0.1 Gauss elimination

Exercise 1

For each of the following linear systems, obtain a solution by graphical methods, if possible. Explain the results from a geometrical standpoint.

a)
$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 - x_2 &= 0 \end{aligned}$$

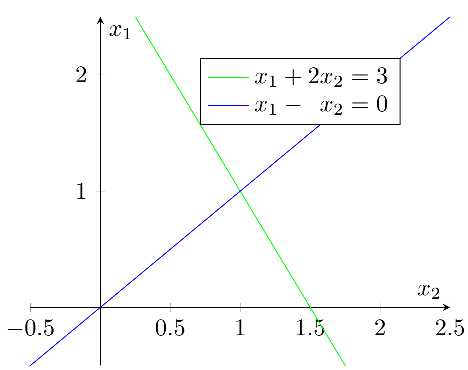
b)
$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 + 4x_2 &= 6 \end{aligned}$$

c)
$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 2x_1 + 4x_2 &= 0 \end{aligned}$$

d)
$$\begin{aligned} 2x_1 + 2x_2 &= -1 \\ 4x_1 + 2x_2 &= -2 \\ x_1 - 3x_2 &= 5 \end{aligned}$$

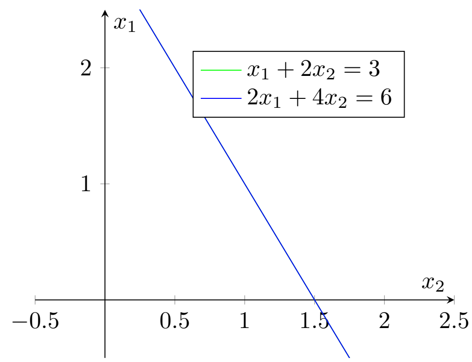
Solution 1

a) The graphs of the equations are as follow:



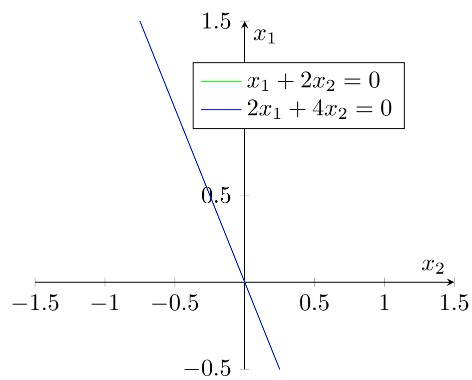
The solution is $x_1 = 1$, $x_2 = 1$ as the lines intersect at $(1, 1)$.

b) The graphs of the equations are as follow:



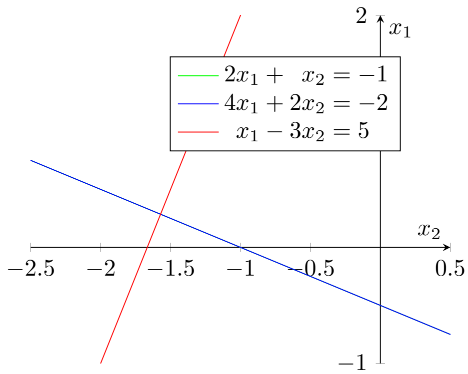
The system of equation has an infinite number of solutions, as the line coincide.

c) The graphs of the equations are as follow:



The system of equation has an infinite number of solutions, as the lines coincide.

d) The graphs of the equations are as follow:



The solution is $x_1 = -\frac{11}{7}$, $x_2 = \frac{2}{7}$ as the lines intersect at $(\frac{2}{7}, -\frac{11}{7})$.

Exercise 2

For each of the following linear systems, obtain a solution by graphical methods, if possible. Explain the results from a geometrical standpoint.

Solution 2

a)

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}$$

b)

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ -2x_1 - 4x_2 &= 6 \end{aligned}$$

c)

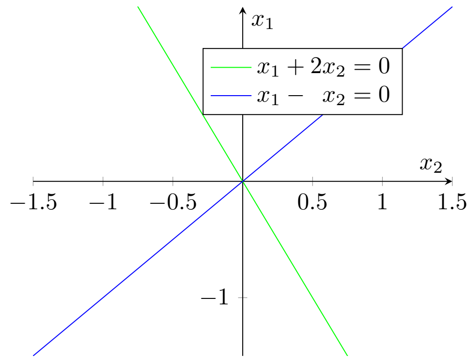
$$\begin{aligned} 2x_1 + x_2 &= -1 \\ x_1 + x_2 &= 2 \\ x_1 - 3x_2 &= 5 \end{aligned}$$

d)

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 - x_3 &= -1 \end{aligned}$$

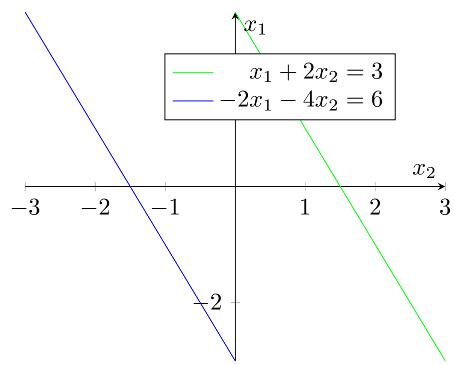
Solution 2

a) The graphs of the equations are as follow:



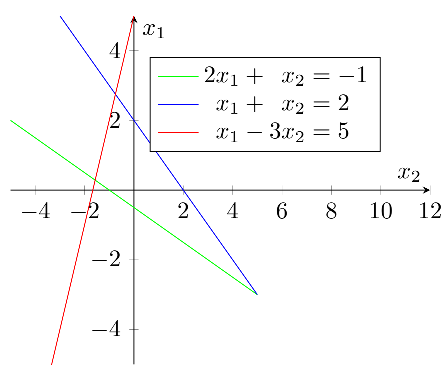
The solution is $x_1 = 0$, $x_2 = 0$ as the lines intersect at $(0, 0)$.

b) The graphs of the equations are as follow:



The system of equation has no solution, as the lines are parallel to each other.

c) The graphs of the equations are as follow:



The system of equation has no solution, as the lines do not intersect.

Exercise 3

Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations.

a)

$$\begin{aligned} 4x_1 - x_2 + x_3 &= 8 \\ 2x_1 + 5x_2 - 2x_3 &= 3 \\ x_1 + 2x_2 - 4x_3 &= 11 \end{aligned}$$

b)

$$\begin{aligned} 4x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 1x_3 &= -5 \\ x_1 + x_2 - 3x_3 &= -9 \end{aligned}$$

Solution 3

a) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{rrr|r} 4 & -1 & 1 & 8 \\ 2 & 5 & 2 & 3 \\ 1 & 2 & 4 & 11 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 0.5E_1;\ E_3 \coloneqq E_3 - 0.25E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{lll|r} 4 & -1 & 1 & 8 \\ 0 & 5.5 & 1.5 & -1 \\ 0 & 2.25 & 3.75 & 9 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - \frac{9}{22}E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{lll|l} 4 & -1 & 1 & 8 \\ 0 & 5.5 & 1.5 & -1 \\ 0 & 0 & 3.136\,36 & 9.409\,09 \end{array}\right)$$

The solution is $x_3 \approx 3$, $x_2 \approx -1$, $x_1 \approx 1$.

b) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{rrr|r} 4 & 1 & 2 & 9 \\ 2 & 4 & -1 & -5 \\ 1 & 1 & -3 & -9 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 0.5E_1;\ E_3 \coloneqq E_3 - 0.25E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 0 & 3.5 & -2 & -9.5 \\ 0 & 0.75 & -3.5 & -11.25 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - \frac{3}{14} E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 0 & 3.5 & -2 & -9.5 \\ 0 & 0 & -3.071\,43 & -9.214\,29 \end{array}\right)$$

The solution is $x_3 \approx 3$, $x_2 \approx -1$, $x_1 \approx 1$.

Exercise 4

Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations.

a)

$$\begin{aligned} -1x_1 + 4\,x_2 + x_3 &= 8 \\ \frac{5}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 11 \end{aligned}$$

b)

$$\begin{aligned} 4x_1 + 2\,x_2 - x_3 &= -5 \\ \frac{1}{9}x_1 + \frac{1}{9}x_2 - \frac{1}{3}x_3 &= -1 \\ 1x_1 + 4\,x_2 + 2x_3 &= 9 \end{aligned}$$

Solution 4

a) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ 1.666\,67 & 0.666\,67 & 0.666\,67 & 1 \\ 2 & 1 & 4 & 11 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - (-1.6667)E_1;\; E_3 \coloneqq E_3 - (-2)E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{ccc|c} -1 & 4 & 1 & 8 \\ 0 & 7.333\,33 & 2.333\,33 & 14.333\,33 \\ 0 & 9 & 6 & 27 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - 1.227\,27 E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{lll:l} -1 & 4 & 1 & 8 \\ 0 & 7.333\,33 & 2.333\,33 & 14.333\,33 \\ 0 & 0 & 3.136\,36 & 9.409\,09 \end{array}\right)$$

The solution is $x_3 \approx 3$, $x_2 \approx 1$, $x_1 \approx -1$.

b) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{lll:r} 4 & 2 & -1 & -5 \\ 0.111\,11 & 0.111\,11 & -0.333\,33 & -1 \\ 1 & 4 & 2 & 9 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 0.027\,78E_1;\ E_3 \coloneqq E_3 - 0.25E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{lll:l} 4 & 2 & -1 & -5 \\ 0 & 0.055\,56 & -0.305\,55 & -0.861\,11 \\ 0 & 3.5 & 2.25 & 10.25 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - 63.000\,63E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{lll:l} 4 & 2 & -1 & -5 \\ 0 & 0.055\,56 & -0.305\,55 & -0.861\,11 \\ 0 & 0 & 21.5 & 64.500\,63 \end{array}\right)$$

The solution is $x_3 \approx 3$, $x_2 \approx 1$, $x_1 \approx -1$.

Exercise 5

Use the Gaussian Elimination Algorithm to solve the following linear systems, if possible, and determine whether row interchanges are necessary:

a)

$$\begin{aligned} x_1 - 1x_2 + 3x_3 &= 2 \\ 3x_1 - 3x_2 + 1x_3 &= -1 \\ x_1 + 1x_2 - &= 3 \end{aligned}$$

b)

$$\begin{aligned} 2x_1 - 1.5x_2 + 3x_3 &= 1 \\ -1x_1 + 2x_3 &= 3 \\ 4x_1 - 4.5x_2 + 5x_3 &= 1 \end{aligned}$$

c)

$$\begin{aligned}
2x_1 \qquad\qquad\qquad\qquad &= \quad 3 \\
x_1 + 1.5x_2 \qquad\qquad &= \quad 4.5 \\
-3x_2 + 0.5x_3 \qquad &= -6.6 \\
2x_1 - 2\ \ x_2 + \quad x_3 + x_4 &= \quad 0.8
\end{aligned}$$

d)

$$\begin{aligned}
x_1 + x_2 \qquad + \ \ x_4 &= \ \ 2 \\
2x_1 + x_2 - \ \ x_3 + \ \ x_4 &= \ \ 1 \\
4 \quad - x_2 - 2x_3 + 2 \quad &= \ \ 0 \\
3x_1 - x_2 - \ \ x_3 + 2x_4 &= -3
\end{aligned}$$

Solution 5

a) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \begin{pmatrix} 1 & -1 & 3 & \vdots & 2 \\ 3 & -3 & 1 & \vdots & -1 \\ 1 & 1 & 0 & \vdots & 3 \end{pmatrix}$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 3E_1;\ E_3 \coloneqq E_3 - 1E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \begin{pmatrix} 1 & -1 & 3 & \vdots & 2 \\ 0 & 0 & -8 & \vdots & -7 \\ 0 & 2 & -3 & \vdots & 1 \end{pmatrix}$$

As $a_{22}^{(2)} = 0$, we have to swap row 2 and 3. Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - 63.000\,63E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \begin{pmatrix} 1 & -1 & 3 & \vdots & 2 \\ 0 & 2 & -3 & \vdots & 1 \\ 0 & 0 & -8 & \vdots & -7 \end{pmatrix}$$

The solution is $x_3 = 0.875$, $x_2 = 1.8125$, $x_1 = 1.1875$.

b) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \begin{pmatrix} 2 & -1.5 & 3 & \vdots & 1 \\ -1 & 0 & 2 & \vdots & 3 \\ 4 & -4.5 & 5 & \vdots & 1 \end{pmatrix}$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - (-0.5)E_1;\ E_3 \coloneqq E_3 - 2E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \begin{pmatrix} 2 & -1.5 & 3 & \vdots & 1 \\ 0 & -0.75 & 3.5 & \vdots & 3.5 \\ 0 & -1.5 & -1 & \vdots & -1 \end{pmatrix}$$

Eliminating x_2 by these transformation

$$E_3 := E_3 - 2E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{ccc:c} 2 & -1.5 & 3 & 1 \\ 0 & -0.75 & 3.5 & 3.5 \\ 0 & 0 & -8 & -8 \end{array}\right)$$

The solution is $x_3 = 1$, $x_2 = 0$, $x_1 = -1$.

c) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{cccc:c} 2 & 0 & 0 & 0 & 3 \\ 1 & 1.5 & 0 & 0 & 4.5 \\ 0 & -3 & 0.5 & 0 & -6.6 \\ 2 & -2 & 1 & 1 & 0.8 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - 0.5E_1;\ E_3 := E_3 - 0E_1;\ E_4 := E_4 - 1E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{cccc:c} 2 & 0 & 0 & 0 & 3 \\ 0 & 1.5 & 0 & 0 & 3 \\ 0 & -3 & 0.5 & 0 & -6.6 \\ 0 & -2 & 1 & 1 & -2.2 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 := E_3 - (-2)E_2;\ E_4 := E_4 - (-1.333\,33)E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{cccc:c} 2 & 0 & 0 & 0 & 3 \\ 0 & 1.5 & 0 & 0 & 3 \\ 0 & 0 & 0.5 & 0 & -0.6 \\ 0 & 0 & 1 & 1 & 1.8 \end{array}\right)$$

Eliminating x_3 by these transformation

$$E_4 := E_4 - 2E_3$$

gives:

$$\tilde{\boldsymbol{A}}^{(4)} = \left(\begin{array}{cccc:c} 2 & 0 & 0 & 0 & 3 \\ 0 & 1.5 & 0 & 0 & 3 \\ 0 & 0 & 0.5 & 0 & -0.6 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

The solution is $x_4 = 3$, $x_3 = -1.2$, $x_2 = 2$, $x_1 = 1.5$.

d) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & -1 & 1 & 1 \\ 4 & -1 & -2 & 2 & 0 \\ 3 & -1 & -1 & 2 & -3 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - 2E_1;\ E_3 := E_3 - 4E_1;\ E_4 := E_4 - 3E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & -5 & -2 & -2 & -8 \\ 0 & -4 & -1 & -1 & -9 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 := E_3 - 5E_2;\ E_4 := E_4 - 4E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 0 & 3 & 3 & 7 \\ 0 & 0 & 3 & 3 & 3 \end{array}\right)$$

Eliminating x_3 by these transformation

$$E_4 := E_4 - E_3$$

gives:

$$\tilde{\boldsymbol{A}}^{(4)} = \left(\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 0 & 3 & 3 & 7 \\ 0 & 0 & 0 & 0 & -4 \end{array}\right)$$

The system has no unique solution.

Exercise 6

Use the Gaussian Elimination Algorithm to solve the following linear systems, if possible, and determine whether row interchanges are necessary:

a)

$$\begin{aligned} x_2 - 2x_3 &= 4 \\ x_1 - 3x_2 + x_3 &= 6 \\ x_1 - x_3 &= 2 \end{aligned}$$

b)

$$\begin{aligned} x_1 - 0.5 + x_3 &= 4 \\ 2x_1 - x_2 - x_3 + x_4 &= 5 \\ x_1 + x_2 + 0.5x_3 &= 2 \\ x_1 - 0.5x_2 + x_3 + x_4 &= 5 \end{aligned}$$

c)

$$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 6 \\ x_2 - x_3 + x_4 &= 5 \\ x_4 &= 5 \\ x_3 - x_4 &= 3 \end{aligned}$$

d)

$$\begin{array}{rcrcrcrcr} x_1 &+& x_2 & & &+& x_4 &=& 2 \\ 2x_1 &+& x_2 &-& x_3 &+& x_4 &=& 1 \\ -1x_1 &+& 2x_2 &+& 3x_3 &-& x_4 &=& 4 \\ 3x_1 &-& x_2 &-& x_3 &+& 2x_4 &=& -3 \end{array}$$

Solution 6

a) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{rrr:r} 0 & 1 & -2 & 4 \\ 1 & -1 & 1 & 6 \\ 1 & 0 & -1 & 2 \end{array}\right)$$

As $a_{11}^{(1)} = 0$, we need to swap row 1 and 2. Eliminating x_1 by these transformation

$$E_3 := E_3 - E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{rrr:r} 1 & -1 & 1 & 6 \\ 0 & 1 & -2 & 4 \\ 0 & 1 & -2 & -4 \end{array}\right)$$

As $a_{22}^{(2)} = 0$, we have to swap row 2 and 3. Eliminating x_2 by these transformation

$$E_3 := E_3 - E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{rrr:r} 1 & -1 & 1 & 6 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & -8 \end{array}\right)$$

The system has no unique solution.

b) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{rrrr:r} 1 & -0.5 & 1 & 0 & 4 \\ 2 & -1 & -1 & 1 & 5 \\ 1 & 1 & 0.5 & 0 & 2 \\ 1 & -0.5 & 1 & 1 & 5 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - 2E_1;\ E_3 := E_3 - E_1;\ E_4 := E_4 - E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{cccc:c} 1 & -0.5 & 1 & 0 & 4 \\ 0 & 0 & -3 & 1 & -3 \\ 0 & 1.5 & -0.5 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

As $a_{22}^{(2)} = 0$, we need to swap row 2 and 3, effectively eliminating x_2 and x_3:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{cccc:c} 1 & -0.5 & 1 & 0 & 4 \\ 0 & 1.5 & -0.5 & 0 & -2 \\ 0 & 0 & -3 & 1 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

The solution is $x_4 = 1$, $x_3 \approx 1.333\,33$, $x_2 \approx -0.888\,89$, $x_1 \approx 2.222\,22$.

c) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{cccc:c} 2 & -1 & 1 & -1 & 6 \\ 0 & 1 & -1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right)$$

x_1 and x_2 are already eliminated. As $a_{33}^{(3)} = 0$, we need to swap row 3 and 4, effectively eliminating x_3:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{cccc:c} 2 & -1 & 1 & -1 & 6 \\ 0 & 1 & -1 & 1 & 5 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right)$$

The solution is $x_4 = 5$, $x_3 = 8$, $x_2 = 8$, $x_1 = 5.5$.

d) Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & -1 & 1 & 1 \\ -1 & 2 & 3 & -1 & 4 \\ 3 & -1 & -1 & 2 & -3 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 \coloneqq E_2 - 2E_1;\ E_3 \coloneqq E_3 - (-1)E_1;\ E_4 \coloneqq E_4 - 3E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & -4 & -1 & -1 & -9 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 \coloneqq E_3 - (-3)E_2;\ E_4 \coloneqq E_4 - 4E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{rrrr:r} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 0 & 0 & -3 & -3 \\ 0 & 0 & 3 & 3 & 3 \end{array}\right)$$

As $a_{33}^{(3)} = 0$, we need to swap row 3 and 4, effectively eliminating x_3:

$$\tilde{\boldsymbol{A}}^{(4)} = \left(\begin{array}{rrrr:r} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right)$$

The solution is $x_4 = 1$, $x_3 = 0$, $x_2 = 2$, $x_1 = -1$.

Exercise 7

Use Algorithm 6.1 and Maple with Digits:= 10 to solve the following linear systems ...

Solution 7

Opps, can't help without Maple license.

Exercise 8

Use Algorithm 6.1 and Maple with Digits:= 10 to solve the following linear systems ...

Solution 8

Opps, can't help without Maple license.

Exercise 9

Given the linear system

$$\begin{aligned} 2x_1 - 6\alpha x_2 &= 3 \\ 3\alpha x_1 - \quad x_2 &= 1.5 \end{aligned}$$

a) Find value(s) of α for which the system has no solutions.

b) Find value(s) of α for which the system has an infinite number of solutions.

c) Assuming a unique solution exists for a given α, find the solution.

Solution 9

Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \begin{pmatrix} 2 & -6\alpha & \vdots & 3 \\ 3\alpha & -1 & \vdots & 1.5 \end{pmatrix}$$

Eliminating x_1 gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \begin{pmatrix} 2 & -6\alpha & \vdots & 3 \\ 0 & 9\alpha^2 - 1 & \vdots & 1.5 - 4.5\alpha \end{pmatrix}$$

The system has no unique solution (either no solution or infinite number of solutions) if and only if:

$$9\alpha^2 - 1 = 0 \iff \alpha = \pm\frac{1}{3}$$

a) The system has no solution if it has no unique solution and

$$1.5(1 - 3\alpha) \neq 0 \iff \alpha = -\frac{1}{3}$$

b) The system has an infinite number of solution if it has no unique solution and

$$1.5(1 - 3\alpha) = 0 \iff \alpha = \frac{1}{3}$$

In this case, the solution assumes a general form:

$$x_2 \in \mathbb{R} \text{ and } x_1 = x_2 + 1.5$$

c) The system has a unique solution if and only if $\alpha \neq \pm\frac{1}{3}$. Then the unique solution is:

$$x_2 = \frac{-1.5}{3\alpha + 1} \text{ and } x_1 = \frac{1.5}{3\alpha + 1}$$

Exercise 10

Given the linear system

$$\begin{aligned} x_1 - x_2 + \alpha x_3 &= -2 \\ -x_1 + 2x_2 - \alpha x_3 &= 3 \\ \alpha x_1 + x_2 + \alpha x_3 &= 2 \end{aligned}$$

a) Find value(s) of α for which the system has no solutions.

b) Find value(s) of α for which the system has an infinite number of solutions.

c) Assuming a unique solution exists for a given α, find the solution.

Solution 10

Let

$$\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}^{(1)} = \left(\begin{array}{ccc:c} 1 & -1 & \alpha & -2 \\ -1 & 2 & -\alpha & 3 \\ \alpha & 1 & \alpha & 2 \end{array}\right)$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - (-1)E_1;\ E_3 := E_3 - \alpha E_1$$

gives:

$$\tilde{\boldsymbol{A}}^{(2)} = \left(\begin{array}{ccc:c} 1 & -1 & \alpha & -2 \\ 0 & 1 & 0 & 1 \\ 0 & \alpha+1 & \alpha-\alpha^2 & 2\alpha+2 \end{array}\right)$$

Eliminating x_2 by these transformation

$$E_3 := E_3 - (\alpha+1)E_2$$

gives:

$$\tilde{\boldsymbol{A}}^{(3)} = \left(\begin{array}{ccc:c} 1 & -1 & \alpha & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \alpha-\alpha^2 & \alpha+1 \end{array}\right)$$

The system has no unique solution (either no solution or infinite number of solutions) if and only if:

$$\alpha - \alpha^2 = 0 \iff \alpha \in \{0, 1\}$$

a) The system has no solution if it has no unique solution and

$$2\alpha + 2 \neq 0 \iff \alpha \in \{0, 1\}$$

b) The system has an infinite number of solution if it has no unique solution and

$$2\alpha + 2 \neq 0 \iff \alpha \in \varnothing$$

c) The system has a unique solution if and only if $\alpha \neq\notin \{0, 1\}$. Then the unique solution is:

$$x_3 = \frac{\alpha+1}{\alpha-\alpha^2}\ ,\ x_2 = 1 \text{ and } x_1 = \frac{2}{\alpha-1}$$

Exercise 11

Show that the 3 elementary row operations do not change the solution set of a linear system.

Solution 11

Let $x_1, x_2, \ldots, x_n$ be the solution of the original system.

When an elementary row operations is applied on row i^{th}, the original solution still satisfies the unchanged rows. We have to proove that it also satisfies the changed row.

a) If i^{th} row is scaled, i^{th} equation is still satisfied by the original solution because both size of it is multiplied with a constant.

b) If a scaled j^{th} row is added to i^{th} row, then the original solution still satisfies the new row, as

   - it satisfies the j^{th} row, therefore satisfies the scaled j^{th} row, as proven above, and
   - it satisfies the original i^{th} row

c) If the rows are swapped, the solution does not change, as the set of the equation does not change.