Chapter 1

Solution approximation

1.1 The Bisection Method

Exercise 1

Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

Solution 1

$f(0) = -1$ and $f(1) \approx 0.459\,697\,694$ have the opposite signs, so there's a root in $[0, 1]$.

Table of iteration for $f(x) = \sqrt{(x)} - \cos x$ on $[0, 1]$:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	$-0.170\,475\,781$
2	0.5	1	0.75	$0.134\,336\,535$
3	0.5	0.75	0.625	$-0.020\,393\,704$

So $p_3 = 0.625$.

Exercise 2

Let $f(x) = 3(x+1)(x-\frac{1}{2})(x-1)$. Use the bisection method to find p_3 in the following intervals:

(a) $[-2, 1.5]$

(b) $[-1.5, 2.5]$

Solution 2

(a) $f(-2) = -22.5$ and $f(1.5) = 3.75$ have the opposite signs, so there's a root in $[-2, 1.5]$.

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-2	1.5	-0.25	2.109 375
2	-2	-0.25	-1.125	$-1.294\,921\,875$
3	-1.125	-0.25	-0.6875	1.878 662 109

So $p_3 = -0.6875$.

(b) $f(-1.25) = -2.953\,125$ and $f(2.5)) = 31.5$ have the opposite signs, so there's a root in $[-1.25, 2.5]$.

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-1.5	2.5	0.5	0

The solution is found in the first iteration so p_3 doesn't exist.

Exercise 3

Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ in the following intervals:

(a) $[0, 1]$

(b) $[1, 3.2]$

(c) $[3.2, 4]$

Solution 3

(a) $f(0) = -6$ and $f(1) = 2$ have the opposite signs, so there's a root in $[0, 1]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-2} \iff n \geq 7$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	−0.625
2	0.5	1	0.75	0.984 375
3	0.5	0.75	0.625	0.259 766
4	0.5	0.625	0.5625	−0.161 865
5	0.5625	0.625	0.593 75	0.054 047
6	0.5625	0.593 75	0.578 125	−0.052 624
7	0.578 125	0.593 75	0.585 937 5	0.001 031

So $p \approx 0.5859$.

(b) $f(1) = 2$ and $f(3.2) = -0.112$ have the opposite signs, so there's a root in $[1, 3.2]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{3.2 - 1}{2^n} < 10^{-2} \iff n \geq 8$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	3.2	2.1	1.791
2	2.1	3.2	2.65	0.552 125
3	2.65	3.2	2.925	0.085 828
4	2.925	3.2	3.0625	−0.054 443
5	2.925	3.0625	2.993 75	0.006 328
6	2.993 75	3.0625	3.028 125	−0.026 521
7	2.993 75	3.028 13	3.010 938	−0.010 697
8	2.993 75	3.010 938	3.002 344	−0.002 333

So $p \approx 3.0023$.

(c) $f(3.2) = -0.112$ and $f(4) = 2$ have the opposite signs, so there's a root in $[3.2, 4]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{4 - 3.2}{2^n} < 10^{-2} \iff n \geq 7$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	3.2	4	3.6	0.336
2	3.2	3.6	3.4	−0.016
3	3.4	3.6	3.5	0.125
4	3.4	3.5	3.45	0.046 125
5	3.4	3.45	3.425	0.013 016
6	3.4	3.425	3.4125	−0.001 998
7	3.4125	3.425	3.418 75	0.005 382

So $p \approx 3.4188$.

Exercise 4

Use the Bisection method to find solutions accurate to within 10^{-2} for $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ for the following intervals:

(a) $[-2, -1]$

(b) $[0, 2]$

(c) $[2, 3]$

(d) $[-1, 0]$

Solution 4

(a) $f(-2) = 12$ and $f(-1) = -1$ have the opposite signs, so there's a root in $[-2, -1]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{-1 - (-2)}{2^n} < 10^{-2} \iff n \geq 7$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	−2	−1	−1.5	0.8125
2	−1.5	−1	−1.25	−0.902 344
3	−1.5	−1.25	−1.375	−0.288 818
4	−1.5	−1.375	−1.4375	0.195 328
5	−1.4375	−1.375	−1.406 25	−0.062 667
6	−1.4375	−1.406 25	−1.421 875	0.062 263
7	−1.421 875	−1.406 25	−1.414 063	−0.001 208

So $p \approx -1.4141$.

(b) $f(0) = 4$ and $f(2) = -4$ have the opposite signs, so there's a root in $[0, 2]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{2-0}{2^n} < 10^{-2} \iff n \geq 8$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	2	1	3
2	1	2	1.5	−0.6875
3	1	1.5	1.25	1.285 156
4	1.25	1.5	1.375	0.312 744
5	1.375	1.5	1.4375	−0.186 508
6	1.375	1.4375	1.406 25	0.063 676
7	1.406 25	1.4375	1.421 875	−0.061 318
8	1.406 25	1.421 875	1.414 063	0.001 208

So $p \approx 1.4141$.

(c) $f(2) = -4$ and $f(3) = 7$ have the opposite signs, so there's a root in $[2, 3]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{3-2}{2^n} < 10^{-2} \iff n \geq 7$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	2	3	2.5	−3.1875
2	2.5	3	2.75	0.347 656
3	2.5	2.75	2.625	−1.757 568
4	2.625	2.75	2.6875	−0.795 639
5	2.6875	2.75	2.718 75	−0.247 466
6	2.718 75	2.75	2.734 375	0.044 125
7	2.718 75	2.734 375	2.726 563	−0.103 151

So $p \approx 2.7266$.

(d) $f(-1) = -1$ and $f(0) = 4$ have the opposite signs, so there's a root in $[-1, 0]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{0-(-1)}{2^n} < 10^{-2} \iff n \geq 7$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-1	0	-0.5	1.3125
2	-1	-0.5	-0.75	$-0.089\,844$
3	-0.75	-0.5	-0.625	$0.578\,369$
4	-0.75	-0.625	-0.6875	$0.232\,681$
5	-0.75	-0.6875	$-0.718\,75$	$0.068\,086$
6	-0.75	$-0.718\,75$	$-0.734\,375$	$-0.011\,768$
7	$-0.734\,375$	$-0.718\,75$	$-0.726\,563$	$0.027\,943$

So $p \approx -0.7266$.

Exercise 5

Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems:

(a) $x - 2^{-x} = 0$, $x \in [0, 1]$

(b) $e^x - x^2 + 3x - 2 = 0$, $x \in [0, 1]$

(c) $2x \cos 2x - (x + 1)^2 = 0$, $x \in [-3, -2]$

(d) $x \cos x - 2x^2 + 3x - 1 = 0$, $x \in [0.2, 0.3]$

Solution 5

(a) $f(0) = -1$ and $f(1) = 0.5$ have the opposite signs, so there's a root in $[0, 1]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-5} \iff n \geq 17$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	$-0.207\,106\,781$
2	0.5	1	0.75	$0.155\,396\,442$
3	0.5	0.75	0.625	$-0.023\,419\,777$
4	0.625	0.75	0.6875	$0.066\,571\,094$

5	0.625	0.6875	0.656 25	0.021 724 521
6	0.625	0.656 25	0.640 625	−0.000 810 008
7	0.640 625	0.656 25	0.648 437 5	0.010 466 611
8	0.640 625	0.648 437 5	0.644 531 25	0.004 830 646
9	0.640 625	0.644 531 25	0.642 578 125	0.002 010 906
10	0.640 625	0.642 578 125	0.641 601 562	0.000 600 596
11	0.640 625	0.641 601 562	0.641 113 281	−0.000 104 669
12	0.641 113 281	0.641 601 562	0.641 357 422	0.000 247 972
13	0.641 113 281	0.641 357 422	0.641 235 352	0.000 071 654
14	0.641 113 281	0.641 235 352	0.641 174 316	−0.000 016 507
15	0.641 174 316	0.641 235 352	0.641 204 834	0.000 027 573
16	0.641 174 316	0.641 204 834	0.641 189 575	0.000 005 533
17	0.641 174 316	0.641 189 575	0.641 181 946	−0.000 005 487

So $p \approx -0.641\,182$.

(b) $f(0) = -1$ and $f(1) = e$ have the opposite signs, so there's a root in $[0, 1]$. The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-5} \iff n \geq 17$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	0.898 721 271
2	0	0.5	0.25	−0.028 474 583
3	0.25	0.5	0.375	0.439 366 415
4	0.25	0.375	0.3125	0.206 681 691
5	0.25	0.3125	0.281 25	0.089 433 196
6	0.25	0.281 25	0.265 625	0.030 564 234
7	0.25	0.265 625	0.257 812 5	0.001 066 368
8	0.25	0.257 812 5	0.253 906 25	−0.013 698 684
9	0.253 906 25	0.257 812 5	0.255 859 375	−0.006 314 807
10	0.255 859 375	0.257 812 5	0.256 835 938	−0.002 623 882
11	0.256 835 938	0.257 812 5	0.257 324 219	−0.000 778 673
12	0.257 324 219	0.257 812 5	0.257 568 359	0.000 143 868
13	0.257 324 219	0.257 568 359	0.257 446 289	−0.000 317 397
14	0.257 446 289	0.257 568 359	0.257 507 324	−0.000 086 763
15	0.257 507 324	0.257 568 359	0.257 537 842	0.000 028 553
16	0.257 507 324	0.257 537 842	0.257 522 583	−0.000 029 105
17	0.257 522 583	0.257 537 842	0.257 530 212	−0.000 000 276

So $p \approx 0.257\,53$.

(c) $f(-3) \approx -9.761\,021\,72$ and $f(-2) = 1.614\,574\,483$ have the opposite signs, so there's a root in $[-3, -2]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{-2-(-3)}{2^n} < 10^{-5} \iff n \geq 17$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-3	-2	-2.5	$-3.668\,310\,93$
2	-2.5	-2	-2.25	$-0.613\,918\,903$
3	-2.25	-2	-2.125	$0.630\,246\,832$
4	-2.25	-2.125	-2.1875	$0.038\,075\,532$
5	-2.25	-2.1875	$-2.218\,75$	$-0.280\,836\,176$
6	$-2.218\,75$	-2.1875	$-2.203\,125$	$-0.119\,556\,815$
7	$-2.203\,125$	-2.1875	$-2.195\,312\,5$	$-0.040\,278\,514$
8	$-2.195\,312\,5$	-2.1875	$-2.191\,406\,25$	$-0.000\,985\,195$
9	$-2.191\,406\,25$	-2.1875	$-2.189\,453\,12$	$0.018\,574\,337$
10	$-2.191\,406\,25$	$-2.189\,453\,12$	$-2.190\,429\,69$	$0.008\,801\,851$
11	$-2.191\,406\,25$	$-2.190\,429\,69$	$-2.190\,917\,97$	$0.003\,910\,147$
12	$-2.191\,406\,25$	$-2.190\,917\,97$	$-2.191\,162\,11$	$0.001\,462\,93$
13	$-2.191\,406\,25$	$-2.191\,162\,11$	$-2.191\,284\,18$	$0.000\,238\,981$
14	$-2.191\,406\,25$	$-2.191\,284\,18$	$-2.191\,345\,21$	$-0.000\,373\,078$
15	$-2.191\,345\,21$	$-2.191\,284\,18$	$-2.191\,314\,7$	$-0.000\,067\,041$
16	$-2.191\,314\,7$	$-2.191\,284\,18$	$-2.191\,299\,44$	$0.000\,085\,972$

So $p \approx -2.191\,299$.

(d) $f(0.2) \approx -0.283\,986\,684$ and $f(0.3) = 0.006\,600\,946$ have the opposite signs, so there's a root in $[0.2, 0.3]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{0.3-0.2}{2^n} < 10^{-5} \iff n \geq 14$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0.2	0.3	0.25	$-0.132\,771\,895$
2	0.25	0.3	0.275	$-0.061\,583\,071$
3	0.275	0.3	0.2875	$-0.027\,112\,719$
4	0.2875	0.3	$0.293\,75$	$-0.010\,160\,959$
5	$0.293\,75$	0.3	$0.296\,875$	$-0.001\,756\,232$

6	0.296 875	0.3	0.298 437 5	0.002 428 306
7	0.296 875	0.298 437 5	0.297 656 25	0.000 337 524
8	0.296 875	0.297 656 25	0.297 265 625	−0.000 708 983
9	0.297 265 625	0.297 656 25	0.297 460 938	−0.000 185 637
10	0.297 460 938	0.297 656 25	0.297 558 594	0.000 075 967
11	0.297 460 938	0.297 558 594	0.297 509 766	−0.000 054 829
12	0.297 509 766	0.297 558 594	0.297 534 18	0.000 010 57
13	0.297 509 766	0.297 534 18	0.297 521 973	−0.000 022 129
14	0.297 521 973	0.297 534 18	0.297 528 076	−0.000 005 779

So $p \approx 0.297\,528$.

Exercise 6

Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems:

(a) $3x - e^x = 0,\ x \in [1, 2]$

(b) $2x + 3\cos x - e^x = 0,\ x \in [0, 1]$

(c) $x^2 - 4x + 4 - \ln x = 0,\ x \in [1, 2]$

(d) $x + 1 - 2\sin \pi x = 0,\ x \in [0, 0.5]$

Solution 6

1. $f(1) \approx 0.281\,718\,172$ and $f(2) = -1.389\,056\,099$ have the opposite signs, so there's a root in $[1, 2]$.

   The number of iteration n needed to approximate p to within 10^{-5} is:

   $$|p_n - p| \leq \frac{2-1}{2^n} < 10^{-5} \iff n \geq 17$$

   We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	0.018 310 93
2	1.5	2	1.75	−0.504 602 676
3	1.5	1.75	1.625	−0.203 419 037
4	1.5	1.625	1.5625	−0.083 233 182
5	1.5	1.5625	1.531 25	−0.030 203 153
6	1.5	1.531 25	1.515 625	−0.005 390 404
7	1.5	1.515 625	1.507 812 5	0.006 598 107
8	1.507 812 5	1.515 625	1.511 718 75	0.000 638 447
9	1.511 718 75	1.515 625	1.513 671 88	−0.002 367 313

10	1.511 718 75	1.513 671 88	1.512 695 31	−0.000 862 268
11	1.511 718 75	1.512 695 31	1.512 207 03	−0.000 111 37
12	1.511 718 75	1.512 207 03	1.511 962 89	0.000 263 674
13	1.511 962 89	1.512 207 03	1.512 084 96	0.000 076 186
14	1.512 084 96	1.512 207 03	1.512 146	−0.000 017 584
15	1.512 084 96	1.512 146	1.512 115 48	0.000 029 303
16	1.512 115 48	1.512 146	1.512 130 74	0.000 005 86
17	1.512 130 74	1.512 146	1.512 138 37	−0.000 005 861

So $p \approx 1.512\,138$.

2. $f(0) = 2$ and $f(1) \approx 0.902\,625\,089$ have the same sign, so there's no root in $[0, 1]$.

3. $f(1) = 1$ and $f(2) = -0.693\,147\,181$ have the opposite signs, so there's a root in $[1, 2]$.

   The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{2-1}{2^n} < 10^{-5} \iff n \geq 17$$

   We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	−0.155 465 108
2	1	1.5	1.25	0.339 356 449
3	1.25	1.5	1.375	0.072 171 269
4	1.375	1.5	1.4375	−0.046 499 244
5	1.375	1.4375	1.406 25	0.011 612 476
6	1.406 25	1.4375	1.421 875	−0.017 747 908
7	1.406 25	1.421 875	1.414 062 5	−0.003 144 013
8	1.406 25	1.414 062 5	1.410 156 25	0.004 215 136
9	1.410 156 25	1.414 062 5	1.412 109 38	0.000 530 79
10	1.412 109 38	1.414 062 5	1.413 085 94	−0.001 307 804
11	1.412 109 38	1.413 085 94	1.412 597 66	−0.000 388 805
12	1.412 109 38	1.412 597 66	1.412 353 52	0.000 070 918
13	1.412 353 52	1.412 597 66	1.412 475 59	−0.000 158 962
14	1.412 353 52	1.412 475 59	1.412 414 55	−0.000 044 027
15	1.412 353 52	1.412 414 55	1.412 384 03	0.000 013 444
16	1.412 384 03	1.412 414 55	1.412 399 29	−0.000 015 292
17	1.412 384 03	1.412 399 29	1.412 391 66	−0.000 000 924

   So $p \approx 1.412\,392$.

4. $f(0) = 1$ and $f(1) = -0.5$ have the opposite signs, so there's a root in

$[0, 0.5]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{0.5 - 0}{2^n} < 10^{-5} \iff n \geq 16$$

We have the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	0.5	0.25	−0.164 213 562
2	0	0.25	0.125	0.359 633 135
3	0.125	0.25	0.1875	0.076 359 534
4	0.1875	0.25	0.218 75	−0.050 036 568
5	0.1875	0.218 75	0.203 125	0.011 726 391
6	0.203 125	0.218 75	0.210 937 5	−0.019 525 681
7	0.203 125	0.210 937 5	0.207 031 25	−0.003 990 833
8	0.203 125	0.207 031 25	0.205 078 125	0.003 845 166
9	0.205 078 125	0.207 031 25	0.206 054 688	−0.000 078 51
10	0.205 078 125	0.206 054 688	0.205 566 406	0.001 881 912
11	0.205 566 406	0.206 054 688	0.205 810 547	0.000 901 347
12	0.205 810 547	0.206 054 688	0.205 932 617	0.000 411 33
13	0.205 932 617	0.206 054 688	0.205 993 652	0.000 166 388
14	0.205 993 652	0.206 054 688	0.206 024 17	0.000 043 934
15	0.206 024 17	0.206 054 688	0.206 039 429	−0.000 017 289
16	0.206 024 17	0.206 039 429	0.206 031 799	0.000 013 322

So $p \approx 0.206\,032$.

Exercise 7

(a) Sketch the graphs of $y = x$ and $y = 2\sin x$.

(b) Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = 2\sin x$.

Solution 7

(a) Graph of $y = x$ and $y = 2\sin x$ is as follow: