0.1 Newton's Method and Its Extensions

Exercise 1

Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.

Solution 1

$f'(x) = 2x$. Therefore, $p_1 = 3.5$, $p_2 = 2.607\,142$.

Exercise 2

Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find p_2. Could $p_0 = 0$ be used?

Solution 2

$f'(x) = -3x^2 + \sin x$. Therefore, $p_1 = -0.880\,333$, $p_2 = -0.865\,684$.
$p_0 = 0$ can't be used, as $f'(p_0) = 0$, therefore p_1 can't be calculated.

Exercise 3

Let $f(x) = x^2 - 6$. With $p_0 = 3$ and $p_1 = 2$, find p_3.

a) Use the Secant method.

b) Use the method of False Position.

c) Which of the above is closer to $\sqrt{6}$?

Solution 3

a) Applying Secant method generates the following table:

n	p_n	$f(p_n)$
0	3	3
1	2	-2
2	2.4	-0.24
3	2.454 545	0.024 793

So $p_3 = 2.454\,545$.

b) Applying False Position method generates the following table:

n	p_n	$f(p_n)$
0	3	3
1	2	-2
2	2.4	-0.24
3	2.454 545	2.444 444

So $p_3 = 2.444\,444$.

c) p_3 produced by Secant method is better.

Exercise 4

Let $f(x) = -x^3 - \cos x$. With $p_0 = -1$ and $p_1 = 0$, find p_3.

a) Use the Secant method.

b) Use the method of False Position.

Solution 4

a) Applying Secant method generates the following table:

n	p_n	$f(p_n)$
0	-1	0.459 697 694
1	0	-1
2	$-0.685\,073\,357$	$-0.452\,850\,234$
3	$-1.252\,076\,489$	1.649 523 592

So $p_3 = -1.252\,076$.

b) Applying False Position method generates the following table:

n	p_n	$f(p_n)$
0	-1	0.459 697 694
1	0	-1
2	$-0.685\,073\,357$	$-0.452\,850\,234$
3	$-0.841\,355\,126$	$-0.070\,875\,968$

So $p_3 = -0.841\,355$.

Exercise 5

Use Newton's method to find solutions accurate to within 10^{-4} for the following problems.

a) $x^3 - 2x^2 - 5 = 0$ in $[1, 4]$

b) $x^3 + 3x^2 - 1 = 0$ in $[-3, -2]$

c) $x - \cos x = 0$ in $[0, \pi/2]$

d) $x - 0.8 - 0.2\sin x = 0$ in $[0, \pi/2]$

Solution 5

a) Let

$$f(x) = x^3 - 2x^2 - 5$$
$$\Rightarrow f'(x) = 3x^2 - 4x$$

Applying Newton's method on f with $p_0 = 2.5$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	2.5	−1.875	8.75
1	2.714 285 714	0.262 390 671	11.244 897 96
2	2.690 951 571	0.003 331 987	10.959 854 13
3	2.690 647 499	0.000 000 561	10.956 161 9
4	2.690 647 448	0	10.956 161 28

We conclude that $p \approx 2.690\,65$ is a solution of the problem.

b) Let

$$f(x) = x^3 + 3x^2 - 1$$
$$\Rightarrow f'(x) = 3x^2 + 6x$$

Applying Newton's method on f with $p_0 = -2.5$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	−2.5	2.125	3.75
1	−3.066 666 67	−1.626 962 963	9.813 333 33
2	−2.900 875 604	−0.165 860 349	7.839 984 184
3	−2.879 719 904	−0.002 542 819	7.600 040 757
4	−2.879 385 325	−0.000 000 631	7.596 267 596
5	−2.879 385 242	0	7.596 266 659

We conclude that $p \approx 2.690\,65$ is a solution of the problem.

c) Let

$$
\begin{aligned}
f(x) &= x - \cos x \\
\Rightarrow f'(x) &= 1 + \sin x
\end{aligned}
$$

Applying Newton's method on f with $p_0 = 0.739$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.739	−0.000 142 477	1.673 549 106
1	0.739 085 135	0.000 000 002	1.673 612 03

We conclude that $p \approx 0.739\,09$ is a solution of the problem.

d) Let

$$
\begin{aligned}
f(x) &= x - 0.8 - 0.2 \sin x \\
\Rightarrow f'(x) &= 1 - 0.2 \cos x
\end{aligned}
$$

Applying Newton's method on f with $p_0 = 0.964$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.964	−0.000 295 817	0.885 952 272
1	0.964 333 898	−0.000 000 009	0.886 007 136
2	0.964 333 888	0	0.886 007 135

We conclude that $p \approx 0.964\,33$ is a solution of the problem.

Exercise 6

Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.

a) $e^x + 2^{-x} + 2\cos x - 6 = 0$ for $x \in [1, 2]$

b) $\ln(x - 1) + \cos(x - 1) = 0$ for $x \in [1.3, 2]$

c) $2x\cos(2x) - (x - 2)^2 = 0$ for $x \in [2, 3]$ and $x \in [3, 4]$

d) $(x - 2)^2 - \ln x = 0$ for $x \in [1, 2]$ and $x \in [e, 4]$

e) $e^x - 3x^2 = 0$ for $x \in [0, 1]$ and $x \in [3, 5]$

f) $\sin x - e^x = 0$ for $x \in [0, 1]$, $x \in [3, 4]$ and $x \in [6, 7]$

Solution 6

a) Let

$$\begin{aligned} f(x) &= e^x + 2^{-x} + 2\cos x - 6 \\ \Rightarrow f'(x) &= e^x - \ln 2 \cdot 2^{-x} - 2\sin x \end{aligned}$$

Applying Newton's method on f with $p_0 = 1.829$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	1.829	−0.001 572 837	4.098 862 489
1	1.829 383 725	0.000 000 506	4.101 500 646
2	1.829 383 602	0	4.101 499 798

We conclude that $p \approx 1.829\,384$ is a solution of the problem.

b) Let

$$\begin{aligned} f(x) &= \ln(x-1) + \cos(x-1) \\ \Rightarrow f'(x) &= \frac{1}{x-1} - \sin(x-1) \end{aligned}$$

Applying Newton's method on f with $p_0 = 1.398$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	1.398	0.000 534 714	1.527 454 989
1	1.397 649 931	−0.000 209 62	1.529 727 16

We conclude that $p \approx 1.397\,65$ is a solution of the problem.

c) Let

$$\begin{aligned} f(x) &= 2x\cos(2x) - (x-2)^2 \\ \Rightarrow f'(x) &= 2(\cos x - x\sin(2x)2) - 2(x-2) \\ &= 2(\cos x - 2x\sin(2x) - x + 2) \end{aligned}$$

Applying Newton's method on f with $p_0 = 2.371$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	2.371	0.002 753 936	7.302 846 51
1	2.370 622 9	−0.000 563 086	7.302 827 46
2	2.3707	0.000 115 071	7.302 831 78
3	2.370 684 24	−0.000 023 518	7.302 830 91

Applying Newton's method on f with $p_0 = 3.722$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	3.722	0.001 838 451	−18.770 682 49
1	3.722 097 943	0.000 241 783	−18.772 292 46
2	3.722 110 823	0.000 031 801	−18.772 504 14
3	3.722 112 517	0.000 004 182	−18.772 531 98

We conclude that $p \approx 2.370\,684$ and $p \approx 3.722\,113$ are solutions of the problem.

d) Let

$$\begin{aligned} f(x) &= (x-2)^2 - \ln x \\ \Rightarrow f'(x) &= 2(x-2) - \frac{1}{x} \end{aligned}$$

Applying Newton's method on f with $p_0 = 1.412$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	1.412	0.000 736 86	−1.884 215 297
1	1.412 391 07	0.000 000 191	−1.883 237 062
2	1.412 391 172	0	−1.883 236 808

Applying Newton's method on f with $p_0 = 3.057$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	3.057	−0.000 185 043	1.786 881 91
1	3.057 103 56	0.000 000 011	1.787 100 1
2	3.057 103 55	0	1.787 100 09

We conclude that $p \approx 1.412\,391$ and $p \approx 3.057\,104$ are solutions of the problem.

e) Let

$$\begin{aligned} f(x) &= e^x - 3x^2 \\ \Rightarrow f'(x) &= e^x - 6x \end{aligned}$$

Applying Newton's method on f with $p_0 = 0.91$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.91	0.000 022 533	−2.975 677 47
1	0.910 007 573	0	−2.975 704 09

Applying Newton's method on f with $p_0 = 3.733$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	3.733	−0.001 533 768	19.406 333 2
1	3.733 079 03	0.000 000 112	19.409 163 1
2	3.733 079 03	0	19.409 162 9

We conclude that $p \approx 0.910\,008$ and $p \approx 3.733\,079$ are solutions of the problem.

f) Let

$$\begin{aligned} f(x) &= \sin x - e^{-x} \\ \Rightarrow f'(x) &= \cos x + e^{-x} \end{aligned}$$

Applying Newton's method on f with $p_0 = 0.588$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.588	−0.000 739 019	1.387 488 79
1	0.588 532 63	−0.000 000 157	1.386 897 46
2	0.588 532 744	0	1.386 897 33

Applying Newton's method on f with $p_0 = 3.096$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	3.096	0.000 347 1	−0.953 731 075
1	3.096 363 94	−0.000 000 601	−0.953 764 054
2	3.096 363 93	0	−0.953 764 053

Applying Newton's method on f with $p_0 = 6.285$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	6.285	−0.000 049 365	1.001 862 41
1	6.285 049 27	0	1.001 862 23
2	6.285 049 27	0	1.001 862 23

We conclude that $p \approx 0.588\,53$, $p \approx 3.096\,36$ and $p = 6.285049$ are solutions of the problem.

Exercise 7

Repeat Exercise 5 using the Secant method.

Solution 7

a) Applying Secant method with $p_0 = 2.6$ and $p_1 = 2.7$ generates the following table:

n	p_n	$f(p_n)$
0	2.6	−0.944
1	2.7	0.103
2	2.690 162 369	−0.005 313 179
3	2.690 644 942	−0.000 027 451
4	2.690 647 449	0.000 000 007

We conclude that $p \approx 2.690\,65$ is a solution of the problem.

b) Applying Secant method with $p_0 = -2.8$ and $p_1 = -2.9$ generates the following table:

n	p_n	$f(p_n)$
0	−2.8	0.568
1	−2.9	−0.159
2	−2.878 129 298	0.009 531 586
3	−2.879 366 233	0.000 144 394
4	−2.879 385 259	−0.000 000 134

We conclude that $p \approx -2.879\,39$ is a solution of the problem.

c) Applying Secant method with $p_0 = 0.73$ and $p_1 = 0.74$ generates the following table:

n	p_n	$f(p_n)$
0	0.73	−0.015 174 402
1	0.74	0.001 531 441
2	0.739 083 29	−0.000 003 084
3	0.739 085 133	0

We conclude that $p \approx 0.739\,09$ is a solution of the problem.

d) Applying Secant method with $p_0 = 0.96$ and $p_1 = 0.97$ generates the following table:

n	p_n	$f(p_n)$
0	0.96	−0.003 838 313
1	0.97	−0.005 022 857
2	0.964 331 61	−0.000 002 018
3	0.964 333 887	−0.000 000 001

We conclude that $p \approx 0.964\,33$ is a solution of the problem.

Exercise 8

Repeat Exercise 6 using the Secant method.

Solution 8

a) Applying Secant method with $p_0 = 1.82$ and $p_1 = 1.83$ generates the following table:

n	p_n	$f(p_n)$
0	1.82	−0.038 185 199
1	1.83	0.002 529 463
2	1.829 378 734	−0.000 019 965
3	1.829 383 599	0.000 000 001

We conclude that $p \approx 1.829\,384$ is a solution of the problem.

b) Applying Secant method with $p_0 = 1.39$ and $p_1 = 1.4$ generates the following table:

n	p_n	$f(p_n)$
0	1.39	−0.016 699 48
1	1.4	0.004 770 262
2	1.397 778 147	0.000 063 1
3	1.397 748 362	−0.000 000 242
4	1.397 748 476	0

We conclude that $p \approx 1.397\,748$ is a solution of the problem.

c) Applying Secant method with $p_0 = 2.37$ and $p_1 = 2.375$ generates the following table:

n	p_n	$f(p_n)$
0	2.37	−0.006 040 395
1	2.375	0.037 985 226
2	2.370 686 009	−0.000 007 99
3	2.370 686 916	−0.000 000 001

Applying Secant method with $p_0 = 3.72$ and $p_1 = 3.73$ generates the following table:

n	p_n	$f(p_n)$
0	3.72	0.034 398 018
1	3.73	−0.129 244 414
2	3.722 102 023	0.000 175 259
3	3.722 112 719	0.000 000 889
4	3.722 112 773	0

We conclude that $p \approx 2.370\,69$ and $p \approx 3.722\,113$ are solutions of the problem.

d) Applying Secant method with $p_0 = 1.41$ and $p_1 = 1.42$ generates the following table:

n	p_n	$f(p_n)$
0	1.41	0.004 510 296
1	1.42	−0.014 256 872
2	1.412 403 29	−0.000 022 822
3	1.412 391 11	0.000 000 116
4	1.412 391 17	0

Applying Secant method with $p_0 = 3.05$ and $p_1 = 3.06$ generates the following table:

n	p_n	$f(p_n)$
0	3.05	−0.012 641 591
1	3.06	0.005 185 084
2	3.057 091 39	−0.000 021 731
3	3.057 103 53	−0.000 000 037
4	3.057 103 55	0

We conclude that $p \approx 1.412\,391$ and $p \approx 3.057\,104$ are solutions of the problem.

e) Applying Secant method with $p_0 = 0.91$ and $p_1 = 0.92$ generates the following table:

n	p_n	$f(p_n)$
0	0.91	0.000 022 533
1	0.92	−0.029 909 61
2	0.910 007 528	0.000 000 132
3	0.910 007 572	0

Applying Secant method with $p_0 = 3.73$ and $p_1 = 3.74$ generates the following table:

n	p_n	$f(p_n)$
0	3.73	−0.059 591 836
1	3.74	0.135 190 165
2	3.733 059 41	−0.000 380 739
3	3.733 078 9	−0.000 002 422
4	3.733 079 03	0

We conclude that $p \approx 0.910\,008$ and $p \approx 3.733\,079$ are solutions of the problem.

f) Applying Secant method with $p_0 = 0.58$ and $p_1 = 0.59$ generates the following table:

n	p_n	$f(p_n)$
0	0.58	−0.011 874 43
1	0.59	0.002 033 738
2	0.588 537 738	0.000 006 927
3	0.588 532 741	−0.000 000 004

Applying Secant method with $p_0 = 3.09$ and $p_1 = 3.1$ generates the following table:

n	p_n	$f(p_n)$
0	3.09	0.006 067 814
1	3.1	−0.003 468 54
2	3.096 362 82	0.000 001 057
3	3.096 363 93	0

Applying Secant method with $p_0 = 6.28$ and $p_1 = 6.29$ generates the following table:

n	p_n	$f(p_n)$
0	6.28	−0.005 058 702
1	6.29	0.004 959 88
2	6.285 049 32	0.000 000 046
3	6.285 049 27	0

We conclude that $p \approx 0.588\,533$, $p \approx 3.096\,364$ and $p \approx 6.285\,049$ are solutions of the problem.

Exercise 9

Repeat Exercise 5 using the method of False Position.

Solution 9

a) Applying False Position method with $p_0 = 2.6$ and $p_1 = 2.7$ generates the following table:

n	p_n	$f(p_n)$
0	2.6	−0.944
1	2.7	0.103
2	2.690 162 369	−0.005 313 179
3	2.690 644 942	−0.000 027 451
4	2.690 647 435	−0.000 000 141

We conclude that $p \approx 2.690\,647$ is a solution of the problem.

b) Applying False Position method with $p_0 = -2.8$ and $p_1 = -2.9$ generates the following table:

n	p_n	$f(p_n)$
0	−2.8	0.568
1	−2.9	−0.159
2	−2.878 129 298	0.009 531 586
3	−2.879 366 233	0.000 144 394
4	−2.879 385 26	−0.000 000 135

We conclude that $p \approx -2.879\,39$ is a solution of the problem.

c) Applying False Position method with $p_0 = 0.73$ and $p_1 = 0.74$ generates the following table:

n	p_n	$f(p_n)$
0	0.73	−0.015 174 402
1	0.74	0.001 531 441
2	0.739 083 29	−0.000 003 084
3	0.739 085 133	0

We conclude that $p \approx 0.739\,09$ is a solution of the problem.

d) Applying False Position method with $p_0 = 0.96$ and $p_1 = 0.97$ generates the following table:

n	p_n	$f(p_n)$
0	0.96	−0.003 838 313
1	0.97	−0.005 022 857
2	0.964 331 61	−0.000 002 018
3	0.964 333 887	−0.000 000 001

We conclude that $p \approx 0.964\,33$ is a solution of the problem.

Exercise 10

Repeat Exercise 6 using the False Position method.

Solution 10

a) Applying False Position method with $p_0 = 1.82$ and $p_1 = 1.83$ generates the following table:

n	p_n	$f(p_n)$
0	1.82	−0.038 185 199
1	1.83	0.002 529 463
2	1.829 378 734	−0.000 019 965
3	1.829 383 599	0.000 000 001

We conclude that $p \approx 1.829\,384$ is a solution of the problem.

b) Applying False Position method with $p_0 = 1.39$ and $p_1 = 1.4$ generates the following table:

n	p_n	$f(p_n)$
0	1.39	−0.016 699 48
1	1.4	0.004 770 262
2	1.397 778 15	0.000 063 1
3	1.397 748 87	0.000 000 831
4	1.397 748 48	0.000 000 001

We conclude that $p \approx 1.397\,748$ is a solution of the problem.

c) Applying False Position method with $p_0 = 2.37$ and $p_1 = 2.375$ generates the following table:

n	p_n	$f(p_n)$
0	2.37	−0.006 040 395
1	2.375	0.037 985 226
2	2.370 686 009	−0.000 007 99
3	2.370 686 916	−0.000 000 001

Applying False Position method with $p_0 = 3.72$ and $p_1 = 3.73$ generates the following table:

n	p_n	$f(p_n)$
0	3.72	0.034 398 018
1	3.73	−0.129 244 414
2	3.722 102 023	0.000 175 259
3	3.722 112 719	0.000 000 889
4	3.722 112 77	0.000 000 001

We conclude that $p \approx 2.370\,69$ and $p \approx 3.722\,113$ are solutions of the problem.

d) Applying False Position method with $p_0 = 1.41$ and $p_1 = 1.42$ generates the following table:

n	p_n	$f(p_n)$
0	1.41	0.004 510 296
1	1.42	−0.014 256 872
2	1.412 403 29	−0.000 022 822
3	1.412 391 19	−0.000 000 036
4	1.412 391 17	0

Applying False Position method with $p_0 = 3.05$ and $p_1 = 3.06$ generates the following table:

n	p_n	$f(p_n)$
0	3.05	−0.012 641 591
1	3.06	0.005 185 084
2	3.057 091 39	−0.000 021 731
3	3.057 103 53	−0.000 000 037
4	3.057 103 55	0

We conclude that $p \approx 1.412\,391$ and $p \approx 3.057\,104$ are solutions of the problem.

e) Applying False Position method with $p_0 = 0.91$ and $p_1 = 0.92$ generates the following table:

n	p_n	$f(p_n)$
0	0.91	0.000 022 533
1	0.92	−0.029 909 61
2	0.910 007 528	0.000 000 132
3	0.910 007 572	0

Applying False Position method with $p_0 = 3.73$ and $p_1 = 3.74$ generates the following table:

n	p_n	$f(p_n)$
0	3.73	−0.059 591 836
1	3.74	0.135 190 165
2	3.733 059 41	−0.000 380 739
3	3.733 078 9	−0.000 002 422
4	3.733 079 03	−0.000 000 015

We conclude that $p \approx 0.910\,008$ and $p \approx 3.733\,079$ are solutions of the problem.

f) Applying False Position method with $p_0 = 0.58$ and $p_1 = 0.59$ generates the following table:

n	p_n	$f(p_n)$
0	0.58	−0.011 874 43
1	0.59	0.002 033 738
2	0.588 537 738	0.000 006 927
3	0.588 532 761	0.000 000 024

Applying False Position method with $p_0 = 3.09$ and $p_1 = 3.1$ generates the following table:

n	p_n	$f(p_n)$
0	3.09	0.006 067 814
1	3.1	−0.003 468 54
2	3.096 362 82	0.000 001 057
3	3.096 363 93	0

Applying False Position method with $p_0 = 6.28$ and $p_1 = 6.29$ generates the following table:

n	p_n	$f(p_n)$
0	6.28	−0.005 058 702
1	6.29	0.004 959 88
2	6.285 049 32	0.000 000 046
3	6.285 049 27	0

We conclude that $p \approx 0.588\,533$, $p \approx 3.096\,364$ and $p \approx 6.285\,049$ are solutions of the problem.

Exercise 11

Use all three methods in this Section to find solutions to within 10^{-5} for the following problems.

a) $3xe^x = 0$ for $x \in [1, 2]$

b) $2x + 3\cos x - e^x$ for $x \in [0, 1]$

Solution 11

a) Such math... much difficult...

b) Let

$$f(x) = 2x + 3\cos x - e^x$$

$$\Rightarrow f'(x) = 2 - 3\sin x - e^x$$

$\sin x$ and e^x are both monotonically increasing in $I = [0, 1]$, therefore $f'(x)$ is monotonically decreasing I. It follows that

$$f'(0) = 2 \geq f'(x) \geq f'(1) \approx -0.524\,412\,954\,4$$

and that $f'(x)$ has exactly one zero p in I. Since the sign of $f'(x)$ changes from positive to negative as x passes p, the local maximum of f in I is at p. Then the minimum value of f in I is achieved at either end:

$$f(x) \geq \min\{f(0), f(1)\} \approx 0.902\,625\,089\,1 > 0$$

Then f has no zero in I.

Exercise 12

Use all three methods in this Section to find solutions to within 10^{-7} for the following problems.

a) $x^2 - 4x + 4 - \ln x = 0$ for $x \in [1, 2]$ and $x \in [2, 4]$

b) $x + 1 - 2\sin \pi x = 0$ for $x \in [0, 1/2]$ and $x \in [1/2, 1]$

Solution 12

a) Let

$$f(x) = x^2 - 4x + 4 - \ln x$$
$$\Rightarrow f'(x) = 2x - 4 - \frac{1}{x}$$

Applying Newton's method on f with $p_0 = 1.41$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	1.41	0.004 510 295 61	−1.889 219 858 16
1	1.412 387 385 24	0.000 007 131 42	−1.883 246 279 86
2	1.412 391 172 01	0.000 000 000 02	−1.883 236 808 04
3	1.412 391 172 02	0	−1.883 236 808 02

Applying Newton's method on f with $p_0 = 3.05$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	3.05	−0.012 641 590 62	1.772 131 147 54
1	3.057 133 552 52	0.000 053 618 47	1.787 163 305 75
2	3.057 103 550 53	0.000 000 000 95	1.787 100 091 6
3	3.057 103 549 99	0	1.787 100 090 48

Applying Secant method with $p_0 = 1.41$ and $p_1 = 1.42$ generates the following table:

n	p_n	$f(p_n)$
0	1.41	0.004 510 295 61
1	1.42	−0.014 256 871 61
2	1.412 403 290 57	−0.000 022 821 92
3	1.412 391 110 52	0.000 000 115 82
4	1.412 391 172 02	0

Applying Secant method with $p_0 = 3.05$ and $p_1 = 3.06$ generates the following table:

n	p_n	$f(p_n)$
0	3.05	−0.012 641 590 62
1	3.06	0.005 185 084 04
2	3.057 091 390 21	−0.000 021 730 59
3	3.057 103 529 27	−0.000 000 037 04
4	3.057 103 549 99	0

Applying False Position method with $p_0 = 1.41$ and $p_1 = 1.42$ generates the following table:

n	p_n	$f(p_n)$
0	1.41	0.004 510 295 61
1	1.42	−0.014 256 871 61
2	1.412 403 290 57	−0.000 022 821 92
3	1.412 391 191 24	−0.000 000 036 19
4	1.412 391 172 05	−0.000 000 000 06

Applying False Position method with $p_0 = 3.05$ and $p_1 = 3.06$ generates the following table:

n	p_n	$f(p_n)$
0	3.05	−0.012 641 590 62
1	3.06	0.005 185 084 04
2	3.057 091 390 21	−0.000 021 730 59
3	3.057 103 529 27	−0.000 000 037 04
4	3.057 103 549 96	0

b) Let

$$\begin{aligned} f(x) &= x + 1 - 2\sin \pi x \\ \Rightarrow f'(x) &= 1 - 2\pi \cos \pi x \end{aligned}$$

Applying Newton's method on f with $p_0 = 0.21$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.21	−0.015 814 107 31	−3.964 690 364 15
1	0.206 011 262 96	0.000 095 722 6	−4.012 556 253 06
2	0.206 035 118 73	0.000 000 003 39	−4.012 272 309 82
3	0.206 035 119 57	0	−4.012 272 299 77

Applying Newton's method on f with $p_0 = 0.68$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.68	−0.008 655 851	4.366 699 045 41
1	0.681 982 241 26	0.000 032 700 17	4.399 670 307 78
2	0.681 974 808 84	0.000 000 000 46	4.399 546 927 47
3	0.681 974 808 74	0	4.399 546 925 74

Applying Secant method with $p_0 = 0.21$ and $p_1 = 0.22$ generates the following table:

n	p_n	$f(p_n)$
0	0.21	−0.015 814 107 31
1	0.22	−0.054 847 979 5
2	0.205 948 619 39	0.000 347 106 82
3	0.206 036 984 68	−0.000 007 483 3
4	0.206 035 119 81	−0.000 000 000 96
5	0.206 035 119 57	0

Applying Secant method with $p_0 = 0.68$ and $p_1 = 0.69$ generates the following table:

n	p_n	$f(p_n)$
0	0.68	−0.008 655 851
1	0.69	0.035 838 851 45
2	0.681 945 366 65	−0.000 129 524 68
3	0.681 974 371 95	−0.000 001 921 66
4	0.681 974 808 76	0.000 000 001 07
5	0.681 974 808 74	0

Applying False Position method with $p_0 = 0.21$ and $p_1 = 0.22$ generates the following table:

n	p_n	$f(p_n)$
0	0.21	−0.015 814 107 31
1	0.22	−0.054 847 979 5
2	0.205 948 619 39	0.000 347 106 82
3	0.206 036 984 68	−0.000 007 483 3
4	0.206 035 119 81	−0.000 000 000 96
5	0.206 035 119 57	0

Applying False Position method with $p_0 = 0.68$ and $p_1 = 0.69$ generates the following table:

n	p_n	$f(p_n)$
0	0.68	−0.008 655 851
1	0.69	0.035 838 851 45
2	0.681 945 366 65	−0.000 129 524 67
3	0.681 974 371 95	−0.000 001 921 66
4	0.681 974 802 26	−0.000 000 028 51
5	0.681 974 808 64	−0.000 000 000 42

Exercise 13

Use Newton's method to approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = x^2$ that is closest to $(1, 0)$.

Solution 13

Let d be the squared distance between the point (x, x^2) of the graph and $(1, 0)$.

$$\begin{aligned}d(x) &= (x-1)^2 + x^4 \\ \Rightarrow d'(x) &= 4x^3 + 2(x-1) \\ \Rightarrow d''(x) &= 12x^2 + 2\end{aligned}$$

We need to find x that minimizes d. First we have to examine d'. As $d''(x) \geq 2 > 0\,\forall x \in \mathbb{R}$, d' is monotonically increasing in $\mathbb{R}$. It follows that d' has at most one zero in $\mathbb{R}$.

Applying Newton's method on d' with $p_0 = 0.59$ generates the following table:

n	p_n	$d'(p_n)$	$d''(p_n)$
0	0.59	0.001 516	6.1772
1	0.589 754 581	0.000 000 426	6.173 725 59
2	0.589 754 512	0	6.173 724 62

Then $p \approx 0.589\,75$ is the only zero of d'. Since the sign of d' changes from negative to positive as x passes p, the global minimum of d is achieved at p.

We conclude that $x \approx 0.589\,75$ produces the point on the graph of $y = x^2$ that is closest to $(1, 0)$.

Exercise 14

Use Newton's method to approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = \frac{1}{x}$ that is closest to $(2, 1)$.

Solution 14

Let d be the squared distance between the point $(x, \frac{1}{x})$ of the graph and $(2, 1)$.

$$\begin{aligned}d(x) &= (x-2)^2 + \left(\frac{1}{x} - 1\right)^2 \\ \Rightarrow d'(x) &= 2(x-2) - 2\left(\frac{1}{x} - 1\right)\frac{1}{x^2} = \frac{2(x^4 - 2x^3 + x - 1)}{x^3} \\ \Rightarrow d''(x) &= 2\left(\frac{3}{x} - 2\right)\frac{1}{x^3} + 2 = \frac{2(x^4 - 2x + 3)}{x^4}\end{aligned}$$

Let

$$\begin{aligned}f(x) &= x^4 - 2x + 3 \\ \Rightarrow f'(x) &= 4x^3 - 2\end{aligned}$$

f' has exactly one zero at $0.5^{1/3}$. Since f' is monotonically increasing in $\mathbb{R}$, the sign of f' changes from negative to positive as x passes $0.5^{1/3}$. It follows that the global minimum of f is achieved at $0.5^{1/3}$:

$$f(x) \geq f(0.5^{1/3}) \approx 1.809\,449\,211 > 0$$

Then, $d''(x) > 0 \,\forall x \in \mathbb{R} \setminus 0$. It follows that d' is monotonically increasing in $D^+ = \mathbb{R}_{>0}$ and $D^- = \mathbb{R}_{<0}$, which means it has at most one zero in D^+ and D^- alike.

Let

$$g(x) = x^4 - 2x^3 + x - 1$$
$$\Rightarrow g'(x) = 4x^3 - 6x^2 + 1$$

Every zero of g is also a zero of d'. Applying Newton's method on g with $p_0 = 1.86$ generates the following table:

n	p_n	$g(p_n)$	$g'(p_n)$
0	1.86	−0.040 879 84	5.981 824
1	1.866 834 01	0.000 449 982	6.113 767 65
2	1.866 760 41	0.000 000 053	6.112 338 49

Applying Newton's method on g with $p_0 = -0.86$ generates the following table:

n	p_n	$g(p_n)$	$g'(p_n)$
0	−0.86	−0.040 879 84	−5.981 824
1	−0.866 834 009	0.000 449 982	−6.113 767 65
2	−0.866 760 408	0.000 000 053	−6.112 338 49

We conclude that $x \approx 1.866\,76$ and $x \approx -0.866\,76$ produce the points on the graph of $y = x^2$ that are closest to $(1, 0)$.

Exercise 15

The following describes Newton's method graphically:

Suppose that $f'(x)$ exists on $[a, b]$ and that $f'(x) \neq 0 \,\forall x \in [a, b]$. Further, suppose there exists one $p \in [a, b]$ such that $f(p) = 0$.

Let $p_0 \in [a, b]$ be arbitrary. Let p_1 be the point at which the tangent line to f at $(p_0, f(p_0))$ crosses the x-axis. For each $n \geq 1$, let p_n be the x-intercept of the line tangent to f at $(p_{n-1}, f(p_{n-1}))$. Derive the formula describing this method.

Solution 15

The equation of the line tangent to f at $(p_{n-1}, f(p_{n-1}))$ is:

$$y = f'(p_{n-1})(x - p_{n-1}) + f(p_{n-1})$$

Then its x-intercept is:

$$x = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Then the formula describing the sequence generated by the procedure is:

$$\{p_n\} \mid p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Exercise 16

Use Newton's method to solve the equation

$$0 = \frac{1}{2} + \frac{1}{4}x^2 - x\sin x - \frac{1}{2}\cos 2x \text{ with } p_0 = \frac{\pi}{2}$$

Iterate using Newton's method until an accuracy of 10^{-5} is obtained. Explain why the result seems unusual for Newton's method. Also, solve the equation with $p_0 = 5\pi$ and $p_0 = 10\pi$.

Solution 16

Let

$$\begin{aligned} f(x) &= \frac{1}{2} + \frac{1}{4}x^2 - x\sin x - \frac{1}{2}\cos 2x \\ \Rightarrow f'(x) &= \frac{1}{2}x - \sin x + x\cos x + \sin 2x \end{aligned}$$

Applying Newton's method on f with $p_0 = \frac{\pi}{2}$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	1.570 796 33	0.046 053 948	−0.214 601 837
1	1.785 398 16	0.007 116 978	−0.120 293 455
2	1.844 561 63	0.001 638 544	−0.062 366 566
3	1.870 834 42	0.000 396 329	−0.031 675 918
4	1.883 346 43	0.000 097 601	−0.015 954 846
5	1.889 463 76	0.000 024 225	−0.008 005 932
6	1.892 489 62	0.000 006 035	−0.004 010 008
7	1.893 994 57	0.000 001 506	−0.002 006 754
8	1.894 745 07	0.000 000 376	−0.001 003 813
9	1.895 119 83	0.000 000 094	−0.000 502 015
10	1.895 307 09	0.000 000 023	−0.000 251 035
11	1.895 400 69	0.000 000 006	−0.000 125 524
12	1.895 447 48	0.000 000 001	−0.000 062 764
13	1.895 470 87	0	−0.000 031 382
14	1.895 482 57	0	−0.000 015 691
15	1.895 488 42	0	−0.000 007 846

It's clear that the number of iteration is unusually large.
Applying Newton's method on f with $p_0 = 5\pi$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	15.707 963 3	61.685 027 5	23.561 944 9
1	13.089 969 4	36.541 84	−4.425 235 93
2	21.347 572	101.479 949	26.190 775 1
3	17.472 927 3	94.433 153 9	5.967 623 72
4	1.648 679 92	0.029 800 649	−0.199 491 346
5	1.798 063 09	0.005 663 214	−0.109 166 251
6	1.849 940 06	0.001 319 265	−0.056 337 315
7	1.873 357 31	0.000 320 334	−0.028 563 789
8	1.884 572	0.000 079 014	−0.014 376 187
9	1.890 068 17	0.000 019 626	−0.007 211 151
10	1.892 789 8	0.000 004 89	−0.003 611 278
11	1.894 144 16	0.000 001 22	−0.001 807 057
12	1.894 819 74	0.000 000 305	−0.000 903 882
13	1.895 157 14	0.000 000 076	−0.000 452 029
14	1.895 325 73	0.000 000 019	−0.000 226 037
15	1.895 410 01	0.000 000 005	−0.000 113 024
16	1.895 452 14	0.000 000 001	−0.000 056 513
17	1.895 473 2	0	−0.000 028 257
18	1.895 483 74	0	−0.000 014 129
19	1.895 489	0	−0.000 007 064

For $p_0 = 10\pi$, the sequence converges and diverges back and forth, then finally stops at $p_{154} \approx -0.000\,006$.

Exercise 17

The fourth-degree polynomial

$$f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$$

has two real zeros, one in $[-1, 0]$ and the other in $[0, 1]$. Attempt to approximate these zeros to within 10^{-6} using the

a) Method of False Position

b) Secant method

c) Newton's method

Use the endpoints of each interval as the initial approximations in a) and b) and the midpoints as the initial approximation in c).

Solution 17

a) Applying False Position method with $p_0 = -1$ and $p_1 = 0$ generates the following table:

n	p_n	$f(p_n)$
0	-1	433
1	0	-9
2	$-0.020\,361\,991$	$-4.496\,380\,93$
3	$-0.030\,430\,247$	$-2.266\,891\,37$
4	$-0.035\,479\,814$	$-1.148\,071\,19$
5	$-0.038\,030\,414$	$-0.582\,770\,74$
6	$-0.039\,323\,38$	$-0.296\,160\,751$
7	$-0.039\,980\,008$	$-0.150\,595\,231$
8	$-0.040\,313\,782$	$-0.076\,599\,144$
9	$-0.040\,483\,524$	$-0.038\,967\,468$
10	$-0.040\,569\,867$	$-0.019\,825\,027$
11	$-0.040\,613\,793$	$-0.010\,086\,543$
12	$-0.040\,636\,141$	$-0.005\,131\,916$
13	$-0.040\,647\,511$	$-0.002\,611\,086$
14	$-0.040\,653\,296$	$-0.001\,328\,51$
15	$-0.040\,656\,24$	$-0.000\,675\,943$
16	$-0.040\,657\,737$	$-0.000\,343\,918$
17	$-0.040\,658\,499$	$-0.000\,174\,985$

Applying False Position method with $p_0 = 0$ and $p_1 = 1$ generates the following table:

n	p_n	$f(p_n)$
0	0	-9
1	1	27
2	0.25	$-62.507\,812\,5$
3	$0.773\,762\,765$	$-83.830\,520\,3$
4	$0.944\,885\,169$	$-11.265\,130\,2$
5	$0.961\,110\,797$	$-0.855\,867\,823$
6	$0.962\,305\,662$	$-0.061\,802\,369$
7	$0.962\,391\,747$	$-0.004\,446\,181$
8	$0.962\,397\,939$	$-0.000\,319\,781$
9	$0.962\,398\,384$	$-0.000\,022\,999$

b) Applying Secant method with $p_0 = -1$ and $p_1 = 0$ generates the following table:

n	p_n	$f(p_n)$
0	-1	433
1	0	-9
2	$-0.020\,361\,991$	$-4.496\,380\,93$
3	$-0.040\,691\,256$	$0.007\,087\,483$
4	$-0.040\,659\,263$	$-0.000\,005\,706$
5	$-0.040\,659\,288$	0

Applying Secant method with $p_0 = 0$ and $p_1 = 1$ generates the following table:

n	p_n	$f(p_n)$
0	0	-9
1	1	27
2	0.25	$-62.507\,812\,5$
3	$0.773\,762\,765$	$-83.830\,520\,3$
4	$-1.285\,417\,78$	$879.638\,986$
5	$0.594\,595\,52$	$-104.691\,389$
6	$0.394\,641\,105$	$-88.128\,940\,4$
7	$-0.669\,318\,136$	$183.713\,16$
8	$0.049\,714\,398$	$-19.961\,021\,6$
9	$-0.020\,754\,151$	$-4.409\,574\,29$
10	$-0.040\,735\,333$	$0.016\,859\,473$
11	$-0.040\,659\,228$	$-0.000\,013\,318$
12	$-0.040\,659\,288$	0

c) Applying Newton's method with $p_0 = -0.5$ generates the following table:

n	p_n	$g(p_n)$	$g'(p_n)$
0	-0.5	115.875	-331.5
1	$-0.150\,452\,489$	$24.510\,271$	$-225.618\,988$
2	$-0.041\,816\,814$	$0.256\,640\,771$	$-221.725\,549$
3	$-0.040\,659\,344$	$0.000\,012\,234$	$-221.704\,436$
4	$-0.040\,659\,288$	0	$-221.704\,435$

Applying Newton's method with $p_0 = 0.5$ generates the following table:

n	p_n	$g(p_n)$	$g'(p_n)$
0	0.5	-100.625	-83.5
1	$-0.705\,089\,82$	$201.836\,304$	$-529.339\,073$
2	$-0.323\,791\,114$	$65.418\,426\,7$	$-252.397\,607$

n	p_n	$g(p_n)$	$g'(p_n)$
3	$-0.064\,603\,131$	$5.314\,007\,07$	$-222.185\,539$
4	$-0.040\,686\,151$	$0.005\,955\,616$	$-221.704\,923$
5	$-0.040\,659\,288$	$0.000\,000\,007$	$-221.704\,435$
6	$-0.040\,659\,288$	0	$-221.704\,435$

Exercise 18

The function $f(x) = \tan \pi x - 6$ has a zero at $\frac{\arctan(6)}{\pi} \approx 0.447\,431\,543$. Let $p_0 = 0$ and $p_1 = 0.48$, and use ten iterations of each of the following methods to approximate this root. Which method is most successful and why?

a) Bisection b) False Position c) Secant

Solution 18

a) Applying Bisection method on f with $a = 0$, $b = 0.48$ generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	0.48	0.24	$-60.509\,683\,2$
2	0.24	0.48	0.36	$-82.690\,675\,2$
3	0.36	0.48	0.42	$-91.741\,915\,2$
4	0.42	0.48	0.45	$-95.555\,812\,5$
5	0.45	0.48	0.465	$-97.255\,924\,1$
6	0.465	0.48	0.4725	$-98.050\,428\,1$
7	0.4725	0.48	$0.476\,25$	$-98.433\,297\,5$
8	$0.476\,25$	0.48	$0.478\,125$	$-98.621\,073\,9$
9	$0.478\,125$	0.48	$0.479\,062\,5$	$-98.714\,039\,5$
10	$0.479\,062\,5$	0.48	$0.479\,531\,25$	$-98.760\,290\,8$

The method indeed does not produce the root in this case, as $f(a_1)$ and $f(b_1)$ have the same sign.

b) Applying method of False Position on f with $p_0 = 0$ and $p_1 = 0.48$ generates the following table:

n	p_n	$f(p_n)$
0	0	-9
1	0.48	$-98.806\,387\,2$
2	$-0.048\,103\,483$	$1.650\,923\,14$
3	$-0.039\,424\,59$	$-0.273\,724\,354$
4	$-0.040\,658\,906$	$-0.000\,084\,697$
5	$-0.040\,659\,288$	$-0.000\,000\,026$

c) Applying Secant method on f with $p_0 = 0$ and $p_1 = 0.48$ generates the following table:

n	p_n	$f(p_n)$
0	0	-9
1	0.48	$-98.806\,387\,2$
2	$-0.048\,103\,483$	$1.650\,923\,14$
3	$-0.039\,424\,59$	$-0.273\,724\,354$
4	$-0.040\,658\,906$	$-0.000\,084\,697$
5	$-0.040\,659\,288$	$0.000\,000\,004$

Clearly, Secant method is the most successful one in this case.

Exercise 19

The iteration equation for the Secant method can be written in the simpler form:

$$p_n = \frac{f(p_{n-1})p_{n-2} - f(p_{n-2})p_{n-1}}{f(p_{n-1}) - f(p_{n-2})}$$

Explain why, in general, this iteration equation is likely to be less accurate than the one given in the text book.

Solution 19

In both formulas, the denominator is close to 0 as consecutive p_n is close to each other.

In the above formula, the numerator is also close to 0 for the same reason. Therefore, both numerator and denominator are close to 0, which can lead to losing digits.

The formula provided in the text book circumvents this situation by having the difference of 2 consecutive p_n multiplied with f *before* dividing.

As a consequence, the formula should be written in the specific way that it is printed in the text book, as it implies the multiplication should be done before division.

Exercise 20

The equation $x^2 - 10\cos x = 0$ has two solutions, $\pm 1.379\,364\,6$. Use Newton's method to approximate the solutions to within 10^{-5} with the following values of p_0.

a) $p_0 = -100$

b) $p_0 = -50$

c) $p_0 = -25$

d) $p_0 = 25$

e) $p_0 = 50$

f) $p_0 = 100$

Solution 20

Let

$$\begin{aligned} f(x) &= x^2 - 10\cos x \\ \Rightarrow f'(x) &= 2x + 10\sin x \end{aligned}$$

a) Applying Newton's method with $p_0 = -100$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	−100	9991.376 811 277 1	−194.936 343 588 9
1	−48.745 438 498 9	2375.610 468 619 5	−87.503 753 248
2	−21.596 769 094	475.652 786 972 2	−47.035 891 967 9
3	−11.484 219 569 1	127.192 997 670 8	−14.138 742 994 8
4	−2.488 158 340 9	14.130 939 015 7	−11.055 485 002 7
5	−1.209 974 795 7	−2.066 390 820 8	−11.776 020 627 6
6	−1.385 449 252 3	0.076 592 885	−12.599 621 987 3
7	−1.379 370 269 5	0.000 071 372 8	−12.576 079 669 9
8	−1.379 364 594 2	0.000 000 000 1	−12.576 057 521 4

b) Applying Newton's method with $p_0 = -50$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	−50	2490.350 339 715 1	−97.376 251 463
1	−24.425 485 656 9	589.002 870 288 5	−42.353 470 822 3
2	−10.518 647 354 1	115.232 454 209 8	−12.153 196 604 1
3	−1.036 989 320 9	−4.012 796 962 4	−10.682 741 185 2
4	−1.412 622 961 5	0.420 357 249 2	−12.700 412 446 9
5	−1.379 525 040 4	0.002 017 830 4	−12.576 683 559 7
6	−1.379 364 598 2	0.000 000 050 2	−12.576 057 537
7	−1.379 364 594 2	0	−12.576 057 521 4

c) Applying Newton's method with $p_0 = -25$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	−25	615.087 971 881 4	−48.676 482 499
1	−12.363 754 727 1	143.066 995 664 8	−22.715 185 535 7
2	−6.065 457 253 8	27.025 864 334 4	−9.970 795 758 7
3	−3.354 955 004 2	21.028 967 802 6	−4.592 438 027 5
4	1.224 087 255 5	−1.899 655 866 7	11.853 135 273 5
5	1.384 353 364 2	0.062 787 419 8	12.595 404 723 1
6	1.379 368 417 7	0.000 048 083 8	12.576 072 442 8
7	1.379 364 594 2	0	12.576 057 521 4

d) Applying Newton's method with $p_0 = 25$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	25	615.087 971 881 4	48.676 482 499
1	12.363 754 727 1	143.066 995 664 8	22.715 185 535 7
2	6.065 457 253 8	27.025 864 334 4	9.970 795 758 7
3	3.354 955 004 2	21.028 967 802 6	4.592 438 027 5
4	−1.224 087 255 5	−1.899 655 866 7	−11.853 135 273 5
5	−1.384 353 364 2	0.062 787 419 8	−12.595 404 723 1
6	−1.379 368 417 7	0.000 048 083 8	−12.576 072 442 8
7	−1.379 364 594 2	0	−12.576 057 521 4

e) Applying Newton's method with $p_0 = 50$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	50	2490.350 339 715 1	97.376 251 463
1	24.425 485 656 9	589.002 870 288 5	42.353 470 822 3
2	10.518 647 354 1	115.232 454 209 8	12.153 196 604 1
3	1.036 989 320 9	−4.012 796 962 4	10.682 741 185 2
4	1.412 622 961 5	0.420 357 249 2	12.700 412 446 9
5	1.379 525 040 4	0.002 017 830 4	12.576 683 559 7
6	1.379 364 598 2	0.000 000 050 2	12.576 057 537
7	1.379 364 594 2	0	12.576 057 521 4

f) Applying Newton's method with $p_0 = 100$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	100	9991.376 811 277 1	194.936 343 588 9
1	48.745 438 498 9	2375.610 468 619 5	87.503 753 248
2	21.596 769 094	475.652 786 972 2	47.035 891 967 9
3	11.484 219 569 1	127.192 997 670 8	14.138 742 994 8
4	2.488 158 340 9	14.130 939 015 7	11.055 485 002 7
5	1.209 974 795 7	−2.066 390 820 8	11.776 020 627 6
6	1.385 449 252 3	0.076 592 885	12.599 621 987 3
7	1.379 370 269 5	0.000 071 372 8	12.576 079 669 9
8	1.379 364 594 2	0.000 000 000 1	12.576 057 521 4

Exercise 21

The equation $4x^2 - e^x - e^{-x} = 0$ has two positive solutions x_1 and x_2. Use Newton's method to approximate the solution to within 10^{-5} with the following values of p_0.

a) $p_0 = -10$ b) $p_0 = -5$ c) $p_0 = -3$

d) $p_0 = -1$ e) $p_0 = 0$ f) $p_0 = 1$

g) $p_0 = 3$ h) $p_0 = 5$ i) $p_0 = 10$

Solution 21

Let

$$f(x) = 4x^2 - e^x - e^{-x}$$
$$\Rightarrow f'(x) = 8x - e^x + e^{-x}$$

a) Applying Newton's method with $p_0 = -10$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	−10	−21 626.465 840 206 6	21 946.465 749 406 8
1	−9.014 580 931 3	−7897.049 455 811 2	8149.983 242 581 3
2	−8.045 615 815 6	−2861.158 494 740 3	3055.720 662 614 5
3	−7.109 287 266 4	−1021.108 321 568 4	1166.400 250 226 2
4	−6.233 851 650 4	−354.273 287 548 9	459.842 176 179 7
5	−5.463 428 000 9	−116.512 778 382 3	192.193 058 460 6
6	−4.857 200 183 3	−34.301 660 964 2	89.798 089 553 3
7	−4.475 213 649 6	−7.714 598 646 1	52.000 262 710 2
8	−4.326 856 732 9	−0.832 400 420 4	41.077 885 300 8
9	−4.306 592 777 8	−0.013 799 244 1	39.721 063 640 1
10	−4.306 245 374 1	−0.000 003 994 3	39.698 069 725 7
11	−4.306 245 273 5	0	39.698 063 067 3

b) Applying Newton's method with $p_0 = -5$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	−5	−48.419 897 049 6	108.406 421 155 6
1	−4.553 348 440 7	−12.028 414 215 9	58.512 491 019 6
2	−4.347 778 416 1	−1.706 755 969 7	42.511 366 227 4
3	−4.307 630 189 4	−0.055 041 972 1	39.789 781 006 6
4	−4.306 246 870 1	−0.000 063 380 9	39.698 168 720 5
5	−4.306 245 273 5	−0.000 000 000 1	39.698 063 067 4

c) Applying Newton's method with $p_0 = -3$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	-3	15.864 676 008 4	$-3.964\,250\,145\,2$
1	1.001 936 161 3	0.924 786 470 1	5.659 107 187 9
2	0.838 520 548 3	0.067 174 591 3	4.827 571 52
3	0.824 605 769 2	0.000 509 551 3	4.754 272 591
4	0.824 498 591 7	0.000 000 030 3	4.753 706 617 5
5	0.824 498 585 3	0	4.753 706 583 8

d) Applying Newton's method with $p_0 = -1$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	-1	0.913 838 730 4	$-5.649\,597\,612\,7$
1	$-0.838\,247\,111\,9$	0.065 854 754	$-4.826\,134\,621\,3$
2	$-0.824\,601\,667$	0.000 490 048 4	$-4.754\,250\,928\,9$
3	$-0.824\,498\,591\,2$	0.000 000 028 1	$-4.753\,706\,615$
4	$-0.824\,498\,585\,3$	0	$-4.753\,706\,583\,8$

e) The method fails in this case as $f'(0) = 0$.

f) Applying Newton's method with $p_0 = 1$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	1	0.913 838 730 4	5.649 597 612 7
1	0.838 247 111 9	0.065 854 754	4.826 134 621 3
2	0.824 601 667	0.000 490 048 4	4.754 250 928 9
3	0.824 498 591 2	0.000 000 028 1	4.753 706 615
4	0.824 498 585 3	0	4.753 706 583 8

g) Applying Newton's method with $p_0 = 3$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	3	15.864 676 008 4	3.964 250 145 2
1	$-1.001\,936\,161\,3$	0.924 786 470 1	$-5.659\,107\,187\,9$
2	$-0.838\,520\,548\,3$	0.067 174 591 3	$-4.827\,571\,52$
3	$-0.824\,605\,769\,2$	0.000 509 551 3	$-4.754\,272\,591$
4	$-0.824\,498\,591\,7$	0.000 000 030 3	$-4.753\,706\,617\,5$
5	$-0.824\,498\,585\,3$	0	$-4.753\,706\,583\,8$

h) Applying Newton's method with $p_0 = 5$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	5	−48.419 897 049 6	−108.406 421 155 6
1	4.553 348 440 7	−12.028 414 215 9	−58.512 491 019 6
2	4.347 778 416 1	−1.706 755 969 7	−42.511 366 227 4
3	4.307 630 189 4	−0.055 041 972 1	−39.789 781 006 6
4	4.306 246 870 1	−0.000 063 380 9	−39.698 168 720 5
5	4.306 245 273 5	−0.000 000 000 1	−39.698 063 067 4

i) Applying Newton's method with $p_0 = 10$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	10	−21 626.465 840 206 6	−21 946.465 749 406 8
1	9.014 580 931 3	−7897.049 455 811 2	−8149.983 242 581 3
2	8.045 615 815 6	−2861.158 494 740 3	−3055.720 662 614 5
3	7.109 287 266 4	−1021.108 321 568 4	−1166.400 250 226 2
4	6.233 851 650 4	−354.273 287 548 9	−459.842 176 179 7
5	5.463 428 000 9	−116.512 778 382 3	−192.193 058 460 6
6	4.857 200 183 3	−34.301 660 964 2	−89.798 089 553 3
7	4.475 213 649 6	−7.714 598 646 1	−52.000 262 710 2
8	4.326 856 732 9	−0.832 400 420 4	−41.077 885 300 8
9	4.306 592 777 8	−0.013 799 244 1	−39.721 063 640 1
10	4.306 245 374 1	−0.000 003 994 3	−39.698 069 725 7
11	4.306 245 273 5	0	−39.698 063 067 3

Exercise 22

Use Maple to determine how many iterations of Newton's method with $p_0 = \pi/4$ are needed to find a root of $f(x) = \cos x - x$ to within 10^{-100}.

Solution 22

Python FTW: 51 iterations.

Exercise 23

The function described by $f(x) = \ln(x^2 + 1) - e^{0.4x} \cos \pi x$ has an infinite number of zeros.

a) Determine, within 10^{-6}, the only negative zero.

b) Determine, within 10^{-6}, the four smallest positive zeros.

c) Determine a reasonable initial approximation to find the n^{th} smallest positive zero of f. [Hint: Sketch an approximate graph of f.]

d) Use part c) to determine, within 10^{-6}, the 25^{th} smallest positive zero of f.

Solution 23

Differentiating f gives:

$$f'(x) = \frac{2x}{x^2+1} - e^{0.4x}(0.4\cos\pi x - \pi\sin\pi x)$$

Consider each term of f:

- $\ln(x^2+1) \geq 0\,\forall x \in \mathbb{R}$
- $e^{0.4x} > 0\,\forall x \in \mathbb{R}$
- $\cos\pi x > 0 \iff -0.5 + 2k < x < 0.5 + 2k$, with $k \in \mathbb{N}$

which means that every zero of f must be in $[2k - 0.5, 2k + 0.5]$, $k \in \mathbb{N}$.

a) e^x is monotonically increasing in $\mathbb{R}$. It follows that:

$$0 < e^{0.4x}\cos\pi x \leq e^{0.4x}1 < e^{0.4\cdot 0} = 1\,\forall x < 0$$

$\ln x$ is monotonically increasing in $\mathbb{R}_{>0}$. Therefore $\ln(x^2+1)$ is monotonically decreasing in $\mathbb{R}_{<0}$. Also, e^x is monotonically increasing in $\mathbb{R}$. Therefore, if f has a negative zero, it must satisfy:

$$\ln(x^2+1) < 1 \iff -\sqrt{e-1} \approx -1.310\,832\,494 < x < 0$$

Combining the above points, it is clear that if f has a negative zero, it must be in $D_1 = [-0.5, 0]$.
As $\ln(x^2+1)$ is monotonically decreasing in D_1, it follows that:

$$\ln(-0.5^2+1) \geq \ln(x^2+1) \geq \ln 1 = 0\,\forall x \in D_1$$

As both e^x and $\cos\pi x$ is monotonically increasing in D_1, it follows that:

$$0 \leq e^{0.4x}\cos\pi x \leq 1\,\forall x \in D_1$$

From the above points, there must be exactly one zero of f in D_1.

Applying Newton method on f with $p_0 = -0.25$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	-0.25	$-0.579\,192\,052$	$-2.797\,220\,033$
1	$-0.457\,059\,883$	$0.077\,693\,927$	$-3.742\,796\,53$
2	$-0.436\,301\,627$	$0.007\,306\,593$	$-3.691\,332\,860$
3	$-0.434\,322\,236$	$0.000\,606\,405$	$-3.685\,958\,212$
4	$-0.434\,157\,718$	$0.000\,049\,647$	$-3.685\,507\,782$
5	$-0.434\,144\,247$	$0.000\,004\,06$	$-3.685\,470\,876$
6	$-0.434\,143\,145$	$0.000\,000\,332$	$-3.685\,467\,857$
7	$-0.434\,143\,055$	$0.000\,000\,027$	$-3.685\,467\,61$

We conclude that the sole negative zero of f is $p \approx -0.434\,143\,1$.

not yet finished

Exercise 24

Find an approximation for λ, accurate to within 10^{-4}, for the population equation

$$1\,564\,000 = 1\,000\,000 e^{\lambda} + \frac{435\,000}{\lambda}(e^{\lambda} - 1)$$

discussed in the introduction to this chapter. Use this value to predict the population at the end of the second year, assuming that the immigration rate during this year remains at 435 000 individuals per year.

Solution 24

Let

$$f(x) = 1000e^{\lambda} + \frac{435}{\lambda}(e^{\lambda} - 1) - 1564$$
$$\Rightarrow f'(x) = 1000e^{\lambda} + 435\left(\frac{1 - e^{\lambda}}{\lambda^2} + \frac{e^{\lambda}}{\lambda}\right)$$

Applying Newton's method on f with $p_0 = 0.1$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.1	−1.335 588 295 3	1337.729 475 414
1	0.100 998 399 4	0.000 628 932	1338.989 559 263 2
2	0.100 997 929 7	0.000 000 000 1	1338.988 966 158

So $\lambda \approx 0.100\,997\,9$.
Since

$$N(t) = N_0 e^{\lambda t} + \frac{v}{\lambda}(e^{\lambda t} - 1)$$

then the population predicted at the end of the second year $N(2) \approx 2187.938\,632 \cdot 1000 = 2\,187\,938.632$.

Exercise 25

The sum of two numbers is 20. If each number is added to its square root, the product of the two sums is 155.55. Determine the two numbers to within 10^{-4}.

Solution 25

Let one number is $x \in [0, 20]$, and the other is $20 - x$. We have:

$$(x + \sqrt{x})(20 - x + \sqrt{20 - x}) = 155.55$$

Let

$$\begin{aligned} f(x) &= (x + \sqrt{x})(20 - x + \sqrt{20 - x}) - 155.55 \\ \Rightarrow f'(x) &= \frac{2\sqrt{x} + 1}{2\sqrt{x}}(20 - x + \sqrt{20 - x}) - \frac{2\sqrt{20 - x} + 1}{2\sqrt{20 - x}}(x + \sqrt{x}) \end{aligned}$$

Applying Newton's method on f with $p_0 = 6.5$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	6.5	−0.131 596 293 5	10.261 387 078
1	6.512 824 415 7	−0.000 248 515 5	10.222 632 862 2
2	6.512 848 726	−0.000 000 000 9	10.222 559 412 4

We conclude that the two numbers are approximately 6.512 85 and 13.487 15.

Exercise 26

The accumulated value of a savings account based on regular periodic payments can be determined from the *annuity due equation*:

$$A = \frac{P}{i}[(1 + i)^n - 1]$$

In this equation, A is the amount in the account, P is the amount regularly deposited, and i is the rate of interest per period for the n deposit periods. An engineer would like to have a savings account valued at \$750 000 upon retirement in 20 years and can afford to put \$1500 per month toward this goal. What is the minimal interest rate at which this amount can be invested, assuming that the interest is compounded monthly?

Solution 26

Replacing symbols with numbers gives:

$$A = \frac{1500}{i}[(1 + i)^{20 \cdot 12} - 1]$$

Find the minimal interest rate is finding $i > 0$ such that $A \geq 750\,000$:

$$\begin{aligned} & \frac{1500}{i}[(1 + i)^{240} - 1] \geq 750\,000 \\ \iff\ & 1500(1 + i)^{240} - 750\,000i - 1500 \geq 0 \qquad (*) \end{aligned}$$

Let

$$\begin{aligned} f(x) &= (1+x)^{240} - 500x - 1 \\ \Rightarrow f'(x) &= 240(x+1)^{239} - 500 \end{aligned}$$

Consider f'.

$$f'(x) = 0 \iff x = A = \sqrt[239]{\frac{25}{12}} - 1$$

As f' is monotonically increasing in $\mathbb{R}^+$, it follows that:

- f is monotonically decreasing in $D_1 = \mathbb{R}_{\leq A} \cap \mathbb{R}^+$
- f is monotonically increasing in $\mathbb{R}_{\geq A}$

Consider the set D_1.

$$f(0) = 0 > f(x)\,\forall x \in D_1$$

Therefore, (*) has no positive zero in D_1.
Consider the set $\mathbb{R}_{\geq A}$.

$$f(A) \approx -0.448\,119 \leq f(x)\,\forall x \in \mathbb{R}_{\geq A}$$

Therefore, f has at most one zero in $\mathbb{R}_{\geq A}$. Applying Newton's method on f with $p_0 = 0.005$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.005	−0.189 795 524 192 6	290.496 591 237 579 4
1	0.005 653 348 541 5	0.042 274 372 099 5	423.327 780 521 256 6
2	0.005 553 486 510 1	0.001 085 579 504 2	401.671 499 784 316 2
3	0.005 550 783 855 1	0.000 000 782 527 8	401.092 480 821 071 4
4	0.005 550 781 904 1	0.000 000 000 000 3	401.092 062 972 948
5	0.005 550 781 904 1	0.000 000 000 000 1	401.092 062 972 805 4

We conclude that the minimal monthly interest rate (assuming that the interest is compounded monthly) is approximately 0.555 078 %.

Exercise 27

Problems involving the amount of money required to pay off a mortgage over a fixed period of time involve the formula

$$A = \frac{P}{i}[1 - (1+i)^{-n}]$$

known as an *ordinary annuity equation*. In this equation, A is the amount of the mortgage, P is the amount of each payment, and i is the interest rate per period for the n payment periods. Suppose that a 30-year home mortgage in the amount of \$135 000 is needed and that the borrower can afford house payments of at most \$1000 per month. What is the maximal interest rate the borrower can afford to pay?

Solution 27

Replacing symbols with numbers gives:

$$A = \frac{1000}{i}[1 - (1+i)^{-(30\cdot 12)}]$$

Find the maximal interest rate is finding i such that $A \leq 135\,000$:

$$\begin{aligned} &\frac{1000}{i}[1 - (1+i)^{-360}] \leq 135\,000 \\ \iff\ &1000[1 - (1+i)^{-360}] - 135\,000i \leq 0 \qquad (*) \end{aligned}$$

Let

$$\begin{aligned} &f(x) = 1 - (1+x)^{-360} - 135x \\ \Rightarrow\ &f'(x) = 360(x+1)^{-361} - 135 \end{aligned}$$

Consider f'.

$$f'(x) = 0 \iff x = A = \sqrt[-361]{0.375} - 1$$

As f' is monotonically decreasing in $\mathbb{R}^+$, it follows that:

- f is monotonically increasing in $D_1 = \mathbb{R}_{\leq A} \cap \mathbb{R}^+$
- f is monotonically decreasing in $\mathbb{R}_{\geq A}$

Consider the set D_1.

$$f(0) = 0 < f(x)\,\forall x \in D_1$$

Therefore, (*) has no positive zero in D_1.
Consider the set $\mathbb{R}_{\geq A}$.

$$f(A) \approx 0.256\,689 \geq f(x)\,\forall x \in \mathbb{R}_{\geq A}$$

Therefore, f has at most one zero in $\mathbb{R}_{\geq A}$. Applying Newton's method on f with $p_0 = 0.0067$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.0067	0.005 140 191 904 9	−102.686 966 410 826 1
1	0.006 750 056 906 8	−0.000 014 430 489 4	−103.261 805 313 492 4
2	0.006 749 917 160 1	−0.000 000 000 111 1	−103.260 214 863 510 3

We conclude that the maximal monthly interest rate is approximately 0.674 992 %.

Exercise 28

A drug administered to a patient produces a concentration in the blood stream given by $c(t) = Ate^{\frac{-t}{3}}$ milligrams per milliliter, t hours after A units have been injected. The maximum safe concentration is 1 mg/mL.

a) What amount should be injected to reach this maximum safe concentration, and when does this maximum occur?

b) An additional amount of this drug is to be administered to the patient after the concentration falls to 0.25 mg/mL. Determine, to the nearest minute, when this second injection should be given.

c) Assume that the concentration from consecutive injections is additive and that 75 % of the amount originally injected is administered in the second injection. When is it time for the third injection?

Solution 28

a) Let

$$f(x) = xe^{\frac{-x}{3}}$$
$$\Rightarrow f'(x) = \left(1 - \frac{x}{3}\right) e^{\frac{-x}{3}}$$

Consider f'.

$$f'(x) = 0 \iff x = 3$$

It's clear that f' is monotonically decreasing in $\mathbb{R}$. It follows that:

- f is monotonically increasing in $\mathbb{R}_{\leq 3}$
- f is monotonically decreasing in $\mathbb{R}_{\geq 3}$
- f has a global maximum at 3

We now know that $\max f = \frac{3}{e}$ is achieved at 3. In other words, the maximum concentration of any injection is reached 3 hours later, regardless of the amount administered.

To reach the maximum safe concentration of 1 mg/mL, the amount should be injected is:

$$A\frac{3}{e} = 1 \iff A = \frac{e}{3} \approx 0.906\,093\,942\,8$$

We conclude that to reach the maximum safe concentration, approximately 0.906 093 942 8 unit should be injected, and the concentration reaches its highest 3 hours after injection.

b) Let

$$g(t) = Ate^{\frac{-t}{3}} - 0.25$$
$$\Rightarrow g'(t) = A\left(1 - \frac{t}{3}\right)e^{\frac{-t}{3}}$$

with $A = \frac{e}{3}$.

We want to inject after the concentration of the first injection already reached its highest, therefore the second injection should be no sooner than 3 hours since the first one.

Applying Newton's method on g with $p_0 = 11.08$ generates the following table:

n	p_n	$g(p_n)$	$g'(p_n)$
0	11.08	−0.000 127 362	−0.060 739 197
1	11.077 903 126	0.000 000 028	−0.060 765 892
2	11.077 903 587	0	−0.060 765 887

We conclude that after about 11 hours and 5 minutes since the first injection, the second one can be administered.

c) Let

$$c_n(t) = \sum_{i=1}^{n} A_i(t - t_i)e^{\frac{-(t-t_i)}{3}}$$
$$\Rightarrow c'_n(t) = \sum_{i=1}^{n} A_i\left(1 - \frac{t - t_i}{3}\right)e^{\frac{-(t-t_i)}{3}}$$

be the function of concentration $t \geq t_n$ hours since the first injection *and* during that time window another $n - 1$ shots are administered. t_n is the number of hours between the first injection and the n^{th} one, and clearly $t_1 = 0$.

From the above parts, we know that $A_1 = \frac{e}{3}$, $A_2 = 0.75A_1 = \frac{e}{4}$, $t_2 = 11.077\,903\,587$.

Consider c_2.

$$c_2(t) = 0$$
$$\iff (1 - \frac{t}{3}) + 0.75(1 - \frac{t - t_2}{3})B = 0 \text{ with } B = e^{\frac{t_2}{3}}$$
$$\iff t - 3 = 2.25(3 - t + t_2)B$$
$$\iff t = \frac{2.25(t_2 + 3)B}{1 + 2.25B} \approx 13.923\,774\,83$$

We want to inject after the total concentration from the previous injections already reached its highest, therefore the third injection should be no sooner than 13.923 774 83 hours since the first one.

Applying Newton's method on $h_2 = c_2 - 0.25$ with $p_0 = 21.25$ generates the following table:

n	p_n	$h_2(p_n)$	$h_2'(p_n)$
0	21.25	−0.000 992 299 872 6	−0.059 350 960 587 8
1	21.233 280 811 923 6	0.000 001 664 222 2	−0.059 550 102 087 8
2	21.233 308 758 511 3	0.000 000 000 004 7	−0.059 549 768 906 2
3	21.233 308 758 589 5	0	−0.059 549 768 905 2

We conclude that after about 21 hours and 14 minutes since the first injection, the third one can be administered.

Exercise 29

Let

$$f(x) = 3^{3x+1} - 7 \cdot 5^{2x}$$

a) Use the Maple commands `solve` and `fsolve` to try to find all roots of f.

b) Plot f to find initial approximations to roots of f.

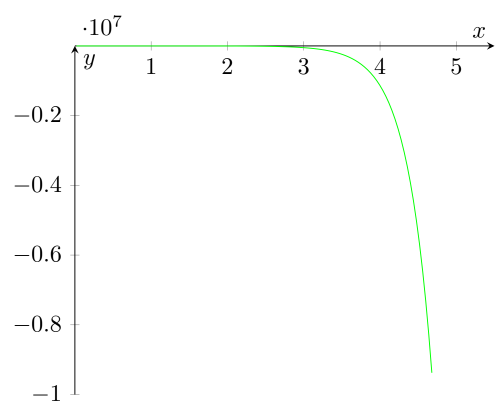
c) Use Newton's method to find roots of f to within 10^{-16}.

d) Find the exact solutions of $f(x) = 0$ without using Maple.

Solution 29

1. Opps, can't help without Maple license.

2. The graph of f is as follow:



No useful initial point found, every where: MATLAB, Maple, gnuplot,...

3. Let:

$$\begin{aligned} f(x) &= 3^{3x+1} - 7 \cdot 5^{2x} \\ \Rightarrow f'(x) &= 3(\ln 3)3^{3x+1} - 14(\ln 5)5^{2x} \end{aligned}$$

Applying Newton's method on f with $p_0 = 11$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	11	−12 118 837 442 806	1 244 484 233 952 568
1	11.009 738 040 155 250 26	396 801 311 654	1 326 632 411 906 544
2	11.009 438 935 966 258 555	386 222 634	1 324 050 511 461 616
3	11.009 438 644 268 449 536	370	1 324 047 995 335 120
4	11.009 438 644 268 170 648	−38	1 324 047 995 332 592
5	11.009 438 644 268 199 07	4	1 324 047 995 332 848
6	11.009 438 644 268 195 517	66	1 324 047 995 333 032
7	11.009 438 644 268 145 779	0	1 324 047 995 332 608

So $p \approx 11.009\,438\,644\,268\,145\,779$.

4. Manipulating $f = 0$ gives:

$$\begin{aligned} f(x) &= 0 \\ \iff 3 \cdot 3^{3x} &= 7 \cdot 5^{2x} \\ \iff \frac{27^x}{25^x} &= \frac{7}{3} \\ \iff x &= \log_{27/25} \frac{7}{3} \end{aligned}$$

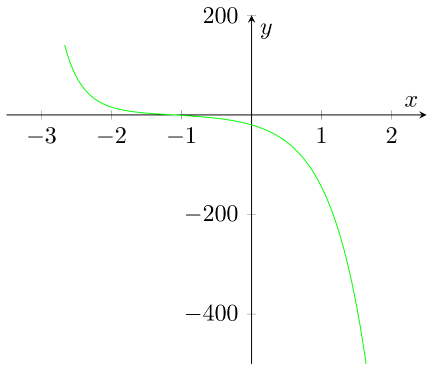
Exercise 30

Repeat Exercise 29 using $f(x) = 2^{x^2} - 3 \cdot 7^{x+1}$.

Solution 30

a) Opps, can't help without Maple license.

b) The graph of f is as follow:



c) Let:

$$f(x) = 2^{x^2} - 3 \cdot 7^{x+1}$$
$$\Rightarrow f'(x) = (\ln 2)2x2^{x^2} - 21(\ln 7)7^x$$

Applying Newton's method on f with $p_0 = 3.92$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	3.919 999 999 999 999 929	−909.989 020 751 884	145 585.672 581 531 893
1	3.926 250 539 662 426 32	22.625 719 019 627	152 874.530 827 350 565
2	3.926 102 537 775 538 082	0.013 028 085 261	152 698.506 017 085 223
3	3.926 102 452 456 528 891	0.000 000 004 293	152 698.404 592 337 756
4	3.926 102 452 456 500 913	0.000 000 000 095	152 698.404 592 304 723
5	3.926 102 452 456 500 469	−0.000 000 000 015	152 698.404 592 304 141

So $p \approx 3.926\,102\,452\,456\,500\,469$.

d) Manipulating $f = 0$ gives:

$$\begin{aligned} & f(x) = 0 \\ \iff\ & 2^{x^2} = 21 \cdot 7^x \\ \iff\ & x^2 = \log_2(21 \cdot 7^x) \\ & \quad = \log_2 21 + x \log_2 7 \\ \iff\ & x^2 - \log_2 7x - \log_2 21 = 0 \\ \iff\ & x = \frac{\log_2 7 \pm \sqrt{\Delta}}{2} \text{ with } \Delta = (\log_2 7)^2 + 4 * \log_2 21 = \log_2 9\,529\,569 \end{aligned}$$

Exercise 31

The logistic population growth model is described by an equation of the form

$$P(t) = \frac{P_L}{1 - ce^{-kt}}$$

where P_L, c, and $k > 0$ are constants, and $P(t)$ is the population at time t. P_L represents the limiting value of the population since $\lim_{t\to\infty} P(t) = P_L$. Use the census data for the years 1950, 1960, and 1970 listed in the table on page 105 to determine the constants P_L, c, and k for a logistic growth model. Use the logistic model to predict the population of the United States in 1980 and in 2010, assuming $t = 0$ at 1950. Compare the 1980 prediction to the actual value.

Solution 31

We have:

$$P(0) = \frac{P_L}{1 - ce^{-k0}} = P_1 \iff ce^0 = 1 - \frac{P_L}{P_1} \tag{1}$$
$$P(10) = \frac{P_L}{1 - ce^{-k10}} = P_2 \iff ce^{-10k} = 1 - \frac{P_L}{P_2} \tag{2}$$
$$P(20) = \frac{P_L}{1 - ce^{-k20}} = P_3 \iff ce^{-20k} = 1 - \frac{P_L}{P_3} \tag{3}$$

Divide (1) by (2) and (2) by (3) gives:

$$e^{10k} = \frac{A - P_2P_L}{A - P_1P_L} \text{ with } A = P_1P_2$$
$$e^{10k} = \frac{B - P_3P_L}{B - P_2P_L} \text{ with } B = P_2P_3$$

Combining both above equations gives:

$$\frac{A - P_2P_L}{A - P_1P_L} = \frac{B - P_3P_L}{B - P_2P_L}$$
$$\iff (A - P_6P_L)(B - P_6P_L) = (A - P_5P_L)(B - P_7P_L)$$
$$\iff (P_6^2 - P_5P_7)P_L^2 + (-AP_6 - BP_6 + AP_7 + BP_5)P_L = 0$$
$$\iff P_L = \frac{A(P_7 - P_6) + B(P_5 - P_6)}{P_5P_7 - P_6^2} \approx 265\,816.4151$$

It follows that $k \approx 0.045\,017\,502\,25$, and $c \approx -0.756\,581\,255\,8$.
We now predict the US population in 1980 and 2010:

$$P_{1980} = P(30) \approx 222\,248.3277$$
$$P_{2010} = P(60) \approx 252\,967.4246$$

It is predicted, using the above model, that the US population in 1980 is 222 248 323 and in 2010 is 252 967 425. However, the actual population in 1980 is larger, so the 1980 prediction undershoots.

Exercise 32

The Gompertz population growth model is described by

$$P(t) = P_L e^{-ce^{-kt}}$$

where P_L, c, and $k > 0$ are constants, and $P(t)$ is the population at time t. Repeat Exercise 31 using the Gompertz growth model in place of the logistic model.

Solution 32

We have:

$$P(0) = P_L e^{-ce^{-k0}} = P_1 \iff e^{-k0} = \log_d \frac{P_1}{P_L} \tag{1}$$

$$P(10) = P_L e^{-ce^{-k10}} = P_2 \iff e^{-k10} = \log_d \frac{P_2}{P_L} \tag{2}$$

$$P(20) = P_L e^{-ce^{-k20}} = P_3 \iff e^{-k20} = \log_d \frac{P_3}{P_L} \tag{3}$$

with $d = e^{-c}$.

From (1), we know that:

$$e^{-k0} = 1 = \log_d \frac{P_1}{P_L} \iff d = \frac{P_1}{P_L}$$

Divide (1) by (2) and (2) by (3) gives:

$$e^{10k} = \frac{\log_d \frac{P_1}{P_L}}{\log_d \frac{P_2}{P_L}} = \frac{\log_d P_1 - \log_d P_L}{\log_d P_2 - \log_d P_L} = \frac{\ln P_1 - \ln P_L}{\ln P_2 - \ln P_L}$$

$$e^{10k} = \frac{\log_d \frac{P_2}{P_L}}{\log_d \frac{P_3}{P_L}} = \frac{\log_d P_2 - \log_d P_L}{\log_d P_3 - \log_d P_L} = \frac{\ln P_2 - \ln P_L}{\ln P_3 - \ln P_L}$$

Combining both above equations gives:

$$\begin{aligned} \frac{\ln P_1 - \ln P_L}{\ln P_2 - \ln P_L} &= \frac{\ln P_2 - \ln P_L}{\ln P_3 - \ln P_L} \\ \iff (\ln P_2 - \ln P_L)^2 &= (\ln P_1 - \ln P_L)(\ln P_3 - \ln P_L) \\ \iff (\ln P_2)^2 - 2\ln P_2 \ln P_L &= \ln P_1 \ln P_3 - \ln(P_1 P_3)\ln P_L \\ \iff \ln P_L &= \frac{(\ln P_2)^2 - \ln P_1 \ln P_3}{2\ln P_2 - \ln(P_1 P_3)} \\ \iff P_L &\approx 290\,227.8618 \end{aligned}$$

It follows that $k \approx 0.030\,200\,281\,3$, $d = 0.521\,404\,110\,1$, $c = 0.651\,229\,894\,7$.

We now predict the US population in 1980 and 2010:

$$P_{1980} = P(30) \approx 223\,069.2173$$
$$P_{2010} = P(60) \approx 260\,943.6839$$

It is predicted, using the above model, that the US population in 1980 is 223 069 217 and in 2010 is 260 943 684. However, the actual population in 1980 is larger, so the 1980 prediction undershoots.

Exercise 33

Player A will shut out (win by a score of 21-0) player B in a game of racquetball with probability

$$P = \frac{1+p}{2}\left(\frac{p}{1-p+p^2}\right)^{21}$$

where p denotes the probability A will win any specific rally (independent of the server). Determine, to within 10^{-3}, the minimal value of p that will ensure that A will shut out B in at least half the matches they play.

Solution 33

Let

$$\begin{aligned} g(x) &= \frac{x}{1-x+x^2} \\ \Rightarrow g'(x) &= \frac{1-x^2}{(1-x+x^2)^2} \end{aligned}$$

$$\begin{aligned} f(x) &= \frac{1+x}{2}\left(\frac{x}{1-x+x^2}\right)^{21} \\ \Rightarrow f'(x) &= \frac{1}{2}\left(\frac{x}{1-x+x^2}\right)^{21} + \frac{1+x}{2}21\left(\frac{x}{1-x+x^2}\right)^{20}\frac{1-x^2}{(1-x+x^2)^2} \\ &= \frac{1}{2}\left(\frac{x}{1-x+x^2}\right)^{20}\left[\frac{x}{1-x+x^2} + \frac{21(1+x)(1-x^2)}{(1-x+x^2)^2}\right] \\ &= \frac{1}{2}\left(\frac{x}{1-x+x^2}\right)^{20}\frac{-20x^3-22x^2+22x+21}{(1-x+x^2)^2} \end{aligned}$$

Finding the minimal value of p that will ensure that A will shut out B in at least half the matches they play is finding the minimal $x \in D = [0,1]$ such that $f(x) \geq 0.5$.

Consider g'.

$$g'(x) = 0 \iff x = \pm 1$$
$$x^2 - x + 1 = x^2 - 2x0.5 + 0.5^2 + 0.75 \geq 0.75 > 0 \, \forall x \in \mathbb{R}$$

It follows that the sign of g' is the sign of $1-x^2$. Therefore, in D, $g' \geq 0$. Therefore, g and then f are monotonically increasing in D:

$$f(0) = 0 \leq f(x) \leq f(1) = 1\,\forall x \in D$$

It's clear that $f(x) \geq 0.5$ is guaranteed to have solution in D.

Applying Newton's method on $h = f - 0.5$ with $p_0 = 0.84$ generates the following table:

n	p_n	$h(p_n)$	$h'(p_n)$
0	0.84	−0.010 231 745 763 236 211	4.430 566 512 699 972 925
1	0.842 309 353 834 076 791	0.000 020 294 149 810 418	4.447 757 674 207 621 47
2	0.842 304 791 051 817 325	0.000 000 000 072 282 402	4.447 725 988 980 080 203
3	0.842 304 791 035 565 77	0.000 000 000 000 000 888	4.447 725 988 867 216 707
4	0.842 304 791 035 565 548	−0.000 000 000 000 000 444	4.447 725 988 867 211 377

We conclude that $p \geq 0.842\,304\,791\,035\,565\,548$ will ensure that A will shut out B in at least half the matches they play.

Exercise 34

In the design of all-terrain vehicles, it is necessary to consider the failure of the vehicle when attempting to negotiate two types of obstacles. One type of failure is called *hang-up failure* and occurs when the vehicle attempts to cross an obstacle that causes the bottom of the vehicle to touch the ground. The other type of failure is called *nose-in failure* and occurs when the vehicle descends into a ditch and its nose touches the ground.

The accompanying figure shows the components associated with the nose-in failure of a vehicle. It is shown that the maximum angle α that can be negotiated by a vehicle when β is the maximum angle at which hang-up failure does *not* occur satisfies the equation

$$A \sin\alpha \cos\alpha + B \sin^2\alpha - C \cos\alpha - E \sin\alpha = 0$$

where

$$\begin{cases} D: \text{wheel diameter} \\ A = l \sin\beta_1 \\ B = l \cos\beta_1 \\ C = (h + 0.5D)\sin\beta_1 - 0.5D \tan\beta_1 \\ E = (h + 0.5D)\cos\beta_1 - 0.5D \end{cases}$$

a) It is stated that when $l = 89\,\text{in}$, $h = 49\,\text{in}$, $D = 55\,\text{in}$, and $\beta_1 = 11.5°$, angle α is approximately $33°$. Verify this result.

b) Find α for the situation when l, h, and β_1 are the same as in part a) but $D = 30\,\text{in}$.

Solution 34

Let

$$\begin{aligned} f(x) &= A \sin x \cos x + B \sin^2 x - C \cos x - E \sin x \\ \Rightarrow f'(x) &= A(\cos^2 x - \sin^2 x) + 2B \sin x \cos x + C \sin x - E \cos x \end{aligned}$$

a) Applying Newton's method on f with $p_0 = 33° \approx 0.575\,958\,653\,158\,13$ generates the following table:

n	p_n	$g(p_n)$	$g'(p_n)$
0	0.575 958 653 158 13	0.025 411 305 811 59	52.342 904 131 061 25
1	0.575 473 175 589 9	0.000 008 546 838 91	52.307 681 811 205 21
2	0.575 473 012 194 42	0.000 000 000 000 97	52.307 669 944 135 87
3	0.575 473 012 194 4	0	52.307 669 944 134 55

So $\alpha \approx 0.575\,473\,012\,194\,4 \approx 32.972\,174\,82°$, which is indeed close to 33°.

b) Applying Newton's method on f with $p_0 = 33° \approx 0.575\,958\,653\,158\,13$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.575 958 653 158 13	−0.154 079 021 971 57	52.160 253 446 542 13
1	0.578 912 607 784 32	0.000 315 645 554 17	52.373 508 587 763 42
2	0.578 906 580 967 27	0.000 000 001 302 72	52.373 076 275 399 87
3	0.578 906 580 942 4	0.000 000 000 000 01	52.373 076 273 615 62

So $\alpha \approx 0.578\,906\,580\,942\,4 \approx 33.168\,903\,82°$.