Bài tập Bài 4.1 Phương pháp khử Gauss

(Trang 368)

1. For each of the following linear systems, obtain a solution by graphical methods, if possible. Explain the results from a geometrical standpoint.

a. $x_1 + 2x_2 = 3,$ $x_1 - x_2 = 0.$

b. $x_1 + 2x_2 = 3,$ $2x_1 + 4x_2 = 6.$

c. $x_1 + 2x_2 = 0,$ $2x_1 + 4x_2 = 0.$

d. $2x_1 + x_2 = -1,$ $4x_1 + 2x_2 = -2,$ $x_1 - 3x_2 = 5.$

2. For each of the following linear systems, obtain a solution by graphical methods, if possible. Explain the results from a geometrical standpoint.

a. $x_1 + 2x_2 = 0,$ $x_1 - x_2 = 0.$

b. $x_1 + 2x_2 = 3,$ $-2x_1 - 4x_2 = 6.$

c. $2x_1 + x_2 = -1,$ $x_1 + x_2 = 2,$ $x_1 - 3x_2 = 5.$

d. $2x_1 + x_2 + x_3 = 1,$ $2x_1 + 4x_2 - x_3 = -1.$

3. Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations. (The exact solution to each system is $x_1 = 1, x_2 = -1, x_3 = 3$.)

a.
$$\begin{aligned} 4x_1 - x_2 + x_3 &= 8, \\ 2x_1 + 5x_2 + 2x_3 &= 3, \\ x_1 + 2x_2 + 4x_3 &= 11. \end{aligned}$$

b.
$$\begin{aligned} 4x_1 + x_2 + 2x_3 &= 9, \\ 2x_1 + 4x_2 - x_3 &= -5, \\ x_1 + x_2 - 3x_3 &= -9. \end{aligned}$$

4. Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations. (The exact solution to each system is $x_1 = -1, x_2 = 1, x_3 = 3$.)

a.
$$\begin{aligned} -x_1 + 4x_2 + x_3 &= 8, \\ \tfrac{5}{3}x_1 + \tfrac{2}{3}x_2 + \tfrac{2}{3}x_3 &= 1, \\ 2x_1 + x_2 + 4x_3 &= 11. \end{aligned}$$

b.
$$\begin{aligned} 4x_1 + 2x_2 - x_3 &= -5, \\ \tfrac{1}{9}x_1 + \tfrac{1}{9}x_2 - \tfrac{1}{3}x_3 &= -1, \\ x_1 + 4x_2 + 2x_3 &= 9. \end{aligned}$$

5. Use the Gaussian Elimination Algorithm to solve the following linear systems, if possible, and determine whether row interchanges are necessary:

a.
$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2,\\ 3x_1 - 3x_2 + x_3 &= -1,\\ x_1 + x_2 &= 3. \end{aligned}$$

b.
$$\begin{aligned} 2x_1 - 1.5x_2 + 3x_3 &= 1,\\ -x_1 + 2x_3 &= 3,\\ 4x_1 - 4.5x_2 + 5x_3 &= 1. \end{aligned}$$

c.
$$\begin{aligned} 2x_1 &= 3,\\ x_1 + 1.5x_2 &= 4.5,\\ -3x_2 + 0.5x_3 &= -6.6,\\ 2x_1 - 2x_2 + x_3 + x_4 &= 0.8. \end{aligned}$$

d.
$$\begin{aligned} x_1 + x_2 + x_4 &= 2,\\ 2x_1 + x_2 - x_3 + x_4 &= 1,\\ 4x_1 - x_2 - 2x_3 + 2x_4 &= 0,\\ 3x_1 - x_2 - x_3 + 2x_4 &= -3. \end{aligned}$$

6. Use the Gaussian Elimination Algorithm to solve the following linear systems, if possible, and determine whether row interchanges are necessary:

a.
$$\begin{aligned} x_2 - 2x_3 &= 4,\\ x_1 - x_2 + x_3 &= 6,\\ x_1 - x_3 &= 2. \end{aligned}$$

b.
$$\begin{aligned} x_1 - \tfrac{1}{2}x_2 + x_3 &= 4,\\ 2x_1 - x_2 - x_3 + x_4 &= 5,\\ x_1 + x_2 + \tfrac{1}{2}x_3 &= 2,\\ x_1 - \tfrac{1}{2}x_2 + x_3 + x_4 &= 5. \end{aligned}$$

c.
$$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 6,\\ x_2 - x_3 + x_4 &= 5,\\ x_4 &= 5,\\ x_3 - x_4 &= 3. \end{aligned}$$

d.
$$\begin{aligned} x_1 + x_2 + x_4 &= 2,\\ 2x_1 + x_2 - x_3 + x_4 &= 1,\\ -x_1 + 2x_2 + 3x_3 - x_4 &= 4,\\ 3x_1 - x_2 - x_3 + 2x_4 &= -3. \end{aligned}$$

7. Use Algorithm 6.1 and Maple with *Digits*:= 10 to solve the following linear systems.

a.
$$\begin{aligned} \tfrac{1}{4}x_1 + \tfrac{1}{5}x_2 + \tfrac{1}{6}x_3 &= 9,\\ \tfrac{1}{3}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{5}x_3 &= 8,\\ \tfrac{1}{2}x_1 + x_2 + 2x_3 &= 8. \end{aligned}$$

b.
$$\begin{aligned} 3.333x_1 + 15920x_2 - 10.333x_3 &= 15913,\\ 2.222x_1 + 16.71x_2 + 9.612x_3 &= 28.544,\\ 1.5611x_1 + 5.1791x_2 + 1.6852x_3 &= 8.4254. \end{aligned}$$

c.
$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 + \tfrac{1}{4}x_4 &= \tfrac{1}{6},\\ \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{4}x_3 + \tfrac{1}{5}x_4 &= \tfrac{1}{7},\\ \tfrac{1}{3}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{5}x_3 + \tfrac{1}{6}x_4 &= \tfrac{1}{8},\\ \tfrac{1}{4}x_1 + \tfrac{1}{5}x_2 + \tfrac{1}{6}x_3 + \tfrac{1}{7}x_4 &= \tfrac{1}{9}. \end{aligned}$$

d.
$$\begin{aligned} 2x_1 + x_2 - x_3 + x_4 - 3x_5 &= 7,\\ x_1 + 2x_3 - x_4 + x_5 &= 2,\\ -2x_2 - x_3 + x_4 - x_5 &= -5,\\ 3x_1 + x_2 - 4x_3 + 5x_5 &= 6,\\ x_1 - x_2 - x_3 - x_4 + x_5 &= 3. \end{aligned}$$

8. Use Algorithm 6.1 and Maple with *Digits*:= 10 to solve the following linear systems.

a.
$$\begin{aligned} \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 - \tfrac{1}{8}x_3 &= 0,\\ \tfrac{1}{3}x_1 - \tfrac{1}{6}x_2 + \tfrac{1}{9}x_3 &= 1,\\ \tfrac{1}{7}x_1 + \tfrac{1}{7}x_2 + \tfrac{1}{10}x_3 &= 2. \end{aligned}$$

b.
$$\begin{aligned} 2.71x_1 + x_2 + 1032x_3 &= 12,\\ 4.12x_1 - x_2 + 500x_3 &= 11.49,\\ 3.33x_1 + 2x_2 - 200x_3 &= 41. \end{aligned}$$

c.
$$\begin{aligned} \pi x_1 + \sqrt{2}x_2 - x_3 + x_4 &= 0,\\ ex_1 - x_2 + x_3 + 2x_4 &= 1,\\ x_1 + x_2 - \sqrt{3}x_3 + x_4 &= 2,\\ -x_1 - x_2 + x_3 - \sqrt{5}x_4 &= 3. \end{aligned}$$

d.
$$\begin{aligned} x_1 + x_2 - x_3 + x_4 - x_5 &= 2,\\ 2x_1 + 2x_2 + x_3 - x_4 + x_5 &= 4,\\ 3x_1 + x_2 - 3x_3 - 2x_4 + 3x_5 &= 8,\\ 4x_1 + x_2 - x_3 + 4x_4 - 5x_5 &= 16,\\ 16x_1 - x_2 + x_3 - x_4 - x_5 &= 32. \end{aligned}$$

9. Given the linear system

$$\begin{aligned} 2x_1 - 6\alpha x_2 &= 3, \\ 3\alpha x_1 - \quad x_2 &= \tfrac{3}{2}. \end{aligned}$$

a. Find value(s) of α for which the system has no solutions.

b. Find value(s) of α for which the system has an infinite number of solutions.

c. Assuming a unique solution exists for a given α, find the solution.

10. Given the linear system

$$\begin{aligned} x_1 - \quad x_2 + \alpha x_3 &= -2, \\ -x_1 + 2x_2 - \alpha x_3 &= 3, \\ \alpha x_1 + \quad x_2 + \quad x_3 &= 2. \end{aligned}$$

a. Find value(s) of α for which the system has no solutions.

b. Find value(s) of α for which the system has an infinite number of solutions.

c. Assuming a unique solution exists for a given α, find the solution.

11. Show that the operations

a. $(\lambda E_i) \to (E_i)$ **b.** $(E_i + \lambda E_j) \to (E_i)$ **c.** $(E_i) \leftrightarrow (E_j)$

do not change the solution set of a linear system.

12. **Gauss-Jordan Method:** This method is described as follows. Use the ith equation to eliminate not only x_i from the equations $E_{i+1}, E_{i+2}, \ldots, E_n$, as was done in the Gaussian elimination method, but also from $E_1, E_2, \ldots, E_{i-1}$. Upon reducing $[A, \mathbf{b}]$ to:

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & 0 & \cdots & 0 & a_{1,n+1}^{(1)} \\ 0 & a_{22}^{(2)} & \ddots & \vdots & a_{2,n+1}^{(2)} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & a_{nn}^{(n)}) & a_{n,n+1}^{(n)} \end{array}\right],$$

the solution is obtained by setting

$$x_i = \frac{a_{i,n+1}^{(i)}}{a_{ii}^{(i)}},$$

for each $i = 1, 2, \ldots, n$. This procedure circumvents the backward substitution in the Gaussian elimination. Construct an algorithm for the Gauss-Jordan procedure patterned after that of Algorithm 6.1.

13. Use the Gauss-Jordan method and two-digit rounding arithmetic to solve the systems in Exercise 3.

14. Repeat Exercise 7 using the Gauss-Jordan method.

15. **a.** Show that the Gauss-Jordan method requires

$$\frac{n^3}{2} + n^2 - \frac{n}{2} \quad \text{multiplications/divisions}$$

and

$$\frac{n^3}{2} - \frac{n}{2} \quad \text{additions/subtractions.}$$

b. Make a table comparing the required operations for the Gauss-Jordan and Gaussian elimination methods for $n = 3, 10, 50, 100$. Which method requires less computation?

16. Consider the following Gaussian-elimination-Gauss-Jordan hybrid method for solving the system (6.4). First, apply the Gaussian-elimination technique to reduce the system to triangular form. Then use the nth equation to eliminate the coefficients of x_n in each of the first $n-1$ rows. After this is completed use the $(n-1)$st equation to eliminate the coefficients of x_{n-1} in the first $n-2$ rows, etc. The system will eventually appear as the reduced system in Exercise 12.

a. Show that this method requires

$$\frac{n^3}{3} + \frac{3}{2}n^2 - \frac{5}{6}n \quad \text{multiplications/divisions}$$

and

$$\frac{n^3}{3} + \frac{n^2}{2} - \frac{5}{6}n \quad \text{additions/subtractions.}$$

b. Make a table comparing the required operations for the Gaussian elimination, Gauss-Jordan, and hybrid methods, for $n = 3, 10, 50, 100$.

17. Use the hybrid method described in Exercise 16 and two-digit rounding arithmetic to solve the systems in Exercise 3.

18. Repeat Exercise 7 using the method described in Exercise 16.

19. Suppose that in a biological system there are n species of animals and m sources of food. Let x_j represent the population of the jth species, for each $j = 1, \cdots, n$; b_i represent the available daily supply of the ith food; and a_{ij} represent the amount of the ith food consumed on the average by a member of the jth species. The linear system

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1,\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2,\\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

represents an equilibrium where there is a daily supply of food to precisely meet the average daily consumption of each species.

a. Let

$$A = [a_{ij}] = \left[\begin{array}{ccc:c} 1 & 2 & 0 & 3 \\ 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right],$$

$\mathbf{x} = (x_j) = [1000, 500, 350, 400]$, and $\mathbf{b} = (b_i) = [3500, 2700, 900]$. Is there sufficient food to satisfy the average daily consumption?

b. What is the maximum number of animals of each species that could be individually added to the system with the supply of food still meeting the consumption?

c. If species 1 became extinct, how much of an individual increase of each of the remaining species could be supported?

d. If species 2 became extinct, how much of an individual increase of each of the remaining species could be supported?

20. A Fredholm integral equation of the second kind is an equation of the form

$$u(x) = f(x) + \int_a^b K(x,t)u(t)\,dt,$$

where a and b and the functions f and K are given. To approximate the function u on the interval $[a,b]$, a partition $x_0 = a < x_1 < \cdots < x_{m-1} < x_m = b$ is selected and the equations

$$u(x_i) = f(x_i) + \int_a^b K(x_i,t)u(t)\,dt, \quad \text{for each } i = 0, \cdots, m,$$

are solved for $u(x_0), u(x_1), \cdots, u(x_m)$. The integrals are approximated using quadrature formulas based on the nodes $x_0, \cdots, x_m$. In our problem, $a = 0$, $b = 1$, $f(x) = x^2$, and $K(x,t) = e^{|x-t|}$.

a. Show that the linear system

$$u(0) = f(0) + \frac{1}{2}[K(0,0)u(0) + K(0,1)u(1)],$$

$$u(1) = f(1) + \frac{1}{2}[K(1,0)u(0) + K(1,1)u(1)]$$

must be solved when the Trapezoidal rule is used.

b. Set up and solve the linear system that results when the Composite Trapezoidal rule is used with $n = 4$.

c. Repeat part (b) using the Composite Simpson's rule.