Bài 2.3 (New) Phương pháp Newton, Phương pháp dây cung và Phương pháp điểm sai

Phương trình f(x) = 0 (1)

Chúng ta đã biết ĐK để pt có nghiệm, ta phải tìm đươc [*a*, *b*], mà trên đó:

- f(x) liên tục trên [*a*, *b*];
- f(a).f(b) < 0.

Để áp dụng PP Newton, ta yêu cầu bổ sung:

Suppose that $f \in C_2\, [a, b]$.

Let $p_0 \in [a, b]$ be an approximation to p such that $f'(p_0) \neq 0$ and $|p - p_0|$ is "small." Consider the first Taylor polynomial for $f(x)$ expanded about $p0$ and evaluated at $x = p$.

$$f(p) = f(p_0) + (p - p_0) f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)),$$

where $\xi(p)$ lies between p and p_0. Since $f(p) = 0$, this equation gives

$$0 = f(p_0) + (p - p_0) f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)).$$

Solving for p gives

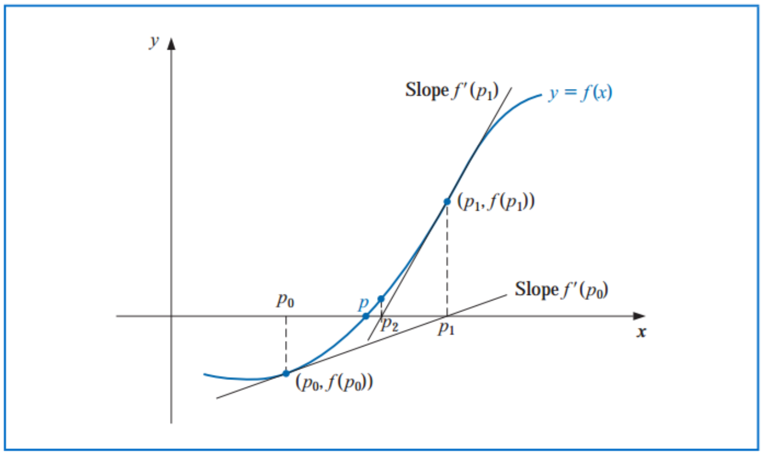
$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1.$$

This sets the stage for Newton's method, which starts with an initial approximation p_0

and generates the sequence { pn}∞ n=0, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1. \tag{2.7}$$

Figure 2.8



Newton's

To find a solution to $f(x) = 0$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \le N_0$ do Steps 3–6.

  Step 3 Set $p = p_0 - f(p_0)/f'(p_0)$. (*Compute* p_i.)

  Step 4 If $|p - p_0| < TOL$ then
  OUTPUT (p); (*The procedure was successful.*)
  STOP.

  Step 5 Set $i = i + 1$.

  Step 6 Set $p_0 = p$. (*Update* p_0.)

Step 7 OUTPUT ('The method failed after N_0 iterations, N_0 =', N_0);
(*The procedure was unsuccessful.*)
STOP. ■

The stopping-technique inequalities given with the Bisection method are applicable to Newton's method. That is, select a tolerance $\varepsilon > 0$, and construct $p1, \ldots pN$ until

$$|p_N - p_{N-1}| < \varepsilon, \tag{2.8}$$

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon, \quad p_N \neq 0, \tag{2.9}$$

$$|f(p_N)| < \varepsilon. \tag{2.10}$$

Theorem 2.6 Let $f \in C^2[a,b]$. If $p \in (a,b)$ is such that $f(p) = 0$ and $f'(p) \neq$ $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ co initial approximation $p_0 \in [p-\delta, p+\delta]$.

Proof The proof is based on analyzing Newton's method as the functional iteration scheme $p_n = g(p_{n-1})$, for $n \geq 1$, with

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Let k be in $(0, 1)$. We first find an interval $[p-\delta, p+\delta]$ that g maps into itself and for which $|g'(x)| \leq k$, for all $x \in (p-\delta, p+\delta)$.

Since f' is continuous and $f'(p) \neq 0$, part (**a**) of Exercise 29 in Section 1.1 implies that there exists a $\delta_1 > 0$, such that $f'(x) \neq 0$ for $x \in [p-\delta_1, p+\delta_1] \subseteq [a, b]$. Thus g is defined and continuous on $[p-\delta_1, p+\delta_1]$. Also

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2},$$

for $x \in [p-\delta_1, p+\delta_1]$, and, since $f \in C^2[a, b]$, we have $g \in C^1[p-\delta_1, p+\delta_1]$.

By assumption, $f(p) = 0$, so

$$g'(p) = \frac{f(p)f''(p)}{[f'(p)]^2} = 0.$$

Since g' is continuous and $0 < k < 1$, part (**b**) of Exercise 29 in Section 1.1 implies that there exists a δ, with $0 < \delta < \delta_1$, and

$$|g'(x)| \leq k, \quad \text{for all } x \in [p-\delta, p+\delta].$$

It remains to show that g maps $[p-\delta, p+\delta]$ into $[p-\delta, p+\delta]$. If $x \in [p-\delta, p+\delta]$, the Mean Value Theorem implies that for some number ξ between x and p, $|g(x) - g(p)| = |g'(\xi)||x-p|$. So

$$|g(x) - p| = |g(x) - g(p)| = |g'(\xi)||x-p| \leq k|x-p| < |x-p|.$$

Since $x \in [p-\delta, p+\delta]$, it follows that $|x-p| < \delta$ and that $|g(x) - p| < \delta$. Hence, g maps $[p-\delta, p+\delta]$ into $[p-\delta, p+\delta]$.

All the hypotheses of the Fixed-Point Theorem 2.4 are now satisfied, so the sequence $\{p_n\}_{n=1}^{\infty}$, defined by

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1,$$

converges to p for any $p_0 \in [p-\delta, p+\delta]$. ▪ ▪ ▪

The Secant Method

Newton's method is an extremely powerful technique, but it has a major weakness: the need to know the value of the derivative of *f* at each approximation. Frequently, *f '(x)* is far more difficult and needs more arithmetic operations to calculate than *f (x)*.

To circumvent the problem of the derivative evaluation in Newton's method, we introduce a slight variation. By definition,

$$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}.$$

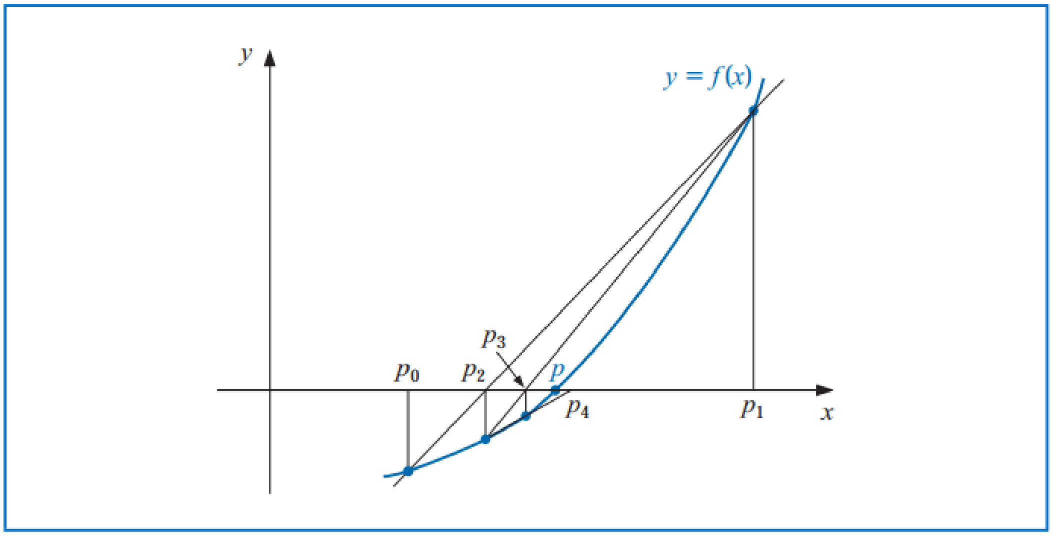
If p_{n-2} is close to p_{n-1}, then

$$f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}} = \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}.$$

Using this approximation for $f'(p_{n-1})$ in Newton's formula gives

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}. \tag{2.12}$$

Figure 2.10



Secant

To find a solution to $f(x) = 0$ given initial approximations p_0 and p_1:

INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of itera

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

Step 2 While $i \le N_0$ do Steps 3–6.

Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute* p_i.)

Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p_1$; (*Update* p_0, q_0, p_1, q_1.)
$q_0 = q_1$;
$p_1 = p$;
$q_1 = f(p)$.

Step 7 OUTPUT ('The method failed after N_0 iterations, N_0 =', N_0);
(*The procedure was unsuccessful.*)
STOP.

The Method of False Position

Each successive pair of approximations in the Bisection method brackets a root *p* of the

equation; that is, for each positive integer *n*, a root lies between *an* and *bn*. This implies that,

for each *n*, the Bisection method iterations satisfy

$$|p_n - p| < \frac{1}{2}|a_n - b_n|,$$

which provides an easily calculated error bound for the approximations

Root bracketing is not guaranteed for either Newton's method or the Secant method.

In Example 1, Newton's method was applied to $f(x) = \cos x - x$, and an

approximate root
was found to be 0.7390851332. Table 2.5 shows that this root is not bracketed by either p_0
and p_1 or p_1 and p_2. The Secant method approximations for this problem are also given in
Table 2.5. In this case the initial approximations p_0 and p_1 bracket the root, but the pair of
approximations p_3 and p_4 fail to do so.

The **method of False Position** (also called *Regula Falsi*) generates approximations
in the same manner as the Secant method, but it includes a test to ensure that the root is
always bracketed between successive

iterations. Although it is not a method we generally
recommend, it illustrates how bracketing can be incorporated.
First choose initial approximations p_0 and p_1 with $f(p_0) \cdot f(p_1) < 0$. The approximation p_2 is chosen in the same manner as in the Secant method, as the x-intercept of the
line joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$. To decide which secant line to use to compute p_3,
consider $f(p_2) \cdot f(p_1)$, or more correctly $\operatorname{sgn} f(p_2) \cdot \operatorname{sgn} f(p_1)$.

• If $\operatorname{sgn} f(p_2) \cdot \operatorname{sgn} f(p_1) < 0$, then p_1 and p_2 bracket a root. Choose p_3 as the x-intercept
of the line joining $(p_1, f(p_1))$ and $(p_2, f$

$(p_2))$.

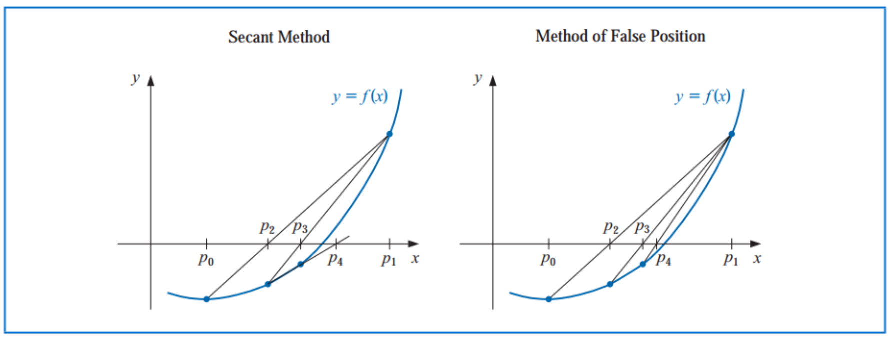
• If not, choose p_3 as the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_2, f(p_2))$, and then interchange the indices on p_0 and p_1.

In a similar manner, once p_3 is found, the sign of $f(p_3) \cdot f(p_2)$ determines whether we use p_2 and p_3 or p_3 and p_1 to compute p_4. In the latter case a relabeling of p_2 and p_1 is performed. The relabeling ensures that the root is bracketed between successive iterations.

The process is described in Algorithm 2.5, and Figure 2.11 shows how the iterations can

differ from those of the Secant method. In this illustration, the first three approximations
are the same, but the fourth approximations differ.

Figure 2.11



False Position

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[p_0, p_1]$ where $f(p_0)$ and $f(p_1)$ have opposite signs:

INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

Step 2 While $i \leq N_0$ do Steps 3–7.

Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute* p_i.)

Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$;
$q = f(p)$.

Step 6 If $q \cdot q_1 < 0$ then set $p_0 = p_1$;
$q_0 = q_1$.

Step 7 Set $p_1 = p$;
$q_1 = q$.

Step 8 OUTPUT ('Method failed after N_0 iterations, N_0 =', N_0);
(*The procedure unsuccessful.*)

Bài tập Phương pháp Newton, PP dây cung và pp điểm sai

1. Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.
2. Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find p_2. Could $p_0 = 0$ be used?
3. Let $f(x) = x^2 - 6$. With $p_0 = 3$ and $p_1 = 2$, find p_3.
   - **a.** Use the Secant method.
   - **b.** Use the method of False Position.
   - **c.** Which of **a.** or **b.** is closer to $\sqrt{6}$?
4. Let $f(x) = -x^3 - \cos x$. With $p_0 = -1$ and $p_1 = 0$, find p_3.
   - **a.** Use the Secant method.
   - **b.** Use the method of False Position.
5. Use Newton's method to find solutions accurate to within 10^{-4} for the following problems.
   - **a.** $x^3 - 2x^2 - 5 = 0, \quad [1, 4]$
   - **b.** $x^3 + 3x^2 - 1 = 0, \quad [-3, -2]$
   - **c.** $x - \cos x = 0, \quad [0, \pi/2]$
   - **d.** $x - 0.8 - 0.2 \sin x = 0, \quad [0, \pi/2]$
6. Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.
   - **a.** $e^x + 2^{-x} + 2\cos x - 6 = 0$ for $1 \le x \le 2$
   - **b.** $\ln(x-1) + \cos(x-1) = 0$ for $1.3 \le x \le 2$
   - **c.** $2x \cos 2x - (x-2)^2 = 0$ for $2 \le x \le 3$ and $3 \le x \le 4$
   - **d.** $(x-2)^2 - \ln x = 0$ for $1 \le x \le 2$ and $e \le x \le 4$
   - **e.** $e^x - 3x^2 = 0$ for $0 \le x \le 1$ and $3 \le x \le 5$
   - **f.** $\sin x - e^{-x} = 0$ for $0 \le x \le 1$ $3 \le x \le 4$ and $6 \le x \le 7$
7. Repeat Exercise 5 using the Secant method.
8. Repeat Exercise 6 using the Secant method.
9. Repeat Exercise 5 using the method of False Position.
10. Repeat Exercise 6 using the method of False Position.
11. Use all three methods in this Section to find solutions to within 10^{-5} for the following problems.
    - **a.** $3xe^x = 0$ for $1 \le x \le 2$
    - **b.** $2x + 3\cos x - e^x = 0$ for $0 \le x \le 1$
12. Use all three methods in this Section to find solutions to within 10^{-7} for the following problems.
    - **a.** $x^2 - 4x + 4 - \ln x = 0$ for $1 \le x \le 2$ and for $2 \le x \le 4$
    - **b.** $x + 1 - 2\sin \pi x = 0$ for $0 \le x \le 1/2$ and for $1/2 \le x \le 1$
13. Use Newton's method to approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = x^2$ that is closest to $(1, 0)$. [*Hint:* Minimize $[d(x)]^2$, where $d(x)$ represents the distance from (x, x^2) to $(1, 0)$.]
14. Use Newton's method to approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = 1/x$ that is closest to (2, 1).
15. The following describes Newton's method graphically: Suppose that $f'(x)$ exists on $[a, b]$ and that $f'(x) \ne 0$ on $[a, b]$. Further, suppose there exists one $p \in [a, b]$ such that $f(p) = 0$, and let $p_0 \in [a, b]$ be arbitrary. Let p_1 be the point at which the tangent line to f at $(p_0, f(p_0))$ crosses the x-axis. For each $n \ge 1$, let p_n be the x-intercept of the line tangent to f at $(p_{n-1}, f(p_{n-1}))$. Derive the formula describing this method.

16. Use Newton's method to solve the equation

$$0 = \frac{1}{2} + \frac{1}{4}x^2 - x\sin x - \frac{1}{2}\cos 2x, \quad \text{with } p_0 = \frac{\pi}{2}.$$

Iterate using Newton's method until an accuracy of 10^{-5} is obtained. Explain why the result seems unusual for Newton's method. Also, solve the equation with $p_0 = 5\pi$ and $p_0 = 10\pi$.

17. The fourth-degree polynomial

$$f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$$

has two real zeros, one in $[-1, 0]$ and the other in $[0, 1]$. Attempt to approximate these zeros to within 10^{-6} using the

a. Method of False Position

b. Secant method

c. Newton's method

Use the endpoints of each interval as the initial approximations in (a) and (b) and the midpoints as the initial approximation in (c).

18. The function $f(x) = \tan \pi x - 6$ has a zero at $(1/\pi)\arctan 6 \approx 0.447431543$. Let $p_0 = 0$ and $p_1 = 0.48$, and use ten iterations of each of the following methods to approximate this root. Which method is most successful and why?

a. Bisection method

b. Method of False Position

c. Secant method

19. The iteration equation for the Secant method can be written in the simpler form

$$p_n = \frac{f(p_{n-1})p_{n-2} - f(p_{n-2})p_{n-1}}{f(p_{n-1}) - f(p_{n-2})}.$$

Explain why, in general, this iteration equation is likely to be less accurate than the one given in Algorithm 2.4.

20. The equation $x^2 - 10\cos x = 0$ has two solutions, ± 1.3793646. Use Newton's method to approximate the solutions to within 10^{-5} with the following values of p_0.

a. $p_0 = -100$ **b.** $p_0 = -50$ **c.** $p_0 = -25$

d. $p_0 = 25$ **e.** $p_0 = 50$ **f.** $p_0 = 100$

21. The equation $4x^2 - e^x - e^{-x} = 0$ has two positive solutions x_1 and x_2. Use Newton's method to approximate the solution to within 10^{-5} with the following values of p_0.

a. $p_0 = -10$ **b.** $p_0 = -5$ **c.** $p_0 = -3$

d. $p_0 = -1$ **e.** $p_0 = 0$ **f.** $p_0 = 1$

g. $p_0 = 3$ **h.** $p_0 = 5$ **i.** $p_0 = 10$

22. Use Maple to determine how many iterations of Newton's method with $p_0 = \pi/4$ are needed to find a root of $f(x) = \cos x - x$ to within 10^{-100}.

23. The function described by $f(x) = \ln(x^2 + 1) - e^{0.4x} \cos \pi x$ has an infinite number of zeros.

a. Determine, within 10^{-6}, the only negative zero.

b. Determine, within 10^{-6}, the four smallest positive zeros.

c. Determine a reasonable initial approximation to find the nth smallest positive zero of f. [*Hint:* Sketch an approximate graph of f.]

d. Use part (c) to determine, within 10^{-6}, the 25th smallest positive zero of f.

24. Find an approximation for λ, accurate to within 10^{-4}, for the population equation

$$1{,}564{,}000 = 1{,}000{,}000 e^{\lambda} + \frac{435{,}000}{\lambda}(e^{\lambda} - 1),$$

discussed in the introduction to this chapter. Use this value to predict the population at the end of the second year, assuming that the immigration rate during this year remains at 435,000 individuals per year.

25. The sum of two numbers is 20. If each number is added to its square root, the product of the two sums is 155.55. Determine the two numbers to within 10^{-4}.

26. The accumulated value of a savings account based on regular periodic payments can be determined from the *annuity due equation*,

$$A = \frac{P}{i}[(1 + i)^n - 1].$$

In this equation, A is the amount in the account, P is the amount regularly deposited, and i is the rate of interest per period for the n deposit periods. An engineer would like to have a savings account valued at \$750,000 upon retirement in 20 years and can afford to put \$1500 per month toward this goal. What is the minimal interest rate at which this amount can be invested, assuming that the interest is compounded monthly?

27. Problems involving the amount of money required to pay off a mortgage over a fixed period of time involve the formula

$$A = \frac{P}{i}[1 - (1 + i)^{-n}],$$

known as an *ordinary annuity equation.* In this equation, A is the amount of the mortgage, P is the amount of each payment, and i is the interest rate per period for the n payment periods. Suppose that a 30-year home mortgage in the amount of \$135,000 is needed and that the borrower can afford house payments of at most \$1000 per month. What is the maximal interest rate the borrower can afford to pay?

28. A drug administered to a patient produces a concentration in the blood stream given by $c(t) = Ate^{-t/3}$ milligrams per milliliter, t hours after A units have been injected. The maximum safe concentration is 1 mg/mL.

a. What amount should be injected to reach this maximum safe concentration, and when does this maximum occur?

b. An additional amount of this drug is to be administered to the patient after the concentration falls to 0.25 mg/mL. Determine, to the nearest minute, when this second injection should be given.

c. Assume that the concentration from consecutive injections is additive and that 75% of the amount originally injected is administered in the second injection. When is it time for the third injection?

29. Let $f(x) = 3^{3x+1} - 7 \cdot 5^{2x}$.

a. Use the Maple commands *solve* and *fsolve* to try to find all roots of f.

b. Plot $f(x)$ to find initial approximations to roots of f.

c. Use Newton's method to find roots of f to within 10^{-16}.

d. Find the exact solutions of $f(x) = 0$ without using Maple.

30. Repeat Exercise 29 using $f(x) = 2^{x^2} - 3 \cdot 7^{x+1}$.

31. The logistic population growth model is described by an equation of the form

$$P(t) = \frac{P_L}{1 - ce^{-kt}},$$

where P_L, c, and $k > 0$ are constants, and $P(t)$ is the population at time t. P_L represents the limiting value of the population since $\lim_{t\to\infty} P(t) = P_L$. Use the census data for the years 1950, 1960, and 1970 listed in the table on page 105 to determine the constants P_L, c, and k for a logistic growth model. Use the logistic model to predict the population of the United States in 1980 and in 2010, assuming $t = 0$ at 1950. Compare the 1980 prediction to the actual value.

32. The Gompertz population growth model is described by

$$P(t) = P_L e^{-ce^{-kt}},$$

where P_L, c, and $k > 0$ are constants, and $P(t)$ is the population at time t. Repeat Exercise 31 using the Gompertz growth model in place of the logistic model.

33. Player A will shut out (win by a score of 21–0) player B in a game of racquetball with probability

$$P = \frac{1+p}{2}\left(\frac{p}{1-p+p^2}\right)^{21},$$

where p denotes the probability A will win any specific rally (independent of the server). (See [Keller, J], p. 267.) Determine, to within 10^{-3}, the minimal value of p that will ensure that A will shut out B in at least half the matches they play.

34. In the design of all-terrain vehicles, it is necessary to consider the failure of the vehicle when attempting to negotiate two types of obstacles. One type of failure is called *hang-up failure* and occurs when the vehicle attempts to cross an obstacle that causes the bottom of the vehicle to touch the ground. The other type of failure is called *nose-in failure* and occurs when the vehicle descends into a ditch and its nose touches the ground.

The accompanying figure, adapted from [Bek], shows the components associated with the nose-in failure of a vehicle. In that reference it is shown that the maximum angle α that can be negotiated by a vehicle when β is the maximum angle at which hang-up failure does *not* occur satisfies the equation

$$A \sin\alpha \cos\alpha + B \sin^2\alpha - C \cos\alpha - E \sin\alpha = 0,$$

where

$$A = l \sin\beta_1, \quad B = l \cos\beta_1, \quad C = (h + 0.5D)\sin\beta_1 - 0.5D \tan\beta_1,$$

and $E = (h + 0.5D)\cos\beta_1 - 0.5D.$

a. It is stated that when $l = 89$ in., $h = 49$ in., $D = 55$ in., and $\beta_1 = 11.5°$, angle α is approximately 33°. Verify this result.

b. Find α for the situation when l, h, and β_1 are the same as in part (a) but $D = 30$ in.