# Phương pháp tính MAT1099

Nguyễn Việt Minh Nghĩa
github.com/nvmnghia

07-09-2020

2

# Chapter 1

# Error analysis

### Exercise 1

Use the Bisection method to find $p_3$ for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

### Solution 1

as hey

# Chapter 2

# Solution approximation

## 2.1 The Bisection Method

### Exercise 1

Use the Bisection method to find $p_3$ for $f(x) = \sqrt{x} - \cos x$ on $[0,1]$.

### Solution 1

$f(0) = -1$ and $f(1) \approx 0.459\,697\,694$ have the opposite signs, so there's a root in $[0,1]$.

Table of iteration for $f(x) = \sqrt{(x)} - \cos x$ on $[0,1]$:

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|---|---|---|---|---|
| 1 | 0 | 1 | 0.5 | $-0.170\,475\,781$ |
| 2 | 0.5 | 1 | 0.75 | $0.134\,336\,535$ |
| 3 | 0.5 | 0.75 | 0.625 | $-0.020\,393\,704$ |

So $p_3 = 0.625$.

### Exercise 2

Let $f(x) = 3(x+1)(x - \dfrac{1}{2})(x-1)$. Use the bisection method to find $p_3$ in the following intervals:

(a) $[-2, 1.5]$

(b) $[-1.5, 2.5]$

## Solution 2

(a) $f(-2) = -22.5$ and $f(1.5) = 3.75$ have the opposite signs, so there's a root in $[-2, 1.5]$.

We have the following table:

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|-----|-------|-------|-------|----------|
| 1 | $-2$ | $1.5$ | $-0.25$ | $2.109\,375$ |
| 2 | $-2$ | $-0.25$ | $-1.125$ | $-1.294\,921\,875$ |
| 3 | $-1.125$ | $-0.25$ | $-0.6875$ | $1.878\,662\,109$ |

So $p_3 = -0.6875$.

(b) $f(-1.25) = -2.953\,125$ and $f(2.5)) = 31.5$ have the opposite signs, so there's a root in $[-1.25, 2.5]$.

We have the following table:

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|-----|-------|-------|-------|----------|
| 1 | $-1.5$ | $2.5$ | $0.5$ | $0$ |

The solution is found in the first iteration so $p_3$ doesn't exist.

## Exercise 3

Use the Bisection method to find solutions accurate to within $10^{-2}$ for $x^3 - 7x^2 + 14x - 6 = 0$ in the following intervals:

(a) $[0, 1]$

(b) $[1, 3.2]$

(c) $[3.2, 4]$

## Solution 3

(a) $f(0) = -6$ and $f(1) = 2$ have the opposite signs, so there's a root in $[0, 1]$.

The number of iteration $n$ needed to approximate $p$ to within $10^{-2}$ is:

$$|p_n - p| \le \frac{1 - 0}{2^n} < 10^{-2} \iff n \ge 7$$

We have the following table:

So $p \approx 0.5859$.

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|---|---|---|---|---|
| 1 | 0 | 1 | 0.5 | $-0.625$ |
| 2 | 0.5 | 1 | 0.75 | 0.984 375 |
| 3 | 0.5 | 0.75 | 0.625 | 0.259 766 |
| 4 | 0.5 | 0.625 | 0.5625 | $-0.161\,865$ |
| 5 | 0.5625 | 0.625 | 0.593 75 | 0.054 047 |
| 6 | 0.5625 | 0.593 75 | 0.578 125 | $-0.052\,624$ |
| 7 | 0.578 125 | 0.593 75 | 0.585 937 5 | 0.001 031 |

(b) $f(1) = 2$ and $f(3.2) = -0.112$ have the opposite signs, so there's a root in $[1, 3.2]$.

The number of iteration $n$ needed to approximate $p$ to within $10^{-2}$ is:

$$|p_n - p| \leq \frac{3.2 - 1}{2^n} < 10^{-2} \iff n \geq 8$$

We have the following table:

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|---|---|---|---|---|
| 1 | 1 | 3.2 | 2.1 | 1.791 |
| 2 | 2.1 | 3.2 | 2.65 | 0.552 125 |
| 3 | 2.65 | 3.2 | 2.925 | 0.085 828 |
| 4 | 2.925 | 3.2 | 3.0625 | $-0.054\,443$ |
| 5 | 2.925 | 3.0625 | 2.993 75 | 0.006 328 |
| 6 | 2.993 75 | 3.0625 | 3.028 125 | $-0.026\,521$ |
| 7 | 2.993 75 | 3.028 13 | 3.010 938 | $-0.010\,697$ |
| 8 | 2.993 75 | 3.010 938 | 3.002 344 | $-0.002\,333$ |

So $p \approx 3.0023$.

(c) $f(3.2) = -0.112$ and $f(4) = 2$ have the opposite signs, so there's a root in $[3.2, 4]$.

The number of iteration $n$ needed to approximate $p$ to within $10^{-2}$ is:

$$|p_n - p| \leq \frac{4 - 3.2}{2^n} < 10^{-2} \iff n \geq 7$$

We have the following table:

So $p \approx 3.4188$.

## Exercise 4

Use the Bisection method to find solutions accurate to within $10^{-2}$ for $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ for the following intervals:

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|-----|-------|-------|-------|----------|
| 1 | 3.2 | 4 | 3.6 | 0.336 |
| 2 | 3.2 | 3.6 | 3.4 | $-0.016$ |
| 3 | 3.4 | 3.6 | 3.5 | 0.125 |
| 4 | 3.4 | 3.5 | 3.45 | 0.046 125 |
| 5 | 3.4 | 3.45 | 3.425 | 0.013 016 |
| 6 | 3.4 | 3.425 | 3.4125 | $-0.001\,998$ |
| 7 | 3.4125 | 3.425 | 3.418 75 | 0.005 382 |

(a) $[-2, -1]$

(b) $[0, 2]$

(c) $[2, 3]$

(d) $[-1, 0]$

## Solution 4

(a) $f(-2) = 12$ and $f(-1) = -1$ have the opposite signs, so there's a root in $[-2, -1]$.

The number of iteration $n$ needed to approximate $p$ to within $10^{-2}$ is:

$$|p_n - p| \leq \frac{-1 - (-2)}{2^n} < 10^{-2} \iff n \geq 7$$

We have the following table:

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|-----|-------|-------|-------|----------|
| 1 | $-2$ | $-1$ | $-1.5$ | 0.8125 |
| 2 | $-1.5$ | $-1$ | $-1.25$ | $-0.902\,344$ |
| 3 | $-1.5$ | $-1.25$ | $-1.375$ | $-0.288\,818$ |
| 4 | $-1.5$ | $-1.375$ | $-1.4375$ | 0.195 328 |
| 5 | $-1.4375$ | $-1.375$ | $-1.406\,25$ | $-0.062\,667$ |
| 6 | $-1.4375$ | $-1.406\,25$ | $-1.421\,875$ | 0.062 263 |
| 7 | $-1.421\,875$ | $-1.406\,25$ | $-1.414\,063$ | $-0.001\,208$ |

So $p \approx -1.4141$.

(b) $f(0) = 4$ and $f(2) = -4$ have the opposite signs, so there's a root in $[0, 2]$.

The number of iteration $n$ needed to approximate $p$ to within $10^{-2}$ is:

$$|p_n - p| \leq \frac{2 - 0}{2^n} < 10^{-2} \iff n \geq 8$$

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|---|---|---|---|---|
| 1 | 0 | 2 | 1 | 3 |
| 2 | 1 | 2 | 1.5 | $-0.6875$ |
| 3 | 1 | 1.5 | 1.25 | 1.285 156 |
| 4 | 1.25 | 1.5 | 1.375 | 0.312 744 |
| 5 | 1.375 | 1.5 | 1.4375 | $-0.186\,508$ |
| 6 | 1.375 | 1.4375 | 1.406 25 | 0.063 676 |
| 7 | 1.406 25 | 1.4375 | 1.421 875 | $-0.061\,318$ |
| 8 | 1.406 25 | 1.421 875 | 1.414 063 | 0.001 208 |

We have the following table:

So $p \approx 1.4141$.

(c) $f(2) = -4$ and $f(3) = 7$ have the opposite signs, so there's a root in $[2, 3]$.

The number of iteration $n$ needed to approximate $p$ to within $10^{-2}$ is:

$$|p_n - p| \leq \frac{3 - 2}{2^n} < 10^{-2} \iff n \geq 7$$

We have the following table:

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|---|---|---|---|---|
| 1 | 2 | 3 | 2.5 | $-3.1875$ |
| 2 | 2.5 | 3 | 2.75 | 0.347 656 |
| 3 | 2.5 | 2.75 | 2.625 | $-1.757\,568$ |
| 4 | 2.625 | 2.75 | 2.6875 | $-0.795\,639$ |
| 5 | 2.6875 | 2.75 | 2.718 75 | $-0.247\,466$ |
| 6 | 2.718 75 | 2.75 | 2.734 375 | 0.044 125 |
| 7 | 2.718 75 | 2.734 375 | 2.726 563 | $-0.103\,151$ |

So $p \approx 2.7266$.

(d) $f(-1) = -1$ and $f(0) = 4$ have the opposite signs, so there's a root in $[-1, 0]$.

The number of iteration $n$ needed to approximate $p$ to within $10^{-2}$ is:

$$|p_n - p| \leq \frac{0 - (-1)}{2^n} < 10^{-2} \iff n \geq 7$$

We have the following table:

So $p \approx -0.7266$.

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|---|---|---|---|---|
| 1 | $-1$ | $0$ | $-0.5$ | $1.3125$ |
| 2 | $-1$ | $-0.5$ | $-0.75$ | $-0.089\,844$ |
| 3 | $-0.75$ | $-0.5$ | $-0.625$ | $0.578\,369$ |
| 4 | $-0.75$ | $-0.625$ | $-0.6875$ | $0.232\,681$ |
| 5 | $-0.75$ | $-0.6875$ | $-0.718\,75$ | $0.068\,086$ |
| 6 | $-0.75$ | $-0.718\,75$ | $-0.734\,375$ | $-0.011\,768$ |
| 7 | $-0.734\,375$ | $-0.718\,75$ | $-0.726\,563$ | $0.027\,943$ |

## Exercise 5

Use the Bisection method to find solutions accurate to within $10^{-5}$ for the following problems:

(a)  $x - 2^{-x} = 0$, $x \in [0, 1]$

(b)  $e^x - x^2 + 3x - 2 = 0$, $x \in [0, 1]$

(c)  $2x \cos 2x - (x + 1)^2 = 0$, $x \in [-3, -2]$

(d)  $x \cos x - 2x^2 + 3x - 1 = 0$, $x \in [0.2, 0.3]$

## Solution 5

(a)  $f(0) = -1$ and $f(1) = 0.5$ have the opposite signs, so there's a root in $[0, 1]$.

The number of iteration $n$ needed to approximate $p$ to within $10^{-5}$ is:

$$|p_n - p| \le \frac{1 - 0}{2^n} < 10^{-5} \iff n \ge 17$$

We have the following table:

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|---|---|---|---|---|
| 1 | 0 | 1 | 0.5 | $-0.207\,106\,781$ |
| 2 | 0.5 | 1 | 0.75 | $0.155\,396\,442$ |
| 3 | 0.5 | 0.75 | 0.625 | $-0.023\,419\,777$ |
| 4 | 0.625 | 0.75 | 0.6875 | $0.066\,571\,094$ |
| 5 | 0.625 | 0.6875 | 0.656\,25 | $0.021\,724\,521$ |
| 6 | 0.625 | 0.656\,25 | 0.640\,625 | $-0.000\,810\,008$ |
| 7 | 0.640\,625 | 0.656\,25 | 0.648\,437\,5 | $0.010\,466\,611$ |
| 8 | 0.640\,625 | 0.648\,437\,5 | 0.644\,531\,25 | $0.004\,830\,646$ |
| 9 | 0.640\,625 | 0.644\,531\,25 | 0.642\,578\,125 | $0.002\,010\,906$ |
| 10 | 0.640\,625 | 0.642\,578\,125 | 0.641\,601\,562 | $0.000\,600\,596$ |
| 11 | 0.640\,625 | 0.641\,601\,562 | 0.641\,113\,281 | $-0.000\,104\,669$ |
| 12 | 0.641\,113\,281 | 0.641\,601\,562 | 0.641\,357\,422 | $0.000\,247\,972$ |

| 13 | 0.641 113 281 | 0.641 357 422 | 0.641 235 352 | 0.000 071 654 |
| 14 | 0.641 113 281 | 0.641 235 352 | 0.641 174 316 | −0.000 016 507 |
| 15 | 0.641 174 316 | 0.641 235 352 | 0.641 204 834 | 0.000 027 573 |
| 16 | 0.641 174 316 | 0.641 204 834 | 0.641 189 575 | 0.000 005 533 |
| 17 | 0.641 174 316 | 0.641 189 575 | 0.641 181 946 | −0.000 005 487 |

So $p \approx -0.641\,182$.

(b) $f(0) = -1$ and $f(1) = e$ have the opposite signs, so there's a root in $[0, 1]$.

The number of iteration $n$ needed to approximate $p$ to within $10^{-5}$ is:

$$|p_n - p| \leq \frac{1 - 0}{2^n} < 10^{-5} \iff n \geq 17$$

We have the following table:

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
| --- | --- | --- | --- | --- |
| 1 | 0 | 1 | 0.5 | 0.898 721 271 |
| 2 | 0 | 0.5 | 0.25 | −0.028 474 583 |
| 3 | 0.25 | 0.5 | 0.375 | 0.439 366 415 |
| 4 | 0.25 | 0.375 | 0.3125 | 0.206 681 691 |
| 5 | 0.25 | 0.3125 | 0.281 25 | 0.089 433 196 |
| 6 | 0.25 | 0.281 25 | 0.265 625 | 0.030 564 234 |
| 7 | 0.25 | 0.265 625 | 0.257 812 5 | 0.001 066 368 |
| 8 | 0.25 | 0.257 812 5 | 0.253 906 25 | −0.013 698 684 |
| 9 | 0.253 906 25 | 0.257 812 5 | 0.255 859 375 | −0.006 314 807 |
| 10 | 0.255 859 375 | 0.257 812 5 | 0.256 835 938 | −0.002 623 882 |
| 11 | 0.256 835 938 | 0.257 812 5 | 0.257 324 219 | −0.000 778 673 |
| 12 | 0.257 324 219 | 0.257 812 5 | 0.257 568 359 | 0.000 143 868 |
| 13 | 0.257 324 219 | 0.257 568 359 | 0.257 446 289 | −0.000 317 397 |
| 14 | 0.257 446 289 | 0.257 568 359 | 0.257 507 324 | −0.000 086 763 |
| 15 | 0.257 507 324 | 0.257 568 359 | 0.257 537 842 | 0.000 028 553 |
| 16 | 0.257 507 324 | 0.257 537 842 | 0.257 522 583 | −0.000 029 105 |
| 17 | 0.257 522 583 | 0.257 537 842 | 0.257 530 212 | −0.000 000 276 |

So $p \approx 0.257\,53$.

(c) $f(-3) \approx -9.761\,021\,72$ and $f(-2) = 1.614\,574\,483$ have the opposite signs, so there's a root in $[-3, -2]$.

The number of iteration $n$ needed to approximate $p$ to within $10^{-5}$ is:

$$|p_n - p| \leq \frac{-2 - (-3)}{2^n} < 10^{-5} \iff n \geq 17$$

We have the following table:

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|---|---|---|---|---|
| 1 | $-3$ | $-2$ | $-2.5$ | $-3.668\,310\,93$ |
| 2 | $-2.5$ | $-2$ | $-2.25$ | $-0.613\,918\,903$ |
| 3 | $-2.25$ | $-2$ | $-2.125$ | $0.630\,246\,832$ |
| 4 | $-2.25$ | $-2.125$ | $-2.1875$ | $0.038\,075\,532$ |
| 5 | $-2.25$ | $-2.1875$ | $-2.218\,75$ | $-0.280\,836\,176$ |
| 6 | $-2.218\,75$ | $-2.1875$ | $-2.203\,125$ | $-0.119\,556\,815$ |
| 7 | $-2.203\,125$ | $-2.1875$ | $-2.195\,312\,5$ | $-0.040\,278\,514$ |
| 8 | $-2.195\,312\,5$ | $-2.1875$ | $-2.191\,406\,25$ | $-0.000\,985\,195$ |
| 9 | $-2.191\,406\,25$ | $-2.1875$ | $-2.189\,453\,12$ | $0.018\,574\,337$ |
| 10 | $-2.191\,406\,25$ | $-2.189\,453\,12$ | $-2.190\,429\,69$ | $0.008\,801\,851$ |
| 11 | $-2.191\,406\,25$ | $-2.190\,429\,69$ | $-2.190\,917\,97$ | $0.003\,910\,147$ |
| 12 | $-2.191\,406\,25$ | $-2.190\,917\,97$ | $-2.191\,162\,11$ | $0.001\,462\,93$ |
| 13 | $-2.191\,406\,25$ | $-2.191\,162\,11$ | $-2.191\,284\,18$ | $0.000\,238\,981$ |
| 14 | $-2.191\,406\,25$ | $-2.191\,284\,18$ | $-2.191\,345\,21$ | $-0.000\,373\,078$ |
| 15 | $-2.191\,345\,21$ | $-2.191\,284\,18$ | $-2.191\,314\,7$ | $-0.000\,067\,041$ |
| 16 | $-2.191\,314\,7$ | $-2.191\,284\,18$ | $-2.191\,299\,44$ | $0.000\,085\,972$ |

So $p \approx -2.191\,299$.

(d) $f(0.2) \approx -0.283\,986\,684$ and $f(0.3) = 0.006\,600\,946$ have the opposite signs, so there's a root in $[0.2, 0.3]$.

The number of iteration $n$ needed to approximate $p$ to within $10^{-5}$ is:

$$|p_n - p| \leq \frac{0.3 - 0.2}{2^n} < 10^{-5} \iff n \geq 14$$

We have the following table:

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|---|---|---|---|---|
| 1 | $0.2$ | $0.3$ | $0.25$ | $-0.132\,771\,895$ |
| 2 | $0.25$ | $0.3$ | $0.275$ | $-0.061\,583\,071$ |
| 3 | $0.275$ | $0.3$ | $0.2875$ | $-0.027\,112\,719$ |
| 4 | $0.2875$ | $0.3$ | $0.293\,75$ | $-0.010\,160\,959$ |
| 5 | $0.293\,75$ | $0.3$ | $0.296\,875$ | $-0.001\,756\,232$ |
| 6 | $0.296\,875$ | $0.3$ | $0.298\,437\,5$ | $0.002\,428\,306$ |
| 7 | $0.296\,875$ | $0.298\,437\,5$ | $0.297\,656\,25$ | $0.000\,337\,524$ |
| 8 | $0.296\,875$ | $0.297\,656\,25$ | $0.297\,265\,625$ | $-0.000\,708\,983$ |
| 9 | $0.297\,265\,625$ | $0.297\,656\,25$ | $0.297\,460\,938$ | $-0.000\,185\,637$ |
| 10 | $0.297\,460\,938$ | $0.297\,656\,25$ | $0.297\,558\,594$ | $0.000\,075\,967$ |
| 11 | $0.297\,460\,938$ | $0.297\,558\,594$ | $0.297\,509\,766$ | $-0.000\,054\,829$ |
| 12 | $0.297\,509\,766$ | $0.297\,558\,594$ | $0.297\,534\,18$ | $0.000\,010\,57$ |
| 13 | $0.297\,509\,766$ | $0.297\,534\,18$ | $0.297\,521\,973$ | $-0.000\,022\,129$ |
| 14 | $0.297\,521\,973$ | $0.297\,534\,18$ | $0.297\,528\,076$ | $-0.000\,005\,779$ |

So $p \approx 0.297\,528$.

## Exercise 6

Use the Bisection method to find solutions accurate to within $10^{-5}$ for the following problems:

(a) $3x - e^x = 0$, $x \in [1, 2]$

(b) $2x + 3\cos x - e^x = 0$, $x \in [0, 1]$

(c) $x^2 - 4x + 4 - \ln x = 0$, $x \in [1, 2]$

(d) $x + 1 - 2\sin \pi x = 0$, $x \in [0, 0.5]$

## Solution 6

1. $f(1) \approx 0.281\,718\,172$ and $f(2) = -1.389\,056\,099$ have the opposite signs, so there's a root in $[1, 2]$.

   The number of iteration $n$ needed to approximate $p$ to within $10^{-5}$ is:

   $$|p_n - p| \le \frac{2 - 1}{2^n} < 10^{-5} \iff n \ge 17$$

   We have the following table:

   | $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
   |---|---|---|---|---|
   | 1 | 1 | 2 | 1.5 | 0.018\,310\,93 |
   | 2 | 1.5 | 2 | 1.75 | $-0.504\,602\,676$ |
   | 3 | 1.5 | 1.75 | 1.625 | $-0.203\,419\,037$ |
   | 4 | 1.5 | 1.625 | 1.5625 | $-0.083\,233\,182$ |
   | 5 | 1.5 | 1.5625 | 1.531\,25 | $-0.030\,203\,153$ |
   | 6 | 1.5 | 1.531\,25 | 1.515\,625 | $-0.005\,390\,404$ |
   | 7 | 1.5 | 1.515\,625 | 1.507\,812\,5 | 0.006\,598\,107 |
   | 8 | 1.507\,812\,5 | 1.515\,625 | 1.511\,718\,75 | 0.000\,638\,447 |
   | 9 | 1.511\,718\,75 | 1.515\,625 | 1.513\,671\,88 | $-0.002\,367\,313$ |
   | 10 | 1.511\,718\,75 | 1.513\,671\,88 | 1.512\,695\,31 | $-0.000\,862\,268$ |
   | 11 | 1.511\,718\,75 | 1.512\,695\,31 | 1.512\,207\,03 | $-0.000\,111\,37$ |
   | 12 | 1.511\,718\,75 | 1.512\,207\,03 | 1.511\,962\,89 | 0.000\,263\,674 |
   | 13 | 1.511\,962\,89 | 1.512\,207\,03 | 1.512\,084\,96 | 0.000\,076\,186 |
   | 14 | 1.512\,084\,96 | 1.512\,207\,03 | 1.512\,146 | $-0.000\,017\,584$ |
   | 15 | 1.512\,084\,96 | 1.512\,146 | 1.512\,115\,48 | 0.000\,029\,303 |
   | 16 | 1.512\,115\,48 | 1.512\,146 | 1.512\,130\,74 | 0.000\,005\,86 |
   | 17 | 1.512\,130\,74 | 1.512\,146 | 1.512\,138\,37 | $-0.000\,005\,861$ |

So $p \approx 1.512\,138$.

2. $f(0) = 2$ and $f(1) \approx 0.902\,625\,089$ have the same sign, so there's no root in $[0, 1]$.

3. $f(1) = 1$ and $f(2) = -0.693\,147\,181$ have the opposite signs, so there's a root in $[1, 2]$.

   The number of iteration $n$ needed to approximate $p$ to within $10^{-5}$ is:

   $$|p_n - p| \le \frac{2 - 1}{2^n} < 10^{-5} \iff n \ge 17$$

We have the following table:

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|---|---|---|---|---|
| 1 | 1 | 2 | 1.5 | $-0.155\,465\,108$ |
| 2 | 1 | 1.5 | 1.25 | $0.339\,356\,449$ |
| 3 | 1.25 | 1.5 | 1.375 | $0.072\,171\,269$ |
| 4 | 1.375 | 1.5 | 1.4375 | $-0.046\,499\,244$ |
| 5 | 1.375 | 1.4375 | 1.406\,25 | $0.011\,612\,476$ |
| 6 | 1.406\,25 | 1.4375 | 1.421\,875 | $-0.017\,747\,908$ |
| 7 | 1.406\,25 | 1.421\,875 | 1.414\,062\,5 | $-0.003\,144\,013$ |
| 8 | 1.406\,25 | 1.414\,062\,5 | 1.410\,156\,25 | $0.004\,215\,136$ |
| 9 | 1.410\,156\,25 | 1.414\,062\,5 | 1.412\,109\,38 | $0.000\,530\,79$ |
| 10 | 1.412\,109\,38 | 1.414\,062\,5 | 1.413\,085\,94 | $-0.001\,307\,804$ |
| 11 | 1.412\,109\,38 | 1.413\,085\,94 | 1.412\,597\,66 | $-0.000\,388\,805$ |
| 12 | 1.412\,109\,38 | 1.412\,597\,66 | 1.412\,353\,52 | $0.000\,070\,918$ |
| 13 | 1.412\,353\,52 | 1.412\,597\,66 | 1.412\,475\,59 | $-0.000\,158\,962$ |
| 14 | 1.412\,353\,52 | 1.412\,475\,59 | 1.412\,414\,55 | $-0.000\,044\,027$ |
| 15 | 1.412\,353\,52 | 1.412\,414\,55 | 1.412\,384\,03 | $0.000\,013\,444$ |
| 16 | 1.412\,384\,03 | 1.412\,414\,55 | 1.412\,399\,29 | $-0.000\,015\,292$ |
| 17 | 1.412\,384\,03 | 1.412\,399\,29 | 1.412\,391\,66 | $-0.000\,000\,924$ |

So $p \approx 1.412\,392$.

4. $f(0) = 1$ and $f(1) = -0.5$ have the opposite signs, so there's a root in $[0, 0.5]$.

   The number of iteration $n$ needed to approximate $p$ to within $10^{-5}$ is:

   $$|p_n - p| \le \frac{0.5 - 0}{2^n} < 10^{-5} \iff n \ge 16$$

   We have the following table:

   So $p \approx 0.206\,032$.

## Exercise 7

(a) Sketch the graphs of $y = x$ and $y = 2\sin x$.

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|-----|-------|-------|-------|----------|
| 1 | 0 | 0.5 | 0.25 | $-0.164\,213\,562$ |
| 2 | 0 | 0.25 | 0.125 | $0.359\,633\,135$ |
| 3 | 0.125 | 0.25 | 0.1875 | $0.076\,359\,534$ |
| 4 | 0.1875 | 0.25 | 0.218\,75 | $-0.050\,036\,568$ |
| 5 | 0.1875 | 0.218\,75 | 0.203\,125 | $0.011\,726\,391$ |
| 6 | 0.203\,125 | 0.218\,75 | 0.210\,937\,5 | $-0.019\,525\,681$ |
| 7 | 0.203\,125 | 0.210\,937\,5 | 0.207\,031\,25 | $-0.003\,990\,833$ |
| 8 | 0.203\,125 | 0.207\,031\,25 | 0.205\,078\,125 | $0.003\,845\,166$ |
| 9 | 0.205\,078\,125 | 0.207\,031\,25 | 0.206\,054\,688 | $-0.000\,078\,51$ |
| 10 | 0.205\,078\,125 | 0.206\,054\,688 | 0.205\,566\,406 | $0.001\,881\,912$ |
| 11 | 0.205\,566\,406 | 0.206\,054\,688 | 0.205\,810\,547 | $0.000\,901\,347$ |
| 12 | 0.205\,810\,547 | 0.206\,054\,688 | 0.205\,932\,617 | $0.000\,411\,33$ |
| 13 | 0.205\,932\,617 | 0.206\,054\,688 | 0.205\,993\,652 | $0.000\,166\,388$ |
| 14 | 0.205\,993\,652 | 0.206\,054\,688 | 0.206\,024\,17 | $0.000\,043\,934$ |
| 15 | 0.206\,024\,17 | 0.206\,054\,688 | 0.206\,039\,429 | $-0.000\,017\,289$ |
| 16 | 0.206\,024\,17 | 0.206\,039\,429 | 0.206\,031\,799 | $0.000\,013\,322$ |

(b) Use the Bisection method to find an approximation to within $10^{-5}$ to the first positive value of $x$ with $x = 2\sin x$.

## Solution 7

(a) Graph of $y = x$ and $y = 2\sin x$ is as follow:
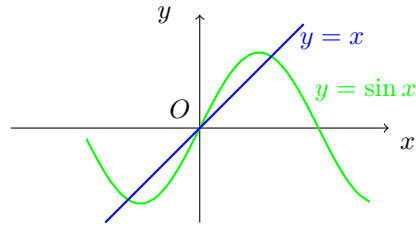


(b) According to the graph, the first positive root $p$ of $f = x - 2\sin x$ is in $[\frac{\pi}{2}, \pi]$.

The number of iteration $n$ needed to approximate $p$ to within $10^{-5}$ in that range is:

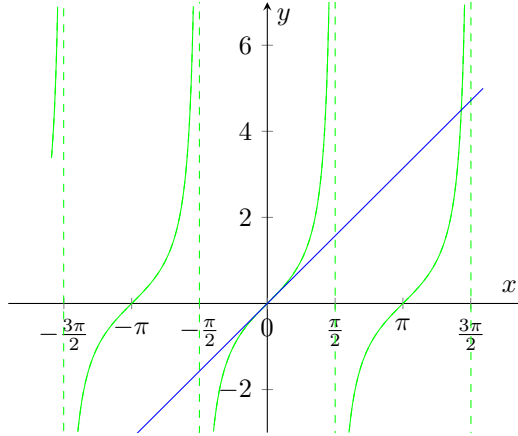$$|p_n - p| \leq \frac{\pi - \frac{\pi}{2}}{2^n} < 10^{-5} \iff n \geq 18$$

We have the following table:

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|---|---|---|---|---|
| 1 | 1.570 796 33 | 3.141 592 65 | 2.356 194 49 | 0.941 980 928 |
| 2 | 1.570 796 33 | 2.356 194 49 | 1.963 495 41 | 0.115 736 343 |
| 3 | 1.570 796 33 | 1.963 495 41 | 1.767 145 87 | −0.194 424 693 |
| 4 | 1.767 145 87 | 1.963 495 41 | 1.865 320 64 | −0.048 560 033 |
| 5 | 1.865 320 64 | 1.963 495 41 | 1.914 408 02 | 0.031 319 893 |
| 6 | 1.865 320 64 | 1.914 408 02 | 1.889 864 33 | −0.009 192 031 |
| 7 | 1.889 864 33 | 1.914 408 02 | 1.902 136 18 | 0.010 921 526 |
| 8 | 1.889 864 33 | 1.902 136 18 | 1.896 000 25 | 0.000 829 072 |
| 9 | 1.889 864 33 | 1.896 000 25 | 1.892 932 29 | −0.004 190 408 |
| 10 | 1.892 932 29 | 1.896 000 25 | 1.894 466 27 | −0.001 682 899 |
| 11 | 1.894 466 27 | 1.896 000 25 | 1.895 233 26 | −0.000 427 471 |
| 12 | 1.895 233 26 | 1.896 000 25 | 1.895 616 76 | 0.000 200 661 |
| 13 | 1.895 233 26 | 1.895 616 76 | 1.895 425 01 | −0.000 113 44 |
| 14 | 1.895 425 01 | 1.895 616 76 | 1.895 520 88 | 0.000 043 602 |
| 15 | 1.895 425 01 | 1.895 520 88 | 1.895 472 95 | −0.000 034 921 |
| 16 | 1.895 472 95 | 1.895 520 88 | 1.895 496 92 | 0.000 004 34 |
| 17 | 1.895 472 95 | 1.895 496 92 | 1.895 484 93 | −0.000 015 291 |
| 18 | 1.895 484 93 | 1.895 496 92 | 1.895 490 92 | −0.000 005 476 |

So $p \approx 1.895\,491$.

## Exercise 8

(a) Graph of $y = x$ and $y = \tan x$ is as follow:



(b) According to the graph, the first positive root $p$ of $f = x - \tan x$ is in $[\pi, \dfrac{3\pi}{2}]$.

The number of iteration $n$ needed to approximate $p$ to within $10^{-5}$ in that range is:

$$|p_n - p| \leq \frac{\dfrac{3\pi}{2} - \pi}{2^n} < 10^{-5} \iff n \geq 18$$

We have the following table:

| $n$ | $a_n$ | $b_n$ | $p_n$ | $f(p_n)$ |
|---|---|---|---|---|
| 1 | 3.141 592 65 | 4.712 388 98 | 3.926 990 82 | 2.926 990 82 |
| 2 | 3.926 990 82 | 4.712 388 98 | 4.319 689 9 | 1.905 476 34 |
| 3 | 4.319 689 9 | 4.712 388 98 | 4.516 039 44 | −0.511 300 053 |
| 4 | 4.319 689 9 | 4.516 039 44 | 4.417 864 67 | 1.121 306 46 |
| 5 | 4.417 864 67 | 4.516 039 44 | 4.466 952 05 | 0.474 728 271 |
| 6 | 4.466 952 05 | 4.516 039 44 | 4.491 495 75 | 0.038 293 523 |
| 7 | 4.491 495 75 | 4.516 039 44 | 4.503 767 59 | −0.219 861 735 |
| 8 | 4.491 495 75 | 4.503 767 59 | 4.497 631 67 | −0.086 980 389 |
| 9 | 4.491 495 75 | 4.497 631 67 | 4.494 563 71 | −0.023 432 692 |
| 10 | 4.491 495 75 | 4.494 563 71 | 4.493 029 73 | 0.007 653 323 |
| 11 | 4.493 029 73 | 4.494 563 71 | 4.493 796 72 | −0.007 833 371 |
| 12 | 4.493 029 73 | 4.493 796 72 | 4.493 413 22 | −0.000 076 02 |
| 13 | 4.493 029 73 | 4.493 413 22 | 4.493 221 48 | 0.003 792 144 |
| 14 | 4.493 221 48 | 4.493 413 22 | 4.493 317 35 | 0.001 858 936 |
| 15 | 4.493 317 35 | 4.493 413 22 | 4.493 365 29 | 0.000 891 677 |
| 16 | 4.493 365 29 | 4.493 413 22 | 4.493 389 25 | 0.000 407 883 |
| 17 | 4.493 389 25 | 4.493 413 22 | 4.493 401 24 | 0.000 165 946 |
| 18 | 4.493 401 24 | 4.493 413 22 | 4.493 407 23 | 0.000 044 966 |

So $p \approx 4.493\,407$.