Phương pháp tính MAT1099

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07-09-2020

Chapter 1

Error analysis

Exercise 1

Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

Solution 1

as hey

Chapter 2

Solution approximation

2.1 The Bisection Method

Exercise 1

Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

Solution 1

$f(0) = -1$ and $f(1) \approx 0.459\,697\,694$ have the opposite signs, so there's a root in $[0, 1]$.

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	$-0.170\,475\,781$
2	0.5	1	0.75	$0.134\,336\,535$
3	0.5	0.75	0.625	$-0.020\,393\,704$

So $p_3 = 0.625$.

Exercise 2

Let $f(x) = 3(x+1)(x-\frac{1}{2})(x-1)$. Use the bisection method to find p_3 in the following intervals:

a) $[-2, 1.5]$

b) $[-1.5, 2.5]$

Solution 2

(a) $f(-2) = -22.5$ and $f(1.5) = 3.75$ have the opposite signs, so there's a root in $[-2, 1.5]$.

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-2	1.5	-0.25	2.109 375
2	-2	-0.25	-1.125	$-1.294\,921\,875$
3	-1.125	-0.25	-0.6875	1.878 662 109

So $p_3 = -0.6875$.

(b) $f(-1.25) = -2.953\,125$ and $f(2.5) = 31.5$ have the opposite signs, so there's a root in $[-1.25, 2.5]$.

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-1.5	2.5	0.5	0

The solution is found in the first iteration so p_3 doesn't exist.

Exercise 3

Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ in the following intervals:

a) $[0, 1]$ b) $[1, 3.2]$ c) $[3.2, 4]$

Solution 3

(a) $f(0) = -6$ and $f(1) = 2$ have the opposite signs, so there's a root in $[0, 1]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	-0.625
2	0.5	1	0.75	0.984 375
3	0.5	0.75	0.625	0.259 766
4	0.5	0.625	0.5625	$-0.161\,865$
5	0.5625	0.625	0.593 75	0.054 047
6	0.5625	0.593 75	0.578 125	$-0.052\,624$
7	0.578 125	0.593 75	0.585 937 5	0.001 031

So $p \approx 0.5859$.

(b) $f(1) = 2$ and $f(3.2) = -0.112$ have the opposite signs, so there's a root in $[1, 3.2]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{3.2 - 1}{2^n} < 10^{-2} \iff n \geq 8$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	3.2	2.1	1.791
2	2.1	3.2	2.65	0.552 125
3	2.65	3.2	2.925	0.085 828
4	2.925	3.2	3.0625	−0.054 443
5	2.925	3.0625	2.993 75	0.006 328
6	2.993 75	3.0625	3.028 125	−0.026 521
7	2.993 75	3.028 13	3.010 938	−0.010 697
8	2.993 75	3.010 938	3.002 344	−0.002 333

So $p \approx 3.0023$.

(c) $f(3.2) = -0.112$ and $f(4) = 2$ have the opposite signs, so there's a root in $[3.2, 4]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{4 - 3.2}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	3.2	4	3.6	0.336
2	3.2	3.6	3.4	−0.016
3	3.4	3.6	3.5	0.125
4	3.4	3.5	3.45	0.046 125
5	3.4	3.45	3.425	0.013 016
6	3.4	3.425	3.4125	−0.001 998
7	3.4125	3.425	3.418 75	0.005 382

So $p \approx 3.4188$.

Exercise 4

Use the Bisection method to find solutions accurate to within 10^{-2} for $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ for the following intervals:

a) $[-2, -1]$ b) $[0, 2]$ c) $[2, 3]$ d) $[-1, 0]$

Solution 4

(a) $f(-2) = 12$ and $f(-1) = -1$ have the opposite signs, so there's a root in $[-2, -1]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{-1 - (-2)}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-2	-1	-1.5	0.8125
2	-1.5	-1	-1.25	-0.902344
3	-1.5	-1.25	-1.375	-0.288818
4	-1.5	-1.375	-1.4375	0.195328
5	-1.4375	-1.375	-1.40625	-0.062667
6	-1.4375	-1.40625	-1.421875	0.062263
7	-1.421875	-1.40625	-1.414063	-0.001208

So $p \approx -1.4141$.

(b) $f(0) = 4$ and $f(2) = -4$ have the opposite signs, so there's a root in $[0, 2]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{2 - 0}{2^n} < 10^{-2} \iff n \geq 8$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	2	1	3
2	1	2	1.5	-0.6875
3	1	1.5	1.25	1.285156
4	1.25	1.5	1.375	0.312744
5	1.375	1.5	1.4375	-0.186508
6	1.375	1.4375	1.40625	0.063676
7	1.40625	1.4375	1.421875	-0.061318
8	1.40625	1.421875	1.414063	0.001208

So $p \approx 1.4141$.

(c) $f(2) = -4$ and $f(3) = 7$ have the opposite signs, so there's a root in $[2, 3]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{3-2}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	2	3	2.5	−3.1875
2	2.5	3	2.75	0.347 656
3	2.5	2.75	2.625	−1.757 568
4	2.625	2.75	2.6875	−0.795 639
5	2.6875	2.75	2.718 75	−0.247 466
6	2.718 75	2.75	2.734 375	0.044 125
7	2.718 75	2.734 375	2.726 563	−0.103 151

So $p \approx 2.7266$.

(d) $f(-1) = -1$ and $f(0) = 4$ have the opposite signs, so there's a root in $[-1, 0]$.

The number of iteration n needed to approximate p to within 10^{-2} is:

$$|p_n - p| \leq \frac{0-(-1)}{2^n} < 10^{-2} \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	−1	0	−0.5	1.3125
2	−1	−0.5	−0.75	−0.089 844
3	−0.75	−0.5	−0.625	0.578 369
4	−0.75	−0.625	−0.6875	0.232 681
5	−0.75	−0.6875	−0.718 75	0.068 086
6	−0.75	−0.718 75	−0.734 375	−0.011 768
7	−0.734 375	−0.718 75	−0.726 563	0.027 943

So $p \approx -0.7266$.

Exercise 5

Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems:

(a) $x - 2^{-x} = 0$, $x \in [0, 1]$

(b) $e^x - x^2 + 3x - 2 = 0$, $x \in [0, 1]$

(c) $2x \cos 2x - (x + 1)^2 = 0$, $x \in [-3, -2]$

(d) $x \cos x - 2x^2 + 3x - 1 = 0$, $x \in [0.2, 0.3]$

Solution 5

(a) $f(0) = -1$ and $f(1) = 0.5$ have the opposite signs, so there's a root in $[0, 1]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	−0.207 106 781
2	0.5	1	0.75	0.155 396 442
3	0.5	0.75	0.625	−0.023 419 777
4	0.625	0.75	0.6875	0.066 571 094
5	0.625	0.6875	0.656 25	0.021 724 521
6	0.625	0.656 25	0.640 625	−0.000 810 008
7	0.640 625	0.656 25	0.648 437 5	0.010 466 611
8	0.640 625	0.648 437 5	0.644 531 25	0.004 830 646
9	0.640 625	0.644 531 25	0.642 578 125	0.002 010 906
10	0.640 625	0.642 578 125	0.641 601 562	0.000 600 596
11	0.640 625	0.641 601 562	0.641 113 281	−0.000 104 669
12	0.641 113 281	0.641 601 562	0.641 357 422	0.000 247 972
13	0.641 113 281	0.641 357 422	0.641 235 352	0.000 071 654
14	0.641 113 281	0.641 235 352	0.641 174 316	−0.000 016 507
15	0.641 174 316	0.641 235 352	0.641 204 834	0.000 027 573
16	0.641 174 316	0.641 204 834	0.641 189 575	0.000 005 533
17	0.641 174 316	0.641 189 575	0.641 181 946	−0.000 005 487

So $p \approx -0.641\,182$.

(b) $f(0) = -1$ and $f(1) = e$ have the opposite signs, so there's a root in $[0, 1]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{1-0}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	0.898 721 271
2	0	0.5	0.25	−0.028 474 583
3	0.25	0.5	0.375	0.439 366 415
4	0.25	0.375	0.3125	0.206 681 691
5	0.25	0.3125	0.281 25	0.089 433 196
6	0.25	0.281 25	0.265 625	0.030 564 234
7	0.25	0.265 625	0.257 812 5	0.001 066 368
8	0.25	0.257 812 5	0.253 906 25	−0.013 698 684
9	0.253 906 25	0.257 812 5	0.255 859 375	−0.006 314 807
10	0.255 859 375	0.257 812 5	0.256 835 938	−0.002 623 882
11	0.256 835 938	0.257 812 5	0.257 324 219	−0.000 778 673
12	0.257 324 219	0.257 812 5	0.257 568 359	0.000 143 868
13	0.257 324 219	0.257 568 359	0.257 446 289	−0.000 317 397
14	0.257 446 289	0.257 568 359	0.257 507 324	−0.000 086 763
15	0.257 507 324	0.257 568 359	0.257 537 842	0.000 028 553
16	0.257 507 324	0.257 537 842	0.257 522 583	−0.000 029 105
17	0.257 522 583	0.257 537 842	0.257 530 212	−0.000 000 276

So $p \approx 0.257\,53$.

(c) $f(-3) \approx -9.761\,021\,72$ and $f(-2) \approx 1.614\,574\,483$ have the opposite signs, so there's a root in $[-3, -2]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{-2-(-3)}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	−3	−2	−2.5	−3.668 310 93
2	−2.5	−2	−2.25	−0.613 918 903
3	−2.25	−2	−2.125	0.630 246 832
4	−2.25	−2.125	−2.1875	0.038 075 532
5	−2.25	−2.1875	−2.218 75	−0.280 836 176
6	−2.218 75	−2.1875	−2.203 125	−0.119 556 815

n	a_n	b_n	p_n	$f(p_n)$
7	−2.203 125	−2.1875	−2.195 312 5	−0.040 278 514
8	−2.195 312 5	−2.1875	−2.191 406 25	−0.000 985 195
9	−2.191 406 25	−2.1875	−2.189 453 12	0.018 574 337
10	−2.191 406 25	−2.189 453 12	−2.190 429 69	0.008 801 851
11	−2.191 406 25	−2.190 429 69	−2.190 917 97	0.003 910 147
12	−2.191 406 25	−2.190 917 97	−2.191 162 11	0.001 462 93
13	−2.191 406 25	−2.191 162 11	−2.191 284 18	0.000 238 981
14	−2.191 406 25	−2.191 284 18	−2.191 345 21	−0.000 373 078
15	−2.191 345 21	−2.191 284 18	−2.191 314 7	−0.000 067 041
16	−2.191 314 7	−2.191 284 18	−2.191 299 44	0.000 085 972

So $p \approx -2.191\,299$.

(d) $f(0.2) \approx -0.283\,986\,684$ and $f(0.3) \approx 0.006\,600\,946$ have the opposite signs, so there's a root in $[0.2, 0.3]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{0.3 - 0.2}{2^n} < 10^{-5} \iff n \geq 14$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0.2	0.3	0.25	−0.132 771 895
2	0.25	0.3	0.275	−0.061 583 071
3	0.275	0.3	0.2875	−0.027 112 719
4	0.2875	0.3	0.293 75	−0.010 160 959
5	0.293 75	0.3	0.296 875	−0.001 756 232
6	0.296 875	0.3	0.298 437 5	0.002 428 306
7	0.296 875	0.298 437 5	0.297 656 25	0.000 337 524
8	0.296 875	0.297 656 25	0.297 265 625	−0.000 708 983
9	0.297 265 625	0.297 656 25	0.297 460 938	−0.000 185 637
10	0.297 460 938	0.297 656 25	0.297 558 594	0.000 075 967
11	0.297 460 938	0.297 558 594	0.297 509 766	−0.000 054 829
12	0.297 509 766	0.297 558 594	0.297 534 18	0.000 010 57
13	0.297 509 766	0.297 534 18	0.297 521 973	−0.000 022 129
14	0.297 521 973	0.297 534 18	0.297 528 076	−0.000 005 779

So $p \approx 0.297\,528$.

Exercise 6

Use the Bisection method to find solutions accurate to within 10^{-5} for the following problems:

a) $3x - e^x = 0,\ x \in [1, 2]$

b) $2x + 3\cos x - e^x = 0,\ x \in [0, 1]$

c) $x^2 - 4x + 4 - \ln x = 0,\ x \in [1, 2]$

d) $x + 1 - 2\sin \pi x = 0,\ x \in [0, 0.5]$

Solution 6

(a) $f(1) \approx 0.281\,718\,172$ and $f(2) \approx -1.389\,056\,099$ have the opposite signs, so there's a root in $[1, 2]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{2-1}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	0.018 310 93
2	1.5	2	1.75	−0.504 602 676
3	1.5	1.75	1.625	−0.203 419 037
4	1.5	1.625	1.5625	−0.083 233 182
5	1.5	1.5625	1.531 25	−0.030 203 153
6	1.5	1.531 25	1.515 625	−0.005 390 404
7	1.5	1.515 625	1.507 812 5	0.006 598 107
8	1.507 812 5	1.515 625	1.511 718 75	0.000 638 447
9	1.511 718 75	1.515 625	1.513 671 88	−0.002 367 313
10	1.511 718 75	1.513 671 88	1.512 695 31	−0.000 862 268
11	1.511 718 75	1.512 695 31	1.512 207 03	−0.000 111 37
12	1.511 718 75	1.512 207 03	1.511 962 89	0.000 263 674
13	1.511 962 89	1.512 207 03	1.512 084 96	0.000 076 186
14	1.512 084 96	1.512 207 03	1.512 146	−0.000 017 584
15	1.512 084 96	1.512 146	1.512 115 48	0.000 029 303
16	1.512 115 48	1.512 146	1.512 130 74	0.000 005 86
17	1.512 130 74	1.512 146	1.512 138 37	−0.000 005 861

So $p \approx 1.512\,138$.

(b) $f(0) = 2$ and $f(1) \approx 0.902\,625\,089$ have the same sign, so there's no root in $[0, 1]$.

(c) $f(1) = 1$ and $f(2) = -0.693\,147\,181$ have the opposite signs, so there's a root in $[1, 2]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{2-1}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	−0.155 465 108
2	1	1.5	1.25	0.339 356 449
3	1.25	1.5	1.375	0.072 171 269
4	1.375	1.5	1.4375	−0.046 499 244
5	1.375	1.4375	1.406 25	0.011 612 476
6	1.406 25	1.4375	1.421 875	−0.017 747 908
7	1.406 25	1.421 875	1.414 062 5	−0.003 144 013
8	1.406 25	1.414 062 5	1.410 156 25	0.004 215 136
9	1.410 156 25	1.414 062 5	1.412 109 38	0.000 530 79
10	1.412 109 38	1.414 062 5	1.413 085 94	−0.001 307 804
11	1.412 109 38	1.413 085 94	1.412 597 66	−0.000 388 805
12	1.412 109 38	1.412 597 66	1.412 353 52	0.000 070 918
13	1.412 353 52	1.412 597 66	1.412 475 59	−0.000 158 962
14	1.412 353 52	1.412 475 59	1.412 414 55	−0.000 044 027
15	1.412 353 52	1.412 414 55	1.412 384 03	0.000 013 444
16	1.412 384 03	1.412 414 55	1.412 399 29	−0.000 015 292
17	1.412 384 03	1.412 399 29	1.412 391 66	−0.000 000 924

So $p \approx 1.412\,392$.

(d) $f(0) = 1$ and $f(1) = -0.5$ have the opposite signs, so there's a root in $[0, 0.5]$.

The number of iteration n needed to approximate p to within 10^{-5} is:

$$|p_n - p| \leq \frac{0.5-0}{2^n} < 10^{-5} \iff n \geq 16$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	0.5	0.25	−0.164 213 562
2	0	0.25	0.125	0.359 633 135
3	0.125	0.25	0.1875	0.076 359 534
4	0.1875	0.25	0.218 75	−0.050 036 568
5	0.1875	0.218 75	0.203 125	0.011 726 391
6	0.203 125	0.218 75	0.210 937 5	−0.019 525 681

n	a_n	b_n	p_n	$f(p_n)$
7	0.203 125	0.210 937 5	0.207 031 25	−0.003 990 833
8	0.203 125	0.207 031 25	0.205 078 125	0.003 845 166
9	0.205 078 125	0.207 031 25	0.206 054 688	−0.000 078 51
10	0.205 078 125	0.206 054 688	0.205 566 406	0.001 881 912
11	0.205 566 406	0.206 054 688	0.205 810 547	0.000 901 347
12	0.205 810 547	0.206 054 688	0.205 932 617	0.000 411 33
13	0.205 932 617	0.206 054 688	0.205 993 652	0.000 166 388
14	0.205 993 652	0.206 054 688	0.206 024 17	0.000 043 934
15	0.206 024 17	0.206 054 688	0.206 039 429	−0.000 017 289
16	0.206 024 17	0.206 039 429	0.206 031 799	0.000 013 322

So $p \approx 0.206\,032$.

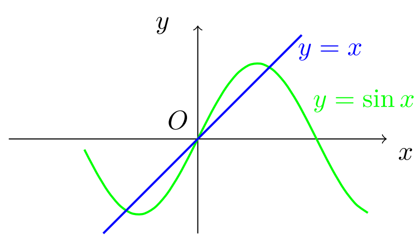
Exercise 7

(a) Sketch the graphs of $y = x$ and $y = 2\sin x$.

(b) Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = 2\sin x$.

Solution 7

(a) Graph of $y = x$ and $y = 2\sin x$ is as follow:



(b) According to the graph, the first positive root p of $f = x - 2\sin x$ is in $[\frac{\pi}{2}, \pi]$.

The number of iteration n needed to approximate p to within 10^{-5} in that interval is:

$$|p_n - p| \leq \frac{\pi - \frac{\pi}{2}}{2^n} < 10^{-5} \iff n \geq 18$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1.570 796 33	3.141 592 65	2.356 194 49	0.941 980 928
2	1.570 796 33	2.356 194 49	1.963 495 41	0.115 736 343
3	1.570 796 33	1.963 495 41	1.767 145 87	−0.194 424 693
4	1.767 145 87	1.963 495 41	1.865 320 64	−0.048 560 033
5	1.865 320 64	1.963 495 41	1.914 408 02	0.031 319 893
6	1.865 320 64	1.914 408 02	1.889 864 33	−0.009 192 031
7	1.889 864 33	1.914 408 02	1.902 136 18	0.010 921 526
8	1.889 864 33	1.902 136 18	1.896 000 25	0.000 829 072
9	1.889 864 33	1.896 000 25	1.892 932 29	−0.004 190 408
10	1.892 932 29	1.896 000 25	1.894 466 27	−0.001 682 899
11	1.894 466 27	1.896 000 25	1.895 233 26	−0.000 427 471
12	1.895 233 26	1.896 000 25	1.895 616 76	0.000 200 661
13	1.895 233 26	1.895 616 76	1.895 425 01	−0.000 113 44
14	1.895 425 01	1.895 616 76	1.895 520 88	0.000 043 602
15	1.895 425 01	1.895 520 88	1.895 472 95	−0.000 034 921
16	1.895 472 95	1.895 520 88	1.895 496 92	0.000 004 34
17	1.895 472 95	1.895 496 92	1.895 484 93	−0.000 015 291
18	1.895 484 93	1.895 496 92	1.895 490 92	−0.000 005 476

So $p \approx 1.895\,491$.

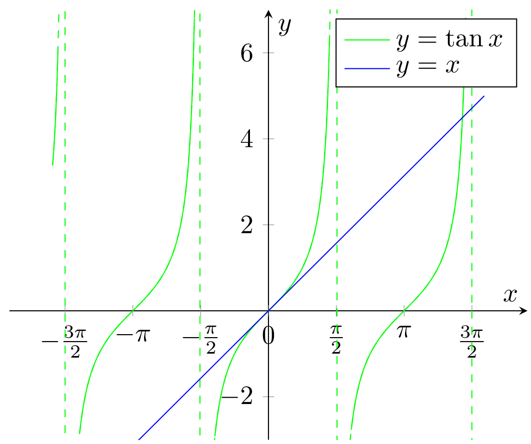
Exercise 8

(a) Sketch the graphs of $y = x$ and $y = \tan x$.

(b) Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $y = \tan x$.

Solution 8

(a) Graph of $y = x$ and $y = \tan x$ is as follow:



(b) According to the graph, the first positive root p of $f = x - \tan x$ is in $[\pi, \dfrac{3\pi}{2}]$.

The number of iteration n needed to approximate p to within 10^{-5} in that interval is:

$$|p_n - p| \leq \frac{\frac{3\pi}{2} - \pi}{2^n} < 10^{-5} \iff n \geq 18$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	3.141 592 65	4.712 388 98	3.926 990 82	2.926 990 82
2	3.926 990 82	4.712 388 98	4.319 689 9	1.905 476 34
3	4.319 689 9	4.712 388 98	4.516 039 44	−0.511 300 053
4	4.319 689 9	4.516 039 44	4.417 864 67	1.121 306 46
5	4.417 864 67	4.516 039 44	4.466 952 05	0.474 728 271
6	4.466 952 05	4.516 039 44	4.491 495 75	0.038 293 523
7	4.491 495 75	4.516 039 44	4.503 767 59	−0.219 861 735
8	4.491 495 75	4.503 767 59	4.497 631 67	−0.086 980 389
9	4.491 495 75	4.497 631 67	4.494 563 71	−0.023 432 692
10	4.491 495 75	4.494 563 71	4.493 029 73	0.007 653 323
11	4.493 029 73	4.494 563 71	4.493 796 72	−0.007 833 371
12	4.493 029 73	4.493 796 72	4.493 413 22	−0.000 076 02
13	4.493 029 73	4.493 413 22	4.493 221 48	0.003 792 144
14	4.493 221 48	4.493 413 22	4.493 317 35	0.001 858 936
15	4.493 317 35	4.493 413 22	4.493 365 29	0.000 891 677
16	4.493 365 29	4.493 413 22	4.493 389 25	0.000 407 883
17	4.493 389 25	4.493 413 22	4.493 401 24	0.000 165 946
18	4.493 401 24	4.493 413 22	4.493 407 23	0.000 044 966

So $p \approx 4.493\,407$.

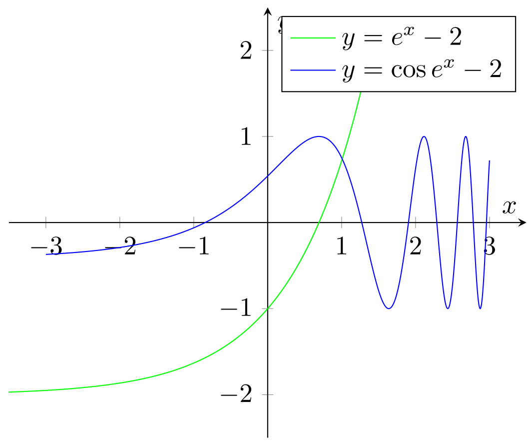
Exercise 9

(a) Sketch the graphs of $y = e^x - 2$ and $y = \cos e^x - 2$.

(b) Use the Bisection method to find an approximation to within 10^{-5} to a value in $[0.5, 1.5]$ with $e^x - 2 = \cos e^x - 2$.

Solution 9

(a) The graphs of the 2 functions are as follow:



(b) Let $f = e^x - 2 - \cos e^x - 2$. $f(0.5) \approx -1.290\,212$ and $f(1.5) \approx 3.271\,74$ have the opposite signs, so there's a root p of f in $[0.5, 1.5]$.

The number of iteration n needed to approximate p to within 10^{-5} in that interval is:

$$|p_n - p| \leq \frac{1.5 - 0.5}{2^n} < 10^{-5} \iff n \geq 17$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0.5	1.5	1	$-0.034\,655\,726$
2	1	1.5	1.25	$1.409\,976\,35$
3	1	1.25	1.125	$0.609\,079\,747$
4	1	1.125	1.0625	$0.266\,982\,288$
5	1	1.0625	1.031 25	$0.111\,147\,764$
6	1	1.031 25	1.015 625	$0.037\,002\,875$
7	1	1.015 625	1.007 812 5	$0.000\,864\,425$

n	a_n	b_n	p_n	$f(p_n)$
8	1	1.007 812 5	1.003 906 25	−0.016 972 716
9	1.003 906 25	1.007 812 5	1.005 859 38	−0.008 073 44
10	1.005 859 38	1.007 812 5	1.006 835 94	−0.003 609 335
11	1.006 835 94	1.007 812 5	1.007 324 22	−0.001 373 662
12	1.007 324 22	1.007 812 5	1.007 568 36	−0.000 254 92
13	1.007 568 36	1.007 812 5	1.007 690 43	0.000 304 677
14	1.007 568 36	1.007 690 43	1.007 629 39	0.000 024 859
15	1.007 568 36	1.007 629 39	1.007 598 88	−0.000 115 035
16	1.007 598 88	1.007 629 39	1.007 614 14	−0.000 045 089

So $p \approx 1.007\,614$.

Exercise 10

Let $f(x) = (x+2)(x+1)^2 x(x-1)^3(x-2)$. To which zero of f does the Bisection method converge when applied on the following intervals?

a) $[-1.5, 2.5]$ b) $[-0.5, 2.4]$ c) $[-0.5, 3]$ d) $[-3, -0.5]$

Solution 10

f has 5 zeros: ± 2, ± 1, 0.

(a) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	−1.5	2.5	0.5	0.527 343 75
2	−1.5	0.5	−0.5	−1.582 031 25
3	−0.5	0.5	0	0

So when applied on $[-1.5, 2.5]$, the Bisection method gives 0.

(b) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	−0.5	2.4	0.95	0.001 398 666
2	−0.5	0.95	0.225	0.620 709 19

At $n = 2$, the interval shrinks to $[-0.5, 0.95]$. So when applied on $[-0.5, 2.4]$, the Bisection method gives 0.

(c) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-0.5	3	1.25	$-0.241\,012\,573$
2	1.25	3	2.125	15.235 282 5

At $n = 2$, the interval shrinks to $[1.25, 3]$. So when applied on $[-0.5, 3]$, the Bisection method gives 2.

(d) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-3	-0.5	-1.75	$-19.192\,428\,6$
2	-3	-1.75	-2.375	283.204 185

At $n = 2$, the interval shrinks to $[3, -1.75]$. So when applied on $[-3, -0.5]$, the Bisection method gives -2.

Exercise 11

Let $f(x) = (x + 2)(x + 1)x(x - 1)^3(x - 2)$. To which zero of f does the Bisection method converge when applied on the following intervals?

a) $[-3, 2.5]$

b) $[-2.5, 3]$

c) $[-1.75, 1.5]$

d) $[-1.5, -1.75]$

Solution 11

f has 5 zeros: ± 2, ± 1, 0.

(a) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-3	2.5	-0.25	$-1.441\,955\,57$
2	-0.25	2.5	1.125	$-0.012\,767\,315$
3	1.125	2.5	1.8125	$-1.954\,572\,48$

At $n = 3$, the interval shrinks to $[1.125, 2.5]$. So when applied on $[-3, 2.5]$, the Bisection method gives 2.

(b) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-2.5	3	0.25	0.519 104 004
2	-2.5	0.25	-1.125	3.689 754 01
3	-2.5	-1.125	-1.8125	23.420 173 2

At $n = 3$, the interval shrinks to $[-2.5, -1.125]$. So when applied on $[-2.5, 3]$, the Bisection method gives -2.

(c) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-1.75	1.5	-0.125	$-0.620\,491\,505$
2	-1.75	-0.125	-0.9375	$-1.330\,096\,78$

At $n = 2$, the interval shrinks to $[-1.75, -0.125]$. So when applied on $[-1.75, 1.5]$, the Bisection method gives -1.

(d) Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	-1.5	1.75	0.125	0.375 359 058
2	0.125	1.75	0.9375	0.001 384 076

At $n = 2$, the interval shrinks to $[0.125, 1.75]$. So when applied on $[-1.5, 1.75]$, the Bisection method gives 1.

Exercise 12

Find an approximation to $\sqrt{3}$ correct to within 10^{-4} using the Bisection Algorithm.

Solution 12

Let $f(x) = x^2 - 3$. The positive zero of f is $\sqrt{3}$, so by approximating that positive zero, we get an approximation of $\sqrt{3}$.

The positive zero of f clearly is inside $[1, 2]$. Using Bisection, the number of iteration n needed to approximate $\sqrt{3}$ to within 10^{-4} in that interval is:

$$\frac{2-1}{2^n} < 10^{-4} \iff n \geq 14$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	−0.75
2	1.5	2	1.75	0.0625
3	1.5	1.75	1.625	−0.359 375
4	1.625	1.75	1.6875	−0.152 343 75
5	1.6875	1.75	1.718 75	−0.045 898 438
6	1.718 75	1.75	1.734 375	0.008 056 641
7	1.718 75	1.734 375	1.726 562 5	−0.018 981 934
8	1.726 562 5	1.734 375	1.730 468 75	−0.005 477 905
9	1.730 468 75	1.734 375	1.732 421 88	0.001 285 553
10	1.730 468 75	1.732 421 88	1.731 445 31	−0.002 097 13
11	1.731 445 31	1.732 421 88	1.731 933 59	−0.000 406 027
12	1.731 933 59	1.732 421 88	1.732 177 73	0.000 439 703
13	1.731 933 59	1.732 177 73	1.732 055 66	0.000 016 823
14	1.731 933 59	1.732 055 66	1.731 994 63	−0.000 194 605

So $\sqrt{3} \approx 1.731\,99$.

Exercise 13

Find an approximation to $\sqrt[3]{25}$ correct to within 10^{-4} using the Bisection Algorithm.

Solution 13

Let $f(x) = x^3 - 25$. The zero of f is $\sqrt[3]{25}$, so by approximating that positive zero, we get an approximation of $\sqrt[3]{25}$.

The positive zero of f clearly is inside $[2, 3]$. Using Bisection, the number of iteration n needed to approximate $\sqrt[3]{25}$ to within 10^{-4} in that interval is:

$$\frac{3-2}{2^n} < 10^{-4} \iff n \geq 14$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	2	3	2.5	−9.375
2	2.5	3	2.75	−4.203 125
3	2.75	3	2.875	−1.236 328 12
4	2.875	3	2.9375	0.347 412 109
5	2.875	2.9375	2.906 25	−0.452 972 412
6	2.906 25	2.9375	2.921 875	−0.054 920 197
7	2.921 875	2.9375	2.929 687 5	0.145 709 515
8	2.921 875	2.929 687 5	2.925 781 25	0.045 260 727
9	2.921 875	2.925 781 25	2.923 828 12	−0.004 863 195

n	a_n	b_n	p_n	$f(p_n)$
10	2.923 828 12	2.925 781 25	2.924 804 69	0.020 190 398
11	2.923 828 12	2.924 804 69	2.924 316 41	0.007 661 51
12	2.923 828 12	2.924 316 41	2.924 072 27	0.001 398 635
13	2.923 828 12	2.924 072 27	2.923 950 2	−0.001 732 411
14	2.923 950 2	2.924 072 27	2.924 011 23	−0.000 166 921

So $\sqrt[3]{25} \approx 2.924\,01$.

Exercise 14

Use Theorem 2.1 (*Định lí 2.2* in the Lectures.pdf of the project) to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-3} to the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$. Find an approximation to the root with this degree of accuracy.

Solution 14

Let $f(x) = x^3 + x - 4$. $f(1) = -2$ and $f(4) = 64$ have the opposite signs, so there's a root p of f in $[1, 4]$.

The number of iteration n needed to approximate p to within 10^{-3} in that interval is:

$$|p_n - p| \leq \frac{4-1}{2^n} < 10^{-3} \iff n \geq 12$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	4	2.5	14.125
2	1	2.5	1.75	3.109 375
3	1	1.75	1.375	−0.025 390 625
4	1.375	1.75	1.5625	1.377 197 27
5	1.375	1.5625	1.468 75	0.637 176 514
6	1.375	1.468 75	1.421 875	0.296 520 233
7	1.375	1.421 875	1.398 437 5	0.133 260 25
8	1.375	1.398 437 5	1.386 718 75	0.053 363 502
9	1.375	1.386 718 75	1.380 859 38	0.013 844 214
10	1.375	1.380 859 38	1.377 929 69	−0.005 808 686
11	1.377 929 69	1.380 859 38	1.379 394 53	0.004 008 885
12	1.377 929 69	1.379 394 53	1.378 662 11	−0.000 902 119

So $p \approx 1.3787$.

Exercise 15

Use Theorem 2.1 (*Định lí 2.2* in the Lectures.pdf of the project) to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-4} to the solution of $x^3 - x - 1 = 0$ lying in the interval $[1, 2]$. Find an approximation to the root with this degree of accuracy.

Solution 15

Let $f(x) = x^3 - x - 1$. $f(1) = -2$ and $f(4) = 64$ have the opposite signs, so there's a root p of f in $[1, 2]$.

The number of iteration n needed to approximate p to within 10^{-4} in that interval is:

$$|p_n - p| \leq \frac{2-1}{2^n} < 10^{-4} \iff n \geq 14$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	1	2	1.5	0.875
2	1	1.5	1.25	−0.296 875
3	1.25	1.5	1.375	0.224 609 375
4	1.25	1.375	1.3125	−0.051 513 672
5	1.3125	1.375	1.343 75	0.082 611 084
6	1.3125	1.343 75	1.328 125	0.014 575 958
7	1.3125	1.328 125	1.320 312 5	−0.018 710 613
8	1.320 312 5	1.328 125	1.324 218 75	−0.002 127 945
9	1.324 218 75	1.328 125	1.326 171 88	0.006 208 83
10	1.324 218 75	1.326 171 88	1.325 195 31	0.002 036 651
11	1.324 218 75	1.325 195 31	1.324 707 03	−0.000 046 595
12	1.324 707 03	1.325 195 31	1.324 951 17	0.000 994 791
13	1.324 707 03	1.324 951 17	1.324 829 1	0.000 474 039
14	1.324 707 03	1.324 829 1	1.324 768 07	0.000 213 707

So $p \approx 1.324\,77$.

Exercise 16

Let $f(x) = (x-1)^{10}$, $p = 1$, and $p_n = 1 + \frac{1}{n}$. Show that $\left|f(p_n)\right| < 10^{-3}$ whenever $n > 1$ but that $|p - p_n| < 10^{-3}$ requires that $n > 1000$.

Solution 16

For $f(p_n) < 10^{-3}$, it is required that $n > 1$ as:

$$f(p_n) < 10^{-3}$$

$$\begin{aligned}
\iff \quad & (p_n - 1)^{10} < 10^{-3} \\
\iff \quad & \frac{1}{n^{10}} < 10^{-3} \\
\iff \quad & n > 1
\end{aligned}$$

For $|p - p_n| < 10^{-3}$, it is required that $n > 1000$ as:

$$\begin{aligned}
& |p - p_n| < 10^{-3} \\
\iff \quad & \frac{1}{n} < 10^{-3} \\
\iff \quad & n > 1000
\end{aligned}$$

□

Exercise 17

Let $\{p_n\}$ be the sequence defined by $p_n = \sum_{k=1}^{n} \frac{1}{k}$. Show that $\{p_n\}$ diverges even though $\lim_{n\to\infty}(p_n - p_{n-1}) = 0$.

Solution 17

It's clear that the difference of 2 consecutive terms goes to zero:

$$\lim_{n\to\infty}(p_n - p_{n-1}) = \lim_{n\to\infty}\frac{1}{n} = 0$$

However, the sequence diverges as:

$$\begin{aligned}
p_n &= \sum_{k=1}^{n}\frac{1}{k} \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \\
&> 1 + (\frac{1}{2}) + (\frac{1}{4} + \frac{1}{4}) + \dots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \dots \\
&= \infty
\end{aligned}$$

Exercise 18

The function defined by $f(x) = \sin \pi x$ has zeros at every integer. Show that when $-1 < a < 0$ and $2 < b < 3$, the Bisection method converges to

a) 0 if $a + b < 2$ b) 2 if $a + b > 2$ c) 1 if $a + b = 2$

Solution 18

Let p be the zero converged by Bisection.
With $-1 < a < 0$ and $2 < b < 3$:

$$\sin \pi a < 0$$
$$\sin \pi b > 0$$
$$1 < a + b < 3$$

(a) If $a + b < 2$, then $0.5 < p_1 = \frac{a+b}{2} < 1$. Then $\sin p_1 > 0$, and the interval shrinks to $[a, p_1]$. 0 is the only zero in that interval, so $p = 0$.

(b) If $a + b > 2$, then $1 < p_1 = \frac{a+b}{2} < 1.5$. Then $\sin p_1 < 0$, and the interval shrinks to $[p_1, b]$. 2 is the only zero in that interval, so $p = 0$.

(c) If $a + b = 2$, then $p_1 = \frac{a+b}{2} = 1$. Then $\sin p_1 = 0$, and a zero $p = 1$ is found.

Exercise 19

A trough of length L has a cross section in the shape of a semicircle with radius r. When filled with water to within a distance h of the top, the volume V of water is:

$$V = L(0.5\pi r^2 - r^2 \arcsin \frac{h}{r} - h\sqrt{r^2 - h^2})$$

Suppose $L = 10\,\text{ft}$, $r = 1\,\text{ft}$, and $V = 12.4\,\text{ft}^3$. Find the depth of water in the trough to within $0.01\,\text{ft}$.

Solution 19

Let d be the depth of the water, so $d = r - h$. Let

$$f(h) = 10(0.5\pi - \arcsin(h) - h\sqrt{1 - h^2}) - 12.4$$

Instead of finding d directly, we find h, also to within $0.01\,\text{ft}$. The number of iteration n needed to approximate h to within 0.01 in $[0, r]$ is:

$$|h - h_n| < \frac{1 - 0}{2^n} < 0.01 \iff n \geq 7$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	−6.258 151 51
2	0	0.5	0.25	−1.639 453 87
3	0	0.25	0.125	0.814 489 029

n	a_n	b_n	p_n	$f(p_n)$
4	0.125	0.25	0.1875	−0.419 946 724
5	0.125	0.1875	0.156 25	0.195 725 903
6	0.156 25	0.1875	0.171 875	−0.112 536 394
7	0.156 25	0.171 875	0.164 062 5	0.041 493 241

So $h \approx 0.1641$, hence $d = r - h \approx 0.8359$.

Exercise 20

A particle starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate ω such that:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega < 0$$

At the end of t seconds, the position of the object is given by:

$$x(t) = -\frac{g}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{x} - \sin \omega t\right)$$

Suppose the particle has moved 1.7 ft in 1 s. Find, to within 10^5, the rate ω at which θ changes. Assume that $g = 32.17\,\mathrm{ft/s^2}$.

Solution 20

As $\omega < 0$, the plane rotates clockwise. After 1 s, the particle still sticks to the plane, so:

$$\theta(1) < \frac{\pi}{2} \iff -\frac{\pi}{2} < \omega < 0$$

After 1 s, the particle has moved 1.7 ft, so that:

$$x(1) = 1.7 = -\frac{32.17}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin \omega t\right)$$

Let

$$f(\omega) = 3.4\omega^2 + 32.17\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin \omega t\right)$$

The root of the above function in $(-\frac{\pi}{2}, 0)$ will be the solution of the problem.

Applying Bisection on f on $[-\frac{\pi}{2}, 0]$ fails as $f(0) = 0$. We need to expand (arbitrarily even) the searching interval a bit for the method to work, and check the solution later on. Hence, we use the interval $[-\frac{\pi}{2}, 1]$.

The number of iteration n needed to approximate ω to within 10^{-5} is:

$$|\omega - \omega_n| < \frac{1 - (-0.5\pi)}{2^n} < 10^{-5} \iff n \geq 18$$

Applying Bisection method generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	$-1.570\,796\,33$	1	$-0.285\,398\,163$	$0.027\,657\,569$
2	$-1.570\,796\,33$	$-0.285\,398\,163$	$-0.928\,097\,245$	$-5.651\,487\,86$
3	$-0.928\,097\,245$	$-0.285\,398\,163$	$-0.606\,747\,704$	$-1.143\,969\,69$
4	$-0.606\,747\,704$	$-0.285\,398\,163$	$-0.446\,072\,934$	$-0.275\,313\,029$
5	$-0.446\,072\,934$	$-0.285\,398\,163$	$-0.365\,735\,549$	$-0.069\,822\,38$
6	$-0.365\,735\,549$	$-0.285\,398\,163$	$-0.325\,566\,856$	$-0.009\,667\,545$
7	$-0.325\,566\,856$	$-0.285\,398\,163$	$-0.305\,482\,51$	$0.011\,587\,981$
8	$-0.325\,566\,856$	$-0.305\,482\,51$	$-0.315\,524\,683$	$0.001\,641\,051$
9	$-0.325\,566\,856$	$-0.315\,524\,683$	$-0.320\,545\,769$	$-0.003\,838\,965$
10	$-0.320\,545\,769$	$-0.315\,524\,683$	$-0.318\,035\,226$	$-0.001\,055\,895$
11	$-0.318\,035\,226$	$-0.315\,524\,683$	$-0.316\,779\,954$	$0.000\,303\,28$
12	$-0.318\,035\,226$	$-0.316\,779\,954$	$-0.317\,407\,59$	$-0.000\,373\,625$
13	$-0.317\,407\,59$	$-0.316\,779\,954$	$-0.317\,093\,772$	$-0.000\,034\,503$
14	$-0.317\,093\,772$	$-0.316\,779\,954$	$-0.316\,936\,863$	$0.000\,134\,556$
15	$-0.317\,093\,772$	$-0.316\,936\,863$	$-0.317\,015\,318$	$0.000\,050\,068$
16	$-0.317\,093\,772$	$-0.317\,015\,318$	$-0.317\,054\,545$	$0.000\,007\,793$
17	$-0.317\,093\,772$	$-0.317\,054\,545$	$-0.317\,074\,159$	$-0.000\,013\,352$
18	$-0.317\,074\,159$	$-0.317\,054\,545$	$-0.317\,064\,352$	$-0.000\,002\,779$

As $-0.317\,064 \in (-\frac{\pi}{2}, 0)$, it is a valid approximation of ω. We conclude that $\omega \approx -0.317\,064$.

2.2 Fixed-Point Iteration

Exercise 1

Use algebraic manipulation to show that each of the following functions has a fixed-point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

a) $g_1(x) = (3 + x - 2x^2)^{1/4}$

b) $g_2(x) = \left(\dfrac{x + 3 - x^4}{2}\right)^{1/2}$

c) $g_3(x) = \left(\dfrac{x + 3}{x^2 + 2}\right)^{1/2}$

d) $g_4(x) = \dfrac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$

Solution 1

a) For $x = p$:

$$g_1(p) = (3 + p - 2p^2)^{\frac{1}{4}} = \left(p^4 - f(p)\right)^{1/4} = |p|$$

So p is a fixed-point of g_1.

b) For $x = p$:

$$
\begin{aligned}
g_2(p) &= \left(\frac{p+3-p^4}{2}\right)^{1/2} \\
&= \left(\frac{2p^2}{2}\right)^{\frac{1}{2}} \\
&= |p|
\end{aligned}
$$

So p is a fixed-point of g_2.

c) For $x = p$:

$$
\begin{aligned}
g_3(p) &= \left(\frac{p+3}{p^2+2}\right)^{1/2} \\
&= \left(\frac{p^4+2p^2}{p^2+2}\right)^{1/2} \\
&= |p|
\end{aligned}
$$

So p is a fixed-point of g_3.

d) For $x = p$:

$$
\begin{aligned}
g_4(p) &= \frac{3p^4+2p^2+3}{4p^3+4p-1} \\
&= \frac{4p^4-(3+p-2p^2)+2p^2+3}{4p^3+4p-1} \\
&= \frac{4p^4+4p^2-p}{4p^3+4p-1} \\
&= p
\end{aligned}
$$

So p is a fixed-point of g_4.

Exercise 2

a) Perform four iterations, if possible, on each of the functions g defined in Exercise 1. Let $p_0 = 1$ and $p_{n+1} = g(p_n)$, for $n = 0, 1, 2, 3$.

b) Which function do you think gives the best approximation to the solution?

Solution 2

a) Applying fixed-point method on the four functions g generates the following table:

n	p_n by g_1	p_n by g_2	p_n by g_3	p_n by g_4
0	1	1	1	1
1	1.189 207 115	1.224 744 871	1.154 700 538	1.142 857 143
2	1.080 057 753	0.993 666 159	1.116 427 41	1.124 481 69
3	1.149 671 431	1.228 568 645	1.126 052 233	1.124 123 164
4	1.107 820 053	0.987 506 429	1.123 638 885	1.124 123 03

b) g_4 gives the best approximation as it generates the smallest difference between p_3 and p_4: $|p_4 - p_3| = -134 \times 10^{-7}$.

Exercise 3

The following four methods are proposed to compute $21^{1/3}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

a) $p_n = \dfrac{20p_{n-1} + {}^{21}/_{p_{n-1}^2}}{21}$

b) $p_n = p_{n-1} - \dfrac{p_{n-1}^3 - 21}{3p_{n-1}^2}$

c) $p_n = p_{n-1} - \dfrac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21}$

d) $p_n = \left(\dfrac{21}{p_{n-1}}\right)^{1/2}$

Solution 3

Applying fixed-point method on the four sequences generate the following table:

n	a)	b)	c)	d)
0	1	1	1	1
1	1.952 380 952	7.666 666 667	0	4.582 575 695
2	2.121 754 174	5.230 203 739	0	2.140 695 143
3	2.242 849 692	3.742 696 919		3.132 075 595
4	2.334 839 673	2.994 853 568		2.589 366 527
5	2.401 093 38	2.777 022 226		2.847 822 274
6	2.465 059 288	2.759 041 866		2.715 521 253
7	2.512 243 463	2.758 924 181		2.780 885 095
8	2.551 057 096	2.758 924 176		2.748 008 838
9	2.583 237 767	2.758 924 176		2.764 398 093
10	2.610 081 445			2.756 191 284
11	2.632 580 301			2.760 291 639
12	2.651 509 504			2.758 240 699

n	a)	b)	c)	d)
13	2.667 484 488			2.759 265 978
14	2.681 000 202			2.758 753 291
15	2.692 458 887			2.759 009 623
16	2.702 190 249			2.758 881 454
17	2.710 466 453			2.758 945 538
18	2.717 513 483			2.758 913 496
19	2.723 519 902			2.758 929 517

Apparently, the speed of convergence is ranked in descending order as follow: b), d), a). c) does not converge.

Exercise 4

The following four methods are proposed to compute $7^{1/5}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

a) $p_n = p_{n-1} - \left(1 + \frac{7-p_{n-1}^5}{p_{n-1}^2}\right)^3$ b) $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{p_{n-1}^2}$

c) $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{5p_{n-1}^4}$ d) $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{12}$

Solution 4

Applying fixed-point method on the four sequences generate the following table:

n	a)	b)	c)	d)
0	1	1	2.2	1
1	343	7	1.819 763 677	1.5
2	-2.25×10^{25}	-335.857	1.583 474 83	1.450 520 833
3		37 884 356	1.489 460 974	1.498 749 661
4			1.476 022 436	1.451 903 535
5			1.475 773 246	1.497 577 067
6			1.475 773 162	1.453 192 29
7			1.475 773 162	1.496 475 364
9				1.454 396 119
8				1.495 438 587
10				1.455 522 81
11				1.494 461 513
12				1.456 579 138
13				1.493 539 533
14				1.457 571 031
15				1.492 668 56

n	a)	b)	c)	d)
16				1.458 803 715
17				1.491 844 948
18				1.459 381 814
19				1.491 065 425

Apparently, the speed of convergence is ranked in descending order as follow: c), d). a) and b) do not converge.

Exercise 5

Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^4 - 3x^2 - 3 = 0$ on $[1, 2]$. Use $p_0 = 1$.

Solution 5

Let $f(x) = x^4 - 3x^2 - 3$. Let p be the root of f in $[1, 2]$. We need to find a function g for which $p = g(p)$ to perform the fixed-point method.

Extract p to RHS gives:

$$p^4 = 3p^2 + 3 \iff |p| = (3p^2 + 3)^{1/4}$$

Then g is chosen as:

$$g(x) = (3x^2 + 3)^{1/4}$$

Applying fixed-point method on g generate the following table:

n	p_n	n	p_n
0	1	4	1.922 847 844
1	1.565 084 58	5	1.937 507 54
2	1.793 572 879	6	1.943 316 93
3	1.885 943 743		

We can try the other obvious option

$$g(x) = \left(\frac{x^4 - 3}{3}\right)^{0.5}$$

which fails on the first iteration. A reasonable explanation for the choice of g is that we need $|g'|$ to be as small as possible. On $[1, 2]$, the $O(x^{0.5})$ of the first choice clearly has an advantage over $O(x^2)$ of the second choice of g.

We conclude that $p \approx 1.943$.

Exercise 6

Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on $[1, 2]$. Use $p_0 = 1$.

Solution 6

Let $f(x) = x^3 - x - 1 = 0$. Let p be the root of f in $[1, 2]$. We need to find a function g for which $p = g(p)$ to perform the fixed-point method.

Extract p to RHS gives:

$$p^3 = p + 1 \iff p = (p + 1)^{1/3}$$

Then g is chosen as:

$$g(x) = (p + 1)^{1/3}$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n
0	1	3	1.322 353 819
1	1.259 921 05	4	1.324 268 745
2	1.312 293 837		

We conclude that $p \approx 1.324$.

Exercise 7

Use Theorem 2.3 (Định lý 2.3 in the accompanying Lectures.pdf) to show that $g(x) = \pi + 0.5 \sin 0.5x$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2}. Use Corollary 2.5 (Hệ quả 2.1) to estimate the number of iterations required to achieve 10^{-2} accuracy, and compare this theoretical estimate to the number actually needed.

Solution 7

From the formula of g:

$$\begin{aligned} g(x) &= \pi + 0.5 \sin 0.5x \\ \Rightarrow g(x) &\in [\pi - 0.5, \pi + 0.5]\, \forall x \end{aligned}$$

Consider the interval $I = [\pi - 0.5, \pi + 0.5] \in [0, 2\pi]$. From the above equations, we know that:

- $g \in CI$

- $g(x) \in I \, \forall x \in I$

According to Theorem 2.3, there exists a fixed point of g on I.
Differentiating g gives:

$$g'(x) = -0.25\cos 0.5x \Rightarrow \left|g'(x)\right| \leq k = 0.25 < 1 \, \forall x$$

Again, according to Theorem 2.3, there exists one and only one fixed point of g on I.
Applying fixed-point method on g, with $p_0 = \pi$, generates the following table:

n	p_n	n	p_n
0	3.141 592 654	2	3.626 048 864
1	3.641 592 654	3	3.626 995 622

Using corollary 2.5, the number of iterations n required to achieve 10^{-2} accuracy is

$$|p_n - p| \leq k^n 0.5 < 10^{-2} \iff n \geq 3$$

which is in line with the number of iteration actually performed.

Exercise 8

Use Theorem 2.3 (Định lý 2.3 in the accompanying Lectures.pdf) to show that $g(x) = 2^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-4}. Use Corollary 2.5 (Hệ quả 2.1) to estimate the number of iterations required to achieve 10^{-4} accuracy, and compare this theoretical estimate to the number actually needed.

Solution 8

From the formula of g:

$$\begin{aligned} g(x) &= 2^{-x} \\ \Rightarrow g'(x) &= -2^{-x}\ln 2 \end{aligned}$$

It is clear that $g \in C^1 R$.
Consider the interval $I = [\frac{1}{3}, 1]$, $I_{open} = (\frac{1}{3}, 1)$:

$$\begin{aligned} & g'(x) < 0 \forall x \in I \\ & \Rightarrow 1 > g(\frac{1}{3}) = 2^{-1/3} \geq g(x) \geq g(1) = 2^{-1} > \frac{1}{3} \\ & \Rightarrow g(x) \in I \, \forall x \in I \end{aligned}$$

So far, we know that:

- $g \in CI$ ($g \in CR$ even)
- $g(x) \in I \, \forall x \in I$

According to Theorem 2.3, there exists a fixed point of g on I.
Consider g':

$$-1 < -\ln 2 \leq g'(x) \leq -\frac{1}{3}\ln 2 < 0 \, \forall x \in I$$
$$\Rightarrow \left|g'(x)\right| \leq k = \ln 2 < 1 \, \forall x \in I$$

Again, according to Theorem 2.3, there exists one and only one fixed point of g on I.

Applying fixed-point method on g, with $p_0 = \frac{2}{3}$, generates the following table:

n	p_n	n	p_n
0	0.666 666 667	5	0.640 746 653
1	0.629 960 525	6	0.641 380 922
2	0.646 194 096	7	0.641 099 006
3	0.638 963 711	8	0.641 224 295
4	0.642 174 057	9	0.641 168 611

Using Corollary 2.5, the number of iterations n required to achieve 10^{-4} accuracy is

$$|p_n - p| \leq k^n \frac{1}{3} < 10^{-4} \iff n \geq 23$$

which is quit a bit higher than the number of iteration actually performed.

Exercise 9

Use a fixed-point iteration method to find an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 12 of Section 2.1.

Solution 9

Let $f(x) = x^2 - 3$, $p > 0$ is a zero of f. Then $p = \sqrt{3}$, and an approximation of p is an approximation of $\sqrt{3}$.

Consider $g(x) = \frac{3}{x}$. It is clear that this is a bad choice, as applying g on any p_0 generates a sequence that jumps between p_0 and $\frac{3}{p_0}$.

From the textbook examples, we choose $g(x) = x - \frac{x^2-3}{x^2}$. Applying fixed-point method on g with $p_0 = 1.5$ generates the following table:

n	p_n	n	p_n
0	1.5	4	1.731 898 58
1	1.833 333 33	5	1.732 074 38
2	1.725 895 32	6	1.732 047 16
3	1.733 041 14		

We conclude that $\sqrt{3} \approx 1.732\,05$. In exercise 12 of section 2.1, 14 iteration is needed, much higher than that of this method.

Exercise 10

Use a fixed-point iteration method to find an approximation to $\sqrt[3]{25}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 13 of Section 2.1.

Solution 10

Let $f(x) = x^3 - 25$, $p > 0$ is a zero of f. Then $p = \sqrt[3]{25}$, and an approximation of p is an approximation of $\sqrt[3]{25}$.

We choose $g(x) = x - \frac{x^3-25}{x^3}$. Applying fixed-point method on g with $p_0 = 2.5$ generates the following table:

n	p_n	n	p_n
0	2.5	3	2.923 783 69
1	3.1	4	2.924 023 86
2	2.939 179 62	5	2.924 017 58

We conclude that $\sqrt[3]{25} \approx 2.924\,02$. In exercise 13 of section 2.1, 14 iteration is needed, much higher than that of this method.

Exercise 11

For each of the following equations, determine an interval $[a, b]$ on which fixed-point iteration converges. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and perform the calculations.

a) $x = \dfrac{2 - e^x + x^2}{3}$

b) $x = \dfrac{5}{x^2} + 2$

c) $x = (e^x/3)^{1/2}$

d) $x = 5^{-x}$

e) $x = 6^{-x}$

f) $x = 0.5(\sin x + \cos x)$

Solution 11

a) Let

$$
\begin{aligned}
& & g(x) &= \frac{2-e^x+x^2}{3} \\
&\Rightarrow & g'(x) &= \frac{2x-e^x}{3} \\
&\Rightarrow & g''(x) &= \frac{2-e^x}{3}
\end{aligned}
$$

It is clear that g is continuous in $\mathbb{R}$.

Consider g'':

- $g''(x) > 0 \iff x < \ln 2$
- $g''(x) = 0 \iff x = \ln 2$
- $g''(x) < 0 \iff x > \ln 2$

So, $\max g'(x) = g'(\ln 2) = \dfrac{\ln 4 - 2}{3} < 0$. So g is monotonically decreasing in $\mathbb{R}$.

Consider the interval $I = [0, 1]$:

$$
\begin{aligned}
1 > g(0) = \frac{1}{3} > &g(x) > g(1) = \frac{3-e}{3} > 0 \, \forall x \in I \\
\Rightarrow &g(x) \in I \, \forall x \in I
\end{aligned}
$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 0.5$ generates the following table:

n	p_n	n	p_n
0	0.5	5	0.257 265 636
1	0.200 426 243	6	0.257 598 985
2	0.272 749 065	7	0.257 512 455
3	0.253 607 157	8	0.257 534 914
4	0.258 550 376	9	0.257 529 084

We conclude that the fixed point $p \approx 0.257\,529$.

b) Let

$$g = \frac{5}{x^2} + 2$$

Consider the interval $I = [2.5, 3]$. $0 \notin I$, so g is continuous in I.

x^2 is monotonically increasing in I, so g is monotonically decreasing in I. So that:

$$3 > g(2.5) = 2.8 > g(x) > g(3) = {}^{23}/_{9} > 2.5 \, \forall x \in I$$
$$\Rightarrow g(x) \in I \, \forall x \in I$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 2.75$ generates the following table:

n	p_n	n	p_n	n	p_n
0	2.75	6	2.691 710 92	12	2.690 666 91
1	2.661 157 02	7	2.690 101 82	13	2.690 637 46
2	2.706 039 5	8	2.690 927 64	14	2.690 652 58
3	2.682 812 93	9	2.690 503 63	15	2.690 644 82
4	2.694 687 08	10	2.690 721 29		
5	2.688 578 29	11	2.690 609 54		

We conclude that the fixed point $p \approx 2.690\,645$.

c) Let

$$g(x) = \left(\frac{e^x}{3}\right)^{1/2}$$

It is clear that g is continuous in $\mathbb{R}$.

g is monotonically increasing in $\mathbb{R}$. Consider the interval $I = [0, 1]$:

$$0 < g(0) = \frac{1}{\sqrt{3}} < g(x) < g(1) = \sqrt{\frac{e}{3}} < 1$$
$$\Rightarrow g(x) \in I \, \forall x \in I$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 0.5$ generates the following table:

n	p_n	n	p_n	n	p_n
0	0.5	5	0.903 281 143	10	0.909 876 791
1	0.741 332 42	6	0.906 952 163	11	0.909 948 068
2	0.836 407 007	7	0.908 618 411	12	0.909 980 498
3	0.877 127 74	8	0.909 375 718	13	0.909 995 254
4	0.895 169 428	9	0.909 720 122	14	0.910 001 967

We conclude that the fixed point $p \approx 0.910\,002$.

d) Let $g(x) = 5^{-x}$. It is clear that g is continuous in $\mathbb{R}$.

5^x is monotonically increasing in $\mathbb{R}$, so g is monotonically decreasing in $\mathbb{R}$.

Consider the interval $I = [0, 1]$:

$$\begin{aligned} 0 < g(1) = 0.2 <&g(x) < g(0) = 1 \\ \Rightarrow&g(x) \in I \, \forall x \in I \end{aligned}$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 0.5$ generates the following table:

n	p_n	n	p_n	n	p_n
0	0.5	11	0.468 245 559	22	0.469 685 261
1	0.447 213 595	12	0.470 663 369	23	0.469 574 052
2	0.486 867 866	13	0.468 835 429	24	0.469 658 106
3	0.456 766 207	14	0.470 216 753	25	0.469 594 575
4	0.479 439 843	15	0.469 172 549	26	0.469 642 593
5	0.462 259 591	16	0.469 961 695	27	0.469 606 3
6	0.475 219 673	17	0.469 365 184	28	0.469 633 731
7	0.465 409 992	18	0.469 816 013	29	0.469 612 998
8	0.472 816 23	19	0.469 475 247	30	0.469 628 669
9	0.467 213 774	20	0.469 732 798	31	0.469 616 824
10	0.471 445 6	21	0.469 538 128	32	0.469 625 777

We conclude that the fixed point $p \approx 0.469\,626$.

e) Let $g(x) = 6^{-x}$. It is clear that g is continuous in $\mathbb{R}$.

6^x is monotonically increasing in $\mathbb{R}$, so g is monotonically decreasing in $\mathbb{R}$.

Consider the interval $I = [0, 1]$:

$$\begin{aligned} 0 < g(1) = \frac{1}{6} <&g(x) < g(0) = 1 \\ \Rightarrow&g(x) \in I \, \forall x \in I \end{aligned}$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = 0.5$ generates the following table:

n	p_n	n	p_n	n	p_n
0	0.5	15	0.446 190 464	30	0.448 132 603
1	0.408 248 29	16	0.449 568 975	31	0.448 007 263
2	0.481 194 974	17	0.446 855 739	32	0.448 107 887
3	0.422 238 208	18	0.449 033 402	33	0.448 027 103
4	0.469 282 988	19	0.447 284 756	34	0.448 091 958

n	p_n	n	p_n	n	p_n
5	0.431 347 074	20	0.448 688 365	35	0.448 039 891
6	0.461 686 032	21	0.447 561 363	36	0.448 081 691
7	0.437 258 678	22	0.448 466 044	37	0.448 048 133
8	0.456 821 582	23	0.447 739 682	38	0.448 075 074
9	0.441 086 448	24	0.448 322 78	39	0.448 053 445
10	0.453 699 216	25	0.447 854 63	40	0.448 070 809
11	0.443 561 035	26	0.448 230 453	41	0.448 056 869
12	0.451 692 029	27	0.447 928 723	42	0.448 068 06
13	0.445 159 128	28	0.448 170 951	43	0.448 059 076
14	0.450 400 504	29	0.447 976 481		

We conclude that the fixed point $p \approx 0.448\,059$.

f) Let $g(x) = 0.5(\sin x + \cos x)$. It is clear that g is continuous in $\mathbb{R}$. Manipulating g gives:

$$\begin{aligned}
\sin x + \cos x &= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) \\
&= \sqrt{2}\left(\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x\right) \\
&= \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) \\
\Rightarrow g(x) &= 0.5(\sin x + \cos x) \\
&= \frac{1}{\sqrt{2}}\sin\left(x + \frac{\pi}{4}\right)
\end{aligned}$$

Consider the interval $I = [0, \frac{\pi}{4}]$. $sinx$ is monotonically increasing in $[0, \frac{\pi}{2}]$, so $\sin x + \frac{\pi}{4}$ also is monotonically increasing in I. It follows that:

$$\begin{aligned}
0 < g(0) = 0.5 < g(x) <& g(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} < \frac{\pi}{4} \\
\Rightarrow& g(x) \in I \, \forall x \in I
\end{aligned}$$

So, I is an interval in which a fixed point p of g exists. Applying fixed-point method on g with $p_0 = \frac{\pi}{8}$ generates the following table:

n	p_n	n	p_n
0	0.392 699 082	4	0.704 799 153
1	0.653 281 482	5	0.704 811 271
2	0.700 944 543	6	0.704 811 96
3	0.704 586 59		

We conclude that the fixed point $p \approx 0.704\,812$.

Exercise 12

For each of the following equations, use the given interval or determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and perform the calculations.

a) $2 + \sin x - x = 0$ on $[2, 3]$

b) $x^3 - 3x - 5 = 0$ on $[2, 3]$

c) $3x^2 - e^x = 0$

d) $x - \cos x = 0$

Solution 12

a) Let $I = [2, 3]$ and

$$\begin{aligned} g(x) &= \sin x + 2 \\ \Rightarrow g'(x) &= \cos x \end{aligned}$$

A fixed point p of g is also a root of the problem.

Consider g. It is clear that g is continuous on $\mathbb{R}$. $\sin x$ is monotonically decreasing in I, so that:

$$2 < g(3) = \sin 3 + 2 < g(x) < g(2) = \sin 2 + 2 < 3$$

Consider g'. $\cos x$ is monotonically decreasing in I, so that:

$$\begin{gathered} \cos 3 \leq g'(x) \leq \cos 2 < 0 \, \forall x \in I \\ \Rightarrow \left|g'(x)\right| \leq k = -\cos 3 < 1 \end{gathered}$$

Therefore, all the conditions in Corollary 2.5 hold. Using Corollary 2.5, with $p_0 = 2.5$, the number of iteration n required to obtain approximations accurate to within 10^{-5} is:

$$|p_n - p| \leq k^n 0.5 < 10^{-5} \iff n \geq 1076$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n	n	p_n
0	2.5	18	2.552 225 43	36	2.554 123 46
1	2.598 472 14	19	2.555 835 11	37	2.554 256 29
2	2.516 809 97	20	2.552 830 8	38	2.554 145 73
3	2.584 921 02	21	2.555 331 77	39	2.554 237 76
4	2.528 363 28	22	2.553 250 15	40	2.554 161 15
5	2.575 511 41	23	2.554 982 97	41	2.554 224 92
6	2.536 328 7	24	2.553 540 68	42	2.554 171 84

n	p_n	n	p_n	n	p_n
7	2.568 979 15	25	2.554 741 28	43	2.554 216 02
8	2.541 830 51	26	2.553 741 95	44	2.554 179 25
9	2.564 446 15	27	2.554 573 8	45	2.554 209 86
10	2.545 634 87	28	2.553 881 4	46	2.554 184 38
11	2.561 301 68	29	2.554 457 76	47	2.554 205 59
12	2.548 267 3	30	2.553 978 01	48	2.554 187 93
13	2.559 121 11	31	2.554 377 35	49	2.554 202 63
14	2.550 089 61	32	2.554 044 95	50	2.554 190 4
15	2.557 609 33	33	2.554 321 64	51	2.554 200 58
16	2.551 351 48	34	2.554 091 33	52	2.554 192 1
17	2.556 561 41	35	2.554 283 04		

So one solution of the problem is $p \approx 2.554\,192$.

b) Let $I = [2, 3]$ and

$$\begin{aligned} g(x) &= \sqrt[3]{2x+5} \\ \Rightarrow g'(x) &= \frac{2}{3}(2x+5)^{-2/3} \end{aligned}$$

A fixed point p of g is also a solution of the problem.

Consider g. It is clear that g is continuous and monotonically increasing on $\mathbb{R}$, so that:

$$\begin{aligned} 2 < g(2) = \sqrt[3]{9} < & g(x) < g(3) = \sqrt[3]{11} < 3 \\ \Rightarrow & g(x) \in I \, \forall x \in I \end{aligned}$$

Consider g'. Since $-2/3 < 0$ and $I > 0$, $g'(x)$ is monotonically decreasing in I, so that:

$$\begin{aligned} g'(2) = \frac{2}{9\sqrt[3]{3}} \geq g'(x) \geq g'(3) = \frac{2}{3\sqrt[3]{121}} \\ \Rightarrow \left|g'(x)\right| \leq k = \frac{2}{9\sqrt[3]{3}} < 1 \end{aligned}$$

Therefore, all the conditions in Corollary 2.5 hold. Using Corollary 2.5, with $p_0 = 2.5$, the number of iteration n required to obtain approximations accurate to within 10^{-5} is:

$$|p_n - p| \leq k^n 0.5 < 10^{-5} \iff n \geq 6$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n
0	2.5	4	2.094 760 55
1	2.154 434 69	5	2.094 583 25
2	2.103 612 03	6	2.094 556 31
3	2.095 927 41	7	2.094 552 22

So one solution of the problem is $p \approx 2.094\,552$.

c) Let $I = [3, 4]$ and

$$g(x) = \ln 3x^2 = 2\ln x + \ln 3$$
$$\Rightarrow g'(x) = \frac{2}{x}$$

A fixed point p of g is also a solution of the problem.

Consider g. It is clear that g is continuous and monotonically increasing on I, so that:

$$3 < g(3) = \ln 27 < g(x) < g(4) = \ln 48 < 4$$
$$\Rightarrow g(x) \in I \,\forall x \in I$$

Consider g'. Since $I > 0$, $g'(x)$ is monotonically decreasing in I, so that:

$$g'(3) = \frac{2}{3} \geq g'(x) \geq g'(4) = \frac{1}{2}$$
$$\Rightarrow \left|g'(x)\right| \leq k = \frac{2}{3} < 1$$

Therefore, all the conditions in Corollary 2.5 hold. Using Corollary 2.5, with $p_0 = 3.5$, the number of iteration n required to obtain approximations accurate to within 10^{-5} is:

$$|p_n - p| \leq k^n 0.5 < 10^{-5} \iff n \geq 27$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n	n	p_n
0	3.5	6	3.727 177 12	12	3.732 939 23
1	3.604 138 23	7	3.729 914 58	13	3.733 004 13
2	3.662 777 67	8	3.731 382 95	14	3.733 038 9
3	3.695 055 86	9	3.732 170 15	15	3.733 057 53
4	3.712 603 63	10	3.732 592 04	16	3.733 067 51
5	3.722 079 13	11	3.732 818 1		

So one solution of the problem is $p \approx 3.733\,068$.

d) Let $I = [0, 1]$ and

$$\begin{aligned} g(x) &= \cos x \\ \Rightarrow g'(x) &= -\sin x \end{aligned}$$

A fixed point p of g is also a solution of the problem.

Consider g. It is clear that g is continuous and monotonically decreasing on I, so that:

$$\begin{aligned} 1 = g(0) \geq& g(x) \geq g(1) = \cos 1 > 0 \\ \Rightarrow& g(x) \in I \, \forall x \in I \end{aligned}$$

Consider g'. Since $I > 0$, $g'(x)$ is monotonically decreasing in I, so that:

$$\begin{aligned} g'(0) = 0 \geq g'(x) \geq g'(1) = -\sin 1 \\ \Rightarrow \left|g'(x)\right| \leq k = \sin 1 < 1 \end{aligned}$$

Therefore, all the conditions in Corollary 2.5 hold. Using Corollary 2.5, with $p_0 = 0.5$, the number of iteration n required to obtain approximations accurate to within 10^{-5} is:

$$|p_n - p| \leq k^n 0.5 < 10^{-5} \iff n \geq 63$$

Applying fixed-point method on g generates the following table:

n	p_n	n	p_n	n	p_n
0	0.5	10	0.735 006 309	20	0.739 006 78
1	0.877 582 562	11	0.741 826 523	21	0.739 137 911
2	0.639 012 494	12	0.737 235 725	22	0.739 049 581
3	0.802 685 101	13	0.740 329 652	23	0.739 109 081
4	0.694 778 027	14	0.738 246 238	24	0.739 069 001
5	0.768 195 831	15	0.739 649 963	25	0.739 096
6	0.719 165 446	16	0.738 704 539	26	0.739 077 813
7	0.752 355 759	17	0.739 341 452	27	0.739 090 064
8	0.730 081 063	18	0.738 912 449	28	0.739 081 812
9	0.745 120 341	19	0.739 201 444		

So one root of the problem is $p \approx 0.739\,082$.

Exercise 13

Find all the zeros of $f(x) = x^2 + 10\cos x$ by using the fixed-point iteration method for an appropriate iteration function g. Find the zeros accurate to within 10^{-4}.

Solution 13

Consider $f = 0$. Since $x^2 \geq 0$, $\cos x$ must be negative for the equation to hold, so that:

$$x \in I_k = [\frac{\pi}{2} + k2\pi, \frac{3\pi}{2} + k2\pi]\, \forall k \in \mathbb{N} \tag{1}$$

Also, since $10\cos x \in [-10, 0]$:

$$x \in [-\sqrt{10}, \sqrt{10}] \tag{2}$$

Combining (1) and (2) gives:

$$x \in I = I_a \cup I_b \text{ where } I_a = [-\sqrt{10}, -\frac{\pi}{2}] \text{ and } I_b = [\frac{\pi}{2}, \sqrt{10}]$$

As x^2 and $\cos x$ take Oy as a symmetry axis, each zero z_b of f in I_b results in another zero $z_a = -z_b$ in I_a. Hence, from now on, we just need to examine on I_b.

Differentiating f gives:

$$f'(x) = 2x - 10\sin x$$

x is monotonically increasing on I_b, $\sin x$ is monotonically decreasing on I_b. It follows that f' is monotonically increasing on I_b, which means:

$$f'(\frac{\pi}{2}) = \pi - 10 \leq f'(x) \leq f'(\sqrt{10}) = 2\sqrt{10} - 10\sin\sqrt{10}$$

Combining with the fact that f' is continuous on I_b, according to Intermediate Value Theorem, f' has one zero in I_b. It follows that f has at most two zeros in I_b.

Let

$$g(x) = x - \frac{-10\cos x}{x^2} + 1 = x + \frac{10\cos x}{x^2} + 1$$

A fixed point of g is also a zero of f. *Try* applying fixed-point method on g with several p_0, we found two fixed points:

- $p_0 = \frac{\pi}{2}$ generates the following table:

n	p_n	n	p_n	n	p_n
0	1.570 796 33	4	1.953 548 67	8	1.968 593 28
1	2.570 796 33	5	1.974 930 8	9	1.968 974 39
2	2.297 575 29	6	1.966 757 33	10	1.968 836 22
3	2.038 843 43	7	1.969 648 71	11	1.968 886 24

- $p_0 = -\sqrt{10}$ generates the following table:

n	p_n
0	$-3.162\,277\,66$
1	$-3.162\,063\,73$
2	$-3.161\,989\,49$

The second fixed point is interesting. It is indeed a fixed point of g, a zero of f, but it belongs to I_a. Due to the symmetry property, we conclude that f has 4 zeros: $\pm 1.968\,89$ and $\pm 3.161\,99$.

Exercise 14

Use a fixed-point iteration method to determine a solution accurate to within 10^{-4} for $x = \tan x$, for $x \in [4, 5]$.

Solution 14

Let

$$g(x) = x - \sqrt[3]{\frac{\tan x}{x}} + 1$$

A fixed point p of g is also a solution of the problem. Applying fixed-point method on g generates the following table:

n	p_n	n	p_n	n	p_n
0	4	4	$4.495\,344\,11$	8	$4.493\,529\,55$
1	$4.338\,504\,07$	5	$4.492\,429\,47$	9	$4.493\,349\,61$
2	$4.500\,975\,94$	6	$4.493\,893\,01$	10	$4.493\,439\,23$
3	$4.489\,378\,73$	7	$4.493\,167\,7$		

So $p \approx 4.493\,44$ is a solution of the problem in $[4, 5]$.

Exercise 15

Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $2 \sin \pi x + x = 0$ on $[1, 2]$. Use $p_0 = 1$.

Solution 15

Consider f:

$$f(x) = 0$$

$$
\begin{aligned}
\iff \quad && 2\sin\pi x &= -x \\
\iff \quad && \pi x &= \arcsin -0.5x + k2\pi \ (k \in \mathbb{N}) \\
\iff \quad && x &= \frac{\arcsin -0.5x}{\pi} + 2k
\end{aligned}
$$

Let

$$g(x) = \frac{\arcsin -0.5x}{\pi} + 2$$

arcsin is chosen as it "behaves" nicer than normal sin. Since arcsin returns values in principal branch $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we need to use $k = 1$ to shift the value to cover $[1, 2]$.

A fixed point p of g is also a solution of the problem. Applying fixed-point method on g generates the following table:

n	p_n	n	p_n
0	1	3	1.696 498
1	1.833 333 33	4	1.677 657 06
2	1.630 869 25	5	1.683 240 99

So $p \approx 1.683$ is a solution of the problem in $[1, 2]$.

Exercise 16

Let A be a given positive constant and $g(x) = 2x - Ax^2$.

a) Show that if fixed-point iteration converges to a nonzero limit, then the limit is $p = 1/A$, so the inverse of a number can be found using only multiplications and subtractions.

b) Find an interval about $1/A$ for which fixed-point iteration converges, provided p_0 is in that interval.

Solution 16

a) If fixed-point iteration converges to a nonzero limit p, then:

$$
\begin{aligned}
p &= \lim_{n\to\infty} p_n \\
&= \lim_{n\to\infty} g(p_{n-1}) \\
&= \lim_{n\to\infty} \left(2p_{n-1} - Ap_{n-1}^2\right) \\
&= 2p - Ap^2 \\
\iff p &= Ap^2 \iff p = \frac{1}{A}
\end{aligned}
$$

b) We try to find $\delta > 0$ such that fixed-point method converges on $I = [1/A - \delta, 1/A + \delta]$ using Fixed Point Theorem.

The condition that g is continuous on I is satisfied with any δ.

Consider g:

$$g(x) = -Ax^2 + 2x = -A\left(x - \frac{1}{A}\right)^2 + \frac{1}{A}$$

So $x = \frac{1}{A}$ is the axis of symmetry for g.

Differentiating g gives:

$$g'(x) = 2 - 2Ax$$

It follows that:

- $g'(x) < 0 \iff x > \frac{1}{A}$
- $g'(x) = 0 \iff x = \frac{1}{A}$
- $g'(x) > 0 \iff x < \frac{1}{A}$

Combining with the fact that $x = \frac{1}{A}$ is the symmetry axis of g gives:

$$\begin{aligned} g\left(\frac{1}{A} + \delta\right) = g\left(\frac{1}{A} - \delta\right) = g\left(\frac{1}{A} \pm \delta\right) \leq g(x) \leq g\left(\frac{1}{A}\right) \ \forall x \in I \\ \iff \frac{2}{A} - A\delta^2 \leq g(x) \leq \frac{1}{A} \end{aligned}$$

Then, to satisfy the condition that $g(x) \in I \, \forall x \in I$, δ must satisfy the following:

$$\begin{aligned} & & \frac{2}{A} - A\delta^2 \geq \frac{1}{A} - \delta \\ \iff & & (A\delta)^2 - A\delta - 1 \leq 0 \\ \iff & & 0 < \delta \leq \frac{1+\sqrt{5}}{2A} \text{ (as } \delta > 0) \qquad (1) \end{aligned}$$

Consider g'. g' is monotonically decreasing on $\mathbb{R}$, so:

$$\begin{aligned} g'\left(\frac{1}{A} - \delta\right) = 2A\delta \geq g'(x) \geq g'\left(\frac{1}{A} - \delta\right) = -2A\delta \\ \iff \left|g'(x)\right| \leq 2A\delta \text{ (equal sign only at either end)} \qquad (2) \end{aligned}$$

Then, to satisfy the condition that $\left|g'(x)\right| < 1 \, \forall x \in I_{open} = (1/A - \delta, 1/A + \delta)$, δ must satisfy the following:

$$2A\delta \le 1 \iff \delta \le \frac{1}{2A}$$

From (1) and (2):

$$0 < \delta < \frac{1}{2A}$$

As all the conditions needed for Fixed Point Theorem hold, we conclude that for any $\delta \in (0, \frac{1}{2A}]$, applying fixed-point method on g with $p_0 \in I$ converges to the fixed point.

Exercise 17

Find a function g defined on $[0,1]$ that satisfies none of the hypotheses of Theorem 2.3 but still has a unique fixed point on $[0,1]$.

Solution 17

Let $I = [0,1]$, $g = \dfrac{1}{x+0.5}$.

Consider g. g is defined on $\mathbb{R} \setminus \{-0.5\}$, so it is defined on I.

$g(x) > 1 \, \forall x \in [-0.5, 0.5]$, so the condition that $g(x) \in I \, \forall x \in I$ does not hold.

Differentiating g gives:

$$g'(x) = -\frac{1}{(x+0.5)^2} < -1 \iff x \in (-1.5, 0.5) \setminus \{-0.5\}$$

So the condition that $\left|g'(x)\right| < 1 \, \forall x \in I$ does not hold.

Yet, g has a fixed point at $x = \dfrac{\sqrt{17}-1}{4}$.

Exercise 18

a) Show that Theorem 2.2 is true if the inequality $\left|g'(x)\right| \le k$ is replaced by $g'(x) \le k$, for all $x \in (a,b)$. [Hint: Only uniqueness is in question.]

b) Show that Theorem 2.3 may not hold if inequality $\left|g'(x)\right| \le k$ is replaced by $g'(x) \le k$.

Solution 18

a) Where the fuck is Theorem 2.2 in the fucking book?

b) In the proof of Theorem 2.3, if $\left|g'(x) \le k\right|$ is replaced with $g'(x) \le k$, then there is a chance that $g'(\xi) = -1$. In that case, the assumption is no longer a contradiction, therefore the proof is invalid, and the theorem doesn't hold.

Exercise 19

a) Use Theorem 2.4 (Định lí 2.5 in the accompanying Lectures.pdf) to show that the sequence defined by:

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \text{ for } n \leq 1$$

converges to $\sqrt{2}$ whenever $x_0 > \sqrt{2}$.

b) Use the fact that $0 < (x_0 - \sqrt{2})^2$ whenever $x_0 \neq \sqrt{2}$ to show that if $0 < x_0 < \sqrt{2}$, then $x_1 > \sqrt{2}$.

c) Use the above results to show that the sequence in (a) converges to $\sqrt{2}$ whenever $x_0 > 0$.

Solution 19

a) Let g be the function that generates the sequence $\{x_n\}$:

$$\begin{aligned} g(x) &= \frac{x}{2} + \frac{1}{x} = \frac{x^2+2}{2x} \\ \Rightarrow g'(x) &= \frac{1}{2} - \frac{1}{x^2} = \frac{x^2-2}{2x^2} \end{aligned}$$

Consider $I = [\sqrt{2}, b]$, for any $b > \sqrt{2}$. It is clear that g and g' exists on I. Since $g'(x) \leq 0 \,\forall x \in I$, g is monotonically increasing on I.

Consider g'. x^2 is strictly increasing on I, so g' is strictly decreasing on I, therefore:

$$\begin{aligned} \frac{1}{2} > g'(x) &\leq g'(\sqrt{2}) = 0 \,\forall x \in I \\ \Rightarrow &\left|g'(x)\right| < 1 \,\forall x \in I \end{aligned}$$

Let

$$f(x) = g(x) - x = \frac{1}{x} - \frac{x}{2}$$

$^1/_x$ is strictly decreasing on I, and so is $-x$. Therefore, f is strictly decreasing on I, so:

$$f(\sqrt{2}) = 0 \leq f(x) \,\forall x \in I$$

In other words, $g(x) \leq x \,\forall x \in I$. It means that for any b, $g(b) < b$. Combining with the fact that $g(\sqrt{2}) = \sqrt{2}$, it is guaranteed that:

$$g(x) \in I \, \forall x \in I$$

All the conditions of Theorem 2.4 hold, so we can apply it here: for any $x_0 \in I$, applying fixed-point method on g converges to the unique fixed point in I, using any $x_0 \in I$.

Trivially, $\sqrt{2}$ is a fixed point of g, therefore it must be the unique fixed point on I.

We can conclude that for any $x_0 > \sqrt{2}$, the sequence converges to $\sqrt{2}$.

b) When $0 < x < \sqrt{2}$, $g'(x) < 0$, which means g is monotonically decreasing. Applying this on $0 < x_0 < \sqrt{2}$ gives:

$$x_1 = g(x_0) > g(\sqrt{2}) = \sqrt{2}$$

c) We have:

- If $x_0 > \sqrt{2}$: proven.
- If $x_0 = \sqrt{2}$: it is exactly the fixed point.
- If $0 < x_0 < \sqrt{2}$: $x_1 = g(x_0) > \sqrt{2}$, then from x_1 onwards, the sequence converges to $\sqrt{2}$, as proven with the case $x_0 > \sqrt{2}$.

Therefore, we can conclude that the sequence converges to $\sqrt{2}$ whenever $x_0 > 0$.

Exercise 20

a) Show that if A is any positive number, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \text{ for } n \geq 1$$

converges to $\sqrt{A}$ whenever $x_0 > 0$.

b) What happens if $x_0 < 0$?

Solution 20

a) Let

$$\begin{aligned} g(x) &= \frac{x}{2} + \frac{A}{2x} = \frac{x^2 + A}{2x} \\ \Rightarrow g'(x) &= \frac{1}{2} - \frac{A}{2x^2} = \frac{x^2 - A}{2x^2} \end{aligned}$$

Trivially, we can find out that $\sqrt{A}$ is a fixed point of g.

Let

$$f(x) = g(x) - x = \frac{A}{2x} - \frac{x}{2} = \frac{A - x^2}{2x}$$
$$\Rightarrow f'(x) = -\frac{A}{2x^2} - \frac{1}{2} = -\frac{x^2 + A}{2x^2}$$

Since $f'(x) < 0 \, \forall x \neq 0$, $f(x)$ is monotonically increasing when $x > 0$.

Consider the sign of g':

- $g'(x) < 0 \iff |x| < \sqrt{A}$
- $g'(x) = 0 \iff |x| = \sqrt{A}$
- $g'(x) > 0 \iff |x| > \sqrt{A}$

If $x > \sqrt{A}$, then:

- $g' > 0$, which means g is monotonically increasing. It follows that:
$$g(x) > g(\sqrt{A}) = \sqrt{A}$$
- $f(x) < f(\sqrt{A}) = 0$, which means $g(x) < x$, making $\{x_n\}$ a decreasing sequence.

From both of the above, we know that $\{x_n\}$ is a lower-bounded decreasing sequence, and therefore must converge:

$$\begin{aligned} x &= \lim_{n\to\infty} x_n \\ &= \lim_{n\to\infty} g(x_{n-1}) \\ &= \lim_{n\to\infty} \frac{x_{n-1}}{2} + \frac{A}{2x_{n-1}} \\ &= \frac{x}{2} + \frac{A}{2x} \\ \iff x &= \sqrt{A} \end{aligned}$$

So, for all $x_0 > \sqrt{A}$, the sequence converges to $\sqrt{A}$.

If $x = \sqrt{A}$, then $g(x) = x = \sqrt{A}$. Hence $x_n = \sqrt{A} \, \forall n \geq 0$. So, for $x_0 = \sqrt{A}$, the sequence converges to $\sqrt{A}$.

If $0 < x < \sqrt{A}$, then $g' < 0$, which means g is monotonically decreasing. It follows that:

$$g(x) > g(\sqrt{A}) = \sqrt{A}$$

So, for $0 < x_0 < \sqrt{A}$, $x_1 = g(x_0) > \sqrt{A}$, then from x_1 onwards, the sequence converges to $\sqrt{A}$, as proven with the case $x_0 > \sqrt{A}$.

We can conclude that the sequence $\{x_n\}$ converges to $\sqrt{2}$ whenever $x_0 > 0$.

b) If $x_0 < 0$, then similar to the above proof, we conclude that the sequence converges to $-\sqrt{A}$.

Exercise 21

Replace the assumption in Theorem 2.4 that "a positive number $k < 1$ exists with $\left|g(x)\right| \leq k$" with "g satisfies a Lipschitz condition on the interval $[a, b]$ with Lipschitz constant $L < 1$" (See Exercise 27, Section 1.1.) Show that the conclusions of this theorem are still valid.

Solution 21

g satisfies a Lipschitz condition on the interval $[a, b]$ with Lipschitz constant $L < 1$ means that:

$$\frac{g(x_1) - g(x_2)}{x_1 - x_2} \leq L \, \forall x_1, x_2 \in [a, b] \tag{*}$$

In the proof of Theorem 2.4, we see that:

$$|p - p_n| = \left|g(p) - g(p_{n-1})\right|$$

From the previous section of the proof, we already proved that p and p_{n-1} is in $[a, b]$. Applying (*) with $x_1 = p$, $x_2 = p_{n-1}$ gives:

$$|p - p_n| = \left|g(p) - g(p_{n-1})\right| \leq L|p - p_{n-1}|$$

Then the proof proceeds normally, replacing k with L.

Exercise 22

Suppose that g is continuously differentiable on some interval (c, d) that contains the fixed point p of g. Show that if $\left|g'(p)\right| < 1$, then there exists a $\delta > 0$ such that if $|p_0 - p| \leq \delta$, then the fixed-point iteration converges.

Solution 22

Since p is a fixed point in (c, d) of g, $g(p) = p$.

Since g' is continuous at p, according to the definition of continuity and limit, for every $\varepsilon > 0$, there exist $\delta > 0$ such that:

$$\begin{aligned} &\left|g'(x) - g'(p)\right| < \varepsilon \, \forall x \in D = [p - \delta, p + \delta] \\ \iff &\; g'(x) \in E = [g'(p) - \varepsilon, g'(p) + \varepsilon] \, \forall x \in D \end{aligned}$$

We can always choose a ε such that $E \subset (-1, 1)$. Then the proof proceeds normally, replacing $[a, b]$ with E.

Exercise 23

An object falling vertically through the air is subjected to viscous resistance as well as to the force of gravity. Assume that an object with mass m is dropped from a height s_0 and that the height of the object after t seconds is:

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}(1 - e^{-kt/m})$$

where $g = 32.17\,\text{ft/s}^2$ and k represents the coefficient of air resistance in lb/s. Suppose $s_0 = 300\,\text{ft}$, $m = 0.25\,\text{lb}$, and $k = 0.1\,\text{lb/s}$. Find, to within $0.01\,\text{s}$, the time it takes this quarter-pounder to hit the ground.

Solution 23

Replacing symbols in $s(t)$ with number gives:

$$s(t) = 501.0625 - 80.425t - 201.0625e^{-0.4t}$$

Let

$$g(t) = \frac{1}{80.425}(501.0625 - 201.0625e^{-0.4t})$$

A fixed point p of g is also a root of $s(t) = 0$, which is the time it takes the quarter-pounder to hit the ground.

Applying fixed-point method on g with $p_0 = 3$ generates the following table:

n	p_n	n	p_n
0	3	3	5.998 865 94
1	5.477 197 87	4	6.003 285 61
2	5.950 637 4		

We conclude that it takes approximately $6.003\,\text{s}$ for the quarter-pounder to hit the ground.

Exercise 24

Let $g \in C^1[a,b]$ and p be in (a,b) with $g(p) = p$ and $\left|g'(p)\right| > 1$. Show that there exists a $\delta > 0$ such that if $0 < |p_0 - p| < \delta$, then $|p_0 - p| < |p_1 - p|$. Thus, no matter how close the initial approximation p_0 is to p, the next iterate p_1 is farther away, so the fixed-point iteration does not converge if $p_0 \neq p$.

Solution 24

This problem is similar to Exercise 22.

Since g' is continuous at p, according to the definition of continuity and limit, for every $\varepsilon > 0$, there exist $\delta > 0$ such that:

$$\begin{aligned} &\left|g'(x) - g'(p)\right| < \varepsilon \, \forall x \in D = [p - \delta, p + \delta] \\ \iff &g'(x) \in E = [g'(p) - \varepsilon, g'(p) + \varepsilon] \, \forall x \in D \end{aligned}$$

We can always choose a ε such that $E \subset (1, \infty)$.

If $p_0 \in D$, then according to Mean Value Theorem, there exist a $\xi \in D$ such that:

$$|p_1 - p| = \left|g(p_0) - g(p)\right| = \left|g'(\xi)\right| |p_0 - p| > |p_0 - p|$$

2.3 Newton's Method and Its Extensions

Exercise 1

Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.

Solution 1

$f'(x) = 2x$. Therefore, $p_1 = 3.5$, $p_2 = 2.607\,142$.

Exercise 2

Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find p_2. Could $p_0 = 0$ be used?

Solution 2

$f'(x) = -3x^2 + \sin x$. Therefore, $p_1 = -0.880\,333$, $p_2 = -0.865\,684$.

$p_0 = 0$ can't be used, as $f'(p_0) = 0$, therefore p_1 can't be calculated.

Exercise 3

Let $f(x) = x^2 - 6$. With $p_0 = 3$ and $p_1 = 2$, find p_3.

a) Use the Secant method.

b) Use the method of False Position.

c) Which of the above is closer to $\sqrt{6}$?

Solution 3

a) Applying Secant method generates the following table:

n	p_n	$f(p_n)$
0	3	3
1	2	−2
2	2.4	−0.24
3	2.454 545	0.024 793

So $p_3 = 2.454\,545$.

b) Applying False Position method generates the following table:

n	p_n	$f(p_n)$
0	3	3
1	2	-2
2	2.4	-0.24
3	2.454 545	2.444 444

So $p_3 = 2.444\,444$.

c) p_3 produced by Secant method is better.

Exercise 4

Let $f(x) = -x^3 - \cos x$. With $p_0 = -1$ and $p_1 = 0$, find p_3.

a) Use the Secant method.

b) Use the method of False Position.

Solution 4

a) Applying Secant method generates the following table:

n	p_n	$f(p_n)$
0	-1	0.459 697 694
1	0	-1
2	$-0.685\,073\,357$	$-0.452\,850\,234$
3	$-1.252\,076\,489$	1.649 523 592

So $p_3 = -1.252\,076$.

b) Applying False Position method generates the following table:

n	p_n	$f(p_n)$
0	-1	0.459 697 694
1	0	-1
2	$-0.685\,073\,357$	$-0.452\,850\,234$
3	$-0.841\,355\,126$	$-0.070\,875\,968$

So $p_3 = -0.841\,355$.

Exercise 5

Use Newton's method to find solutions accurate to within 10^{-4} for the following problems.

a) $x^3 - 2x^2 - 5 = 0$ in $[1, 4]$

b) $x^3 + 3x^2 - 1 = 0$ in $[-3, -2]$

c) $x - \cos x = 0$ in $[0, \pi/2]$

d) $x - 0.8 - 0.2 \sin x = 0$ in $[0, \pi/2]$

Solution 5

a) Let

$$\begin{aligned} f(x) &= x^3 - 2x^2 - 5 \\ \Rightarrow f'(x) &= 3x^2 - 4x \end{aligned}$$

Applying Newton's method on f with $p_0 = 2.5$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	2.5	−1.875	8.75
1	2.714 285 714	0.262 390 671	11.244 897 96
2	2.690 951 571	0.003 331 987	10.959 854 13
3	2.690 647 499	0.000 000 561	10.956 161 9
4	2.690 647 448	0	10.956 161 28

We conclude that $p \approx 2.690\,65$ is a solution of the problem.

b) Let

$$\begin{aligned} f(x) &= x^3 + 3x^2 - 1 \\ \Rightarrow f'(x) &= 3x^2 + 6x \end{aligned}$$

Applying Newton's method on f with $p_0 = -2.5$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	−2.5	2.125	3.75
1	−3.066 666 67	−1.626 962 963	9.813 333 33
2	−2.900 875 604	−0.165 860 349	7.839 984 184
3	−2.879 719 904	−0.002 542 819	7.600 040 757
4	−2.879 385 325	−0.000 000 631	7.596 267 596
5	−2.879 385 242	0	7.596 266 659

We conclude that $p \approx 2.690\,65$ is a solution of the problem.

c) Let

$$f(x) = x - \cos x$$
$$\Rightarrow f'(x) = 1 + \sin x$$

Applying Newton's method on f with $p_0 = 0.739$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.739	−0.000 142 477	1.673 549 106
1	0.739 085 135	0.000 000 002	1.673 612 03

We conclude that $p \approx 0.739\,09$ is a solution of the problem.

d) Let

$$f(x) = x - 0.8 - 0.2 \sin x$$
$$\Rightarrow f'(x) = 1 - 0.2 \cos x$$

Applying Newton's method on f with $p_0 = 0.964$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.964	−0.000 295 817	0.885 952 272
1	0.964 333 898	−0.000 000 009	0.886 007 136
2	0.964 333 888	0	0.886 007 135

We conclude that $p \approx 0.964\,33$ is a solution of the problem.

Exercise 6

Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.

a) $e^x + 2^{-x} + 2\cos x - 6 = 0$ for $x \in [1, 2]$

b) $\ln(x-1) + \cos(x-1) = 0$ for $x \in [1.3, 2]$

c) $2x\cos(2x) - (x-2)^2 = 0$ for $x \in [2, 3]$ and $x \in [3, 4]$

d) $(x-2)^2 - \ln x = 0$ for $x \in [1, 2]$ and $x \in [e, 4]$

e) $e^x - 3x^2 = 0$ for $x \in [0, 1]$ and $x \in [3, 5]$

f) $\sin x - e^x = 0$ for $x \in [0, 1]$, $x \in [3, 4]$ and $x \in [6, 7]$

Solution 6

a) Let

$$f(x) = e^x + 2^{-x} + 2\cos x - 6$$
$$\Rightarrow f'(x) = e^x - \ln 2 \cdot 2^{-x} - 2\sin x$$

Applying Newton's method on f with $p_0 = 1.829$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	1.829	−0.001 572 837	4.098 862 489
1	1.829 383 725	0.000 000 506	4.101 500 646
2	1.829 383 602	0	4.101 499 798

We conclude that $p \approx 1.829\,384$ is a solution of the problem.

b) Let

$$f(x) = \ln(x-1) + \cos(x-1)$$
$$\Rightarrow f'(x) = \frac{1}{x-1} - \sin(x-1)$$

Applying Newton's method on f with $p_0 = 1.398$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	1.398	0.000 534 714	1.527 454 989
1	1.397 649 931	−0.000 209 62	1.529 727 16

We conclude that $p \approx 1.397\,65$ is a solution of the problem.

c) Let

$$\begin{aligned} f(x) &= 2x\cos(2x) - (x-2)^2 \\ \Rightarrow f'(x) &= 2(\cos x - x\sin(2x)2) - 2(x-2) \\ &= 2(\cos x - 2x\sin(2x) - x + 2) \end{aligned}$$

Applying Newton's method on f with $p_0 = 2.371$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	2.371	0.002 753 936	7.302 846 51
1	2.370 622 9	−0.000 563 086	7.302 827 46
2	2.3707	0.000 115 071	7.302 831 78
3	2.370 684 24	−0.000 023 518	7.302 830 91

Applying Newton's method on f with $p_0 = 3.722$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	3.722	0.001 838 451	−18.770 682 49
1	3.722 097 943	0.000 241 783	−18.772 292 46
2	3.722 110 823	0.000 031 801	−18.772 504 14
3	3.722 112 517	0.000 004 182	−18.772 531 98

We conclude that $p \approx 2.370\,684$ and $p \approx 3.722\,113$ are solutions of the problem.

d) Let

$$\begin{aligned} f(x) &= (x-2)^2 - \ln x \\ \Rightarrow f'(x) &= 2(x-2) - \frac{1}{x} \end{aligned}$$

Applying Newton's method on f with $p_0 = 1.412$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	1.412	0.000 736 86	−1.884 215 297
1	1.412 391 07	0.000 000 191	−1.883 237 062
2	1.412 391 172	0	−1.883 236 808

Applying Newton's method on f with $p_0 = 3.057$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	3.057	−0.000 185 043	1.786 881 91
1	3.057 103 56	0.000 000 011	1.787 100 1
2	3.057 103 55	0	1.787 100 09

We conclude that $p \approx 1.412\,391$ and $p \approx 3.057\,104$ are solutions of the problem.

e) Let

$$\begin{aligned} f(x) &= e^x - 3x^2 \\ \Rightarrow f'(x) &= e^x - 6x \end{aligned}$$

Applying Newton's method on f with $p_0 = 0.91$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.91	0.000 022 533	−2.975 677 47
1	0.910 007 573	0	−2.975 704 09

Applying Newton's method on f with $p_0 = 3.733$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	3.733	−0.001 533 768	19.406 333 2
1	3.733 079 03	0.000 000 112	19.409 163 1
2	3.733 079 03	0	19.409 162 9

We conclude that $p \approx 0.910\,008$ and $p \approx 3.733\,079$ are solutions of the problem.

f) Let

$$f(x) = \sin x - e^{-x}$$
$$\Rightarrow f'(x) = \cos x + e^{-x}$$

Applying Newton's method on f with $p_0 = 0.588$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.588	−0.000 739 019	1.387 488 79
1	0.588 532 63	−0.000 000 157	1.386 897 46
2	0.588 532 744	0	1.386 897 33

Applying Newton's method on f with $p_0 = 3.096$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	3.096	0.000 347 1	−0.953 731 075
1	3.096 363 94	−0.000 000 601	−0.953 764 054
2	3.096 363 93	0	−0.953 764 053

Applying Newton's method on f with $p_0 = 6.285$ gives:

n	p_n	$f(p_n)$	$f'(p_n)$
0	6.285	−0.000 049 365	1.001 862 41
1	6.285 049 27	0	1.001 862 23
2	6.285 049 27	0	1.001 862 23

We conclude that $p \approx 0.588\,53$, $p \approx 3.096\,36$ and $p = 6.285049$ are solutions of the problem.

Exercise 7

Repeat Exercise 5 using the Secant method.

Solution 7

a) Applying Secant method with $p_0 = 2.6$ and $p_1 = 2.7$ generates the following table:

n	p_n	$f(p_n)$
0	2.6	−0.944
1	2.7	0.103
2	2.690 162 369	−0.005 313 179
3	2.690 644 942	−0.000 027 451
4	2.690 647 449	0.000 000 007

We conclude that $p \approx 2.690\,65$ is a solution of the problem.

b) Applying Secant method with $p_0 = -2.8$ and $p_1 = -2.9$ generates the following table:

n	p_n	$f(p_n)$
0	−2.8	0.568
1	−2.9	−0.159
2	−2.878 129 298	0.009 531 586
3	−2.879 366 233	0.000 144 394
4	−2.879 385 259	−0.000 000 134

We conclude that $p \approx -2.879\,39$ is a solution of the problem.

c) Applying Secant method with $p_0 = 0.73$ and $p_1 = 0.74$ generates the following table:

n	p_n	$f(p_n)$
0	0.73	−0.015 174 402
1	0.74	0.001 531 441
2	0.739 083 29	−0.000 003 084
3	0.739 085 133	0

We conclude that $p \approx 0.739\,09$ is a solution of the problem.

d) Applying Secant method with $p_0 = 0.96$ and $p_1 = 0.97$ generates the following table:

n	p_n	$f(p_n)$
0	0.96	−0.003 838 313
1	0.97	−0.005 022 857
2	0.964 331 61	−0.000 002 018
3	0.964 333 887	−0.000 000 001

We conclude that $p \approx 0.964\,33$ is a solution of the problem.

Exercise 8

Repeat Exercise 6 using the Secant method.

Solution 8

a) Applying Secant method with $p_0 = 1.82$ and $p_1 = 1.83$ generates the following table:

n	p_n	$f(p_n)$
0	1.82	−0.038 185 199
1	1.83	0.002 529 463
2	1.829 378 734	−0.000 019 965
3	1.829 383 599	0.000 000 001

We conclude that $p \approx 1.829\,384$ is a solution of the problem.

b) Applying Secant method with $p_0 = 1.39$ and $p_1 = 1.4$ generates the following table:

n	p_n	$f(p_n)$
0	1.39	−0.016 699 48
1	1.4	0.004 770 262
2	1.397 778 147	0.000 063 1
3	1.397 748 362	−0.000 000 242
4	1.397 748 476	0

We conclude that $p \approx 1.397\,748$ is a solution of the problem.

c) Applying Secant method with $p_0 = 2.37$ and $p_1 = 2.375$ generates the following table:

n	p_n	$f(p_n)$
0	2.37	−0.006 040 395
1	2.375	0.037 985 226
2	2.370 686 009	−0.000 007 99
3	2.370 686 916	−0.000 000 001

Applying Secant method with $p_0 = 3.72$ and $p_1 = 3.73$ generates the following table:

n	p_n	$f(p_n)$
0	3.72	0.034 398 018
1	3.73	−0.129 244 414
2	3.722 102 023	0.000 175 259
3	3.722 112 719	0.000 000 889
4	3.722 112 773	0

We conclude that $p \approx 2.370\,69$ and $p \approx 3.722\,113$ are solutions of the problem.

d) Applying Secant method with $p_0 = 1.41$ and $p_1 = 1.42$ generates the following table:

n	p_n	$f(p_n)$
0	1.41	0.004 510 296
1	1.42	−0.014 256 872
2	1.412 403 29	−0.000 022 822
3	1.412 391 11	0.000 000 116
4	1.412 391 17	0

Applying Secant method with $p_0 = 3.05$ and $p_1 = 3.06$ generates the following table:

n	p_n	$f(p_n)$
0	3.05	−0.012 641 591
1	3.06	0.005 185 084
2	3.057 091 39	−0.000 021 731
3	3.057 103 53	−0.000 000 037
4	3.057 103 55	0

We conclude that $p \approx 1.412\,391$ and $p \approx 3.057\,104$ are solutions of the problem.

e) Applying Secant method with $p_0 = 0.91$ and $p_1 = 0.92$ generates the following table:

n	p_n	$f(p_n)$
0	0.91	0.000 022 533
1	0.92	−0.029 909 61
2	0.910 007 528	0.000 000 132
3	0.910 007 572	0

Applying Secant method with $p_0 = 3.73$ and $p_1 = 3.74$ generates the following table:

n	p_n	$f(p_n)$
0	3.73	−0.059 591 836
1	3.74	0.135 190 165
2	3.733 059 41	−0.000 380 739
3	3.733 078 9	−0.000 002 422
4	3.733 079 03	0

We conclude that $p \approx 0.910\,008$ and $p \approx 3.733\,079$ are solutions of the problem.

f) Applying Secant method with $p_0 = 0.58$ and $p_1 = 0.59$ generates the following table:

n	p_n	$f(p_n)$
0	0.58	−0.011 874 43
1	0.59	0.002 033 738
2	0.588 537 738	0.000 006 927
3	0.588 532 741	−0.000 000 004

Applying Secant method with $p_0 = 3.09$ and $p_1 = 3.1$ generates the following table:

n	p_n	$f(p_n)$
0	3.09	0.006 067 814
1	3.1	−0.003 468 54
2	3.096 362 82	0.000 001 057
3	3.096 363 93	0

Applying Secant method with $p_0 = 6.28$ and $p_1 = 6.29$ generates the following table:

n	p_n	$f(p_n)$
0	6.28	$-0.005\,058\,702$
1	6.29	$0.004\,959\,88$
2	6.285 049 32	$0.000\,000\,046$
3	6.285 049 27	0

We conclude that $p \approx 0.588\,533$, $p \approx 3.096\,364$ and $p \approx 6.285\,049$ are solutions of the problem.

Exercise 9

Repeat Exercise 5 using the method of False Position.

Solution 9

a) Applying False Position method with $p_0 = 2.6$ and $p_1 = 2.7$ generates the following table:

n	p_n	$f(p_n)$
0	2.6	-0.944
1	2.7	0.103
2	2.690 162 369	$-0.005\,313\,179$
3	2.690 644 942	$-0.000\,027\,451$
4	2.690 647 435	$-0.000\,000\,141$

We conclude that $p \approx 2.690\,647$ is a solution of the problem.

b) Applying False Position method with $p_0 = -2.8$ and $p_1 = -2.9$ generates the following table:

n	p_n	$f(p_n)$
0	-2.8	0.568
1	-2.9	-0.159
2	$-2.878\,129\,298$	$0.009\,531\,586$
3	$-2.879\,366\,233$	$0.000\,144\,394$
4	$-2.879\,385\,26$	$-0.000\,000\,135$

We conclude that $p \approx -2.879\,39$ is a solution of the problem.

c) Applying False Position method with $p_0 = 0.73$ and $p_1 = 0.74$ generates the following table:

n	p_n	$f(p_n)$
0	0.73	−0.015 174 402
1	0.74	0.001 531 441
2	0.739 083 29	−0.000 003 084
3	0.739 085 133	0

We conclude that $p \approx 0.739\,09$ is a solution of the problem.

d) Applying False Position method with $p_0 = 0.96$ and $p_1 = 0.97$ generates the following table:

n	p_n	$f(p_n)$
0	0.96	−0.003 838 313
1	0.97	−0.005 022 857
2	0.964 331 61	−0.000 002 018
3	0.964 333 887	−0.000 000 001

We conclude that $p \approx 0.964\,33$ is a solution of the problem.

Exercise 10

Repeat Exercise 6 using the False Position method.

Solution 10

a) Applying False Position method with $p_0 = 1.82$ and $p_1 = 1.83$ generates the following table:

n	p_n	$f(p_n)$
0	1.82	−0.038 185 199
1	1.83	0.002 529 463
2	1.829 378 734	−0.000 019 965
3	1.829 383 599	0.000 000 001

We conclude that $p \approx 1.829\,384$ is a solution of the problem.

b) Applying False Position method with $p_0 = 1.39$ and $p_1 = 1.4$ generates the following table:

n	p_n	$f(p_n)$
0	1.39	−0.016 699 48
1	1.4	0.004 770 262
2	1.397 778 15	0.000 063 1
3	1.397 748 87	0.000 000 831
4	1.397 748 48	0.000 000 001

We conclude that $p \approx 1.397\,748$ is a solution of the problem.

c) Applying False Position method with $p_0 = 2.37$ and $p_1 = 2.375$ generates the following table:

n	p_n	$f(p_n)$
0	2.37	−0.006 040 395
1	2.375	0.037 985 226
2	2.370 686 009	−0.000 007 99
3	2.370 686 916	−0.000 000 001

Applying False Position method with $p_0 = 3.72$ and $p_1 = 3.73$ generates the following table:

n	p_n	$f(p_n)$
0	3.72	0.034 398 018
1	3.73	−0.129 244 414
2	3.722 102 023	0.000 175 259
3	3.722 112 719	0.000 000 889
4	3.722 112 77	0.000 000 001

We conclude that $p \approx 2.370\,69$ and $p \approx 3.722\,113$ are solutions of the problem.

d) Applying False Position method with $p_0 = 1.41$ and $p_1 = 1.42$ generates the following table:

n	p_n	$f(p_n)$
0	1.41	0.004 510 296
1	1.42	−0.014 256 872
2	1.412 403 29	−0.000 022 822
3	1.412 391 19	−0.000 000 036
4	1.412 391 17	0

Applying False Position method with $p_0 = 3.05$ and $p_1 = 3.06$ generates the following table:

n	p_n	$f(p_n)$
0	3.05	−0.012 641 591
1	3.06	0.005 185 084
2	3.057 091 39	−0.000 021 731
3	3.057 103 53	−0.000 000 037
4	3.057 103 55	0

We conclude that $p \approx 1.412\,391$ and $p \approx 3.057\,104$ are solutions of the problem.

e) Applying False Position method with $p_0 = 0.91$ and $p_1 = 0.92$ generates the following table:

n	p_n	$f(p_n)$
0	0.91	0.000 022 533
1	0.92	−0.029 909 61
2	0.910 007 528	0.000 000 132
3	0.910 007 572	0

Applying False Position method with $p_0 = 3.73$ and $p_1 = 3.74$ generates the following table:

n	p_n	$f(p_n)$
0	3.73	−0.059 591 836
1	3.74	0.135 190 165
2	3.733 059 41	−0.000 380 739
3	3.733 078 9	−0.000 002 422
4	3.733 079 03	−0.000 000 015

We conclude that $p \approx 0.910\,008$ and $p \approx 3.733\,079$ are solutions of the problem.

f) Applying False Position method with $p_0 = 0.58$ and $p_1 = 0.59$ generates the following table:

n	p_n	$f(p_n)$
0	0.58	−0.011 874 43
1	0.59	0.002 033 738
2	0.588 537 738	0.000 006 927
3	0.588 532 761	0.000 000 024

Applying False Position method with $p_0 = 3.09$ and $p_1 = 3.1$ generates the following table:

n	p_n	$f(p_n)$
0	3.09	0.006 067 814
1	3.1	−0.003 468 54
2	3.096 362 82	0.000 001 057
3	3.096 363 93	0

Applying False Position method with $p_0 = 6.28$ and $p_1 = 6.29$ generates the following table:

n	p_n	$f(p_n)$
0	6.28	−0.005 058 702
1	6.29	0.004 959 88
2	6.285 049 32	0.000 000 046
3	6.285 049 27	0

We conclude that $p \approx 0.588\,533$, $p \approx 3.096\,364$ and $p \approx 6.285\,049$ are solutions of the problem.

Exercise 11

Use all three methods in this Section to find solutions to within 10^{-5} for the following problems.

a) $3xe^x = 0$ for $x \in [1, 2]$

b) $2x + 3\cos x - e^x$ for $x \in [0, 1]$

Solution 11

a) Such math... much difficult...

b) Let

$$f(x) = 2x + 3\cos x - e^x$$

$$\Rightarrow f'(x) = 2 - 3\sin x - e^x$$

$\sin x$ and e^x are both monotonically increasing in $I = [0, 1]$, therefore $f'(x)$ is monotonically decreasing I. It follows that

$$f'(0) = 2 \geq f'(x) \geq f'(1) \approx -0.524\,412\,954\,4$$

and that $f'(x)$ has exactly one zero p in I. Since the sign of $f'(x)$ changes from positive to negative as x passes p, the local maximum of f in I is at p. Then the minimum value of f in I is achieved at either end:

$$f(x) \geq \min\{f(0), f(1)\} \approx 0.902\,625\,089\,1 > 0$$

Then f has no zero in I.

Exercise 12

Use all three methods in this Section to find solutions to within 10^{-7} for the following problems.

a) $x^2 - 4x + 4 - \ln x = 0$ for $x \in [1, 2]$ and $x \in [2, 4]$

b) $x + 1 - 2\sin \pi x = 0$ for $x \in [0, 1/2]$ and $x \in [1/2, 1]$

Solution 12

a) Let

$$f(x) = x^2 - 4x + 4 - \ln x$$
$$\Rightarrow f'(x) = 2x - 4 - \frac{1}{x}$$

Applying Newton's method on f with $p_0 = 1.41$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	1.41	0.004 510 295 61	−1.889 219 858 16
1	1.412 387 385 24	0.000 007 131 42	−1.883 246 279 86
2	1.412 391 172 01	0.000 000 000 02	−1.883 236 808 04
3	1.412 391 172 02	0	−1.883 236 808 02

Applying Newton's method on f with $p_0 = 3.05$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	3.05	−0.012 641 590 62	1.772 131 147 54
1	3.057 133 552 52	0.000 053 618 47	1.787 163 305 75
2	3.057 103 550 53	0.000 000 000 95	1.787 100 091 6
3	3.057 103 549 99	0	1.787 100 090 48

Applying Secant method with $p_0 = 1.41$ and $p_1 = 1.42$ generates the following table:

n	p_n	$f(p_n)$
0	1.41	0.004 510 295 61
1	1.42	−0.014 256 871 61
2	1.412 403 290 57	−0.000 022 821 92
3	1.412 391 110 52	0.000 000 115 82
4	1.412 391 172 02	0

Applying Secant method with $p_0 = 3.05$ and $p_1 = 3.06$ generates the following table:

n	p_n	$f(p_n)$
0	3.05	−0.012 641 590 62
1	3.06	0.005 185 084 04
2	3.057 091 390 21	−0.000 021 730 59
3	3.057 103 529 27	−0.000 000 037 04
4	3.057 103 549 99	0

Applying False Position method with $p_0 = 1.41$ and $p_1 = 1.42$ generates the following table:

n	p_n	$f(p_n)$
0	1.41	0.004 510 295 61
1	1.42	−0.014 256 871 61
2	1.412 403 290 57	−0.000 022 821 92
3	1.412 391 191 24	−0.000 000 036 19
4	1.412 391 172 05	−0.000 000 000 06

Applying False Position method with $p_0 = 3.05$ and $p_1 = 3.06$ generates the following table:

n	p_n	$f(p_n)$
0	3.05	$-0.012\,641\,590\,62$
1	3.06	$0.005\,185\,084\,04$
2	$3.057\,091\,390\,21$	$-0.000\,021\,730\,59$
3	$3.057\,103\,529\,27$	$-0.000\,000\,037\,04$
4	$3.057\,103\,549\,96$	0

b) Let

$$\begin{aligned} f(x) &= x + 1 - 2\sin \pi x \\ \Rightarrow f'(x) &= 1 - 2\pi \cos \pi x \end{aligned}$$

Applying Newton's method on f with $p_0 = 0.21$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.21	$-0.015\,814\,107\,31$	$-3.964\,690\,364\,15$
1	$0.206\,011\,262\,96$	$0.000\,095\,722\,6$	$-4.012\,556\,253\,06$
2	$0.206\,035\,118\,73$	$0.000\,000\,003\,39$	$-4.012\,272\,309\,82$
3	$0.206\,035\,119\,57$	0	$-4.012\,272\,299\,77$

Applying Newton's method on f with $p_0 = 0.68$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.68	$-0.008\,655\,851$	$4.366\,699\,045\,41$
1	$0.681\,982\,241\,26$	$0.000\,032\,700\,17$	$4.399\,670\,307\,78$
2	$0.681\,974\,808\,84$	$0.000\,000\,000\,46$	$4.399\,546\,927\,47$
3	$0.681\,974\,808\,74$	0	$4.399\,546\,925\,74$

Applying Secant method with $p_0 = 0.21$ and $p_1 = 0.22$ generates the following table:

n	p_n	$f(p_n)$
0	0.21	$-0.015\,814\,107\,31$
1	0.22	$-0.054\,847\,979\,5$
2	$0.205\,948\,619\,39$	$0.000\,347\,106\,82$
3	$0.206\,036\,984\,68$	$-0.000\,007\,483\,3$
4	$0.206\,035\,119\,81$	$-0.000\,000\,000\,96$
5	$0.206\,035\,119\,57$	0

Applying Secant method with $p_0 = 0.68$ and $p_1 = 0.69$ generates the following table:

n	p_n	$f(p_n)$
0	0.68	−0.008 655 851
1	0.69	0.035 838 851 45
2	0.681 945 366 65	−0.000 129 524 68
3	0.681 974 371 95	−0.000 001 921 66
4	0.681 974 808 76	0.000 000 001 07
5	0.681 974 808 74	0

Applying False Position method with $p_0 = 0.21$ and $p_1 = 0.22$ generates the following table:

n	p_n	$f(p_n)$
0	0.21	−0.015 814 107 31
1	0.22	−0.054 847 979 5
2	0.205 948 619 39	0.000 347 106 82
3	0.206 036 984 68	−0.000 007 483 3
4	0.206 035 119 81	−0.000 000 000 96
5	0.206 035 119 57	0

Applying False Position method with $p_0 = 0.68$ and $p_1 = 0.69$ generates the following table:

n	p_n	$f(p_n)$
0	0.68	−0.008 655 851
1	0.69	0.035 838 851 45
2	0.681 945 366 65	−0.000 129 524 67
3	0.681 974 371 95	−0.000 001 921 66
4	0.681 974 802 26	−0.000 000 028 51
5	0.681 974 808 64	−0.000 000 000 42

Exercise 13

Use Newton's method to approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = x^2$ that is closest to $(1, 0)$.

Solution 13

Let d be the squared distance between the point (x, x^2) of the graph and $(1, 0)$.

$$\begin{aligned}
d(x) &= (x-1)^2 + x^4 \\
\Rightarrow d'(x) &= 4x^3 + 2(x-1) \\
\Rightarrow d''(x) &= 12x^2 + 2
\end{aligned}$$

We need to find x that minimizes d. First we have to examine d'. As $d''(x) \geq 2 > 0\,\forall x \in \mathbb{R}$, d' is monotonically increasing in $\mathbb{R}$. It follows that d' has at most one zero in $\mathbb{R}$.

Applying Newton's method on d' with $p_0 = 0.59$ generates the following table:

n	p_n	$d'(p_n)$	$d''(p_n)$
0	0.59	0.001 516	6.1772
1	0.589 754 581	0.000 000 426	6.173 725 59
2	0.589 754 512	0	6.173 724 62

Then $p \approx 0.589\,75$ is the only zero of d'. Since the sign of d' changes from negative to positive as x passes p, the global minimum of d is achieved at p.

We conclude that $x \approx 0.589\,75$ produces the point on the graph of $y = x^2$ that is closest to $(1, 0)$.

Exercise 14

Use Newton's method to approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = \frac{1}{x}$ that is closest to $(2, 1)$.

Solution 14

Let d be the squared distance between the point $(x, \frac{1}{x})$ of the graph and $(2, 1)$.

$$\begin{aligned}
d(x) &= (x-2)^2 + \left(\frac{1}{x} - 1\right)^2 \\
\Rightarrow d'(x) &= 2(x-2) - 2\left(\frac{1}{x} - 1\right)\frac{1}{x^2} = \frac{2(x^4 - 2x^3 + x - 1)}{x^3} \\
\Rightarrow d''(x) &= 2\left(\frac{3}{x} - 2\right)\frac{1}{x^3} + 2 = \frac{2(x^4 - 2x + 3)}{x^4}
\end{aligned}$$

Let

$$\begin{aligned}
f(x) &= x^4 - 2x + 3 \\
\Rightarrow f'(x) &= 4x^3 - 2
\end{aligned}$$

f' has exactly one zero at $0.5^{1/3}$. Since f' is monotonically increasing in $\mathbb{R}$, the sign of f' changes from negative to positive as x passes $0.5^{1/3}$. It follows that the global minimum of f is achieved at $0.5^{1/3}$:

$$f(x) \geq f(0.5^{1/3}) \approx 1.809\,449\,211 > 0$$

Then, $d''(x) > 0\,\forall x \in \mathbb{R} \setminus 0$. It follows that d' is monotonically increasing in $D^+ = \mathbb{R}_{>0}$ and $D^- = \mathbb{R}_{<0}$, which means it has at most one zero in D^+ and D^- alike.

Let

$$g(x) = x^4 - 2x^3 + x - 1$$
$$\Rightarrow g'(x) = 4x^3 - 6x^2 + 1$$

Every zero of g is also a zero of d'. Applying Newton's method on g with $p_0 = 1.86$ generates the following table:

n	p_n	$g(p_n)$	$g'(p_n)$
0	1.86	−0.040 879 84	5.981 824
1	1.866 834 01	0.000 449 982	6.113 767 65
2	1.866 760 41	0.000 000 053	6.112 338 49

Applying Newton's method on g with $p_0 = -0.86$ generates the following table:

n	p_n	$g(p_n)$	$g'(p_n)$
0	−0.86	−0.040 879 84	−5.981 824
1	−0.866 834 009	0.000 449 982	−6.113 767 65
2	−0.866 760 408	0.000 000 053	−6.112 338 49

We conclude that $x \approx 1.866\,76$ and $x \approx -0.866\,76$ produce the points on the graph of $y = x^2$ that are closest to $(1, 0)$.

Exercise 15

The following describes Newton's method graphically:

Suppose that $f'(x)$ exists on $[a, b]$ and that $f'(x) \neq 0\,\forall x \in [a, b]$. Further, suppose there exists one $p \in [a, b]$ such that $f(p) = 0$.

Let $p_0 \in [a, b]$ be arbitrary. Let p_1 be the point at which the tangent line to f at $(p_0, f(p_0))$ crosses the x-axis. For each $n \geq 1$, let p_n be the x-intercept of the line tangent to f at $(p_{n-1}, f(p_{n-1}))$. Derive the formula describing this method.

Solution 15

The equation of the line tangent to f at $(p_{n-1}, f(p_{n-1}))$ is:

$$y = f'(p_{n-1})(x - p_{n-1}) + f(p_{n-1})$$

Then its x-intercept is:

$$x = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Then the formula describing the sequence generated by the procedure is:

$$\{p_n\} \mid p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Exercise 16

Use Newton's method to solve the equation

$$0 = \frac{1}{2} + \frac{1}{4}x^2 - x\sin x - \frac{1}{2}\cos 2x \text{ with } p_0 = \frac{\pi}{2}$$

Iterate using Newton's method until an accuracy of 10^{-5} is obtained. Explain why the result seems unusual for Newton's method. Also, solve the equation with $p_0 = 5\pi$ and $p_0 = 10\pi$.

Solution 16

Let

$$\begin{aligned} f(x) &= \frac{1}{2} + \frac{1}{4}x^2 - x\sin x - \frac{1}{2}\cos 2x \\ \Rightarrow f'(x) &= \frac{1}{2}x - \sin x + x\cos x + \sin 2x \end{aligned}$$

Applying Newton's method on f with $p_0 = \frac{\pi}{2}$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	1.570 796 33	0.046 053 948	−0.214 601 837
1	1.785 398 16	0.007 116 978	−0.120 293 455
2	1.844 561 63	0.001 638 544	−0.062 366 566
3	1.870 834 42	0.000 396 329	−0.031 675 918
4	1.883 346 43	0.000 097 601	−0.015 954 846
5	1.889 463 76	0.000 024 225	−0.008 005 932
6	1.892 489 62	0.000 006 035	−0.004 010 008
7	1.893 994 57	0.000 001 506	−0.002 006 754
8	1.894 745 07	0.000 000 376	−0.001 003 813
9	1.895 119 83	0.000 000 094	−0.000 502 015
10	1.895 307 09	0.000 000 023	−0.000 251 035
11	1.895 400 69	0.000 000 006	−0.000 125 524
12	1.895 447 48	0.000 000 001	−0.000 062 764
13	1.895 470 87	0	−0.000 031 382
14	1.895 482 57	0	−0.000 015 691
15	1.895 488 42	0	−0.000 007 846

It's clear that the number of iteration is unusually large.
Applying Newton's method on f with $p_0 = 5\pi$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	15.707 963 3	61.685 027 5	23.561 944 9
1	13.089 969 4	36.541 84	−4.425 235 93
2	21.347 572	101.479 949	26.190 775 1
3	17.472 927 3	94.433 153 9	5.967 623 72
4	1.648 679 92	0.029 800 649	−0.199 491 346
5	1.798 063 09	0.005 663 214	−0.109 166 251
6	1.849 940 06	0.001 319 265	−0.056 337 315
7	1.873 357 31	0.000 320 334	−0.028 563 789
8	1.884 572	0.000 079 014	−0.014 376 187
9	1.890 068 17	0.000 019 626	−0.007 211 151
10	1.892 789 8	0.000 004 89	−0.003 611 278
11	1.894 144 16	0.000 001 22	−0.001 807 057
12	1.894 819 74	0.000 000 305	−0.000 903 882
13	1.895 157 14	0.000 000 076	−0.000 452 029
14	1.895 325 73	0.000 000 019	−0.000 226 037
15	1.895 410 01	0.000 000 005	−0.000 113 024
16	1.895 452 14	0.000 000 001	−0.000 056 513
17	1.895 473 2	0	−0.000 028 257
18	1.895 483 74	0	−0.000 014 129
19	1.895 489	0	−0.000 007 064

For $p_0 = 10\pi$, the sequence converges and diverges back and forth, then finally stops at $p_{154} \approx -0.000\,006$.

Exercise 17

The fourth-degree polynomial

$$f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$$

has two real zeros, one in $[-1, 0]$ and the other in $[0, 1]$. Attempt to approximate these zeros to within 10^{-6} using the

a) Method of False Position

b) Secant method

c) Newton's method

Use the endpoints of each interval as the initial approximations in a) and b) and the midpoints as the initial approximation in c).

Solution 17

a) Applying False Position method with $p_0 = -1$ and $p_1 = 0$ generates the following table:

n	p_n	$f(p_n)$
0	-1	433
1	0	-9
2	$-0.020\,361\,991$	$-4.496\,380\,93$
3	$-0.030\,430\,247$	$-2.266\,891\,37$
4	$-0.035\,479\,814$	$-1.148\,071\,19$
5	$-0.038\,030\,414$	$-0.582\,770\,74$
6	$-0.039\,323\,38$	$-0.296\,160\,751$
7	$-0.039\,980\,008$	$-0.150\,595\,231$
8	$-0.040\,313\,782$	$-0.076\,599\,144$
9	$-0.040\,483\,524$	$-0.038\,967\,468$
10	$-0.040\,569\,867$	$-0.019\,825\,027$
11	$-0.040\,613\,793$	$-0.010\,086\,543$
12	$-0.040\,636\,141$	$-0.005\,131\,916$
13	$-0.040\,647\,511$	$-0.002\,611\,086$
14	$-0.040\,653\,296$	$-0.001\,328\,51$
15	$-0.040\,656\,24$	$-0.000\,675\,943$
16	$-0.040\,657\,737$	$-0.000\,343\,918$
17	$-0.040\,658\,499$	$-0.000\,174\,985$

Applying False Position method with $p_0 = 0$ and $p_1 = 1$ generates the following table:

n	p_n	$f(p_n)$
0	0	-9
1	1	27
2	0.25	$-62.507\,812\,5$
3	$0.773\,762\,765$	$-83.830\,520\,3$
4	$0.944\,885\,169$	$-11.265\,130\,2$
5	$0.961\,110\,797$	$-0.855\,867\,823$
6	$0.962\,305\,662$	$-0.061\,802\,369$
7	$0.962\,391\,747$	$-0.004\,446\,181$
8	$0.962\,397\,939$	$-0.000\,319\,781$
9	$0.962\,398\,384$	$-0.000\,022\,999$

b) Applying Secant method with $p_0 = -1$ and $p_1 = 0$ generates the following table:

n	p_n	$f(p_n)$
0	-1	433
1	0	-9
2	$-0.020\,361\,991$	$-4.496\,380\,93$
3	$-0.040\,691\,256$	$0.007\,087\,483$
4	$-0.040\,659\,263$	$-0.000\,005\,706$
5	$-0.040\,659\,288$	0

Applying Secant method with $p_0 = 0$ and $p_1 = 1$ generates the following table:

n	p_n	$f(p_n)$
0	0	-9
1	1	27
2	0.25	$-62.507\,812\,5$
3	$0.773\,762\,765$	$-83.830\,520\,3$
4	$-1.285\,417\,78$	$879.638\,986$
5	$0.594\,595\,52$	$-104.691\,389$
6	$0.394\,641\,105$	$-88.128\,940\,4$
7	$-0.669\,318\,136$	$183.713\,16$
8	$0.049\,714\,398$	$-19.961\,021\,6$
9	$-0.020\,754\,151$	$-4.409\,574\,29$
10	$-0.040\,735\,333$	$0.016\,859\,473$
11	$-0.040\,659\,228$	$-0.000\,013\,318$
12	$-0.040\,659\,288$	0

c) Applying Newton's method with $p_0 = -0.5$ generates the following table:

n	p_n	$g(p_n)$	$g'(p_n)$
0	-0.5	115.875	-331.5
1	$-0.150\,452\,489$	$24.510\,271$	$-225.618\,988$
2	$-0.041\,816\,814$	$0.256\,640\,771$	$-221.725\,549$
3	$-0.040\,659\,344$	$0.000\,012\,234$	$-221.704\,436$
4	$-0.040\,659\,288$	0	$-221.704\,435$

Applying Newton's method with $p_0 = 0.5$ generates the following table:

n	p_n	$g(p_n)$	$g'(p_n)$
0	0.5	-100.625	-83.5
1	$-0.705\,089\,82$	$201.836\,304$	$-529.339\,073$
2	$-0.323\,791\,114$	$65.418\,426\,7$	$-252.397\,607$

n	p_n	$g(p_n)$	$g'(p_n)$
3	−0.064 603 131	5.314 007 07	−222.185 539
4	−0.040 686 151	0.005 955 616	−221.704 923
5	−0.040 659 288	0.000 000 007	−221.704 435
6	−0.040 659 288	0	−221.704 435

Exercise 18

The function $f(x) = \tan \pi x - 6$ has a zero at $\frac{\arctan(6)}{\pi} \approx 0.447\,431\,543$. Let $p_0 = 0$ and $p_1 = 0.48$, and use ten iterations of each of the following methods to approximate this root. Which method is most successful and why?

a) Bisection b) False Position c) Secant

Solution 18

a) Applying Bisection method on f with $a = 0$, $b = 0.48$ generates the following table:

n	a_n	b_n	p_n	$f(p_n)$
1	0	0.48	0.24	−60.509 683 2
2	0.24	0.48	0.36	−82.690 675 2
3	0.36	0.48	0.42	−91.741 915 2
4	0.42	0.48	0.45	−95.555 812 5
5	0.45	0.48	0.465	−97.255 924 1
6	0.465	0.48	0.4725	−98.050 428 1
7	0.4725	0.48	0.476 25	−98.433 297 5
8	0.476 25	0.48	0.478 125	−98.621 073 9
9	0.478 125	0.48	0.479 062 5	−98.714 039 5
10	0.479 062 5	0.48	0.479 531 25	−98.760 290 8

The method indeed does not produce the root in this case, as $f(a_1)$ and $f(b_1)$ have the same sign.

b) Applying method of False Position on f with $p_0 = 0$ and $p_1 = 0.48$ generates the following table:

n	p_n	$f(p_n)$
0	0	−9
1	0.48	−98.806 387 2
2	−0.048 103 483	1.650 923 14
3	−0.039 424 59	−0.273 724 354
4	−0.040 658 906	−0.000 084 697
5	−0.040 659 288	−0.000 000 026

c) Applying Secant method on f with $p_0 = 0$ and $p_1 = 0.48$ generates the following table:

n	p_n	$f(p_n)$
0	0	-9
1	0.48	$-98.806\,387\,2$
2	$-0.048\,103\,483$	$1.650\,923\,14$
3	$-0.039\,424\,59$	$-0.273\,724\,354$
4	$-0.040\,658\,906$	$-0.000\,084\,697$
5	$-0.040\,659\,288$	$0.000\,000\,004$

Clearly, Secant method is the most successful one in this case.

Exercise 19

The iteration equation for the Secant method can be written in the simpler form:

$$p_n = \frac{f(p_{n-1})p_{n-2} - f(p_{n-2})p_{n-1}}{f(p_{n-1}) - f(p_{n-2})}$$

Explain why, in general, this iteration equation is likely to be less accurate than the one given in the text book.

Solution 19

In both formulas, the denominator is close to 0 as consecutive p_n is close to each other.

In the above formula, the numerator is also close to 0 for the same reason. Therefore, both numerator and denominator are close to 0, which can lead to losing digits.

The formula provided in the text book circumvents this situation by having the difference of 2 consecutive p_n multiplied with f *before* dividing.

As a consequence, the formula should be written in the specific way that it is printed in the text book, as it implies the multiplication should be done before division.

Exercise 20

The equation $x^2 - 10\cos x = 0$ has two solutions, $\pm 1.379\,364\,6$. Use Newton's method to approximate the solutions to within 10^{-5} with the following values of p_0.

a) $p_0 = -100$

b) $p_0 = -50$

c) $p_0 = -25$

d) $p_0 = 25$

e) $p_0 = 50$

f) $p_0 = 100$

Solution 20

Let

$$
\begin{aligned}
f(x) &= x^2 - 10\cos x \\
\Rightarrow f'(x) &= 2x + 10\sin x
\end{aligned}
$$

a) Applying Newton's method with $p_0 = -100$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	−100	9991.376 811 277 1	−194.936 343 588 9
1	−48.745 438 498 9	2375.610 468 619 5	−87.503 753 248
2	−21.596 769 094	475.652 786 972 2	−47.035 891 967 9
3	−11.484 219 569 1	127.192 997 670 8	−14.138 742 994 8
4	−2.488 158 340 9	14.130 939 015 7	−11.055 485 002 7
5	−1.209 974 795 7	−2.066 390 820 8	−11.776 020 627 6
6	−1.385 449 252 3	0.076 592 885	−12.599 621 987 3
7	−1.379 370 269 5	0.000 071 372 8	−12.576 079 669 9
8	−1.379 364 594 2	0.000 000 000 1	−12.576 057 521 4

b) Applying Newton's method with $p_0 = -50$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	−50	2490.350 339 715 1	−97.376 251 463
1	−24.425 485 656 9	589.002 870 288 5	−42.353 470 822 3
2	−10.518 647 354 1	115.232 454 209 8	−12.153 196 604 1
3	−1.036 989 320 9	−4.012 796 962 4	−10.682 741 185 2
4	−1.412 622 961 5	0.420 357 249 2	−12.700 412 446 9
5	−1.379 525 040 4	0.002 017 830 4	−12.576 683 559 7
6	−1.379 364 598 2	0.000 000 050 2	−12.576 057 537
7	−1.379 364 594 2	0	−12.576 057 521 4

c) Applying Newton's method with $p_0 = -25$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	−25	615.087 971 881 4	−48.676 482 499
1	−12.363 754 727 1	143.066 995 664 8	−22.715 185 535 7
2	−6.065 457 253 8	27.025 864 334 4	−9.970 795 758 7
3	−3.354 955 004 2	21.028 967 802 6	−4.592 438 027 5
4	1.224 087 255 5	−1.899 655 866 7	11.853 135 273 5
5	1.384 353 364 2	0.062 787 419 8	12.595 404 723 1
6	1.379 368 417 7	0.000 048 083 8	12.576 072 442 8
7	1.379 364 594 2	0	12.576 057 521 4

d) Applying Newton's method with $p_0 = 25$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	25	615.087 971 881 4	48.676 482 499
1	12.363 754 727 1	143.066 995 664 8	22.715 185 535 7
2	6.065 457 253 8	27.025 864 334 4	9.970 795 758 7
3	3.354 955 004 2	21.028 967 802 6	4.592 438 027 5
4	−1.224 087 255 5	−1.899 655 866 7	−11.853 135 273 5
5	−1.384 353 364 2	0.062 787 419 8	−12.595 404 723 1
6	−1.379 368 417 7	0.000 048 083 8	−12.576 072 442 8
7	−1.379 364 594 2	0	−12.576 057 521 4

e) Applying Newton's method with $p_0 = 50$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	50	2490.350 339 715 1	97.376 251 463
1	24.425 485 656 9	589.002 870 288 5	42.353 470 822 3
2	10.518 647 354 1	115.232 454 209 8	12.153 196 604 1
3	1.036 989 320 9	−4.012 796 962 4	10.682 741 185 2
4	1.412 622 961 5	0.420 357 249 2	12.700 412 446 9
5	1.379 525 040 4	0.002 017 830 4	12.576 683 559 7
6	1.379 364 598 2	0.000 000 050 2	12.576 057 537
7	1.379 364 594 2	0	12.576 057 521 4

f) Applying Newton's method with $p_0 = 100$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	100	9991.376 811 277 1	194.936 343 588 9
1	48.745 438 498 9	2375.610 468 619 5	87.503 753 248
2	21.596 769 094	475.652 786 972 2	47.035 891 967 9
3	11.484 219 569 1	127.192 997 670 8	14.138 742 994 8
4	2.488 158 340 9	14.130 939 015 7	11.055 485 002 7
5	1.209 974 795 7	−2.066 390 820 8	11.776 020 627 6
6	1.385 449 252 3	0.076 592 885	12.599 621 987 3
7	1.379 370 269 5	0.000 071 372 8	12.576 079 669 9
8	1.379 364 594 2	0.000 000 000 1	12.576 057 521 4

Exercise 21

The equation $4x^2 - e^x - e^{-x} = 0$ has two positive solutions x_1 and x_2. Use Newton's method to approximate the solution to within 10^{-5} with the following values of p_0.

a) $p_0 = -10$
b) $p_0 = -5$
c) $p_0 = -3$
d) $p_0 = -1$
e) $p_0 = 0$
f) $p_0 = 1$
g) $p_0 = 3$
h) $p_0 = 5$
i) $p_0 = 10$

Solution 21

Let

$$f(x) = 4x^2 - e^x - e^{-x}$$
$$\Rightarrow f'(x) = 8x - e^x + e^{-x}$$

a) Applying Newton's method with $p_0 = -10$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	−10	−21 626.465 840 206 6	21 946.465 749 406 8
1	−9.014 580 931 3	−7897.049 455 811 2	8149.983 242 581 3
2	−8.045 615 815 6	−2861.158 494 740 3	3055.720 662 614 5
3	−7.109 287 266 4	−1021.108 321 568 4	1166.400 250 226 2
4	−6.233 851 650 4	−354.273 287 548 9	459.842 176 179 7
5	−5.463 428 000 9	−116.512 778 382 3	192.193 058 460 6
6	−4.857 200 183 3	−34.301 660 964 2	89.798 089 553 3
7	−4.475 213 649 6	−7.714 598 646 1	52.000 262 710 2
8	−4.326 856 732 9	−0.832 400 420 4	41.077 885 300 8
9	−4.306 592 777 8	−0.013 799 244 1	39.721 063 640 1
10	−4.306 245 374 1	−0.000 003 994 3	39.698 069 725 7
11	−4.306 245 273 5	0	39.698 063 067 3

b) Applying Newton's method with $p_0 = -5$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	−5	−48.419 897 049 6	108.406 421 155 6
1	−4.553 348 440 7	−12.028 414 215 9	58.512 491 019 6
2	−4.347 778 416 1	−1.706 755 969 7	42.511 366 227 4
3	−4.307 630 189 4	−0.055 041 972 1	39.789 781 006 6
4	−4.306 246 870 1	−0.000 063 380 9	39.698 168 720 5
5	−4.306 245 273 5	−0.000 000 000 1	39.698 063 067 4

c) Applying Newton's method with $p_0 = -3$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	-3	15.864 676 008 4	$-3.964\,250\,145\,2$
1	1.001 936 161 3	0.924 786 470 1	5.659 107 187 9
2	0.838 520 548 3	0.067 174 591 3	4.827 571 52
3	0.824 605 769 2	0.000 509 551 3	4.754 272 591
4	0.824 498 591 7	0.000 000 030 3	4.753 706 617 5
5	0.824 498 585 3	0	4.753 706 583 8

d) Applying Newton's method with $p_0 = -1$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	-1	0.913 838 730 4	$-5.649\,597\,612\,7$
1	$-0.838\,247\,111\,9$	0.065 854 754	$-4.826\,134\,621\,3$
2	$-0.824\,601\,667$	0.000 490 048 4	$-4.754\,250\,928\,9$
3	$-0.824\,498\,591\,2$	0.000 000 028 1	$-4.753\,706\,615$
4	$-0.824\,498\,585\,3$	0	$-4.753\,706\,583\,8$

e) The method fails in this case as $f'(0) = 0$.

f) Applying Newton's method with $p_0 = 1$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	1	0.913 838 730 4	5.649 597 612 7
1	0.838 247 111 9	0.065 854 754	4.826 134 621 3
2	0.824 601 667	0.000 490 048 4	4.754 250 928 9
3	0.824 498 591 2	0.000 000 028 1	4.753 706 615
4	0.824 498 585 3	0	4.753 706 583 8

g) Applying Newton's method with $p_0 = 3$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	3	15.864 676 008 4	3.964 250 145 2
1	$-1.001\,936\,161\,3$	0.924 786 470 1	$-5.659\,107\,187\,9$
2	$-0.838\,520\,548\,3$	0.067 174 591 3	$-4.827\,571\,52$
3	$-0.824\,605\,769\,2$	0.000 509 551 3	$-4.754\,272\,591$
4	$-0.824\,498\,591\,7$	0.000 000 030 3	$-4.753\,706\,617\,5$
5	$-0.824\,498\,585\,3$	0	$-4.753\,706\,583\,8$

h) Applying Newton's method with $p_0 = 5$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	5	−48.419 897 049 6	−108.406 421 155 6
1	4.553 348 440 7	−12.028 414 215 9	−58.512 491 019 6
2	4.347 778 416 1	−1.706 755 969 7	−42.511 366 227 4
3	4.307 630 189 4	−0.055 041 972 1	−39.789 781 006 6
4	4.306 246 870 1	−0.000 063 380 9	−39.698 168 720 5
5	4.306 245 273 5	−0.000 000 000 1	−39.698 063 067 4

i) Applying Newton's method with $p_0 = 10$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	10	−21 626.465 840 206 6	−21 946.465 749 406 8
1	9.014 580 931 3	−7897.049 455 811 2	−8149.983 242 581 3
2	8.045 615 815 6	−2861.158 494 740 3	−3055.720 662 614 5
3	7.109 287 266 4	−1021.108 321 568 4	−1166.400 250 226 2
4	6.233 851 650 4	−354.273 287 548 9	−459.842 176 179 7
5	5.463 428 000 9	−116.512 778 382 3	−192.193 058 460 6
6	4.857 200 183 3	−34.301 660 964 2	−89.798 089 553 3
7	4.475 213 649 6	−7.714 598 646 1	−52.000 262 710 2
8	4.326 856 732 9	−0.832 400 420 4	−41.077 885 300 8
9	4.306 592 777 8	−0.013 799 244 1	−39.721 063 640 1
10	4.306 245 374 1	−0.000 003 994 3	−39.698 069 725 7
11	4.306 245 273 5	0	−39.698 063 067 3

Exercise 22

Use Maple to determine how many iterations of Newton's method with $p_0 = \pi/4$ are needed to find a root of $f(x) = \cos x - x$ to within 10^{-100}.

Solution 22

Python FTW: 51 iterations.

Exercise 23

The function described by $f(x) = \ln(x^2 + 1) - e^{0.4x} \cos \pi x$ has an infinite number of zeros.

a) Determine, within 10^{-6}, the only negative zero.

b) Determine, within 10^{-6}, the four smallest positive zeros.

c) Determine a reasonable initial approximation to find the n^{th} smallest positive zero of f. [Hint: Sketch an approximate graph of f.]

d) Use part c) to determine, within 10^{-6}, the 25^{th} smallest positive zero of f.

Solution 23

Differentiating f gives:

$$f'(x) = \frac{2x}{x^2+1} - e^{0.4x}(0.4\cos\pi x - \pi\sin\pi x)$$

Consider each term of f:

- $\ln(x^2+1) \geq 0\,\forall x \in \mathbb{R}$
- $e^{0.4x} > 0\,\forall x \in \mathbb{R}$
- $\cos\pi x > 0 \iff -0.5+2k < x < 0.5+2k$, with $k \in \mathbb{N}$

which means that every zero of f must be in $[2k-0.5, 2k+0.5]$, $k \in \mathbb{N}$.

a) e^x is monotonically increasing in $\mathbb{R}$. It follows that:

$$0 < e^{0.4x}\cos\pi x \leq e^{0.4x}1 < e^{0.4\cdot 0} = 1\,\forall x < 0$$

$\ln x$ is monotonically increasing in $\mathbb{R}_{>0}$. Therefore $\ln(x^2+1)$ is monotonically decreasing in $\mathbb{R}_{<0}$. Also, e^x is monotonically increasing in $\mathbb{R}$. Therefore, if f has a negative zero, it must satisfy:

$$\ln(x^2+1) < 1 \iff -\sqrt{e-1} \approx -1.310\,832\,494 < x < 0$$

Combining the above points, it is clear that if f has a negative zero, it must be in $D_1 = [-0.5, 0]$.
As $\ln(x^2+1)$ is monotonically decreasing in D_1, it follows that:

$$\ln(-0.5^2+1) \geq \ln(x^2+1) \geq \ln 1 = 0\,\forall x \in D_1$$

As both e^x and $\cos\pi x$ is monotonically increasing in D_1, it follows that:

$$0 \leq e^{0.4x}\cos\pi x \leq 1\,\forall x \in D_1$$

From the above points, there must be exactly one zero of f in D_1.

Applying Newton method on f with $p_0 = -0.25$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	-0.25	$-0.579\,192\,052$	$-2.797\,220\,033$
1	$-0.457\,059\,883$	$0.077\,693\,927$	$-3.742\,796\,53$
2	$-0.436\,301\,627$	$0.007\,306\,593$	$-3.691\,332\,860$
3	$-0.434\,322\,236$	$0.000\,606\,405$	$-3.685\,958\,212$
4	$-0.434\,157\,718$	$0.000\,049\,647$	$-3.685\,507\,782$
5	$-0.434\,144\,247$	$0.000\,004\,06$	$-3.685\,470\,876$
6	$-0.434\,143\,145$	$0.000\,000\,332$	$-3.685\,467\,857$
7	$-0.434\,143\,055$	$0.000\,000\,027$	$-3.685\,467\,61$

We conclude that the sole negative zero of f is $p \approx -0.434\,143\,1$.

not yet finished

Exercise 24

Find an approximation for λ, accurate to within 10^{-4}, for the population equation

$$1\,564\,000 = 1\,000\,000e^{\lambda} + \frac{435\,000}{\lambda}(e^{\lambda} - 1)$$

discussed in the introduction to this chapter. Use this value to predict the population at the end of the second year, assuming that the immigration rate during this year remains at 435 000 individuals per year.

Solution 24

Let

$$f(x) = 1000e^{\lambda} + \frac{435}{\lambda}(e^{\lambda} - 1) - 1564$$
$$\Rightarrow f'(x) = 1000e^{\lambda} + 435\left(\frac{1 - e^{\lambda}}{\lambda^2} + \frac{e^{\lambda}}{\lambda}\right)$$

Applying Newton's method on f with $p_0 = 0.1$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.1	−1.335 588 295 3	1337.729 475 414
1	0.100 998 399 4	0.000 628 932	1338.989 559 263 2
2	0.100 997 929 7	0.000 000 000 1	1338.988 966 158

So $\lambda \approx 0.100\,997\,9$.
Since

$$N(t) = N_0 e^{\lambda t} + \frac{v}{\lambda}(e^{\lambda t} - 1)$$

then the population predicted at the end of the second year $N(2) \approx 2187.938\,632 \cdot 1000 = 2\,187\,938.632$.

Exercise 25

The sum of two numbers is 20. If each number is added to its square root, the product of the two sums is 155.55. Determine the two numbers to within 10^{-4}.

Solution 25

Let one number is $x \in [0, 20]$, and the other is $20 - x$. We have:

$$(x + \sqrt{x})(20 - x + \sqrt{20 - x}) = 155.55$$

Let

$$\begin{aligned} f(x) &= (x + \sqrt{x})(20 - x + \sqrt{20 - x}) - 155.55 \\ \Rightarrow f'(x) &= \frac{2\sqrt{x} + 1}{2\sqrt{x}}(20 - x + \sqrt{20 - x}) - \frac{2\sqrt{20 - x} + 1}{2\sqrt{20 - x}}(x + \sqrt{x}) \end{aligned}$$

Applying Newton's method on f with $p_0 = 6.5$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	6.5	−0.131 596 293 5	10.261 387 078
1	6.512 824 415 7	−0.000 248 515 5	10.222 632 862 2
2	6.512 848 726	−0.000 000 000 9	10.222 559 412 4

We conclude that the two numbers are approximately 6.512 85 and 13.487 15.

Exercise 26

The accumulated value of a savings account based on regular periodic payments can be determined from the *annuity due equation*:

$$A = \frac{P}{i}[(1 + i)^n - 1]$$

In this equation, A is the amount in the account, P is the amount regularly deposited, and i is the rate of interest per period for the n deposit periods. An engineer would like to have a savings account valued at \$750 000 upon retirement in 20 years and can afford to put \$1500 per month toward this goal. What is the minimal interest rate at which this amount can be invested, assuming that the interest is compounded monthly?

Solution 26

Replacing symbols with numbers gives:

$$A = \frac{1500}{i}[(1 + i)^{20 \cdot 12} - 1]$$

Find the minimal interest rate is finding $i > 0$ such that $A \geq 750\,000$:

$$\begin{aligned} &\frac{1500}{i}[(1 + i)^{240} - 1] \geq 750\,000 \\ \iff &1500(1 + i)^{240} - 750\,000i - 1500 \geq 0 \qquad (*) \end{aligned}$$

Let

$$\begin{aligned} f(x) &= (1+x)^{240} - 500x - 1 \\ \Rightarrow f'(x) &= 240(x+1)^{239} - 500 \end{aligned}$$

Consider f'.

$$f'(x) = 0 \iff x = A = \sqrt[239]{\frac{25}{12}} - 1$$

As f' is monotonically increasing in $\mathbb{R}^+$, it follows that:

- f is monotonically decreasing in $D_1 = \mathbb{R}_{\leq A} \cap \mathbb{R}^+$
- f is monotonically increasing in $\mathbb{R}_{\geq A}$

Consider the set D_1.

$$f(0) = 0 > f(x) \, \forall x \in D_1$$

Therefore, (*) has no positive zero in D_1.
Consider the set $\mathbb{R}_{\geq A}$.

$$f(A) \approx -0.448\,119 \leq f(x) \, \forall x \in \mathbb{R}_{\geq A}$$

Therefore, f has at most one zero in $\mathbb{R}_{\geq A}$. Applying Newton's method on f with $p_0 = 0.005$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.005	−0.189 795 524 192 6	290.496 591 237 579 4
1	0.005 653 348 541 5	0.042 274 372 099 5	423.327 780 521 256 6
2	0.005 553 486 510 1	0.001 085 579 504 2	401.671 499 784 316 2
3	0.005 550 783 855 1	0.000 000 782 527 8	401.092 480 821 071 4
4	0.005 550 781 904 1	0.000 000 000 000 3	401.092 062 972 948
5	0.005 550 781 904 1	0.000 000 000 000 1	401.092 062 972 805 4

We conclude that the minimal monthly interest rate (assuming that the interest is compounded monthly) is approximately 0.555 078 %.

Exercise 27

Problems involving the amount of money required to pay off a mortgage over a fixed period of time involve the formula

$$A = \frac{P}{i}[1 - (1+i)^{-n}]$$

known as an *ordinary annuity equation*. In this equation, A is the amount of the mortgage, P is the amount of each payment, and i is the interest rate per period for the n payment periods. Suppose that a 30-year home mortgage in the amount of \$135 000 is needed and that the borrower can afford house payments of at most \$1000 per month. What is the maximal interest rate the borrower can afford to pay?

Solution 27

Replacing symbols with numbers gives:

$$A = \frac{1000}{i}[1 - (1+i)^{-(30\cdot 12)}]$$

Find the maximal interest rate is finding i such that $A \leq 135\,000$:

$$\begin{aligned} & \frac{1000}{i}[1-(1+i)^{-360}] \leq 135\,000 \\ \iff \; & 1000[1-(1+i)^{-360}] - 135\,000i \leq 0 \qquad (*) \end{aligned}$$

Let

$$\begin{aligned} f(x) &= 1 - (1+x)^{-360} - 135x \\ \Rightarrow f'(x) &= 360(x+1)^{-361} - 135 \end{aligned}$$

Consider f'.

$$f'(x) = 0 \iff x = A = \sqrt[-361]{0.375} - 1$$

As f' is monotonically decreasing in $\mathbb{R}^+$, it follows that:

- f is monotonically increasing in $D_1 = \mathbb{R}_{\leq A} \cap \mathbb{R}^+$
- f is monotonically decreasing in $\mathbb{R}_{\geq A}$

Consider the set D_1.

$$f(0) = 0 < f(x)\, \forall x \in D_1$$

Therefore, (*) has no positive zero in D_1.
Consider the set $\mathbb{R}_{\geq A}$.

$$f(A) \approx 0.256\,689 \geq f(x)\, \forall x \in \mathbb{R}_{\geq A}$$

Therefore, f has at most one zero in $\mathbb{R}_{\geq A}$. Applying Newton's method on f with $p_0 = 0.0067$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.0067	0.005 140 191 904 9	−102.686 966 410 826 1
1	0.006 750 056 906 8	−0.000 014 430 489 4	−103.261 805 313 492 4
2	0.006 749 917 160 1	−0.000 000 000 111 1	−103.260 214 863 510 3

We conclude that the maximal monthly interest rate is approximately 0.674 992 %.

Exercise 28

A drug administered to a patient produces a concentration in the blood stream given by $c(t) = Ate^{\frac{-t}{3}}$ milligrams per milliliter, t hours after A units have been injected. The maximum safe concentration is 1 mg/mL.

a) What amount should be injected to reach this maximum safe concentration, and when does this maximum occur?

b) An additional amount of this drug is to be administered to the patient after the concentration falls to 0.25 mg/mL. Determine, to the nearest minute, when this second injection should be given.

c) Assume that the concentration from consecutive injections is additive and that 75 % of the amount originally injected is administered in the second injection. When is it time for the third injection?

Solution 28

a) Let

$$f(x) = xe^{\frac{-x}{3}}$$
$$\Rightarrow f'(x) = \left(1 - \frac{x}{3}\right) e^{\frac{-x}{3}}$$

Consider f'.

$$f'(x) = 0 \iff x = 3$$

It's clear that f' is monotonically decreasing in $\mathbb{R}$. It follows that:

- f is monotonically increasing in $\mathbb{R}_{\leq 3}$
- f is monotonically decreasing in $\mathbb{R}_{\geq 3}$
- f has a global maximum at 3

We now know that $\max f = \frac{3}{e}$ is achieved at 3. In other words, the maximum concentration of any injection is reached 3 hours later, regardless of the amount administered.

To reach the maximum safe concentration of 1 mg/mL, the amount should be injected is:

$$A\frac{3}{e} = 1 \iff A = \frac{e}{3} \approx 0.906\,093\,942\,8$$

We conclude that to reach the maximum safe concentration, approximately 0.906 093 942 8 unit should be injected, and the concentration reaches its highest 3 hours after injection.

b) Let

$$g(t) = Ate^{\frac{-t}{3}} - 0.25$$
$$\Rightarrow g'(t) = A\left(1 - \frac{t}{3}\right)e^{\frac{-t}{3}}$$

with $A = \frac{e}{3}$.

We want to inject after the concentration of the first injection already reached its highest, therefore the second injection should be no sooner than 3 hours since the first one.

Applying Newton's method on g with $p_0 = 11.08$ generates the following table:

n	p_n	$g(p_n)$	$g'(p_n)$
0	11.08	−0.000 127 362	−0.060 739 197
1	11.077 903 126	0.000 000 028	−0.060 765 892
2	11.077 903 587	0	−0.060 765 887

We conclude that after about 11 hours and 5 minutes since the first injection, the second one can be administered.

c) Let

$$c_n(t) = \sum_{i=1}^{n} A_i(t - t_i)e^{\frac{-(t-t_i)}{3}}$$
$$\Rightarrow c_n'(t) = \sum_{i=1}^{n} A_i\left(1 - \frac{t - t_i}{3}\right)e^{\frac{-(t-t_i)}{3}}$$

be the function of concentration $t \geq t_n$ hours since the first injection *and* during that time window another $n - 1$ shots are administered. t_n is the number of hours between the first injection and the n^{th} one, and clearly $t_1 = 0$.

From the above parts, we know that $A_1 = \frac{e}{3}$, $A_2 = 0.75A_1 = \frac{e}{4}$, $t_2 = 11.077\,903\,587$.

Consider c_2.

$$c_2(t) = 0$$
$$\iff (1 - \frac{t}{3}) + 0.75(1 - \frac{t - t_2}{3})B = 0 \text{ with } B = e^{\frac{t_2}{3}}$$
$$\iff t - 3 = 2.25(3 - t + t_2)B$$
$$\iff t = \frac{2.25(t_2 + 3)B}{1 + 2.25B} \approx 13.923\,774\,83$$

We want to inject after the total concentration from the previous injections already reached its highest, therefore the third injection should be no sooner than 13.923 774 83 hours since the first one.

Applying Newton's method on $h_2 = c_2 - 0.25$ with $p_0 = 21.25$ generates the following table:

n	p_n	$h_2(p_n)$	$h_2'(p_n)$
0	21.25	−0.000 992 299 872 6	−0.059 350 960 587 8
1	21.233 280 811 923 6	0.000 001 664 222 2	−0.059 550 102 087 8
2	21.233 308 758 511 3	0.000 000 000 004 7	−0.059 549 768 906 2
3	21.233 308 758 589 5	0	−0.059 549 768 905 2

We conclude that after about 21 hours and 14 minutes since the first injection, the third one can be administered.

Exercise 29

Let

$$f(x) = 3^{3x+1} - 7 \cdot 5^{2x}$$

a) Use the Maple commands `solve` and `fsolve` to try to find all roots of f.

b) Plot f to find initial approximations to roots of f.

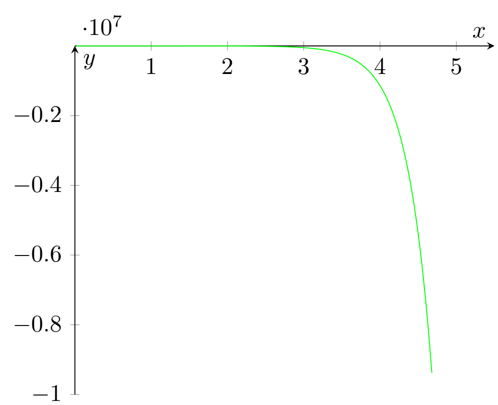
c) Use Newton's method to find roots of f to within 10^{-16}.

d) Find the exact solutions of $f(x) = 0$ without using Maple.

Solution 29

1. Opps, can't help without Maple license.

2. The graph of f is as follow:



No useful initial point found, every where: MATLAB, Maple, gnuplot,...

3. Let:

$$f(x) = 3^{3x+1} - 7 \cdot 5^{2x}$$
$$\Rightarrow f'(x) = 3(\ln 3)3^{3x+1} - 14(\ln 5)5^{2x}$$

Applying Newton's method on f with $p_0 = 11$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	11	−12 118 837 442 806	1 244 484 233 952 568
1	11.009 738 040 155 250 26	396 801 311 654	1 326 632 411 906 544
2	11.009 438 935 966 258 555	386 222 634	1 324 050 511 461 616
3	11.009 438 644 268 449 536	370	1 324 047 995 335 120
4	11.009 438 644 268 170 648	−38	1 324 047 995 332 592
5	11.009 438 644 268 199 07	4	1 324 047 995 332 848
6	11.009 438 644 268 195 517	66	1 324 047 995 333 032
7	11.009 438 644 268 145 779	0	1 324 047 995 332 608

So $p \approx 11.009\,438\,644\,268\,145\,779$.

4. Manipulating $f = 0$ gives:

$$\begin{aligned} f(x) &= 0 \\ \iff 3 \cdot 3^{3x} &= 7 \cdot 5^{2x} \\ \iff \frac{27^x}{25^x} &= \frac{7}{3} \\ \iff x &= \log_{27/25} \frac{7}{3} \end{aligned}$$

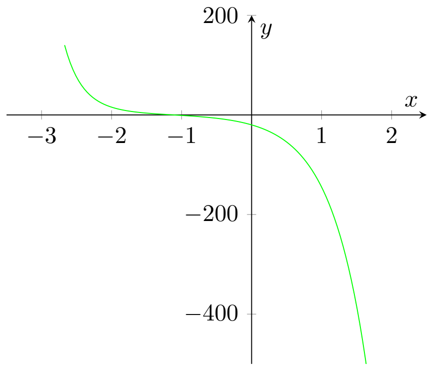
Exercise 30

Repeat Exercise 29 using $f(x) = 2^{x^2} - 3 \cdot 7^{x+1}$.

Solution 30

a) Opps, can't help without Maple license.

b) The graph of f is as follow:



c) Let:

$$f(x) = 2^{x^2} - 3 \cdot 7^{x+1}$$
$$\Rightarrow f'(x) = (\ln 2)2x2^{x^2} - 21(\ln 7)7^x$$

Applying Newton's method on f with $p_0 = 3.92$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	3.919 999 999 999 999 929	−909.989 020 751 884	145 585.672 581 531 893
1	3.926 250 539 662 426 32	22.625 719 019 627	152 874.530 827 350 565
2	3.926 102 537 775 538 082	0.013 028 085 261	152 698.506 017 085 223
3	3.926 102 452 456 528 891	0.000 000 004 293	152 698.404 592 337 756
4	3.926 102 452 456 500 913	0.000 000 000 095	152 698.404 592 304 723
5	3.926 102 452 456 500 469	−0.000 000 000 015	152 698.404 592 304 141

So $p \approx 3.926\,102\,452\,456\,500\,469$.

d) Manipulating $f = 0$ gives:

$$\begin{aligned} f(x) &= 0 \\ \iff 2^{x^2} &= 21 \cdot 7^x \\ \iff x^2 &= \log_2(21 \cdot 7^x) \\ &= \log_2 21 + x \log_2 7 \\ \iff x^2 - \log_2 7x - \log_2 21 &= 0 \\ \iff x &= \frac{\log_2 7 \pm \sqrt{\Delta}}{2} \text{ with } \Delta = (\log_2 7)^2 + 4 * \log_2 21 = \log_2 9\,529\,569 \end{aligned}$$

Exercise 31

The logistic population growth model is described by an equation of the form

$$P(t) = \frac{P_L}{1 - ce^{-kt}}$$

where P_L, c, and $k > 0$ are constants, and $P(t)$ is the population at time t. P_L represents the limiting value of the population since $\lim_{t\to\infty} P(t) = P_L$. Use the census data for the years 1950, 1960, and 1970 listed in the table on page 105 to determine the constants P_L, c, and k for a logistic growth model. Use the logistic model to predict the population of the United States in 1980 and in 2010, assuming $t = 0$ at 1950. Compare the 1980 prediction to the actual value.

Solution 31

We have:

$$P(0) = \frac{P_L}{1 - ce^{-k0}} = P_1 \iff ce^0 = 1 - \frac{P_L}{P_1} \tag{1}$$
$$P(10) = \frac{P_L}{1 - ce^{-k10}} = P_2 \iff ce^{-10k} = 1 - \frac{P_L}{P_2} \tag{2}$$
$$P(20) = \frac{P_L}{1 - ce^{-k20}} = P_3 \iff ce^{-20k} = 1 - \frac{P_L}{P_3} \tag{3}$$

Divide (1) by (2) and (2) by (3) gives:

$$e^{10k} = \frac{A - P_2 P_L}{A - P_1 P_L} \text{ with } A = P_1 P_2$$
$$e^{10k} = \frac{B - P_3 P_L}{B - P_2 P_L} \text{ with } B = P_2 P_3$$

Combining both above equations gives:

$$\frac{A - P_2 P_L}{A - P_1 P_L} = \frac{B - P_3 P_L}{B - P_2 P_L}$$
$$\iff (A - P_6 P_L)(B - P_6 P_L) = (A - P_5 P_L)(B - P_7 P_L)$$
$$\iff (P_6^2 - P_5 P_7)P_L^2 + (-AP_6 - BP_6 + AP_7 + BP_5)P_L = 0$$
$$\iff P_L = \frac{A(P_7 - P_6) + B(P_5 - P_6)}{P_5 P_7 - P_6^2} \approx 265\,816.4151$$

It follows that $k \approx 0.045\,017\,502\,25$, and $c \approx -0.756\,581\,255\,8$.
We now predict the US population in 1980 and 2010:

$$P_{1980} = P(30) \approx 222\,248.3277$$
$$P_{2010} = P(60) \approx 252\,967.4246$$

It is predicted, using the above model, that the US population in 1980 is 222 248 323 and in 2010 is 252 967 425. However, the actual population in 1980 is larger, so the 1980 prediction undershoots.

Exercise 32

The Gompertz population growth model is described by

$$P(t) = P_L e^{-ce^{-kt}}$$

where P_L, c, and $k > 0$ are constants, and $P(t)$ is the population at time t. Repeat Exercise 31 using the Gompertz growth model in place of the logistic model.

Solution 32

We have:

$$P(0) = P_L e^{-ce^{-k0}} = P_1 \iff e^{-k0} = \log_d \frac{P_1}{P_L} \tag{1}$$

$$P(10) = P_L e^{-ce^{-k10}} = P_2 \iff e^{-k10} = \log_d \frac{P_2}{P_L} \tag{2}$$

$$P(20) = P_L e^{-ce^{-k20}} = P_3 \iff e^{-k20} = \log_d \frac{P_3}{P_L} \tag{3}$$

with $d = e^{-c}$.

From (1), we know that:

$$e^{-k0} = 1 = \log_d \frac{P_1}{P_L} \iff d = \frac{P_1}{P_L}$$

Divide (1) by (2) and (2) by (3) gives:

$$e^{10k} = \frac{\log_d \frac{P_1}{P_L}}{\log_d \frac{P_2}{P_L}} = \frac{\log_d P_1 - \log_d P_L}{\log_d P_2 - \log_d P_L} = \frac{\ln P_1 - \ln P_L}{\ln P_2 - \ln P_L}$$

$$e^{10k} = \frac{\log_d \frac{P_2}{P_L}}{\log_d \frac{P_3}{P_L}} = \frac{\log_d P_2 - \log_d P_L}{\log_d P_3 - \log_d P_L} = \frac{\ln P_2 - \ln P_L}{\ln P_3 - \ln P_L}$$

Combining both above equations gives:

$$\begin{aligned} \frac{\ln P_1 - \ln P_L}{\ln P_2 - \ln P_L} &= \frac{\ln P_2 - \ln P_L}{\ln P_3 - \ln P_L} \\ \iff (\ln P_2 - \ln P_L)^2 &= (\ln P_1 - \ln P_L)(\ln P_3 - \ln P_L) \\ \iff (\ln P_2)^2 - 2\ln P_2 \ln P_L &= \ln P_1 \ln P_3 - \ln(P_1 P_3)\ln P_L \\ \iff \ln P_L &= \frac{(\ln P_2)^2 - \ln P_1 \ln P_3}{2\ln P_2 - \ln(P_1 P_3)} \\ \iff P_L &\approx 290\,227.8618 \end{aligned}$$

It follows that $k \approx 0.030\,200\,281\,3$, $d = 0.521\,404\,110\,1$, $c = 0.651\,229\,894\,7$.

We now predict the US population in 1980 and 2010:

$$P_{1980} = P(30) \approx 223\,069.2173$$
$$P_{2010} = P(60) \approx 260\,943.6839$$

It is predicted, using the above model, that the US population in 1980 is 223 069 217 and in 2010 is 260 943 684. However, the actual population in 1980 is larger, so the 1980 prediction undershoots.

Exercise 33

Player A will shut out (win by a score of 21-0) player B in a game of racquetball with probability

$$P = \frac{1+p}{2}\left(\frac{p}{1-p+p^2}\right)^{21}$$

where p denotes the probability A will win any specific rally (independent of the server). Determine, to within 10^{-3}, the minimal value of p that will ensure that A will shut out B in at least half the matches they play.

Solution 33

Let

$$g(x) = \frac{x}{1-x+x^2}$$
$$\Rightarrow g'(x) = \frac{1-x^2}{(1-x+x^2)^2}$$

$$\begin{aligned}
f(x) &= \frac{1+x}{2}\left(\frac{x}{1-x+x^2}\right)^{21} \\
\Rightarrow f'(x) &= \frac{1}{2}\left(\frac{x}{1-x+x^2}\right)^{21} + \frac{1+x}{2}21\left(\frac{x}{1-x+x^2}\right)^{20}\frac{1-x^2}{(1-x+x^2)^2} \\
&= \frac{1}{2}\left(\frac{x}{1-x+x^2}\right)^{20}\left[\frac{x}{1-x+x^2} + \frac{21(1+x)(1-x^2)}{(1-x+x^2)^2}\right] \\
&= \frac{1}{2}\left(\frac{x}{1-x+x^2}\right)^{20}\frac{-20x^3-22x^2+22x+21}{(1-x+x^2)^2}
\end{aligned}$$

Finding the minimal value of p that will ensure that A will shut out B in at least half the matches they play is finding the minimal $x \in D = [0,1]$ such that $f(x) \geq 0.5$.

Consider g'.

$$g'(x) = 0 \iff x = \pm 1$$
$$x^2 - x + 1 = x^2 - 2x0.5 + 0.5^2 + 0.75 \geq 0.75 > 0\,\forall x \in \mathbb{R}$$

It follows that the sign of g' is the sign of $1-x^2$. Therefore, in D, $g' \geq 0$. Therefore, g and then f are monotonically increasing in D:

$$f(0) = 0 \leq f(x) \leq f(1) = 1 \,\forall x \in D$$

It's clear that $f(x) \geq 0.5$ is guaranteed to have solution in D.

Applying Newton's method on $h = f - 0.5$ with $p_0 = 0.84$ generates the following table:

n	p_n	$h(p_n)$	$h'(p_n)$
0	0.84	−0.010 231 745 763 236 211	4.430 566 512 699 972 925
1	0.842 309 353 834 076 791	0.000 020 294 149 810 418	4.447 757 674 207 621 47
2	0.842 304 791 051 817 325	0.000 000 000 072 282 402	4.447 725 988 980 080 203
3	0.842 304 791 035 565 77	0.000 000 000 000 000 888	4.447 725 988 867 216 707
4	0.842 304 791 035 565 548	−0.000 000 000 000 000 444	4.447 725 988 867 211 377

We conclude that $p \geq 0.842\,304\,791\,035\,565\,548$ will ensure that A will shut out B in at least half the matches they play.

Exercise 34

In the design of all-terrain vehicles, it is necessary to consider the failure of the vehicle when attempting to negotiate two types of obstacles. One type of failure is called *hang-up failure* and occurs when the vehicle attempts to cross an obstacle that causes the bottom of the vehicle to touch the ground. The other type of failure is called *nose-in failure* and occurs when the vehicle descends into a ditch and its nose touches the ground.

The accompanying figure shows the components associated with the nose-in failure of a vehicle. It is shown that the maximum angle α that can be negotiated by a vehicle when β is the maximum angle at which hang-up failure does *not* occur satisfies the equation

$$A \sin\alpha \cos\alpha + B \sin^2\alpha - C \cos\alpha - E \sin\alpha = 0$$

where

$$\begin{cases} D : \text{wheel diameter} \\ A = l \sin\beta_1 \\ B = l \cos\beta_1 \\ C = (h + 0.5D)\sin\beta_1 - 0.5D\tan\beta_1 \\ E = (h + 0.5D)\cos\beta_1 - 0.5D \end{cases}$$

a) It is stated that when $l = 89\,\text{in}$, $h = 49\,\text{in}$, $D = 55\,\text{in}$, and $\beta_1 = 11.5°$, angle α is approximately $33°$. Verify this result.

b) Find α for the situation when l, h, and β_1 are the same as in part a) but $D = 30\,\text{in}$.

Solution 34

Let

$$\begin{aligned} f(x) &= A\sin x\cos x + B\sin^2 x - C\cos x - E\sin x \\ \Rightarrow f'(x) &= A(\cos^2 x - \sin^2 x) + 2B\sin x\cos x + C\sin x - E\cos x \end{aligned}$$

a) Applying Newton's method on f with $p_0 = 33° \approx 0.575\,958\,653\,158\,13$ generates the following table:

n	p_n	$g(p_n)$	$g'(p_n)$
0	0.575 958 653 158 13	0.025 411 305 811 59	52.342 904 131 061 25
1	0.575 473 175 589 9	0.000 008 546 838 91	52.307 681 811 205 21
2	0.575 473 012 194 42	0.000 000 000 000 97	52.307 669 944 135 87
3	0.575 473 012 194 4	0	52.307 669 944 134 55

So $\alpha \approx 0.575\,473\,012\,194\,4 \approx 32.972\,174\,82°$, which is indeed close to 33°.

b) Applying Newton's method on f with $p_0 = 33° \approx 0.575\,958\,653\,158\,13$ generates the following table:

n	p_n	$f(p_n)$	$f'(p_n)$
0	0.575 958 653 158 13	−0.154 079 021 971 57	52.160 253 446 542 13
1	0.578 912 607 784 32	0.000 315 645 554 17	52.373 508 587 763 42
2	0.578 906 580 967 27	0.000 000 001 302 72	52.373 076 275 399 87
3	0.578 906 580 942 4	0.000 000 000 000 01	52.373 076 273 615 62

So $\alpha \approx 0.578\,906\,580\,942\,4 \approx 33.168\,903\,82°$.