0.1 The Bisection Method

Exercise 0.1.1

Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

Solution 0.1.1

$f(0) = -1$ and $f(1) \approx 0.459\,697\,694$ have the opposite signs, so there's a root in $[0, 1]$.

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	$-0.170\,475\,781$
2	0.5	1	0.75	$0.134\,336\,535$
3	0.5	0.75	0.625	$-0.020\,393\,704$

So $p_3 = 0.625$.

Exercise 0.1.2

Let $f(x) = 3(x+1)(x-\frac{1}{2})(x-1)$. Use the bisection method to find p_3 in the following intervals:

a) $[-2, 1.5]$

b) $[-1.5, 2.5]$

Solution 0.1.2

(a) $f(-2) = -22.5$ and $f(1.5) = 3.75$ have the opposite signs, so there's a root in $[-2, 1.5]$.

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-2	1.5	-0.25	$2.109\,375$
2	-2	-0.25	-1.125	$-1.294\,921\,875$
3	-1.125	-0.25	-0.6875	$1.878\,662\,109$

So $p_3 = -0.6875$.

(b) $f(-1.25) = -2.953\,125$ and $f(2.5) = 31.5$ have the opposite signs, so there's a root in $[-1.25, 2.5]$.

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-1.5	2.5	0.5	0

The solution is found in the first iteration so p_3 doesn't exist.

Exercise 0.1.3

Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ in the following intervals:

a) $[0, 1]$ b) $[1, 3.2]$ c) $[3.2, 4]$

Solution 0.1.3

(a) $f(0) = -6$ and $f(1) = 2$ have the opposite signs, so there's a root in $[0, 1]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	-0.625
2	0.5	1	0.75	0.984 375
3	0.5	0.75	0.625	0.259 766
4	0.5	0.625	0.5625	$-0.161\,865$
5	0.5625	0.625	0.593 75	0.054 047
6	0.5625	0.593 75	0.578 125	$-0.052\,624$
7	0.578 125	0.593 75	0.585 937 5	0.001 031

So $p \approx 0.5859$.

(b) $f(1) = 2$ and $f(3.2) = -0.112$ have the opposite signs, so there's a root in $[1, 3.2]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{3.2-1}{2^n} < 10^{-2} \iff n \geq 8$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	3.2	2.1	1.791
2	2.1	3.2	2.65	0.552 125
3	2.65	3.2	2.925	0.085 828
4	2.925	3.2	3.0625	−0.054 443
5	2.925	3.0625	2.993 75	0.006 328
6	2.993 75	3.0625	3.028 125	−0.026 521
7	2.993 75	3.028 13	3.010 938	−0.010 697
8	2.993 75	3.010 938	3.002 344	−0.002 333

So $p \approx 3.0023$.

(c) $f(3.2) = -0.112$ and $f(4) = 2$ have the opposite signs, so there's a root in $[3.2, 4]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{4 - 3.2}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	3.2	4	3.6	0.336
2	3.2	3.6	3.4	−0.016
3	3.4	3.6	3.5	0.125
4	3.4	3.5	3.45	0.046 125
5	3.4	3.45	3.425	0.013 016
6	3.4	3.425	3.4125	−0.001 998
7	3.4125	3.425	3.418 75	0.005 382

So $p \approx 3.4188$.

Exercise 0.1.4

Use the Bisection method to find solutions accurate to within 10^{-2} for $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ for the following intervals:

a) $[-2, -1]$ b) $[0, 2]$ c) $[2, 3]$ d) $[-1, 0]$

Solution 0.1.4

(a) $f(-2) = 12$ and $f(-1) = -1$ have the opposite signs, so there's a root in $[-2, -1]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{-1-(-2)}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-2	-1	-1.5	0.8125
2	-1.5	-1	-1.25	$-0.902\,344$
3	-1.5	-1.25	-1.375	$-0.288\,818$
4	-1.5	-1.375	-1.4375	$0.195\,328$
5	-1.4375	-1.375	$-1.406\,25$	$-0.062\,667$
6	-1.4375	$-1.406\,25$	$-1.421\,875$	$0.062\,263$
7	$-1.421\,875$	$-1.406\,25$	$-1.414\,063$	$-0.001\,208$

So $p \approx -1.4141$.

(b) $f(0) = 4$ and $f(2) = -4$ have the opposite signs, so there's a root in $[0, 2]$. The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{2-0}{2^n} < 10^{-2} \iff n \geq 8$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	2	1	3
2	1	2	1.5	-0.6875
3	1	1.5	1.25	$1.285\,156$
4	1.25	1.5	1.375	$0.312\,744$
5	1.375	1.5	1.4375	$-0.186\,508$
6	1.375	1.4375	$1.406\,25$	$0.063\,676$
7	$1.406\,25$	1.4375	$1.421\,875$	$-0.061\,318$
8	$1.406\,25$	$1.421\,875$	$1.414\,063$	$0.001\,208$

So $p \approx 1.4141$.

(c) $f(2) = -4$ and $f(3) = 7$ have the opposite signs, so there's a root in $[2, 3]$. The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{3-2}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	2	3	2.5	−3.1875
2	2.5	3	2.75	0.347 656
3	2.5	2.75	2.625	−1.757 568
4	2.625	2.75	2.6875	−0.795 639
5	2.6875	2.75	2.718 75	−0.247 466
6	2.718 75	2.75	2.734 375	0.044 125
7	2.718 75	2.734 375	2.726 563	−0.103 151

So $p \approx 2.7266$.

(d) $f(-1) = -1$ and $f(0) = 4$ have the opposite signs, so there's a root in $[-1, 0]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{0-(-1)}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	−1	0	−0.5	1.3125
2	−1	−0.5	−0.75	−0.089 844
3	−0.75	−0.5	−0.625	0.578 369
4	−0.75	−0.625	−0.6875	0.232 681
5	−0.75	−0.6875	−0.718 75	0.068 086
6	−0.75	−0.718 75	−0.734 375	−0.011 768
7	−0.734 375	−0.718 75	−0.726 563	0.027 943

So $p \approx -0.7266$.

Exercise 0.1.5

Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems:

(a) $x - 2^{-x} = 0,\ x \in [0, 1]$

(b) $e^x - x^2 + 3x - 2 = 0,\ x \in [0, 1]$

(c) $2x \cos 2x - (x+1)^2 = 0,\ x \in [-3, -2]$

(d) $x \cos x - 2x^2 + 3x - 1 = 0,\ x \in [0.2, 0.3]$

Solution 0.1.5

(a) $f(0) = -1$ and $f(1) = 0.5$ have the opposite signs, so there's a root in $[0, 1]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	−0.207 106 781
2	0.5	1	0.75	0.155 396 442
3	0.5	0.75	0.625	−0.023 419 777
4	0.625	0.75	0.6875	0.066 571 094
5	0.625	0.6875	0.656 25	0.021 724 521
6	0.625	0.656 25	0.640 625	−0.000 810 008
7	0.640 625	0.656 25	0.648 437 5	0.010 466 611
8	0.640 625	0.648 437 5	0.644 531 25	0.004 830 646
9	0.640 625	0.644 531 25	0.642 578 125	0.002 010 906
10	0.640 625	0.642 578 125	0.641 601 562	0.000 600 596
11	0.640 625	0.641 601 562	0.641 113 281	−0.000 104 669
12	0.641 113 281	0.641 601 562	0.641 357 422	0.000 247 972
13	0.641 113 281	0.641 357 422	0.641 235 352	0.000 071 654
14	0.641 113 281	0.641 235 352	0.641 174 316	−0.000 016 507
15	0.641 174 316	0.641 235 352	0.641 204 834	0.000 027 573
16	0.641 174 316	0.641 204 834	0.641 189 575	0.000 005 533
17	0.641 174 316	0.641 189 575	0.641 181 946	−0.000 005 487

So $p \approx -0.641\,182$.

(b) $f(0) = -1$ and $f(1) = e$ have the opposite signs, so there's a root in $[0, 1]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	0.898 721 271
2	0	0.5	0.25	−0.028 474 583
3	0.25	0.5	0.375	0.439 366 415

n	a_n	b_n	p_n	$f(p_n)$
4	0.25	0.375	0.3125	0.206 681 691
5	0.25	0.3125	0.281 25	0.089 433 196
6	0.25	0.281 25	0.265 625	0.030 564 234
7	0.25	0.265 625	0.257 812 5	0.001 066 368
8	0.25	0.257 812 5	0.253 906 25	−0.013 698 684
9	0.253 906 25	0.257 812 5	0.255 859 375	−0.006 314 807
10	0.255 859 375	0.257 812 5	0.256 835 938	−0.002 623 882
11	0.256 835 938	0.257 812 5	0.257 324 219	−0.000 778 673
12	0.257 324 219	0.257 812 5	0.257 568 359	0.000 143 868
13	0.257 324 219	0.257 568 359	0.257 446 289	−0.000 317 397
14	0.257 446 289	0.257 568 359	0.257 507 324	−0.000 086 763
15	0.257 507 324	0.257 568 359	0.257 537 842	0.000 028 553
16	0.257 507 324	0.257 537 842	0.257 522 583	−0.000 029 105
17	0.257 522 583	0.257 537 842	0.257 530 212	−0.000 000 276

So $p \approx 0.257\,53$.

(c) $f(-3) \approx -9.761\,021\,72$ and $f(-2) \approx 1.614\,574\,483$ have the opposite signs, so there's a root in $[-3, -2]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{-2-(-3)}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	−3	−2	−2.5	−3.668 310 93
2	−2.5	−2	−2.25	−0.613 918 903
3	−2.25	−2	−2.125	0.630 246 832
4	−2.25	−2.125	−2.1875	0.038 075 532
5	−2.25	−2.1875	−2.218 75	−0.280 836 176
6	−2.218 75	−2.1875	−2.203 125	−0.119 556 815
7	−2.203 125	−2.1875	−2.195 312 5	−0.040 278 514
8	−2.195 312 5	−2.1875	−2.191 406 25	−0.000 985 195
9	−2.191 406 25	−2.1875	−2.189 453 12	0.018 574 337
10	−2.191 406 25	−2.189 453 12	−2.190 429 69	0.008 801 851
11	−2.191 406 25	−2.190 429 69	−2.190 917 97	0.003 910 147
12	−2.191 406 25	−2.190 917 97	−2.191 162 11	0.001 462 93
13	−2.191 406 25	−2.191 162 11	−2.191 284 18	0.000 238 981
14	−2.191 406 25	−2.191 284 18	−2.191 345 21	−0.000 373 078
15	−2.191 345 21	−2.191 284 18	−2.191 314 7	−0.000 067 041

n	a_n	b_n	p_n	$f(p_n)$
16	−2.191 314 7	−2.191 284 18	−2.191 299 44	0.000 085 972

So $p \approx -2.191\,299$.

(d) $f(0.2) \approx -0.283\,986\,684$ and $f(0.3) \approx 0.006\,600\,946$ have the opposite signs, so there's a root in $[0.2, 0.3]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{0.3 - 0.2}{2^n} < 10^{-5} \iff n \geq 14$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0.2	0.3	0.25	−0.132 771 895
2	0.25	0.3	0.275	−0.061 583 071
3	0.275	0.3	0.2875	−0.027 112 719
4	0.2875	0.3	0.293 75	−0.010 160 959
5	0.293 75	0.3	0.296 875	−0.001 756 232
6	0.296 875	0.3	0.298 437 5	0.002 428 306
7	0.296 875	0.298 437 5	0.297 656 25	0.000 337 524
8	0.296 875	0.297 656 25	0.297 265 625	−0.000 708 983
9	0.297 265 625	0.297 656 25	0.297 460 938	−0.000 185 637
10	0.297 460 938	0.297 656 25	0.297 558 594	0.000 075 967
11	0.297 460 938	0.297 558 594	0.297 509 766	−0.000 054 829
12	0.297 509 766	0.297 558 594	0.297 534 18	0.000 010 57
13	0.297 509 766	0.297 534 18	0.297 521 973	−0.000 022 129
14	0.297 521 973	0.297 534 18	0.297 528 076	−0.000 005 779

So $p \approx 0.297\,528$.

Exercise 0.1.6

Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems:

a) $3x - e^x = 0,\ x \in [1, 2]$

b) $2x + 3\cos x - e^x = 0,\ x \in [0, 1]$

c) $x^2 - 4x + 4 - \ln x = 0,\ x \in [1, 2]$

d) $x + 1 - 2\sin \pi x = 0,\ x \in [0, 0.5]$

Solution 0.1.6

(a) $f(1) \approx 0.281\,718\,172$ and $f(2) \approx -1.389\,056\,099$ have the opposite signs, so there's a root in $[1, 2]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{2-1}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	0.018 310 93
2	1.5	2	1.75	−0.504 602 676
3	1.5	1.75	1.625	−0.203 419 037
4	1.5	1.625	1.5625	−0.083 233 182
5	1.5	1.5625	1.531 25	−0.030 203 153
6	1.5	1.531 25	1.515 625	−0.005 390 404
7	1.5	1.515 625	1.507 812 5	0.006 598 107
8	1.507 812 5	1.515 625	1.511 718 75	0.000 638 447
9	1.511 718 75	1.515 625	1.513 671 88	−0.002 367 313
10	1.511 718 75	1.513 671 88	1.512 695 31	−0.000 862 268
11	1.511 718 75	1.512 695 31	1.512 207 03	−0.000 111 37
12	1.511 718 75	1.512 207 03	1.511 962 89	0.000 263 674
13	1.511 962 89	1.512 207 03	1.512 084 96	0.000 076 186
14	1.512 084 96	1.512 207 03	1.512 146	−0.000 017 584
15	1.512 084 96	1.512 146	1.512 115 48	0.000 029 303
16	1.512 115 48	1.512 146	1.512 130 74	0.000 005 86
17	1.512 130 74	1.512 146	1.512 138 37	−0.000 005 861

So $p \approx 1.512\,138$.

(b) $f(0) = 2$ and $f(1) \approx 0.902\,625\,089$ have the same sign, so there's no root in $[0, 1]$.

(c) $f(1) = 1$ and $f(2) = -0.693\,147\,181$ have the opposite signs, so there's a root in $[1, 2]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{2-1}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	−0.155 465 108
2	1	1.5	1.25	0.339 356 449
3	1.25	1.5	1.375	0.072 171 269
4	1.375	1.5	1.4375	−0.046 499 244
5	1.375	1.4375	1.406 25	0.011 612 476
6	1.406 25	1.4375	1.421 875	−0.017 747 908
7	1.406 25	1.421 875	1.414 062 5	−0.003 144 013
8	1.406 25	1.414 062 5	1.410 156 25	0.004 215 136
9	1.410 156 25	1.414 062 5	1.412 109 38	0.000 530 79
10	1.412 109 38	1.414 062 5	1.413 085 94	−0.001 307 804
11	1.412 109 38	1.413 085 94	1.412 597 66	−0.000 388 805
12	1.412 109 38	1.412 597 66	1.412 353 52	0.000 070 918
13	1.412 353 52	1.412 597 66	1.412 475 59	−0.000 158 962
14	1.412 353 52	1.412 475 59	1.412 414 55	−0.000 044 027
15	1.412 353 52	1.412 414 55	1.412 384 03	0.000 013 444
16	1.412 384 03	1.412 414 55	1.412 399 29	−0.000 015 292
17	1.412 384 03	1.412 399 29	1.412 391 66	−0.000 000 924

So $p \approx 1.412\,392$.

(d) $f(0) = 1$ and $f(1) = -0.5$ have the opposite signs, so there's a root in $[0, 0.5]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{0.5 - 0}{2^n} < 10^{-5} \iff n \geq 16$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	0.5	0.25	−0.164 213 562
2	0	0.25	0.125	0.359 633 135
3	0.125	0.25	0.1875	0.076 359 534
4	0.1875	0.25	0.218 75	−0.050 036 568
5	0.1875	0.218 75	0.203 125	0.011 726 391
6	0.203 125	0.218 75	0.210 937 5	−0.019 525 681
7	0.203 125	0.210 937 5	0.207 031 25	−0.003 990 833
8	0.203 125	0.207 031 25	0.205 078 125	0.003 845 166
9	0.205 078 125	0.207 031 25	0.206 054 688	−0.000 078 51
10	0.205 078 125	0.206 054 688	0.205 566 406	0.001 881 912
11	0.205 566 406	0.206 054 688	0.205 810 547	0.000 901 347
12	0.205 810 547	0.206 054 688	0.205 932 617	0.000 411 33
13	0.205 932 617	0.206 054 688	0.205 993 652	0.000 166 388

n	a_n	b_n	p_n	$f(p_n)$
14	0.205 993 652	0.206 054 688	0.206 024 17	0.000 043 934
15	0.206 024 17	0.206 054 688	0.206 039 429	−0.000 017 289
16	0.206 024 17	0.206 039 429	0.206 031 799	0.000 013 322

So $p \approx 0.206\,032$.

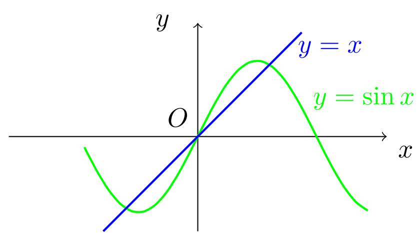
Exercise 0.1.7

(a) Sketch the graphs of $y = x$ and $y = 2\sin x$.

(b) Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = 2\sin x$.

Solution 0.1.7

(a) Graph of $y = x$ and $y = 2\sin x$ is as follow:



(b) According to the graph, the first positive root p of $f = x - 2\sin x$ is in $[\frac{\pi}{2}, \pi]$.

The number of iteration n needed to approximate p to within 10^{-5} in that interval is:

$$|p_n - p| \leq \frac{\pi - \frac{\pi}{2}}{2^n} < 10^{-5} \iff n \geq 18$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1.570 796 33	3.141 592 65	2.356 194 49	0.941 980 928
2	1.570 796 33	2.356 194 49	1.963 495 41	0.115 736 343
3	1.570 796 33	1.963 495 41	1.767 145 87	−0.194 424 693
4	1.767 145 87	1.963 495 41	1.865 320 64	−0.048 560 033
5	1.865 320 64	1.963 495 41	1.914 408 02	0.031 319 893

n	a_n	b_n	p_n	$f(p_n)$
6	1.865 320 64	1.914 408 02	1.889 864 33	−0.009 192 031
7	1.889 864 33	1.914 408 02	1.902 136 18	0.010 921 526
8	1.889 864 33	1.902 136 18	1.896 000 25	0.000 829 072
9	1.889 864 33	1.896 000 25	1.892 932 29	−0.004 190 408
10	1.892 932 29	1.896 000 25	1.894 466 27	−0.001 682 899
11	1.894 466 27	1.896 000 25	1.895 233 26	−0.000 427 471
12	1.895 233 26	1.896 000 25	1.895 616 76	0.000 200 661
13	1.895 233 26	1.895 616 76	1.895 425 01	−0.000 113 44
14	1.895 425 01	1.895 616 76	1.895 520 88	0.000 043 602
15	1.895 425 01	1.895 520 88	1.895 472 95	−0.000 034 921
16	1.895 472 95	1.895 520 88	1.895 496 92	0.000 004 34
17	1.895 472 95	1.895 496 92	1.895 484 93	−0.000 015 291
18	1.895 484 93	1.895 496 92	1.895 490 92	−0.000 005 476

So $p \approx 1.895\,491$.

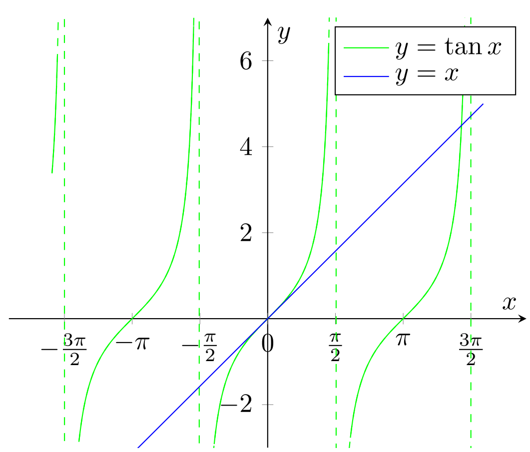
Exercise 0.1.8

(a) Sketch the graphs of $y = x$ and $y = \tan x$.

(b) Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $y = \tan x$.

Solution 0.1.8

(a) Graph of $y = x$ and $y = \tan x$ is as follow:



(b) According to the graph, the first positive root p of $f = x - \tan x$ is in $[\pi, \dfrac{3\pi}{2}]$.

The number of iteration n needed to approximate p to within 10^{-5} in that interval is:

$$|p_n - p| \leq \frac{\frac{3\pi}{2} - \pi}{2^n} < 10^{-5} \iff n \geq 18$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	3.141 592 65	4.712 388 98	3.926 990 82	2.926 990 82
2	3.926 990 82	4.712 388 98	4.319 689 9	1.905 476 34
3	4.319 689 9	4.712 388 98	4.516 039 44	−0.511 300 053
4	4.319 689 9	4.516 039 44	4.417 864 67	1.121 306 46
5	4.417 864 67	4.516 039 44	4.466 952 05	0.474 728 271
6	4.466 952 05	4.516 039 44	4.491 495 75	0.038 293 523
7	4.491 495 75	4.516 039 44	4.503 767 59	−0.219 861 735
8	4.491 495 75	4.503 767 59	4.497 631 67	−0.086 980 389
9	4.491 495 75	4.497 631 67	4.494 563 71	−0.023 432 692
10	4.491 495 75	4.494 563 71	4.493 029 73	0.007 653 323
11	4.493 029 73	4.494 563 71	4.493 796 72	−0.007 833 371
12	4.493 029 73	4.493 796 72	4.493 413 22	−0.000 076 02
13	4.493 029 73	4.493 413 22	4.493 221 48	0.003 792 144
14	4.493 221 48	4.493 413 22	4.493 317 35	0.001 858 936
15	4.493 317 35	4.493 413 22	4.493 365 29	0.000 891 677
16	4.493 365 29	4.493 413 22	4.493 389 25	0.000 407 883
17	4.493 389 25	4.493 413 22	4.493 401 24	0.000 165 946
18	4.493 401 24	4.493 413 22	4.493 407 23	0.000 044 966

So $p \approx 4.493\,407$.

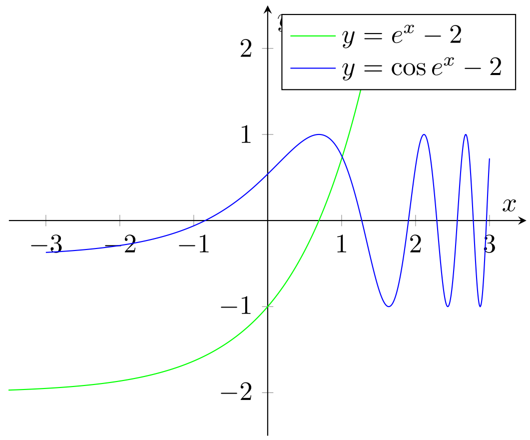
Exercise 0.1.9

(a) Sketch the graphs of $y = e^x - 2$ and $y = \cos e^x - 2$.

(b) Use the Bisection method to find an approximation to within 10^{-5} to a value in $[0.5, 1.5]$ with $e^x - 2 = \cos e^x - 2$.

Solution 0.1.9

(a) The graphs of the 2 functions are as follow:



(b) Let $f = e^x - 2 - \cos e^x - 2$. $f(0.5) \approx -1.290\,212$ and $f(1.5) \approx 3.271\,74$ have the opposite signs, so there's a root p of f in $[0.5, 1.5]$.

The number of iteration n needed to approximate p to within 10^{-5} in that interval is:

$$|p_n - p| \leq \frac{1.5 - 0.5}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0.5	1.5	1	−0.034 655 726
2	1	1.5	1.25	1.409 976 35
3	1	1.25	1.125	0.609 079 747
4	1	1.125	1.0625	0.266 982 288
5	1	1.0625	1.031 25	0.111 147 764
6	1	1.031 25	1.015 625	0.037 002 875
7	1	1.015 625	1.007 812 5	0.000 864 425
8	1	1.007 812 5	1.003 906 25	−0.016 972 716
9	1.003 906 25	1.007 812 5	1.005 859 38	−0.008 073 44
10	1.005 859 38	1.007 812 5	1.006 835 94	−0.003 609 335
11	1.006 835 94	1.007 812 5	1.007 324 22	−0.001 373 662
12	1.007 324 22	1.007 812 5	1.007 568 36	−0.000 254 92
13	1.007 568 36	1.007 812 5	1.007 690 43	0.000 304 677
14	1.007 568 36	1.007 690 43	1.007 629 39	0.000 024 859
15	1.007 568 36	1.007 629 39	1.007 598 88	−0.000 115 035
16	1.007 598 88	1.007 629 39	1.007 614 14	−0.000 045 089

So $p \approx 1.007\,614$.

Exercise 0.1.10

Let $f(x) = (x+2)(x+1)^2x(x-1)^3(x-2)$. To which zero of f does the Bisection method converge when applied on the following intervals?

a) $[-1.5, 2.5]$ b) $[-0.5, 2.4]$ c) $[-0.5, 3]$ d) $[-3, -0.5]$

Solution 0.1.10

f has 5 zeros: ± 2, ± 1, 0.

(a) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-1.5	2.5	0.5	0.527 343 75
2	-1.5	0.5	-0.5	$-1.582\,031\,25$
3	-0.5	0.5	0	0

So when applied on $[-1.5, 2.5]$, the Bisection method gives 0.

(b) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-0.5	2.4	0.95	0.001 398 666
2	-0.5	0.95	0.225	0.620 709 19

At $n = 2$, the interval shrinks to $[-0.5, 0.95]$. So when applied on $[-0.5, 2.4]$, the Bisection method gives 0.

(c) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-0.5	3	1.25	$-0.241\,012\,573$
2	1.25	3	2.125	15.235 282 5

At $n = 2$, the interval shrinks to $[1.25, 3]$. So when applied on $[-0.5, 3]$, the Bisection method gives 2.

(d) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-3	-0.5	-1.75	$-19.192\,428\,6$
2	-3	-1.75	-2.375	283.204 185

At $n = 2$, the interval shrinks to $[3, -1.75]$. So when applied on $[-3, -0.5]$, the Bisection method gives -2.

Exercise 0.1.11

Let $f(x) = (x + 2)(x + 1)x(x - 1)^3(x - 2)$. To which zero of f does the Bisection method converge when applied on the following intervals?

a) $[-3, 2.5]$

b) $[-2.5, 3]$

c) $[-1.75, 1.5]$

d) $[-1.5, -1.75]$

Solution 0.1.11

f has 5 zeros: ± 2, ± 1, 0.

(a) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-3	2.5	-0.25	$-1.441\,955\,57$
2	-0.25	2.5	1.125	$-0.012\,767\,315$
3	1.125	2.5	1.8125	$-1.954\,572\,48$

At $n = 3$, the interval shrinks to $[1.125, 2.5]$. So when applied on $[-3, 2.5]$, the Bisection method gives 2.

(b) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-2.5	3	0.25	$0.519\,104\,004$
2	-2.5	0.25	-1.125	$3.689\,754\,01$
3	-2.5	-1.125	-1.8125	$23.420\,173\,2$

At $n = 3$, the interval shrinks to $[-2.5, -1.125]$. So when applied on $[-2.5, 3]$, the Bisection method gives -2.

(c) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-1.75	1.5	-0.125	$-0.620\,491\,505$
2	-1.75	-0.125	-0.9375	$-1.330\,096\,78$

At $n = 2$, the interval shrinks to $[-1.75, -0.125]$. So when applied on $[-1.75, 1.5]$, the Bisection method gives -1.

(d) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-1.5	1.75	0.125	0.375 359 058
2	0.125	1.75	0.9375	0.001 384 076

At $n = 2$, the interval shrinks to $[0.125, 1.75]$. So when applied on $[-1.5, 1.75]$, the Bisection method gives 1.

Exercise 0.1.12

Find an approximation to $\sqrt{3}$ correct to within 10^{-4} using the Bisection Algorithm.

Solution 0.1.12

Let $f(x) = x^2 - 3$. The positive zero of f is $\sqrt{3}$, so by approximating that positive zero, we get an approximation of $\sqrt{3}$.

The positive zero of f clearly is inside $[1, 2]$. Using Bisection, the number of iteration n needed to approximate $\sqrt{3}$ to within 10^{-4} in that interval is:

$$\frac{2-1}{2^n} < 10^{-4} \iff n \geq 14$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	-0.75
2	1.5	2	1.75	0.0625
3	1.5	1.75	1.625	$-0.359 375$
4	1.625	1.75	1.6875	$-0.152 343 75$
5	1.6875	1.75	1.718 75	$-0.045 898 438$
6	1.718 75	1.75	1.734 375	0.008 056 641
7	1.718 75	1.734 375	1.726 562 5	$-0.018 981 934$
8	1.726 562 5	1.734 375	1.730 468 75	$-0.005 477 905$
9	1.730 468 75	1.734 375	1.732 421 88	0.001 285 553
10	1.730 468 75	1.732 421 88	1.731 445 31	$-0.002 097 13$
11	1.731 445 31	1.732 421 88	1.731 933 59	$-0.000 406 027$
12	1.731 933 59	1.732 421 88	1.732 177 73	0.000 439 703
13	1.731 933 59	1.732 177 73	1.732 055 66	0.000 016 823
14	1.731 933 59	1.732 055 66	1.731 994 63	$-0.000 194 605$

So $\sqrt{3} \approx 1.731 99$.

Exercise 0.1.13

Find an approximation to $\sqrt[3]{25}$ correct to within 10^{-4} using the Bisection Algorithm.

Solution 0.1.13

Let $f(x) = x^3 - 25$. The zero of f is $\sqrt[3]{25}$, so by approximating that positive zero, we get an approximation of $\sqrt[3]{25}$.

The positive zero of f clearly is inside $[2, 3]$. Using Bisection, the number of iteration n needed to approximate $\sqrt[3]{25}$ to within 10^{-4} in that interval is:

$$\frac{3-2}{2^n} < 10^{-4} \iff n \geq 14$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	2	3	2.5	−9.375
2	2.5	3	2.75	−4.203 125
3	2.75	3	2.875	−1.236 328 12
4	2.875	3	2.9375	0.347 412 109
5	2.875	2.9375	2.906 25	−0.452 972 412
6	2.906 25	2.9375	2.921 875	−0.054 920 197
7	2.921 875	2.9375	2.929 687 5	0.145 709 515
8	2.921 875	2.929 687 5	2.925 781 25	0.045 260 727
9	2.921 875	2.925 781 25	2.923 828 12	−0.004 863 195
10	2.923 828 12	2.925 781 25	2.924 804 69	0.020 190 398
11	2.923 828 12	2.924 804 69	2.924 316 41	0.007 661 51
12	2.923 828 12	2.924 316 41	2.924 072 27	0.001 398 635
13	2.923 828 12	2.924 072 27	2.923 950 2	−0.001 732 411
14	2.923 950 2	2.924 072 27	2.924 011 23	−0.000 166 921

So $\sqrt[3]{25} \approx 2.924\,01$.

Exercise 0.1.14

Use Theorem 2.1 (*Định lí 2.2* in the Lectures.pdf of the project) to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-3} to the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$. Find an approximation to the root with this degree of accuracy.

Solution 0.1.14

Let $f(x) = x^3 + x - 4$. $f(1) = -2$ and $f(4) = 64$ have the opposite signs, so there's a root p of f in $[1, 4]$.

The number of iteration n needed to approximate p to within 10^{-3} in that interval is:

$$|p_n - p| \leq \frac{4-1}{2^n} < 10^{-3} \iff n \geq 12$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	4	2.5	14.125
2	1	2.5	1.75	3.109 375
3	1	1.75	1.375	−0.025 390 625
4	1.375	1.75	1.5625	1.377 197 27
5	1.375	1.5625	1.468 75	0.637 176 514
6	1.375	1.468 75	1.421 875	0.296 520 233
7	1.375	1.421 875	1.398 437 5	0.133 260 25
8	1.375	1.398 437 5	1.386 718 75	0.053 363 502
9	1.375	1.386 718 75	1.380 859 38	0.013 844 214
10	1.375	1.380 859 38	1.377 929 69	−0.005 808 686
11	1.377 929 69	1.380 859 38	1.379 394 53	0.004 008 885
12	1.377 929 69	1.379 394 53	1.378 662 11	−0.000 902 119

So $p \approx 1.3787$.

Exercise 0.1.15

Use Theorem 2.1 (*Định lí 2.2* in the Lectures.pdf of the project) to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-4} to the solution of $x^3 - x - 1 = 0$ lying in the interval $[1, 2]$. Find an approximation to the root with this degree of accuracy.

Solution 0.1.15

Let $f(x) = x^3 - x - 1$. $f(1) = -2$ and $f(4) = 64$ have the opposite signs, so there's a root p of f in $[1, 2]$.

The number of iteration n needed to approximate p to within 10^{-4} in that interval is:

$$|p_n - p| \leq \frac{2-1}{2^n} < 10^{-4} \iff n \geq 14$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	0.875
2	1	1.5	1.25	−0.296 875

n	a_n	b_n	p_n	$f(p_n)$
3	1.25	1.5	1.375	0.224 609 375
4	1.25	1.375	1.3125	−0.051 513 672
5	1.3125	1.375	1.343 75	0.082 611 084
6	1.3125	1.343 75	1.328 125	0.014 575 958
7	1.3125	1.328 125	1.320 312 5	−0.018 710 613
8	1.320 312 5	1.328 125	1.324 218 75	−0.002 127 945
9	1.324 218 75	1.328 125	1.326 171 88	0.006 208 83
10	1.324 218 75	1.326 171 88	1.325 195 31	0.002 036 651
11	1.324 218 75	1.325 195 31	1.324 707 03	−0.000 046 595
12	1.324 707 03	1.325 195 31	1.324 951 17	0.000 994 791
13	1.324 707 03	1.324 951 17	1.324 829 1	0.000 474 039
14	1.324 707 03	1.324 829 1	1.324 768 07	0.000 213 707

So $p \approx 1.324\,77$.

Exercise 0.1.16

Let $f(x) = (x-1)^{10}$, $p = 1$, and $p_n = 1 + \frac{1}{n}$. Show that $\left|f(p_n)\right| < 10^{-3}$ whenever $n > 1$ but that $|p - p_n| < 10^{-3}$ requires that $n > 1000$.

Solution 0.1.16

For $f(p_n) < 10^{-3}$, it is required that $n > 1$ as:

$$\begin{aligned} & f(p_n) < 10^{-3} \\ \iff\quad & (p_n - 1)^{10} < 10^{-3} \\ \iff\quad & \frac{1}{n^{10}} < 10^{-3} \\ \iff\quad & n > 1 \end{aligned}$$

For $|p - p_n| < 10^{-3}$, it is required that $n > 1000$ as:

$$\begin{aligned} & |p - p_n| < 10^{-3} \\ \iff\quad & \frac{1}{n} < 10^{-3} \\ \iff\quad & n > 1000 \end{aligned}$$

$\square$

Exercise 0.1.17

Let $\{p_n\}$ be the sequence defined by $p_n = \sum_{k=1}^{n} \frac{1}{k}$. Show that $\{p_n\}$ diverges even though $\lim_{n\to\infty}(p_n - p_{n-1}) = 0$.

Solution 0.1.17

It's clear that the difference of 2 consecutive terms goes to zero:

$$\lim_{n\to\infty}(p_n - p_{n-1}) = \lim_{n\to\infty}\frac{1}{n} = 0$$

However, the sequence diverges as:

$$\begin{aligned} p_n &= \sum_{k=1}^{n}\frac{1}{k} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \\ &> 1 + (\frac{1}{2}) + (\frac{1}{4} + \frac{1}{4}) + \dots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \dots \\ &= \infty \end{aligned}$$

Exercise 0.1.18

The function defined by $f(x) = \sin \pi x$ has zeros at every integer. Show that when $-1 < a < 0$ and $2 < b < 3$, the Bisection method converges to

a) 0 if $a + b < 2$ b) 2 if $a + b > 2$ c) 1 if $a + b = 2$

Solution 0.1.18

Let p be the zero converged by Bisection.
With $-1 < a < 0$ and $2 < b < 3$:

$$\begin{gathered} \sin \pi a < 0 \\ \sin \pi b > 0 \\ 1 < a + b < 3 \end{gathered}$$

(a) If $a + b < 2$, then $0.5 < p_1 = \frac{a+b}{2} < 1$. Then $\sin p_1 > 0$, and the interval shrinks to $[a, p_1]$. 0 is the only zero in that interval, so $p = 0$.

(b) If $a + b > 2$, then $1 < p_1 = \frac{a+b}{2} < 1.5$. Then $\sin p_1 < 0$, and the interval shrinks to $[p_1, b]$. 2 is the only zero in that interval, so $p = 0$.

(c) If $a + b = 2$, then $p_1 = \frac{a+b}{2} = 1$. Then $\sin p_1 = 0$, and a zero $p = 1$ is found.

Exercise 0.1.19

A trough of length L has a cross section in the shape of a semicircle with radius r. When filled with water to within a distance h of the top, the volume V of water is:

$$V = L(0.5\pi r^2 - r^2 \arcsin \frac{h}{r} - h\sqrt{r^2 - h^2})$$

Suppose $L = 10\,\mathrm{ft}$, $r = 1\,\mathrm{ft}$, and $V = 12.4\,\mathrm{ft}^3$. Find the depth of water in the trough to within $0.01\,\mathrm{ft}$.

Solution 0.1.19

Let d be the depth of the water, so $d = r - h$. Let

$$f(h) = 10(0.5\pi - \arcsin(h) - h\sqrt{1 - h^2}) - 12.4$$

Instead of finding d directly, we find h, also to within $0.01\,\mathrm{ft}$. The number of iteration n needed to approximate h to within 0.01 in $[0, r]$ is:

$$|h - h_n| < \frac{1 - 0}{2^n} < 0.01 \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	−6.258 151 51
2	0	0.5	0.25	−1.639 453 87
3	0	0.25	0.125	0.814 489 029
4	0.125	0.25	0.1875	−0.419 946 724
5	0.125	0.1875	0.156 25	0.195 725 903
6	0.156 25	0.1875	0.171 875	−0.112 536 394
7	0.156 25	0.171 875	0.164 062 5	0.041 493 241

So $h \approx 0.1641$, hence $d = r - h \approx 0.8359$.

Exercise 0.1.20

A particle starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate ω such that:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega < 0$$

At the end of t seconds, the position of the object is given by:

$$x(t) = -\frac{g}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{x} - \sin \omega t\right)$$

Suppose the particle has moved $1.7\,\mathrm{ft}$ in $1\,\mathrm{s}$. Find, to within 10^5, the rate ω at which θ changes. Assume that $g = 32.17\,\mathrm{ft/s^2}$.

Solution 0.1.20

As $\omega < 0$, the plane rotates clockwise. After 1 s, the particle still sticks to the plane, so:

$$\theta(1) < \frac{\pi}{2} \iff -\frac{\pi}{2} < \omega < 0$$

After 1 s, the particle has moved 1.7 ft, so that:

$$x(1) = 1.7 = -\frac{32.17}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin\omega t\right)$$

Let

$$f(\omega) = 3.4\omega^2 + 32.17\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin\omega t\right)$$

The root of the above function in $(-\frac{\pi}{2}, 0)$ will be the solution of the problem.

Applying Bisection on f on $[-\frac{\pi}{2}, 0]$ fails as $f(0) = 0$. We need to expand (arbitrarily even) the searching interval a bit for the method to work, and check the solution later on. Hence, we use the interval $[-\frac{\pi}{2}, 1]$.

The number of iteration n needed to approximate ω to within 10^{-5} is:

$$|\omega - \omega_n| < \frac{1 - (-0.5\pi)}{2^n} < 10^{-5} \iff n \geq 18$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	−1.570 796 33	1	−0.285 398 163	0.027 657 569
2	−1.570 796 33	−0.285 398 163	−0.928 097 245	−5.651 487 86
3	−0.928 097 245	−0.285 398 163	−0.606 747 704	−1.143 969 69
4	−0.606 747 704	−0.285 398 163	−0.446 072 934	−0.275 313 029
5	−0.446 072 934	−0.285 398 163	−0.365 735 549	−0.069 822 38
6	−0.365 735 549	−0.285 398 163	−0.325 566 856	−0.009 667 545
7	−0.325 566 856	−0.285 398 163	−0.305 482 51	0.011 587 981
8	−0.325 566 856	−0.305 482 51	−0.315 524 683	0.001 641 051
9	−0.325 566 856	−0.315 524 683	−0.320 545 769	−0.003 838 965
10	−0.320 545 769	−0.315 524 683	−0.318 035 226	−0.001 055 895
11	−0.318 035 226	−0.315 524 683	−0.316 779 954	0.000 303 28
12	−0.318 035 226	−0.316 779 954	−0.317 407 59	−0.000 373 625
13	−0.317 407 59	−0.316 779 954	−0.317 093 772	−0.000 034 503
14	−0.317 093 772	−0.316 779 954	−0.316 936 863	0.000 134 556
15	−0.317 093 772	−0.316 936 863	−0.317 015 318	0.000 050 068
16	−0.317 093 772	−0.317 015 318	−0.317 054 545	0.000 007 793
17	−0.317 093 772	−0.317 054 545	−0.317 074 159	−0.000 013 352

n	a_n	b_n	p_n	$f(p_n)$
18	$-0.317\,074\,159$	$-0.317\,054\,545$	$-0.317\,064\,352$	$-0.000\,002\,779$

As $-0.317\,064 \in (-\frac{\pi}{2}, 0)$, it is a valid approximation of ω. We conclude that $\omega \approx -0.317\,064$.