Bài 2.1 Phương pháp chia đôi

Giả sử f là hàm số xác định và liên tục trên khoảng [a, b], với f (a) và f (b) của dấu trái. Định lý giá trị trung gian nói rằng tồn tại một số p trong (a, b) với f (p) = 0.

Mặc dù có thể xảy ra nhiều hơn một nghiệm trong khoảng (a, b), nhưng để thuận lơi trong việc việc nghiên cứu, chúng ta giả thiết chỉ có duy nhất một nghiệm trong khoảng này.

Phương thức tiến hành:

Chia đôi [a, b], nếu điểm giữa $p = (a + b)/2$ thỏa mãn $f(p)=0$, thì p là nghiệm cần tìm.

Nếu $f(p) \neq 0$, thì nghiệm nằm trong khoảng [a, p] hoặc [p, b].

Để bắt đầu, ta đặt $a_1 = a$ và $b_1=b$, và đặt p_1 là điểm giữa của $[a, b]$; nghĩa là:

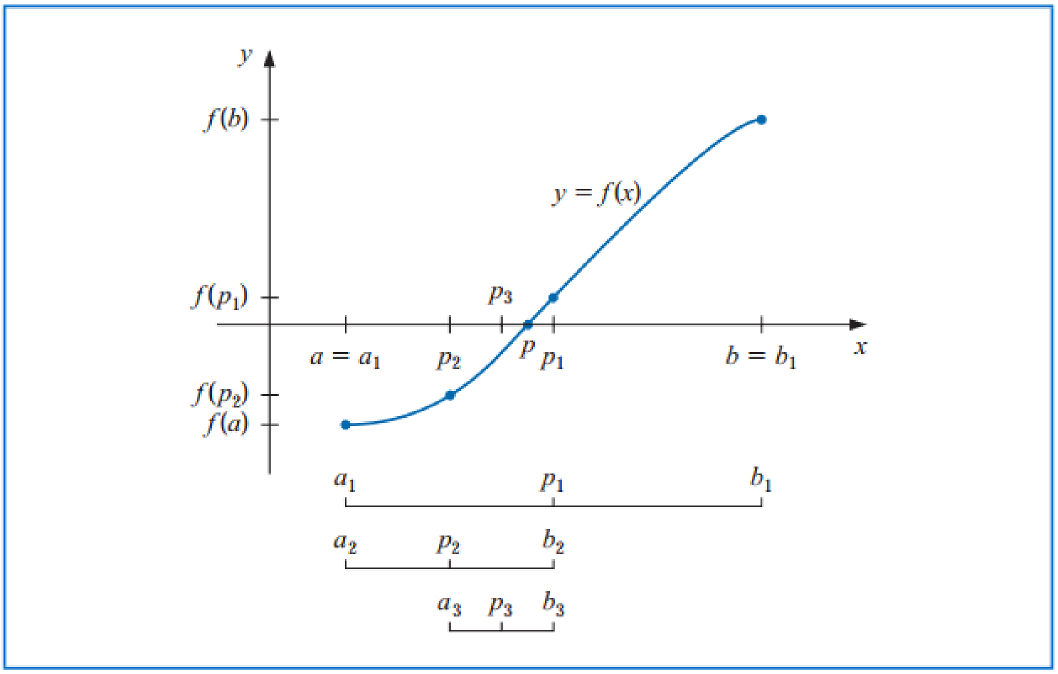
$$p_1 = a_1 + \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2}.$$

- If $f(p1) = 0$, then $p = p1$, and we are done.

- If $f(p_1) \neq 0$, then $f(p_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.
- If $f(p_1)$ and $f(a_1)$ have the same sign, $p \in (p_1, b_1)$. Set $a_2 = p_1$ and $b_2 = b_1$.
- If $f(p_1)$ and $f(a_1)$ have opposite signs, $p \in (a_1, p_1)$. Set $a_2 = a_1$ and $b_2 = p_1$.

Then reapply the process to the interval $[a_2, b_2]$. This produces the method described in Algorithm 2.1. (See Figure 2.1.

Figure 2.1



Bisection

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs:

INPUT endpoints a, b; tolerance TOL; maximum number o

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$;
$FA = f(a)$.

Step 2 While $i \le N_0$ do Steps 3–6.

Step 3 Set $p = a + (b - a)/2$; (*Compute* p_i.)
$FP = f(p)$.

Step 4 If $FP = 0$ or $(b - a)/2 < TOL$ then
OUTPUT (p); (*Procedure completed suc*
STOP.

Step 5 Set $i = i + 1$.

Step 6 If $FA \cdot FP > 0$ then set $a = p$; (*Compute* a_i
$FA = FP$
else set $b = p$. (*FA is uncha*

Step 7 OUTPUT ('Method failed after N_0 iterations, $N_0 =$
(*The procedure was unsuccessful.*)
STOP.

Other stopping procedures can be applied in Step 4 of

Algorithm 2.1 or in any of the iterative techniques in this chapter. For example, we can select a tolerance $\varepsilon > 0$ and generate $p1, \ldots, pN$ until one of the following conditions is met:

Các thủ tục dừng khác có thể được áp dụng trong Bước 4 của Thuật toán 2.1 hoặc trong bất kỳ các kỹ thuật lặp lại trong chương này. Ví dụ, chúng ta có thể chọn một dung sai ε> 0 và tạo p1,. . . , pN cho đến khi đáp ứng một trong các điều kiện sau:

$$|p_N - p_{N-1}| < \varepsilon,$$

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon, \quad p_N \neq 0, \quad \text{or}$$

$$|f(p_N)| < \varepsilon.$$

Unfortunately, difficulties can arise using any of these stopping criteria. For example, there are sequences { *pn*}∞ *n*=0 with the property that the differences *pn* - *pn*-1 converge

to
zero while the sequence itself diverges. (See Exercise 17.) It is also possible for $f(p_n)$ to be close to zero while p_n differs significantly from p. (See Exercise 16.) Without additional knowledge about f or p, Inequality (2.2) is the best stopping criterion to apply because it comes closest to testing relative error.

When using a computer to generate approximations, it is good practice to set an upper bound on the number of iterations. This eliminates the possibility of entering an infinite loop, a situation that can arise when the sequence diverges (and also when the program is incorrectly coded). This was done in Step 2 of Algorithm 2.1 where the bound $N0$ was set and the procedure terminated if $i > N0$.

Note that to start the Bisection Algorithm, an interval [a, b] must be found with $f(a) \cdot f(b) < 0$. At each step the length of the interval known to contain a zero of f is reduced by a factor of 2; hence it is advantageous to choose the initial interval [a, b] as small as possible. For example, if $f(x) = 2x3 - x2 + x - 1$, we have both

Thật không may, khó khăn có thể phát sinh bằng cách sử dụng bất kỳ tiêu chí dừng nào. Ví dụ, có các chuỗi {pn} ∞ n = 0 với thuộc tính mà sự khác biệt pn - pn -1 hội tụ về 0 trong khi trình tự phân kỳ. (Xem Bài tập 17.) Cũng có thể cho f (pn) gần bằng không trong khi pn khác đáng kể so với p. (Xem Bài tập 16.) Nếu không có kiến thức bổ sung về f hoặc p, bất bình đẳng (2.2) là tiêu chuẩn dừng tốt nhất để áp dụng vì nó đến gần nhất để kiểm tra lỗi tương đối.

Khi sử dụng một máy tính để tạo ra xấp xỉ, nó là thực hành tốt để thiết lập một giới hạn trên về số lần lặp lại. Điều này giúp loại bỏ khả năng nhập một vòng lặp vô hạn, một tình huống có thể phát sinh khi trình tự phân kỳ (và cả khi chương trình được mã hóa sai). Điều này đã được thực hiện trong Bước 2 của Thuật toán 2.1 trong đó N0 bị ràng buộc được thiết lập và thủ tục chấm dứt nếu i> N0.

Lưu ý rằng để bắt đầu thuật toán bisection, một khoảng [a, b] phải được tìm thấy với f (a) · f (b) <0. Ở mỗi bước, độ dài của khoảng được biết là chứa số không f được giảm theo hệ số 2; do đó rất thuận lợi để chọn khoảng thời gian ban đầu [a, b] càng nhỏ càng tốt. Ví dụ: nếu f (x) = 2x3 - x2 + x - 1, chúng tôi có cả hai

$$f(-4) \cdot f(4) < 0 \quad \text{and} \quad f(0) \cdot f(1) < 0,$$

so the Bisection Algorithm could be used on [-4, 4] or on [0, 1]. Starting the Bisection Algorithm on [0, 1] instead of [-4, 4] will reduce by 3 the number of iterations required to achieve a specified accuracy. The following example illustrates the Bisection Algorithm. The iteration in this example
is terminated when a bound for the relative error is less than

0.0001. This is ensured by having

$$\frac{|p - p_n|}{\min\{|a_n|, |b_n|\}} < 10^{-4}.$$

Example 1 Show that $f(x) = x3 + 4x2 - 10 = 0$ has a root in [1, 2], and use the Bisection method to
determine an approximation to the root that is accurate to at least within 10-4.

Solution Because $f(1) = -5$ and $f(2) = 14$ the Intermediate

Value Theorem 1.11 ensures that this continuous function has a root in [1, 2].
For the first iteration of the Bisection method we use the fact that at the midpoint of [1, 2] we have $f(1.5) = 2.375 > 0$. This indicates that we should select the interval [1, 1.5] for our second iteration. Then we find that $f(1.25) = -1.796875$ so our new interval becomes [1.25, 1.5], whose midpoint is 1.375. Continuing in this manner gives

the values in Table 2.1. After 13 iterations, $p13 = 1.365112305$ approximates the root p with an error

$$|p - p_{13}| < |b_{14} - a_{14}| = |1.365234375 - 1.365112305| = 0.000122070.$$

Since $|a14| < |p|$, we have

$$\frac{|p - p_{13}|}{|p|} < \frac{|b_{14} - a_{14}|}{|a_{14}|} \leq 9.0 \times 10^{-5},$$

Table 2.1

n	a_n	b_n	p_n	$f(p_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	−1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	−0.84839
5	1.3125	1.375	1.34375	−0.35098
6	1.34375	1.375	1.359375	−0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	−0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	−0.01605
11	1.364257813	1.365234375	1.364746094	−0.00799
12	1.364746094	1.365234375	1.364990235	−0.00396
13	1.364990235	1.365234375	1.365112305	−0.00194

so the approximation is correct to at least within 10-4. The correct value of p to nine decimal

places is $p = 1.365230013$. Note that $p9$ is closer to p than is the final approximation $p13$.

You might suspect this is true because $|f(p9)| < |f(p13)|$, but we cannot be sure of this unless the true answer is known.

The Bisection method, though conceptually clear, has significant drawbacks. It is relatively slow to converge (that is, N may become quite large before $| p - p_N|$ is sufficiently small), and a good intermediate approximation might be inadvertently discarded. However, the method has the important property that it always converges to a solution, and for that

reason it is often used as a starter for the more efficient methods we will see later in this chapter.

Theorem 2.1 Suppose that $f \in C[a, b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \leq \frac{b-a}{2^n}, \quad \text{when} \quad n \geq 1.$$

Proof For each $n \geq 1$, we have

$$b_n - a_n = \frac{1}{2^{n-1}}(b-a) \quad \text{and} \quad p \in (a_n, b_n).$$

Since

$$p_n = \frac{1}{2}(a_n + b_n)$$

for all $n \geq 1$, it follows that

$$|p_n - p| \leq \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^n}.$$

the sequence { *pn*}∞ *n*=1 converges to *p* with rate of convergence *O* 21 *n* ; that is,

$$p_n = p + O\left(\frac{1}{2^n}\right).$$

It is important to realize that Theorem 2.1 gives only a bound for approximation error
and that this bound might be

quite conservative. For example, this bound applied to the
problem in Example 1 ensures only that

$$|p - p_9| \le \frac{2-1}{2^9} \approx 2 \times 10^{-3},$$

but the actual error is much smaller:

$$|p - p_9| = |1.365230013 - 1.365234375| \approx 4.4 \times 10^{-6}.$$

Example 2 Determine the number of iterations necessary to solve $f(x) = x3 + 4x2 - 10 = 0$

with
accuracy 10-3 using $a1 = 1$ and $b1 = 2$.

Solution We we will use logarithms to find an integer N that satisfies

$$|p_N - p| \leq 2^{-N}(b - a) = 2^{-N} < 10^{-3}.$$

Logarithms to any base would suffice, but we will use base-10 logarithms because the tolerance is given as a power of 10. Since

$$2^{-N} < 10^{-3}$$

implies that

$$\log_{10} 2^{-N} < \log_{10} 10^{-3} = -3,$$

we have

$$-N \log_{10} 2 < -3 \quad \text{and} \quad N > \frac{3}{\log_{10} 2} \approx 9.96.$$

Hence, ten iterations will ensure an approximation accurate to within 10-3.

Table 2.1 shows that the value of *p*9 = 1.365234375 is accurate to within 10-4. Again, it is important to keep in mind that the error analysis gives only a bound for the number of iterations. In many cases this bound is much larger than the actual number required.

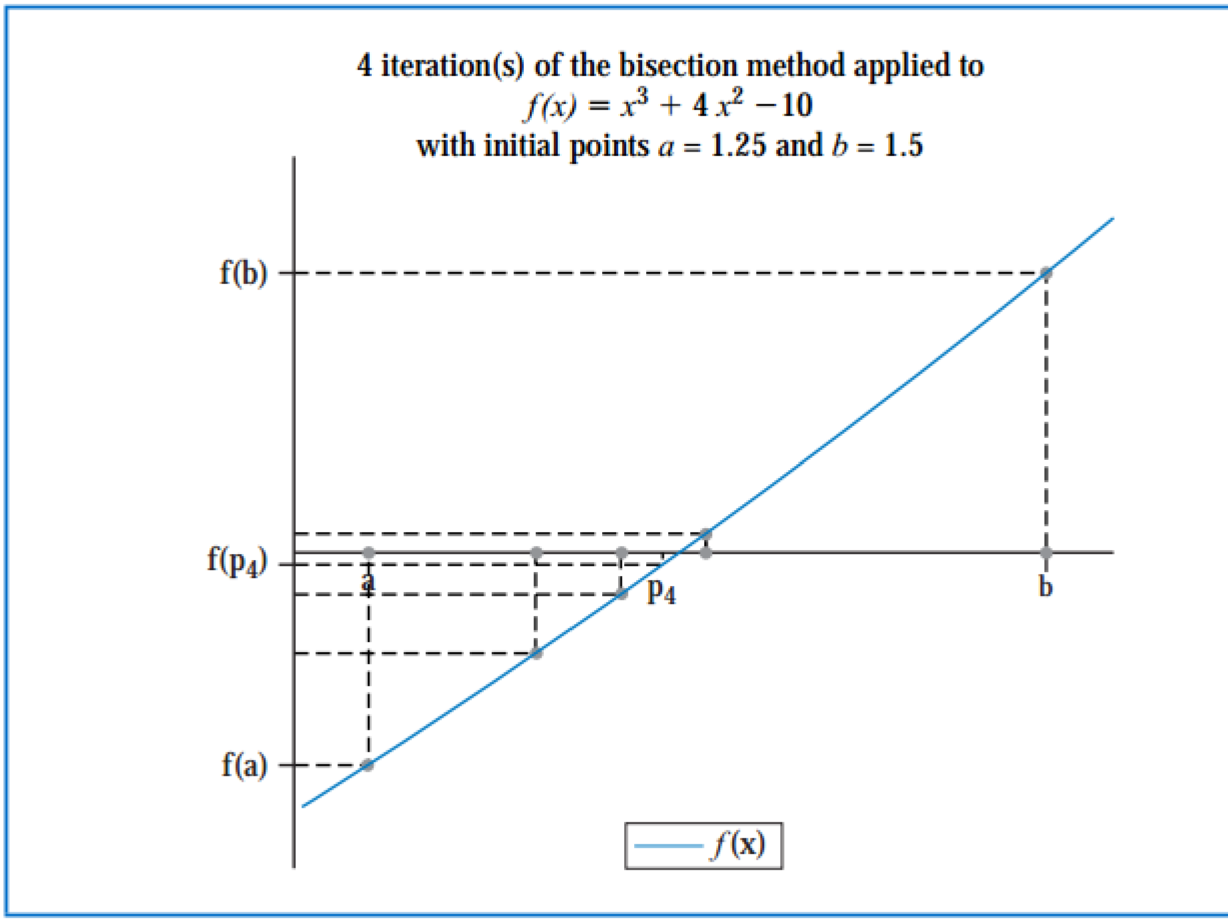
Maple has a *NumericalAnalysis* package that implements many of the techniques we will discuss, and the presentation and examples in the package are closely aligned with this text. The Bisection method in this package has a number of options, some of which we will now consider. In what follows, Maple code is given in *black italic* type and Maple response in cyan.

Load the *NumericalAnalysis* package with the command *with*(*Student*[*NumericalAnalysis*])

which gives access to the procedures in the package.

Define the function with

Figure 2.2


4 iteration(s) of the bisection method applied to
$f(x) = x^3 + 4x^2 - 10$
with initial points $a = 1.25$ and $b = 1.5$
f(b)
f(p4)
f(a)
a
p4
b
f(x)

Bài tập Phương pháp chia đôi

1. Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

2. Let $f(x) = 3(x+1)(x-\frac{1}{2})(x-1)$. Use the Bisection method on the following intervals to find p_3.

a. $[-2, 1.5]$ **b.** $[-1.25, 2.5]$

3. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ on each interval.

a. $[0, 1]$ **b.** $[1, 3.2]$ **c.** $[3.2, 4]$

4. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ on each interval.

a. $[-2, -1]$ **b.** $[0, 2]$ **c.** $[2, 3]$ **d.** $[-1, 0]$

5. Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems.

a. $x - 2^{-x} = 0$ for $0 \le x \le 1$

b. $e^x - x^2 + 3x - 2 = 0$ for $0 \le x \le 1$

c. $2x\cos(2x) - (x+1)^2 = 0$ for $-3 \le x \le -2$ and $-1 \le x \le 0$

d. $x\cos x - 2x^2 + 3x - 1 = 0$ for $0.2 \le x \le 0.3$ and $1.2 \le x \le 1.3$

6. Use the Bisection method to find solutions, accurate to within 10^{-5} for the following problems.

a. $3x - e^x = 0$ for $1 \le x \le 2$

b. $2x + 3\cos x - e^x = 0$ for $0 \le x \le 1$

c. $x^2 - 4x + 4 - \ln x = 0$ for $1 \le x \le 2$ and $2 \le x \le 4$

d. $x + 1 - 2\sin \pi x = 0$ for $0 \le x \le 0.5$ and $0.5 \le x \le 1$

7. **a.** Sketch the graphs of $y = x$ and $y = 2\sin x$.

b. Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = 2\sin x$.

8. **a.** Sketch the graphs of $y = x$ and $y = \tan x$.

b. Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = \tan x$.

9. **a.** Sketch the graphs of $y = e^x - 2$ and $y = \cos(e^x - 2)$.

b. Use the Bisection method to find an approximation to within 10^{-5} to a value in $[0.5, 1.5]$ with $e^x - 2 = \cos(e^x - 2)$.

10. Let $f(x) = (x+2)(x+1)^2x(x-1)^3(x-2)$. To which zero of f does the Bisection method converge when applied on the following intervals?

a. $[-1.5, 2.5]$ **b.** $[-0.5, 2.4]$ **c.** $[-0.5, 3]$ **d.** $[-3, -0.5]$

11. Let $f(x) = (x+2)(x+1)x(x-1)^3(x-2)$. To which zero of f does the Bisection method converge when applied on the following intervals?

a. $[-3, 2.5]$ **b.** $[-2.5, 3]$ **c.** $[-1.75, 1.5]$ **d.** $[-1.5, 1.75]$

12. Find an approximation to $\sqrt{3}$ correct to within 10^{-4} using the Bisection Algorithm. [*Hint:* Consider $f(x) = x^2 - 3$.]

13. Find an approximation to $\sqrt[3]{25}$ correct to within 10^{-4} using the Bisection Algorithm.

14. Use Theorem 2.1 to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-3} to the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$. Find an approximation to the root with this degree of accuracy.

15. Use Theorem 2.1 to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-4} to the solution of $x^3 - x - 1 = 0$ lying in the interval $[1, 2]$. Find an approximation to the root with this degree of accuracy.

16. Let $f(x) = (x-1)^{10}$, $p = 1$, and $p_n = 1 + 1/n$. Show that $|f(p_n)| < 10^{-3}$ whenever $n > 1$ but that $|p - p_n| < 10^{-3}$ requires that $n > 1000$.

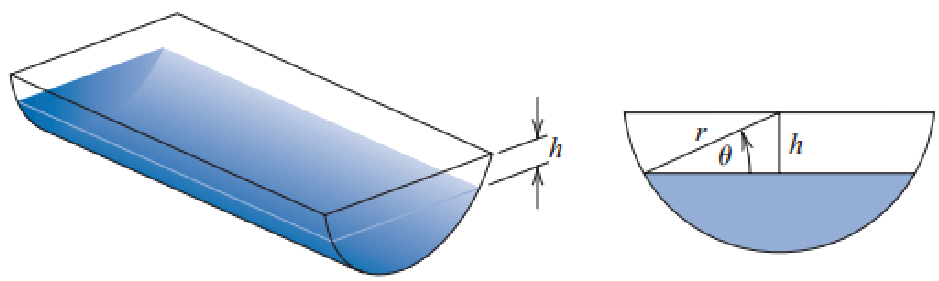
17. Let $\{p_n\}$ be the sequence defined by $p_n = \sum_{k=1}^{n} \frac{1}{k}$. Show that $\{p_n\}$ diverges even though $\lim_{n\to\infty}(p_n - p_{n-1}) = 0$.

18. The function defined by $f(x) = \sin \pi x$ has zeros at every integer. Show that when $-1 < a < 0$ and $2 < b < 3$, the Bisection method converges to

a. 0, if $a + b < 2$ **b.** 2, if $a + b > 2$ **c.** 1, if $a + b = 2$

19. A trough of length L has a cross section in the shape of a semicircle with radius r. (See the accompanying figure.) When filled with water to within a distance h of the top, the volume V of water is

$$V = L\left[0.5\pi r^2 - r^2 \arcsin(h/r) - h(r^2 - h^2)^{1/2}\right].$$



Suppose $L = 10$ ft, $r = 1$ ft, and $V = 12.4$ ft^3. Find the depth of water in the trough to within 0.01 ft.

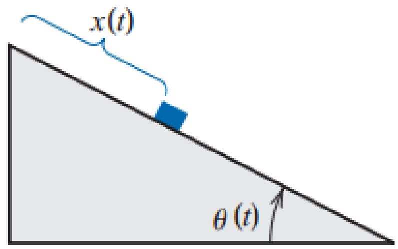
20. A particle starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate

$$\frac{d\theta}{dt} = \omega < 0.$$

At the end of t seconds, the position of the object is given by

$$x(t) = -\frac{g}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin \omega t\right).$$

Suppose the particle has moved 1.7 ft in 1 s. Find, to within 10^{-5}, the rate ω at which θ changes. Assume that $g = 32.17$ ft/s^2.



The Latin word signum means "token" or "sign". So the signum function quite naturally returns the sign of a number (unless the number is 0)

Chữ signum Latin có nghĩa là "token" hoặc "sign". Vì vậy, chức năng signum khá tự nhiên trả về dấu của một số (trừ khi con số là 0)