

Chapter 1

Solving System of Equations

1.1 Gauss elimination

Exercise 1

For each of the following linear systems, obtain a solution by graphical methods, if possible. Explain the results from a geometrical standpoint.

a)

$$\begin{aligned}x_1 + 2x_2 &= 3 \\x_1 - x_2 &= 0\end{aligned}$$

b)

$$\begin{aligned}x_1 + 2x_2 &= 3 \\2x_1 + 4x_2 &= 6\end{aligned}$$

c)

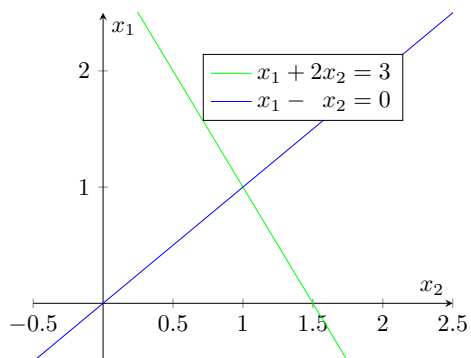
$$\begin{aligned}x_1 + 2x_2 &= 0 \\2x_1 + 4x_2 &= 0\end{aligned}$$

d)

$$\begin{aligned}2x_1 + 2x_2 &= -1 \\4x_1 + 2x_2 &= -2 \\x_1 - 3x_2 &= 5\end{aligned}$$

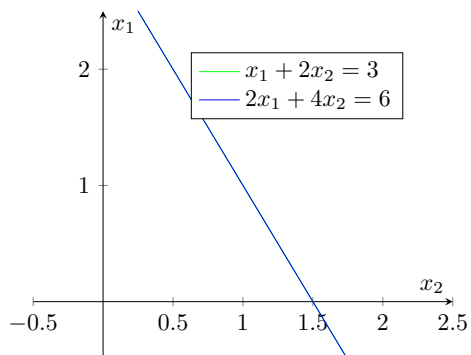
Solution 1

a) The graphs of the equations are as follow:



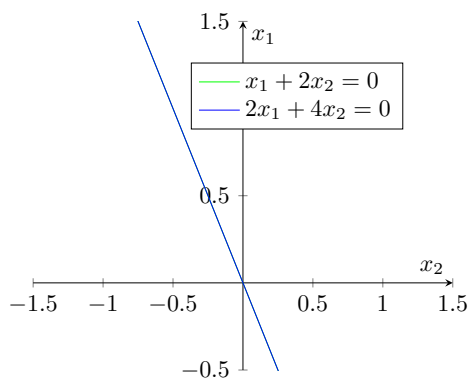
The solution is $x_1 = 1$, $x_2 = 1$ as the lines intersect at $(1, 1)$.

b) The graphs of the equations are as follow:



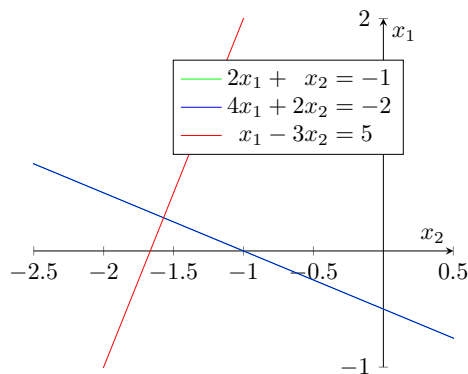
The system of equation has an infinite number of solutions, as the line coincide.

c) The graphs of the equations are as follow:



The system of equation has an infinite number of solutions, as the lines coincide.

d) The graphs of the equations are as follow:



The solution is $x_1 = -\frac{11}{7}$, $x_2 = \frac{2}{7}$ as the lines intersect at $(\frac{2}{7}, -\frac{11}{7})$.

Exercise 2

For each of the following linear systems, obtain a solution by graphical methods, if possible. Explain the results from a geometrical standpoint.

Solution 2

a)

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}$$

b)

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ -2x_1 - 4x_2 &= 6 \end{aligned}$$

c)

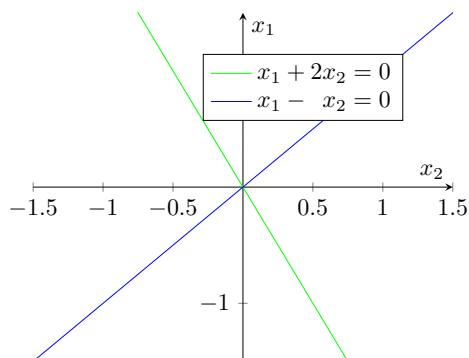
$$\begin{aligned} 2x_1 + x_2 &= -1 \\ x_1 + x_2 &= 2 \\ x_1 - 3x_2 &= 5 \end{aligned}$$

d)

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 - x_3 &= -1 \end{aligned}$$

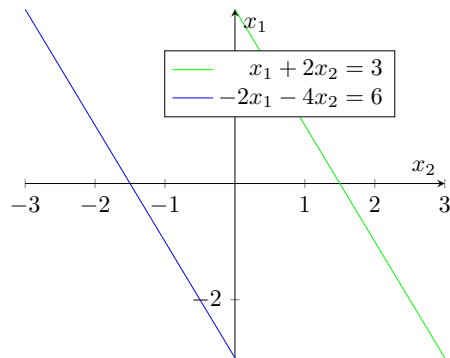
Solution 2

a) The graphs of the equations are as follow:



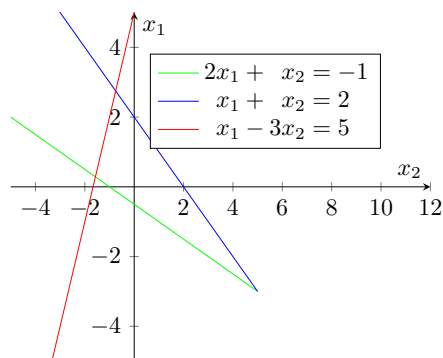
The solution is $x_1 = 0$, $x_2 = 0$ as the lines intersect at $(0, 0)$.

b) The graphs of the equations are as follow:



The system of equation has no solution, as the lines are parallel to each other.

c) The graphs of the equations are as follow:



The system of equation has no solution, as the lines do not intersect.

Exercise 3

Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations.

a)

$$\begin{aligned} 4x_1 - x_2 + x_3 &= 8 \\ 2x_1 + 5x_2 - 2x_3 &= 3 \\ x_1 + 2x_2 - 4x_3 &= 11 \end{aligned}$$

b)

$$\begin{aligned} 4x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 1x_3 &= -5 \\ x_1 + x_2 - 3x_3 &= -9 \end{aligned}$$

Solution 3

a) Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} 4 & -1 & 1 & \vdots & 8 \\ 2 & 5 & 2 & \vdots & 3 \\ 1 & 2 & 4 & \vdots & 11 \end{pmatrix}$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - 0.5E_1; E_3 := E_3 - 0.25E_1$$

gives:

$$\tilde{\mathbf{A}}^{(2)} = \begin{pmatrix} 4 & -1 & 1 & \vdots & 8 \\ 0 & 5.5 & 1.5 & \vdots & -1 \\ 0 & 2.25 & 3.75 & \vdots & 9 \end{pmatrix}$$

Eliminating x_2 by these transformation

$$E_3 := E_3 - \frac{9}{22}E_2$$

gives:

$$\tilde{\mathbf{A}}^{(3)} = \begin{pmatrix} 4 & -1 & 1 & \vdots & 8 \\ 0 & 5.5 & 1.5 & \vdots & -1 \\ 0 & 0 & 3.13636 & \vdots & 9.40909 \end{pmatrix}$$

The solution is $x_3 \approx 3$, $x_2 \approx -1$, $x_1 \approx 1$.

b) Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} 4 & 1 & 2 & \vdots & 9 \\ 2 & 4 & -1 & \vdots & -5 \\ 1 & 1 & -3 & \vdots & -9 \end{pmatrix}$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - 0.5E_1; E_3 := E_3 - 0.25E_1$$

gives:

$$\tilde{\mathbf{A}}^{(2)} = \begin{pmatrix} 4 & 1 & 2 & \vdots & 9 \\ 0 & 3.5 & -2 & \vdots & -9.5 \\ 0 & 0.75 & -3.5 & \vdots & -11.25 \end{pmatrix}$$

Eliminating x_2 by these transformation

$$E_3 := E_3 - \frac{3}{14}E_2$$

gives:

$$\tilde{\mathbf{A}}^{(3)} = \begin{pmatrix} 4 & 1 & 2 & \vdots & 9 \\ 0 & 3.5 & -2 & \vdots & -9.5 \\ 0 & 0 & -3.07143 & \vdots & -9.21429 \end{pmatrix}$$

The solution is $x_3 \approx 3$, $x_2 \approx -1$, $x_1 \approx 1$.

Exercise 4

Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations.

a)

$$\begin{aligned} -1x_1 + 4x_2 + x_3 &= 8 \\ \frac{5}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 11 \end{aligned}$$

b)

$$\begin{aligned} 4x_1 + 2x_2 - x_3 &= -5 \\ \frac{1}{9}x_1 + \frac{1}{9}x_2 - \frac{1}{3}x_3 &= -1 \\ 1x_1 + 4x_2 + 2x_3 &= 9 \end{aligned}$$

Solution 4

a) Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} -1 & 4 & 1 & \vdots & 8 \\ 1.66667 & 0.66667 & 0.66667 & \vdots & 1 \\ 2 & 1 & 4 & \vdots & 11 \end{pmatrix}$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - (-1.66667)E_1; E_3 := E_3 - (-2)E_1$$

gives:

$$\tilde{\mathbf{A}}^{(2)} = \begin{pmatrix} -1 & 4 & 1 & \vdots & 8 \\ 0 & 7.33333 & 2.33333 & \vdots & 14.33333 \\ 0 & 9 & 6 & \vdots & 27 \end{pmatrix}$$

Eliminating x_2 by these transformation

$$E_3 := E_3 - 1.22727E_2$$

gives:

$$\tilde{\mathbf{A}}^{(3)} = \begin{pmatrix} -1 & 4 & 1 & \vdots & 8 \\ 0 & 7.333\,33 & 2.333\,33 & \vdots & 14.333\,33 \\ 0 & 0 & 3.136\,36 & \vdots & 9.409\,09 \end{pmatrix}$$

The solution is $x_3 \approx 3$, $x_2 \approx 1$, $x_1 \approx -1$.

b) Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} 4 & 2 & -1 & \vdots & -5 \\ 0.111\,11 & 0.111\,11 & -0.333\,33 & \vdots & -1 \\ 1 & 4 & 2 & \vdots & 9 \end{pmatrix}$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - 0.027\,78E_1; \quad E_3 := E_3 - 0.25E_1$$

gives:

$$\tilde{\mathbf{A}}^{(2)} = \begin{pmatrix} 4 & 2 & -1 & \vdots & -5 \\ 0 & 0.055\,56 & -0.305\,55 & \vdots & -0.861\,11 \\ 0 & 3.5 & 2.25 & \vdots & 10.25 \end{pmatrix}$$

Eliminating x_2 by these transformation

$$E_3 := E_3 - 63.000\,63E_2$$

gives:

$$\tilde{\mathbf{A}}^{(3)} = \begin{pmatrix} 4 & 2 & -1 & \vdots & -5 \\ 0 & 0.055\,56 & -0.305\,55 & \vdots & -0.861\,11 \\ 0 & 0 & 21.5 & \vdots & 64.500\,63 \end{pmatrix}$$

The solution is $x_3 \approx 3$, $x_2 \approx 1$, $x_1 \approx -1$.

Exercise 5

Use the Gaussian Elimination Algorithm to solve the following linear systems, if possible, and determine whether row interchanges are necessary:

a)

$$\begin{aligned} x_1 - 1x_2 + 3x_3 &= 2 \\ 3x_1 - 3x_2 + 1x_3 &= -1 \\ x_1 + 1x_2 - &= 3 \end{aligned}$$

b)

$$\begin{aligned} 2x_1 - 1.5x_2 + 3x_3 &= 1 \\ -1x_1 &+ 2x_3 = 3 \\ 4x_1 - 4.5x_2 + 5x_3 &= 1 \end{aligned}$$

c)

$$\begin{array}{rcl}
2x_1 & = & 3 \\
x_1 + 1.5x_2 & = & 4.5 \\
-3x_2 + 0.5x_3 & = & -6.6 \\
2x_1 - 2x_2 + x_3 + x_4 & = & 0.8
\end{array}$$

d)

$$\begin{array}{rcl}
x_1 + x_2 + x_4 & = & 2 \\
2x_1 + x_2 - x_3 + x_4 & = & 1 \\
4 - x_2 - 2x_3 + 2 & = & 0 \\
3x_1 - x_2 - x_3 + 2x_4 & = & -3
\end{array}$$

Solution 5

a) Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} 1 & -1 & 3 & \vdots & 2 \\ 3 & -3 & 1 & \vdots & -1 \\ 1 & 1 & 0 & \vdots & 3 \end{pmatrix}$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - 3E_1; E_3 := E_3 - 1E_1$$

gives:

$$\tilde{\mathbf{A}}^{(2)} = \begin{pmatrix} 1 & -1 & 3 & \vdots & 2 \\ 0 & 0 & -8 & \vdots & -7 \\ 0 & 2 & -3 & \vdots & 1 \end{pmatrix}$$

As $a_{22}^{(2)} = 0$, we have to swap row 2 and 3. Eliminating x_2 by these transformation

$$E_3 := E_3 - 63.00063E_2$$

gives:

$$\tilde{\mathbf{A}}^{(3)} = \begin{pmatrix} 1 & -1 & 3 & \vdots & 2 \\ 0 & 2 & -3 & \vdots & 1 \\ 0 & 0 & -8 & \vdots & -7 \end{pmatrix}$$

The solution is $x_3 = 0.875$, $x_2 = 1.8125$, $x_1 = 1.1875$.

b) Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} 2 & -1.5 & 3 & \vdots & 1 \\ -1 & 0 & 2 & \vdots & 3 \\ 4 & -4.5 & 5 & \vdots & 1 \end{pmatrix}$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - (-0.5)E_1; E_3 := E_3 - 2E_1$$

gives:

$$\tilde{\mathbf{A}}^{(2)} = \begin{pmatrix} 2 & -1.5 & 3 & \vdots & 1 \\ 0 & -0.75 & 3.5 & \vdots & 3.5 \\ 0 & -1.5 & -1 & \vdots & -1 \end{pmatrix}$$

Eliminating x_2 by these transformation

$$E_3 := E_3 - 2E_2$$

gives:

$$\tilde{\mathbf{A}}^{(3)} = \begin{pmatrix} 2 & -1.5 & 3 & \vdots & 1 \\ 0 & -0.75 & 3.5 & \vdots & 3.5 \\ 0 & 0 & -8 & \vdots & -8 \end{pmatrix}$$

The solution is $x_3 = 1$, $x_2 = 0$, $x_1 = -1$.

c) Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} 2 & 0 & 0 & 0 \vdots & 3 \\ 1 & 1.5 & 0 & 0 \vdots & 4.5 \\ 0 & -3 & 0.5 & 0 \vdots & -6.6 \\ 2 & -2 & 1 & 1 \vdots & 0.8 \end{pmatrix}$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - 0.5E_1; E_3 := E_3 - 0E_1; E_4 := E_4 - 1E_1$$

gives:

$$\tilde{\mathbf{A}}^{(2)} = \begin{pmatrix} 2 & 0 & 0 & 0 \vdots & 3 \\ 0 & 1.5 & 0 & 0 \vdots & 3 \\ 0 & -3 & 0.5 & 0 \vdots & -6.6 \\ 0 & -2 & 1 & 1 \vdots & -2.2 \end{pmatrix}$$

Eliminating x_2 by these transformation

$$E_3 := E_3 - (-2)E_2; E_4 := E_4 - (-1.33333)E_2$$

gives:

$$\tilde{\mathbf{A}}^{(3)} = \begin{pmatrix} 2 & 0 & 0 & 0 \vdots & 3 \\ 0 & 1.5 & 0 & 0 \vdots & 3 \\ 0 & 0 & 0.5 & 0 \vdots & -0.6 \\ 0 & 0 & 1 & 1 \vdots & 1.8 \end{pmatrix}$$

Eliminating x_3 by these transformation

$$E_4 := E_4 - 2E_3$$

gives:

$$\tilde{\mathbf{A}}^{(4)} = \begin{pmatrix} 2 & 0 & 0 & 0 \vdots & 3 \\ 0 & 1.5 & 0 & 0 \vdots & 3 \\ 0 & 0 & 0.5 & 0 \vdots & -0.6 \\ 0 & 0 & 0 & 1 \vdots & 3 \end{pmatrix}$$

The solution is $x_4 = 3$, $x_3 = -1.2$, $x_2 = 2$, $x_1 = 1.5$.

d) Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} 1 & 1 & 0 & 1 & \vdots & 2 \\ 2 & 1 & -1 & 1 & \vdots & 1 \\ 4 & -1 & -2 & 2 & \vdots & 0 \\ 3 & -1 & -1 & 2 & \vdots & -3 \end{pmatrix}$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - 2E_1; E_3 := E_3 - 4E_1; E_4 := E_4 - 3E_1$$

gives:

$$\tilde{\mathbf{A}}^{(2)} = \begin{pmatrix} 1 & 1 & 0 & 1 & \vdots & 2 \\ 0 & -1 & -1 & -1 & \vdots & -3 \\ 0 & -5 & -2 & -2 & \vdots & -8 \\ 0 & -4 & -1 & -1 & \vdots & -9 \end{pmatrix}$$

Eliminating x_2 by these transformation

$$E_3 := E_3 - 5E_2; E_4 := E_4 - 4E_2$$

gives:

$$\tilde{\mathbf{A}}^{(3)} = \begin{pmatrix} 1 & 1 & 0 & 1 & \vdots & 2 \\ 0 & -1 & -1 & -1 & \vdots & -3 \\ 0 & 0 & 3 & 3 & \vdots & 7 \\ 0 & 0 & 3 & 3 & \vdots & 3 \end{pmatrix}$$

Eliminating x_3 by these transformation

$$E_4 := E_4 - E_3$$

gives:

$$\tilde{\mathbf{A}}^{(4)} = \begin{pmatrix} 1 & 1 & 0 & 1 & \vdots & 2 \\ 0 & -1 & -1 & -1 & \vdots & -3 \\ 0 & 0 & 3 & 3 & \vdots & 7 \\ 0 & 0 & 0 & 0 & \vdots & -4 \end{pmatrix}$$

The system has no unique solution.

Exercise 6

Use the Gaussian Elimination Algorithm to solve the following linear systems, if possible, and determine whether row interchanges are necessary:

a)

$$\begin{aligned} x_2 - 2x_3 &= 4 \\ x_1 - 3x_2 + x_3 &= 6 \\ x_1 - x_3 &= 2 \end{aligned}$$

b)

$$\begin{aligned} x_1 - 0.5x_3 &= 4 \\ 2x_1 - x_2 - x_3 + x_4 &= 5 \\ x_1 + x_2 + 0.5x_3 &= 2 \\ x_1 - 0.5x_2 + x_3 + x_4 &= 5 \end{aligned}$$

c)

$$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 6 \\ x_2 - x_3 + x_4 &= 5 \\ x_4 &= 5 \\ x_3 - x_4 &= 3 \end{aligned}$$

d)

$$\begin{aligned} x_1 + x_2 + x_4 &= 2 \\ 2x_1 + x_2 - x_3 + x_4 &= 1 \\ -1x_1 + 2x_2 + 3x_3 - x_4 &= 4 \\ 3x_1 - x_2 - x_3 + 2x_4 &= -3 \end{aligned}$$

Solution 6

a) Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} 0 & 1 & -2 & : & 4 \\ 1 & -1 & 1 & : & 6 \\ 1 & 0 & -1 & : & 2 \end{pmatrix}$$

As $a_{11}^{(1)} = 0$, we need to swap row 1 and 2. Eliminating x_1 by these transformation

$$E_3 := E_3 - E_1$$

gives:

$$\tilde{\mathbf{A}}^{(2)} = \begin{pmatrix} 1 & -1 & 1 & : & 6 \\ 0 & 1 & -2 & : & 4 \\ 0 & 1 & -2 & : & -4 \end{pmatrix}$$

As $a_{22}^{(2)} = 0$, we have to swap row 2 and 3. Eliminating x_2 by these transformation

$$E_3 := E_3 - E_2$$

gives:

$$\tilde{\mathbf{A}}^{(3)} = \begin{pmatrix} 1 & -1 & 1 & : & 6 \\ 0 & 1 & -2 & : & 4 \\ 0 & 0 & 0 & : & -8 \end{pmatrix}$$

The system has no unique solution.

b) Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} 1 & -0.5 & 1 & 0 & : & 4 \\ 2 & -1 & -1 & 1 & : & 5 \\ 1 & 1 & 0.5 & 0 & : & 2 \\ 1 & -0.5 & 1 & 1 & : & 5 \end{pmatrix}$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - 2E_1; E_3 := E_3 - E_1; E_4 := E_4 - E_1$$

gives:

$$\tilde{\mathbf{A}}^{(2)} = \left(\begin{array}{cccc|c} 1 & -0.5 & 1 & 0 & 4 \\ 0 & 0 & -3 & 1 & -3 \\ 0 & 1.5 & -0.5 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right)$$

As $a_{22}^{(2)} = 0$, we need to swap row 2 and 3, effectively eliminating x_2 and x_3 :

$$\tilde{\mathbf{A}}^{(3)} = \left(\begin{array}{cccc|c} 1 & -0.5 & 1 & 0 & 4 \\ 0 & 1.5 & -0.5 & 0 & -2 \\ 0 & 0 & -3 & 1 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right)$$

The solution is $x_4 = 1$, $x_3 \approx 1.333\,33$, $x_2 \approx -0.888\,89$, $x_1 \approx 2.222\,22$.

c) Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \left(\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 6 \\ 0 & 1 & -1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & -1 & 3 \end{array} \right)$$

x_1 and x_2 are already eliminated. As $a_{33}^{(3)} = 0$, we need to swap row 3 and 4, effectively eliminating x_3 :

$$\tilde{\mathbf{A}}^{(2)} = \left(\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 6 \\ 0 & 1 & -1 & 1 & 5 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right)$$

The solution is $x_4 = 5$, $x_3 = 8$, $x_2 = 8$, $x_1 = 5.5$.

d) Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & -1 & 1 & 1 \\ -1 & 2 & 3 & -1 & 4 \\ 3 & -1 & -1 & 2 & -3 \end{array} \right)$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - 2E_1; \quad E_3 := E_3 - (-1)E_1; \quad E_4 := E_4 - 3E_1$$

gives:

$$\tilde{\mathbf{A}}^{(2)} = \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & -4 & -1 & -1 & -9 \end{array} \right)$$

Eliminating x_2 by these transformation

$$E_3 := E_3 - (-3)E_2; E_4 := E_4 - 4E_2$$

gives:

$$\tilde{A}^{(3)} = \begin{pmatrix} 1 & 1 & 0 & 1 & \vdots & 2 \\ 0 & -1 & -1 & -1 & \vdots & -3 \\ 0 & 0 & 0 & -3 & \vdots & -3 \\ 0 & 0 & 3 & 3 & \vdots & 3 \end{pmatrix}$$

As $a_{33}^{(3)} = 0$, we need to swap row 3 and 4, effectively eliminating x_3 :

$$\tilde{A}^{(4)} = \begin{pmatrix} 1 & 1 & 0 & 1 & \vdots & 2 \\ 0 & -1 & -1 & -1 & \vdots & -3 \\ 0 & 0 & 3 & 3 & \vdots & 3 \\ 0 & 0 & 0 & -3 & \vdots & -3 \end{pmatrix}$$

The solution is $x_4 = 1$, $x_3 = 0$, $x_2 = 2$, $x_1 = -1$.

Exercise 7

Use Algorithm 6.1 and Maple with Digits:= 10 to solve the following linear systems ...

Solution 7

Opps, can't help without Maple license.

Exercise 8

Use Algorithm 6.1 and Maple with Digits:= 10 to solve the following linear systems ...

Solution 8

Opps, can't help without Maple license.

Exercise 9

Given the linear system

$$\begin{aligned} 2x_1 - 6\alpha x_2 &= 3 \\ 3\alpha x_1 - x_2 &= 1.5 \end{aligned}$$

- Find value(s) of α for which the system has no solutions.
- Find value(s) of α for which the system has an infinite number of solutions.
- Assuming a unique solution exists for a given α , find the solution.

Solution 9

Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} 2 & -6\alpha & : & 3 \\ 3\alpha & -1 & : & 1.5 \end{pmatrix}$$

Eliminating x_1 gives:

$$\tilde{\mathbf{A}}^{(2)} = \begin{pmatrix} 2 & -6\alpha & : & 3 \\ 0 & 9\alpha^2 - 1 & : & 1.5 - 4.5\alpha \end{pmatrix}$$

The system has no unique solution (either no solution or infinite number of solutions) if and only if:

$$9\alpha^2 - 1 = 0 \iff \alpha = \pm \frac{1}{3}$$

- a) The system has no solution if it has no unique solution and

$$1.5(1 - 3\alpha) \neq 0 \iff \alpha = -\frac{1}{3}$$

- b) The system has an infinite number of solution if it has no unique solution and

$$1.5(1 - 3\alpha) = 0 \iff \alpha = \frac{1}{3}$$

In this case, the solution assumes a general form:

$$x_2 \in \mathbb{R} \text{ and } x_1 = x_2 + 1.5$$

- c) The system has a unique solution if and only if $\alpha \neq \pm \frac{1}{3}$. Then the unique solution is:

$$x_2 = \frac{-1.5}{3\alpha + 1} \text{ and } x_1 = \frac{1.5}{3\alpha + 1}$$

Exercise 10

Given the linear system

$$\begin{aligned} x_1 - x_2 + \alpha x_3 &= -2 \\ -x_1 + 2x_2 - \alpha x_3 &= 3 \\ \alpha x_1 + x_2 + \alpha x_3 &= 2 \end{aligned}$$

- Find value(s) of α for which the system has no solutions.
- Find value(s) of α for which the system has an infinite number of solutions.
- Assuming a unique solution exists for a given α , find the solution.

Solution 10

Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} 1 & -1 & \alpha & \vdots & -2 \\ -1 & 2 & -\alpha & \vdots & 3 \\ \alpha & 1 & \alpha & \vdots & 2 \end{pmatrix}$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - (-1)E_1; E_3 := E_3 - \alpha E_1$$

gives:

$$\tilde{\mathbf{A}}^{(2)} = \begin{pmatrix} 1 & -1 & \alpha & \vdots & -2 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & \alpha + 1 & \alpha - \alpha^2 & \vdots & 2\alpha + 2 \end{pmatrix}$$

Eliminating x_2 by these transformation

$$E_3 := E_3 - (\alpha + 1)E_2$$

gives:

$$\tilde{\mathbf{A}}^{(3)} = \begin{pmatrix} 1 & -1 & \alpha & \vdots & -2 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & \alpha - \alpha^2 & \vdots & \alpha + 1 \end{pmatrix}$$

The system has no unique solution (either no solution or infinite number of solutions) if and only if:

$$\alpha - \alpha^2 = 0 \iff \alpha \in \{0, 1\}$$

a) The system has no solution if it has no unique solution and

$$2\alpha + 2 \neq 0 \iff \alpha \in \{0, 1\}$$

b) The system has an infinite number of solution if it has no unique solution and

$$2\alpha + 2 \neq 0 \iff \alpha \in \emptyset$$

c) The system has a unique solution if and only if $\alpha \notin \{0, 1\}$. Then the unique solution is:

$$x_3 = \frac{\alpha + 1}{\alpha - \alpha^2}, x_2 = 1 \text{ and } x_1 = \frac{2}{\alpha - 1}$$

Exercise 11

Show that the 3 elementary row operations do not change the solution set of a linear system.

Solution 11

Let x_1, x_2, \dots, x_n be the solution of the original system.

When an elementary row operations is applied on row i^{th} , the original solution still satisfies the unchanged rows. We have to prove that it also satisfies the changed row.

- a) If i^{th} row is scaled, i^{th} equation is still satisfied by the original solution because both size of it is multiplied with a constant.
- b) If a scaled j^{th} row is added to i^{th} row, then the original solution still satisfies the new row, as
 - it satisfies the j^{th} row, therefore satisfies the scaled j^{th} row, as proven above, and
 - it satisfies the original i^{th} row
- c) If the rows are swapped, the solution does not change, as the set of the equation does not change.

Exercise 12

Gauss-Jordan Method: This method is described as follows. Use the i^{th} equation to eliminate not only x_i from the equations $E_{>i}$ as was done in the Gaussian elimination method, but also from $E_{<i}$. Upon reducing $[A, b]$ to:

$$\left(\begin{array}{ccccccc} a_{11}^{(1)} & & & & & & b_1^{(1)} \\ & a_{22}^{(2)} & & & & & b_2^{(2)} \\ & & \ddots & & & & \vdots \\ & & & a_{nn}^{(n)} & & & b_n^{(n)} \end{array} \right)$$

the solution can be obtained by

$$x_i = \frac{b_i^{(i)}}{a_{ii}^{(i)}}$$

This procedure circumvents the backward substitution in the Gaussian elimination. Construct an algorithm for the Gauss-Jordan procedure patterned after that of Algorithm 6.1.

Solution 12

In Step 4, change j from $j > i$ to $j \neq i$. In Step 8, calculate for all i :

$$x_i = \frac{b_i}{a_{ii}}$$

Remove Step 9.

Exercise 13

Use the Gauss-Jordan method and two-digit rounding arithmetic to solve the systems in Exercise 3.

Solution 13

a) Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} 4 & -1 & 1 & \vdots & 8 \\ 2 & 5 & 2 & \vdots & 3 \\ 1 & 2 & 4 & \vdots & 11 \end{pmatrix}$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - 0.5E_1; E_3 := E_3 - 0.25E_1$$

gives:

$$\tilde{\mathbf{A}}^{(2)} = \begin{pmatrix} 4 & -1 & 1 & \vdots & 8 \\ 0 & 5.5 & 1.5 & \vdots & -1 \\ 0 & 2.25 & 3.75 & \vdots & 9 \end{pmatrix}$$

Eliminating x_2 by these transformation

$$E_1 := E_1 - (-0.18182)E_2; E_3 := E_3 - 0.40909E_2$$

gives:

$$\tilde{\mathbf{A}}^{(3)} = \begin{pmatrix} 4 & 0 & 1.27273 & \vdots & 7.81818 \\ 0 & 5.5 & 1.5 & \vdots & -1 \\ 0 & 0 & 3.13636 & \vdots & 9.40909 \end{pmatrix}$$

Eliminating x_3 by these transformation

$$E_1 := E_1 - 0.40580E_3; E_2 := E_2 - 0.47826E_3$$

gives:

$$\tilde{\mathbf{A}}^{(4)} = \begin{pmatrix} 4 & 0 & 0 & \vdots & 4 \\ 0 & 5.5 & 0 & \vdots & -5.5 \\ 0 & 0 & 3.13636 & \vdots & 9.40909 \end{pmatrix}$$

The solution is $x_3 \approx 3$, $x_2 \approx -1$, $x_1 \approx 1$.

b) Let

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}^{(1)} = \begin{pmatrix} 4 & 1 & 2 & \vdots & 9 \\ 2 & 4 & -1 & \vdots & -5 \\ 1 & 1 & -3 & \vdots & -9 \end{pmatrix}$$

Eliminating x_1 by these transformation

$$E_2 := E_2 - 0.50000E_1; E_3 := E_3 - 0.25000E_1$$

gives:

$$\tilde{\mathbf{A}}^{(2)} = \begin{pmatrix} 4 & 1 & 2 & \vdots & 9 \\ 0 & 3.5 & -2 & \vdots & -9.5 \\ 0 & 0.75 & -3.5 & \vdots & -11.25 \end{pmatrix}$$

Eliminating x_2 by these transformation

$$E_1 := E_1 - 0.28571E_2; E_3 := E_3 - 0.21429E_2$$

gives:

$$\tilde{\mathbf{A}}^{(3)} = \begin{pmatrix} 4 & 0 & 2.57143 & \vdots & 11.71429 \\ 0 & 3.5 & -2 & \vdots & -9.5 \\ 0 & 0 & -3.07143 & \vdots & -9.21429 \end{pmatrix}$$

Eliminating x_3 by these transformation

$$E_1 := E_1 - (-0.83721)E_3; E_2 := E_2 - 0.65116E_3$$

gives:

$$\tilde{\mathbf{A}}^{(4)} = \begin{pmatrix} 4 & 0 & 0 & \vdots & 4 \\ 0 & 3.5 & 0 & \vdots & -3.5 \\ 0 & 0 & -3.07143 & \vdots & -9.21429 \end{pmatrix}$$

The solution is $x_3 \approx 3$, $x_2 \approx -1$, $x_1 \approx 1$.

Exercise 14

Repeat Exercise 7 using the Gauss-Jordan method.

Solution 14

Opps, can't help without Maple license.