & 1 & 1 \\ 4 & -1 & -2 & 2 & 0 \\ 3 & -1 & -1 & 2 & -3 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 2E_1;\ E_3 \coloneqq E_3 - 4E_1;\ E_4 \coloneqq E_4 - 3E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{rrrr|r} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & -5 & -2 & -2 & -8 \\ 0 & -4 & -1 & -1 & -9 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - 5E_2;\ E_4 \coloneqq E_4 - 4E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{rrrr|r} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 0 & 3 & 3 & 7 \\ 0 & 0 & 3 & 3 & 3 \end{array}\right)$$

Eliminating x_3 by these transformation

$$E_4 \coloneqq E_4 - E_3$$

gives:

$$\tilde{\boldsymbol{A}}^{(4)} = \left(\begin{array}{rrrr|r} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 0 & 3 & 3 & 7 \\ 0 & 0 & 0 & 0 & -4 \end{array}\right)$$

The system has no unique solution.

Exercise 6

Use the Gaussian Elimination Algorithm to solve the following linear systems, if possible, and determine whether row interchanges are necessary:

a)

$$\begin{aligned} x_2 - 2x_3 &= 4 \\ x_1 - 3x_2 + x_3 &= 6 \\ x_1 \qquad - x_3 &= 2 \end{aligned}$$

b)

$$\begin{aligned} x_1 - 0.5 \quad + \quad x_3 \qquad &= 4 \\ 2x_1 - x_2 - x_3 + x_4 &= 5 \\ x_1 + x_2 + 0.5x_3 \qquad &= 2 \\ x_1 - 0.5x_2 + x_3 + x_4 &= 5 \end{aligned}$$

c)

$$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 6 \\ x_2 - x_3 + x_4 &= 5 \\ x_4 &= 5 \\ x_3 - x_4 &= 3 \end{aligned}$$

d)

$$\begin{aligned} x_1 + \ x_2 \qquad + \ x_4 &= \ \ 2 \\ 2x_1 + \ x_2 - \ x_3 + \ x_4 &= \ \ 1 \\ -1x_1 + 2x_2 + 3x_3 - \ x_4 &= \ \ 4 \\ 3x_1 - \ x_2 - \ x_3 + 2x_4 &= -3 \end{aligned}$$

Solution 6

a) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{rrr:r} 0 & 1 & -2 & 4 \\ 1 & -1 & 1 & 6 \\ 1 & 0 & -1 & 2 \end{array}\right)$$

As $a_{11}^{(1)} = 0$, we need to swap row 1 and 2. Eliminating x_1 by these transformation

$$E_3 \coloneqq E_3 - E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{rrr:r} 1 & -1 & 1 & 6 \\ 0 & 1 & -2 & 4 \\ 0 & 1 & -2 & -4 \end{array}\right)$$

As $a_{22}^{(2)} = 0$, we have to swap row 2 and 3. Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{rrr:r} 1 & -1 & 1 & 6 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & -8 \end{array}\right)$$

The system has no unique solution.

b) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{rrrr:r} 1 & -0.5 & 1 & 0 & 4 \\ 2 & -1 & -1 & 1 & 5 \\ 1 & 1 & 0.5 & 0 & 2 \\ 1 & -0.5 & 1 & 1 & 5 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 2E_1;\ E_3 \coloneqq E_3 - E_1;\ E_4 \coloneqq E_4 - E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{cccc:c} 1 & -0.5 & 1 & 0 & 4 \\ 0 & 0 & -3 & 1 & -3 \\ 0 & 1.5 & -0.5 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

As $a_{22}^{(2)} = 0$, we need to swap row 2 and 3, effectively eliminating x_2 and x_3:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{cccc:c} 1 & -0.5 & 1 & 0 & 4 \\ 0 & 1.5 & -0.5 & 0 & -2 \\ 0 & 0 & -3 & 1 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

The solution is $x_4 = 1$, $x_3 \approx 1.333\,33$, $x_2 \approx -0.888\,89$, $x_1 \approx 2.222\,22$.

c) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{cccc:c} 2 & -1 & 1 & -1 & 6 \\ 0 & 1 & -1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right)$$

x_1 and x_2 are already eliminated. As $a_{33}^{(3)} = 0$, we need to swap row 3 and 4, effectively eliminating x_3:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{cccc:c} 2 & -1 & 1 & -1 & 6 \\ 0 & 1 & -1 & 1 & 5 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right)$$

The solution is $x_4 = 5$, $x_3 = 8$, $x_2 = 8$, $x_1 = 5.5$.

d) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & -1 & 1 & 1 \\ -1 & 2 & 3 & -1 & 4 \\ 3 & -1 & -1 & 2 & -3 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 2E_1;\ E_3 \coloneqq E_3 - (-1)E_1;\ E_4 \coloneqq E_4 - 3E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & -4 & -1 & -1 & -9 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - (-3)E_2;\ E_4 \coloneqq E_4 - 4E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{rrrr:r} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 0 & 0 & -3 & -3 \\ 0 & 0 & 3 & 3 & 3 \end{array}\right)$$

As $a_{33}^{(3)} = 0$, we need to swap row 3 and 4, effectively eliminating x_3:

$$\tilde{\boldsymbol{A}}^{(4)} = \left(\begin{array}{rrrr:r} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right)$$

The solution is $x_4 = 1$, $x_3 = 0$, $x_2 = 2$, $x_1 = -1$.

Exercise 7

Use Algorithm 6.1 and Maple with Digits:= 10 to solve the following linear systems ...

Solution 7

Opps, can't help without Maple license.

Exercise 8

Use Algorithm 6.1 and Maple with Digits:= 10 to solve the following linear systems ...

Solution 8

Opps, can't help without Maple license.

Exercise 9

Given the linear system

$$\begin{aligned} 2x_1 - 6\alpha x_2 &= 3 \\ 3\alpha x_1 - \quad x_2 &= 1.5 \end{aligned}$$

a) Find value(s) of α for which the system has no solutions.

b) Find value(s) of α for which the system has an infinite number of solutions.

c) Assuming a unique solution exists for a given α, find the solution.

Solution 9

Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \begin{pmatrix} 2 & -6\alpha & \vdots & 3 \\ 3\alpha & -1 & \vdots & 1.5 \end{pmatrix}$$

Eliminating x_1 gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \begin{pmatrix} 2 & -6\alpha & \vdots & 3 \\ 0 & 9\alpha^2 - 1 & \vdots & 1.5 - 4.5\alpha \end{pmatrix}$$

The system has no unique solution (either no solution or infinite number of solutions) if and only if:

$$9\alpha^2 - 1 = 0 \iff \alpha = \pm\frac{1}{3}$$

a) The system has no solution if it has no unique solution and

$$1.5(1 - 3\alpha) \neq 0 \iff \alpha = -\frac{1}{3}$$

b) The system has an infinite number of solution if it has no unique solution and

$$1.5(1 - 3\alpha) = 0 \iff \alpha = \frac{1}{3}$$

In this case, the solution assumes a general form:

$$x_2 \in \mathbb{R} \text{ and } x_1 = x_2 + 1.5$$

c) The system has a unique solution if and only if $\alpha \neq \pm\frac{1}{3}$. Then the unique solution is:

$$x_2 = \frac{-1.5}{3\alpha + 1} \text{ and } x_1 = \frac{1.5}{3\alpha + 1}$$

Exercise 10

Given the linear system

$$\begin{aligned} x_1 - x_2 + \alpha x_3 &= -2 \\ -x_1 + 2x_2 - \alpha x_3 &= 3 \\ \alpha x_1 + x_2 + \alpha x_3 &= 2 \end{aligned}$$

a) Find value(s) of α for which the system has no solutions.

b) Find value(s) of α for which the system has an infinite number of solutions.

c) Assuming a unique solution exists for a given α, find the solution.

Solution 10

Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{ccc|c} 1 & -1 & \alpha & -2 \\ -1 & 2 & -\alpha & 3 \\ \alpha & 1 & \alpha & 2 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - (-1)E_1;\ E_3 := E_3 - \alpha E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{ccc|c} 1 & -1 & \alpha & -2 \\ 0 & 1 & 0 & 1 \\ 0 & \alpha+1 & \alpha-\alpha^2 & 2\alpha+2 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 := E_3 - (\alpha+1)E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{ccc|c} 1 & -1 & \alpha & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \alpha-\alpha^2 & \alpha+1 \end{array}\right)$$

The system has no unique solution (either no solution or infinite number of solutions) if and only if:

$$\alpha - \alpha^2 = 0 \iff \alpha \in \{0, 1\}$$

a) The system has no solution if it has no unique solution and

$$2\alpha + 2 \neq 0 \iff \alpha \in \{0, 1\}$$

b) The system has an infinite number of solution if it has no unique solution and

$$2\alpha + 2 \neq 0 \iff \alpha \in \varnothing$$

c) The system has a unique solution if and only if $\alpha \notin \{0, 1\}$. Then the unique solution is:

$$x_3 = \frac{\alpha+1}{\alpha-\alpha^2}\ ,\ x_2 = 1 \text{ and } x_1 = \frac{2}{\alpha-1}$$

Exercise 11

Show that the 3 elementary row operations do not change the solution set of a linear system.

Solution 11

Let $x_1, x_2, \ldots, x_n$ be the solution of the original system.

When an elementary row operations is applied on row i^{th}, the original solution still satisfies the unchanged rows. We have to proove that it also satisfies the changed row.

a) If i^{th} row is scaled, i^{th} equation is still satisfied by the original solution because both size of it is multiplied with a constant.

b) If a scaled j^{th} row is added to i^{th} row, then the original solution still satisfies the new row, as

- it satisfies the j^{th} row, therefore satisfies the scaled j^{th} row, as proven above, and
- it satisfies the original i^{th} row

c) If the rows are swapped, the solution does not change, as the set of the equation does not change.

Exercise 12

Gauss-Jordan Method: This method is described as follows. Use the i^{th} equation to eliminate not only x_i from the equations $E_{>i}$ as was done in the Gaussian elimination method, but also from $E_{<i}$. Upon reducing $[\boldsymbol{A}, \boldsymbol{b}]$ to:

$$\left(\begin{array}{cccc:c} a_{11}^{(1)} & & & & b_1^{(1)} \\ & a_{22}^{(2)} & & & b_2^{(2)} \\ & & \ddots & & \vdots \\ & & & a_{nn}^{(n)} & b_n^{(n)} \end{array}\right)$$

the solution can be obtained by

$$x_i = \frac{b_i^{(i)}}{a_{ii}^{(i)}}$$

This procedure circumvents the backward substitution in the Gaussian elimination. Construct an algorithm for the Gauss-Jordan procedure patterned after that of Algorithm 6.1.

Solution 12

In Step 4, change j from $j > i$ to $j \neq i$.

In Step 8, calculate for all i:

$$x_i = \frac{b_i}{a_{ii}}$$

Remove Step 9.

Exercise 13

Use the Gauss-Jordan method and two-digit rounding arithmetic to solve the systems in Exercise 3.

Solution 13

a) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{lrr|r} 4 & -1 & 1 & 8 \\ 2 & 5 & 2 & 3 \\ 1 & 2 & 4 & 11 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 0.5E_1;\ E_3 \coloneqq E_3 - 0.25E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{lrl|r} 4 & -1 & 1 & 8 \\ 0 & 5.5 & 1.5 & -1 \\ 0 & 2.25 & 3.75 & 9 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_1 \coloneqq E_1 - (-0.18182)E_2;\ E_3 \coloneqq E_3 - 0.40909E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{lll|l} 4 & 0 & 1.272\,73 & 7.818\,18 \\ 0 & 5.5 & 1.5 & -1 \\ 0 & 0 & 3.136\,36 & 9.409\,09 \end{array}\right)$$

Eliminating x_3 by these transformation

$$E_1 \coloneqq E_1 - 0.40580E_3;\ E_2 \coloneqq E_2 - 0.47826E_3$$

gives:

$$\tilde{\boldsymbol{A}}^{(4)} = \left(\begin{array}{lll|l} 4 & 0 & 0 & 4 \\ 0 & 5.5 & 0 & -5.5 \\ 0 & 0 & 3.136\,36 & 9.409\,09 \end{array}\right)$$

The solution is $x_3 \approx 3$, $x_2 \approx -1$, $x_1 \approx 1$.

b) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{llr|r} 4 & 1 & 2 & 9 \\ 2 & 4 & -1 & -5 \\ 1 & 1 & -3 & -9 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 0.50000E_1;\ E_3 \coloneqq E_3 - 0.25000E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{ccc:c} 4 & 1 & 2 & 9 \\ 0 & 3.5 & -2 & -9.5 \\ 0 & 0.75 & -3.5 & -11.25 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_1 \coloneqq E_1 - 0.28571E_2;\ E_3 \coloneqq E_3 - 0.21429E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{ccc:c} 4 & 0 & 2.571\,43 & 11.714\,29 \\ 0 & 3.5 & -2 & -9.5 \\ 0 & 0 & -3.071\,43 & -9.214\,29 \end{array}\right)$$

Eliminating x_3 by these transformation

$$E_1 \coloneqq E_1 - (-0.83721)E_3;\ E_2 \coloneqq E_2 - 0.65116E_3$$

gives:

$$\tilde{\boldsymbol{A}}^{(4)} = \left(\begin{array}{ccc:c} 4 & 0 & 0 & 4 \\ 0 & 3.5 & 0 & -3.5 \\ 0 & 0 & -3.071\,43 & -9.214\,29 \end{array}\right)$$

The solution is $x_3 \approx 3$, $x_2 \approx -1$, $x_1 \approx 1$.

Exercise 14

Repeat Exercise 7 using the Gauss-Jordan method.

Solution 14

Opps, can't help without Maple license.

Exercise 15

a) Show that the Gauss-Jordan method requires

$$\frac{n^3}{2} + n^2 - \frac{n}{2} \text{ multiplications/divisions}$$

and

$$\frac{n^3}{2} - \frac{n}{2} \text{ additions/subtractions}$$

b) Make a table comparing the required operations for the Gauss-Jordan and Gaussian elimination methods for $n = 3, 10, 50, 100$. Which method requires less computation?

Solution 15

a) We have the following analysis:

- In Step 1, i iterates from 1 to n, so there is n iterations. Inside each iteration:
  - In Step 4, j iterates from 1 to n but skips i, so there is $n-1$ iterations. Inside each iteration:
    * In Step 5: 1 divisions
    * In Step 6: $n+1$ multiplications; $n+1$ subtractions. However, some operations with or known to results in 0 could be skipped, therefore, Step 6 requires $n+1-i$ multiplications and $n+1-i$ subtractions.

    Therefore, in each Step 4 iteration, there are $n-i+2$ multiplications/divisions and $n-i+1$ subtractions.

  Therefore, in each Step 1 iteration, there are $(n-1)(n-i+2)$ multiplications/divisions and $(n-1)(n-i+1)$ subtractions

  Therefore, in all Step 1 iterations, there are

  $$\sum_{i=1}^{n}(n-1)(n-i+2) = (n-1)\left[n(n+2) - \sum_{i=1}^{n} i\right]$$
  $$= \frac{n^3+2n^2-3n}{2} \text{ multiplications/divisions}$$

  and

  $$\sum_{i=1}^{n}(n-1)(n-i+1) = (n-1)\left[n(n+1) - \sum_{i=1}^{n} i\right]$$
  $$= \frac{n^3-n}{2} \text{ subtractions}$$

- In Step 9, i iterates from 1 to n, so there is n iterations. Inside each iteration, there is only 1 divisions. Therefore, in all Step 9 divisions, there are n divisions.

We can now conclude that Gauss-Jordan requires

$$\frac{n^3+2n^2-3n}{2} + n = \frac{n^3}{2} + n^2 - \frac{n}{2} \text{ multiplications/divisions}$$

and

$$\frac{n^3}{2} - \frac{n}{2} \text{ additions/subtractions}$$

Note that in most simple implementation, the cost of branching code to skip operations (for example in Step 6 of this analysis) is greater than the save from skipping operations itself. Therefore, a well-vectorized implementation, though requiring even more computation, turns out to outperform a "skip" implementation.

b) We have the following table:

	Gauss Elimination		Gauss-Jordan	
n	M/D	A/S	M/D	A/S
3	17	11	21	12
10	430	375	595	495
50	44150	42875	64975	62475
100	343300	338250	509950	499950

Obviously, Gauss Elimination requires less computation.

Exercise 16

Consider the following Gaussian-elimination-Gauss-Jordan hybrid method for solving system of equations. First, apply the Gaussian-elimination technique to reduce the system to triangular form. Then use the n^{th} equation to eliminate the coefficients of x_n in each of the first $n-1$ rows. After this is completed use the $(n-1)^{th}$ equation to eliminate the coefficients of x_{n-1} in the first $n-2$ rows, etc. The system will eventually appear as the reduced system in Exercise 12.

a) Show that this method requires

$$\frac{n^3}{3} + \frac{3n^2}{2} - \frac{5n}{6} \text{ multiplications/divisions}$$

and

$$\frac{n^3}{2} + \frac{n^2}{2} - \frac{5n}{6} \text{ additions/subtractions}$$

b) Make a table comparing the required operations for the Gaussian elimination, Gauss-Jordan, and hybrid methods, for $n = 3, 10, 50, 100$. Which method requires less computation?

Solution 16

a) We have the following analysis:

- Gauss elimination to upper triangular form: takes

$$\frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6} \text{ multiplications/divisions}$$

and

$$\frac{n^3}{3} - \frac{n}{3}\text{additions/subtractions}$$

- Use the i^{th} equation to eliminate x_i in each of the first $i-1$ rows, starting with $i = n$: Let i iterates from n to 2, so there is $n-1$ iterations.

  Inside each iteration, x_i is eliminated from $(i-1)^{th}$ equation to the first one. We only need to update the last column, as most operations with, or results in 0 is skipped. So, there is 1 division (for multiplier), 1 multiplication (scale row, or in fact last element of the row), 1 subtraction (elimination).

  Therefore, in all iterations of this step, there are

  $$\sum_{i=2}^{n} 2(i-1) = 2\sum_{i=1}^{n-1} i = n(n-1) \text{ multiplications/divisions}$$

  and

  $$\sum_{i=2}^{n} (2i) = \frac{n(n-1)}{2} \text{ multiplications/divisions}$$

- The last step of solving diagonal matrix takes n divisions

We can now conclude that the hybrid methods takes

$$\frac{n^3}{3} + \frac{3n^2}{2} - \frac{5n}{6} \text{ multiplications/divisions}$$

and

$$\frac{n^3}{2} + \frac{n^2}{2} - \frac{5n}{6} \text{ additions/subtractions}$$

b) We have the following table:

	Gauss Elimination		Gauss-Jordan		Hybrid	
n	M/D	A/S	M/D	A/S	M/D	A/S
3	17	11	21	12	20	11
10	430	375	595	495	475	375
50	44150	42875	64975	62475	45375	42875
100	343300	338250	509950	499950	348250	338250