Chapter 1

Solution approximation

1.1 The Bisection Method

Exercise 1

Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

Solution 1

Table of iteration for $f(x) = \sqrt(x) - \cos x$ on $[0, 1]$:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	−0.170 475 781
2	0.5	1	0.75	0.134 336 535
3	0.5	0.75	0.625	−0.020 393 704

So $p_3 = 0.625$.

Exercise 2

Let $f(x) = 3(x+1)(x - \frac{1}{2})(x-1)$. Use the bisection method to find p_3 in the following intervals:

(a) $[-2, 1.5]$

(b) $[-1.5, 2.5]$

Solution 2

(a) We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-2	1.5	-0.25	2.109 375
2	-2	-0.25	-1.125	$-1.294\,921\,875$
3	-1.125	-0.25	-0.6875	1.878 662 109

So $p_3 = -0.6875$.

(b) We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-1.5	2.5	0.5	0

The solution is found in the first iteration so p_3 doesn't exist.

Exercise 3

Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ in the following intervals:

(a) $[0, 1]$

(b) $[1, 3.2]$

(c) $[3.2, 4]$

Solution 3

(a) The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-2} \iff n \geq 7$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	-0.625
2	0.5	1	0.75	0.984 375
3	0.5	0.75	0.625	0.259 766
4	0.5	0.625	0.5625	$-0.161\,865$
5	0.5625	0.625	0.593 75	0.054 047
6	0.5625	0.593 75	0.578 125	$-0.052\,624$
7	0.578 125	0.593 75	0.585 937 5	0.001 031

So $p \approx 0.5859$.

(b) The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{3.2 - 1}{2^n} < 10^{-2} \iff n \geq 8$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	3.2	2.1	1.791
2	2.1	3.2	2.65	0.552 125
3	2.65	3.2	2.925	0.085 828
4	2.925	3.2	3.0625	−0.054 443
5	2.925	3.0625	2.993 75	0.006 328
6	2.993 75	3.0625	3.028 125	−0.026 521
7	2.993 75	3.028 13	3.010 938	−0.010 697
8	2.993 75	3.010 938	3.002 344	−0.002 333

So $p \approx 3.0023$.

(c) The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{4 - 3.2}{2^n} < 10^{-2} \iff n \geq 7$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	3.2	4	3.6	0.336
2	3.2	3.6	3.4	−0.016
3	3.4	3.6	3.5	0.125
4	3.4	3.5	3.45	0.046 125
5	3.4	3.45	3.425	0.013 016
6	3.4	3.425	3.4125	−0.001 998
7	3.4125	3.425	3.418 75	0.005 382

So $p \approx 3.4188$.

Exercise 4

Use the Bisection method to find solutions accurate to within 10^{-2} for $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ for the following intervals:

(a) $[-2, -1]$

(b) $[0, 2]$

(c) $[2, 3]$

(d) $[-1, 0]$

Solution 4

(a) The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{-1-(-2)}{2^n} < 10^{-2} \iff n \geq 7$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-2	-1	-1.5	0.8125
2	-1.5	-1	-1.25	$-0.902\,344$
3	-1.5	-1.25	-1.375	$-0.288\,818$
4	-1.5	-1.375	-1.4375	$0.195\,328$
5	-1.4375	-1.375	$-1.406\,25$	$-0.062\,667$
6	-1.4375	$-1.406\,25$	$-1.421\,875$	$0.062\,263$
7	$-1.421\,875$	$-1.406\,25$	$-1.414\,063$	$-0.001\,208$

So $p \approx -1.4141$.

(b) The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{2-0}{2^n} < 10^{-2} \iff n \geq 8$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	2	1	3
2	1	2	1.5	-0.6875
3	1	1.5	1.25	$1.285\,156$
4	1.25	1.5	1.375	$0.312\,744$
5	1.375	1.5	1.4375	$-0.186\,508$
6	1.375	1.4375	$1.406\,25$	$0.063\,676$
7	$1.406\,25$	1.4375	$1.421\,875$	$-0.061\,318$
8	$1.406\,25$	$1.421\,875$	$1.414\,063$	$0.001\,208$

So $p \approx 1.4141$.

(c) The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{3-2}{2^n} < 10^{-2} \iff n \geq 7$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	2	3	2.5	−3.1875
2	2.5	3	2.75	0.347 656
3	2.5	2.75	2.625	−1.757 568
4	2.625	2.75	2.6875	−0.795 639
5	2.6875	2.75	2.718 75	−0.247 466
6	2.718 75	2.75	2.734 375	0.044 125
7	2.718 75	2.734 375	2.726 563	−0.103 151

So $p \approx 2.7266$.

(d) The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{0-(-1)}{2^n} < 10^{-2} \iff n \geq 7$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	−1	0	−0.5	1.3125
2	−1	−0.5	−0.75	−0.089 844
3	−0.75	−0.5	−0.625	0.578 369
4	−0.75	−0.625	−0.6875	0.232 681
5	−0.75	−0.6875	−0.718 75	0.068 086
6	−0.75	−0.718 75	−0.734 375	−0.011 768
7	−0.734 375	−0.718 75	−0.726 563	0.027 943

So $p \approx -0.7266$.

Exercise 5

Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems:

(a) $x - 2^{-x} = 0$, $x \in [0, 1]$

(b) $e^x - x^2 + 3x - 2 = 0$, $x \in [0, 1]$

(c) $2x\cos 2x - (x+1)^2 = 0,\ x \in [-3,-2]$

(d) $x\cos x - 2x^2 + 3x - 1 = 0,\ x \in [0.2, 0.3]$

Solution 5

(a) The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-5} \iff n \geq 17$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	−0.207 107
2	0.5	1	0.75	0.155 396
3	0.5	0.75	0.625	−0.023 42
4	0.625	0.75	0.6875	0.066 571
5	0.625	0.6875	0.656 25	0.021 725
6	0.625	0.656 25	0.640 625	−0.000 81
7	0.640 625	0.656 25	0.648 438	0.010 467
8	0.640 625	0.648 438	0.644 531	0.004 831
9	0.640 625	0.644 531	0.642 578	0.002 011
10	0.640 625	0.642 578	0.641 602	0.000 601
11	0.640 625	0.641 602	0.641 113	−0.000 105
12	0.641 113	0.641 602	0.641 357	0.000 248
13	0.641 113	0.641 357	0.641 235	0.000 072
14	0.641 113	0.641 235	0.641 174	−0.000 017
15	0.641 174	0.641 235	0.641 205	0.000 028
16	0.641 174	0.641 205	0.641 19	0.000 006
17	0.641 174	0.641 19	0.641 182	−0.000 005

So $p \approx -0.641\,182$.

(b) The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-5} \iff n \geq 17$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	0.898 721
2	0	0.5	0.25	−0.028 474 6
3	0.25	0.5	0.375	0.439 366
4	0.25	0.375	0.3125	0.206 682
5	0.25	0.3125	0.281 25	0.089 433 2
6	0.25	0.281 25	0.265 625	0.030 564 2
7	0.25	0.265 625	0.257 812	0.001 066 37
8	0.25	0.257 812	0.253 906	−0.013 698 7
9	0.253 906	0.257 812	0.255 859	−0.006 314 81
10	0.255 859	0.257 812	0.256 836	−0.002 623 88
11	0.256 836	0.257 812	0.257 324	−0.000 778 673
12	0.257 324	0.257 812	0.257 568	0.000 143 868
13	0.257 324	0.257 568	0.257 446	−0.000 317 397
14	0.257 446	0.257 568	0.257 507	−0.000 086 763
15	0.257 507	0.257 568	0.257 538	0.000 028 553
16	0.257 507	0.257 538	0.257 523	−0.000 029 105
17	0.257 523	0.257 538	0.257 53	−0.000 000 276

So $p \approx 0.257\,53$.

(c) The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{-2-(-3)}{2^n} < 10^{-5} \iff n \geq 17$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	−3	−2	−2.5	−3.668 31
2	−2.5	−2	−2.25	−0.613 919
3	−2.25	−2	−2.125	0.630 247
4	−2.25	−2.125	−2.1875	0.038 075 5
5	−2.25	−2.1875	−2.218 75	−0.280 836
6	−2.218 75	−2.1875	−2.203 12	−0.119 557
7	−2.203 12	−2.1875	−2.195 31	−0.040 278 5
8	−2.195 31	−2.1875	−2.191 41	−0.000 985 195
9	−2.191 41	−2.1875	−2.189 45	0.018 574 3
10	−2.191 41	−2.189 45	−2.190 43	0.008 801 85
11	−2.191 41	−2.190 43	−2.190 92	0.003 910 15
12	−2.191 41	−2.190 92	−2.191 16	0.001 462 93
13	−2.191 41	−2.191 16	−2.191 28	0.000 238 981
14	−2.191 41	−2.191 28	−2.191 35	−0.000 373 078

15	$-2.191\,35$	$-2.191\,28$	$-2.191\,31$	$-0.000\,067\,041$
16	$-2.191\,31$	$-2.191\,28$	-2.1913	$0.000\,085\,972$
17	$-2.191\,31$	-2.1913	$-2.191\,31$	$0.000\,009\,466$

So $p \approx -2.191\,31$.

(d) The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{0.3 - 0.2}{2^n} < 10^{-5} \iff n \geq 14$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0.2	0.3	0.25	$-0.132\,772$
2	0.25	0.3	0.275	$-0.061\,583\,1$
3	0.275	0.3	0.2875	$-0.027\,112\,7$
4	0.2875	0.3	$0.293\,75$	$-0.010\,161$
5	$0.293\,75$	0.3	$0.296\,875$	$-0.001\,756\,23$
6	$0.296\,875$	0.3	$0.298\,438$	$0.002\,428\,31$
7	$0.296\,875$	$0.298\,438$	$0.297\,656$	$0.000\,337\,524$
8	$0.296\,875$	$0.297\,656$	$0.297\,266$	$-0.000\,708\,983$
9	$0.297\,266$	$0.297\,656$	$0.297\,461$	$-0.000\,185\,637$
10	$0.297\,461$	$0.297\,656$	$0.297\,559$	$0.000\,075\,967$
11	$0.297\,461$	$0.297\,559$	$0.297\,51$	$-0.000\,054\,829$
12	$0.297\,51$	$0.297\,559$	$0.297\,534$	$0.000\,010\,570$
13	$0.297\,51$	$0.297\,534$	$0.297\,522$	$-0.000\,022\,129$
14	$0.297\,522$	$0.297\,534$	$0.297\,528$	$-0.000\,005\,779$

So $p \approx 0.297\,528$.