

# POROELASTICITY 1D CODE

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GitHub Link : <https://github.com/nvohra0016/Biot1D-MATLAB>

## 1. INTRODUCTION

In this document we provide the details of a one dimensional (1D) implementation of Biot's poroelasticity system [3] in MATLAB. We use a 3 field, mixed finite element scheme [7] in which the displacement is approximated in the space of piecewise linear polynomials, pressure in the space of piecewise constants, and fluxes in the lowest order Raviart-Thomas space.

**1.1. Credits and Use.** This code is part of the MPower toolbox:

<http://sites.science.oregonstate.edu/~mpesz/mpower/>

The code is publicly available through GitHub: <https://github.com/nvohra0016/Biot1D-MATLAB>. The implementation is licensed under the Creative Commons CC BY-NC-ND 4.0 Attribution-NonCommercial-NoDerivatives 4.0 International license and the GNU GPL license. To view the full license, please see the file License.md in the GitHub repository. To view the funding sources, please see the Acknowledgement section (Section 7.1) in this document. For any questions, please feel free to contact the authors via email:

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## 2. GOVERNING EQUATIONS

Let  $\Omega = (a, b) \subset \mathbb{R}$  represent a porous medium that is fully saturated with a fluid. For any  $x \in \Omega$ ,  $t > 0$ , let  $u(x, t)$  denote the displacement of the medium and  $p(x, t)$  denote the pressure. Then,  $\forall x \in \Omega$ ,  $t > 0$  as in [5](Pg. 114)

$$-\frac{\partial}{\partial x} \left[ (\lambda + 2\mu) \frac{\partial u}{\partial x} \right] + \alpha \frac{\partial p}{\partial x} = f + \bar{\rho} G \frac{\partial D}{\partial x}, \quad (1a)$$

$$\frac{\partial \eta_f}{\partial t} + \frac{\partial q_f}{\partial x} = \frac{g}{\rho_f}, \quad (1b)$$

$$q_f = -\frac{\kappa}{\mu_f} \left[ \frac{\partial p}{\partial x} - \rho_f G \frac{\partial D}{\partial x} \right], \quad (1c)$$

where  $\eta_f$  denotes the total fluid content, given by

$$\eta_f(x, t) = \beta_f \phi p + \alpha \frac{\partial u}{\partial x}, \quad \forall x \in \Omega, t > 0, \quad (2)$$

Variable	Description/ SI Unit	Preferred units
$u$	Displacement [m]	[m]
$p$	Pressure [Pa]	[MPa]
$q_f$	Flux [m/s]	[m/hr]
$\eta_f$	Total fluid content [–]	[–]
$D$	Depth [m]	[m]
$f$	Volumetric body force [N/m <sup>3</sup> ]	[MPa/m]
$g$	Mass source rate [kg/m <sup>3</sup> s]	[MPa hr/m <sup>2</sup> ]
Parameter		Typical value
$E$	Young's modulus [Pa]	Sand (medium/fine): 12 – 20[MPa] [4](Pg. 407) Silt: 2 – 20[MPa] [4](Pg. 407) Clay (medium/firm): 15 – 50[MPa] [4](Pg. 406)
$\nu$	Poisson's ratio [–]	Sand (medium/ fine): 0.25 [4](Pg. 407) Silt : 0.30 – 0.35 [4](Pg. 407) Clay(medium/firm): 0.30 [4](Pg. 406)
$\lambda$	Lamé parameter [Pa]; $\frac{E\nu}{(1+\nu)(1-2\nu)}$	
$\mu$	Lamé parameter [Pa]; $\frac{E}{2(1+\nu)}$	
$\rho_f$	Fluid density [kg/m <sup>3</sup> ]	998.21[kg/m <sup>3</sup> ] (at 20° C) [6]
$\rho_s$	Solid particles density [kg/m <sup>3</sup> ]	Sand: 2650[kg/m <sup>3</sup> ] [1](Pg. 22) Clay: 2700[kg/m <sup>3</sup> ] [1](Pg. 22)
$\phi$	Porosity [–]	Sand(medium/fine): 0.30 – 0.35 [2](Pg. 74) Silt: 0.4 – 0.5 [2](Pg. 74) Clay: 0.45 – 0.55 [2](Pg. 74)
$\bar{\rho}$	Average density of media [kg/m <sup>3</sup> ]; $\rho_f\phi + \rho_s(1 - \phi)$	–[kg/m <sup>3</sup> ]
$\beta_f$	Fluid compressibility [1/Pa]; $\frac{1}{\rho_f} \frac{\partial \rho_f}{\partial p}$	$4.16 \times 10^{-4}$ [1/MPa] [6]
$\kappa$	Permeability [m <sup>2</sup> ]	Sand: $10^{-13} - 10^{-11}$ [m <sup>2</sup> ] (saturated) [4](Pg. 373) Silt: $10^{-15} - 10^{-13}$ [m <sup>2</sup> ] (saturated) [4](Pg. 373) Clay: $10^{-18} - 10^{-15}$ [m <sup>2</sup> ] (saturated) [4](Pg. 373)
$\mu_f$	Fluid viscosity [Pa s]	$2.7822 \times 10^{-13}$ [MPa hr] (at 20°C) [6]
$G$	Acceleration due to gravity [m/s <sup>2</sup> ]	$1.27290528 \times 10^8$ [m/hr <sup>2</sup> ] (at Anchorage, Alaska) [6]
Conversion factors: $1[\text{kg/m}^3] = \frac{10^{-6}}{3600^2} [\text{MPa hr}^2/\text{m}^2]$ , $1[\text{m/s}^2] = 3600^2 [\text{m/hr}^2]$		

TABLE 1. Variables and parameters used throughout the article.

$q_f$  is the flux, and  $D$  is the depth. The other variables and physical parameters used in (1) are described in Table 1. Mechanical gravitational effects are included in (1a) using the term  $\bar{\rho}G \frac{\partial D}{\partial x}$  and the hydrological gravitational effects are included in Darcy's law (1c) using  $\rho_f G \frac{\partial D}{\partial x}$ . Here, the depth  $D$  is a linear function which depends on the physical scenario. For example, if  $\Omega = (a, b)$  represents a vertical column of soil with  $x = a$  being the top and  $x = b$  being the bottom, then

$$D(x) = x - a, \quad \forall x \in \Omega. \quad (3)$$

If  $x = a$  is the bottom then

$$D(x) = b - x, \quad \forall x \in \Omega. \quad (4)$$

For a horizontal domain we have  $D = 0$ .

**2.1. Boundary and initial conditions.** The two distinct sets of boundary conditions correspond to mechanical deformation [M] and hydrological flow [H]. Mixed boundary conditions corresponding to [M] are

$$u = u_D, \text{ on } \Gamma_{MD}, \quad (5a)$$

$$\tilde{\sigma}n = \sigma_N, \text{ on } \Gamma_{MN}, \quad (5b)$$

where  $\Gamma_{MD}, \Gamma_{MN} \subset \{a, b\}$ ,  $\Gamma_{MD} \cup \Gamma_{MN} = \{a, b\}$ , and  $\Gamma_{MD} \neq \emptyset$ , and  $n$  is the outward unit normal to  $\Gamma_{MN}$ . The boundary conditions corresponding to [H] are

$$p = p_D, \text{ on } \Gamma_{HD}, \quad (6a)$$

$$q_f n = q_N, \text{ on } \Gamma_{HN}, \quad (6b)$$

where  $\Gamma_{HD}, \Gamma_{HN} \subset \{a, b\}$ .

The initial condition is given on the fluid content  $\eta_f$  as

$$\eta_f(x, 0) = \eta_{f_{init}}(x) \quad \forall x \in (a, b), \quad (7)$$

although in practise it is sometimes computed using initial conditions on  $p$  and  $\frac{\partial u}{\partial x}$ .

We test multiple examples with different combinations of the Dirichlet (D) and Neumann (N) boundary conditions. In our notation, we specify the deformation boundary conditions followed by the flow boundary conditions: [MMHH]. An example of our notation is as follows: a case with Dirichlet boundary conditions for [M] and mixed boundary conditions for [H], with  $\Gamma_{HD} = \{a\}$  and  $\Gamma_{HN} = \{b\}$  is denoted as [DDDN].

### 3. DISCRETIZATION

Consider the discretization of  $\Omega = (a, b)$  into  $M$  cells using a grid  $\mathcal{T}_h = (\omega_j)_j$  so that  $\Omega = \cup_{j=1}^M \omega_j$ , where the cell  $\omega_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$  has center  $x_j$ ,  $1 \leq j \leq M$  and size  $|\omega_j| = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} = h_j$ . Consider a uniform time step  $\tau > 0$  so that the  $n^{th}$  time step is given by  $t_n = n\tau$ .

**3.1. Approximation spaces.** Here we describe the finite dimensional function spaces built on  $\mathcal{T}_h$  in which we seek an approximate solution to (1). Let

$$V_h = \{\phi_h \in C^0(\Omega) \mid \phi_h|_{\omega_j} = a_j x + b_j, \quad a_j, b_j \in \mathbb{R}, \quad 1 \leq j \leq M\}, \quad (8a)$$

be the space of continuous, piecewise linear functions on  $\Omega$  and

$$V_{h,0} \subset V_h, \quad V_{h,0} = \{\phi_h \in V_h \mid \phi_h|_{\Gamma_{MD}} = 0\}, \quad (8b)$$

be the subspace of  $V_h$  containing functions which vanish on  $\Gamma_{MD}$ . Let

$$M_h = \{\eta_h \mid \eta_h|_{\omega_j} = \text{const}, \quad \forall 1 \leq j \leq M\}, \quad (8c)$$

be the space of piecewise constants on  $\Omega$  and

$$X_h = \{\psi_h \in C^0(\Omega) \mid \psi_h|_{\omega_j} = a_j x + b_j, \quad a_j, b_j \in \mathbb{R}, \quad 1 \leq j \leq M\}, \quad (8d)$$

be the lowest order Raviart-Thomas space of functions with

$$X_{h,0} \subset X_h, \quad X_{h,0} = \{\psi_h \in X_h \mid \psi_h|_{\Gamma_{HN}} = 0\}, \quad (9)$$

its subspace consisting of functions which vanish on  $\Gamma_{HN}$ . The basis functions of these spaces are described in Section 7.

**3.2. Discrete problem.** Let  $\tilde{u}_D^n \in V_h$ , be such that

$$\tilde{u}_D^n(x_{i-\frac{1}{2}}) = u_D(x_{i-\frac{1}{2}}, t_n), \quad \forall x_{i-\frac{1}{2}} \in \Gamma_{MD}, \quad n \geq 1, \quad (10a)$$

and  $\tilde{q}_N^n \in X_h$  be such that

$$\tilde{q}_N^n(x_{i-\frac{1}{2}}) = q_N(x_{i-\frac{1}{2}}, t_n), \quad \forall x_{i-\frac{1}{2}} \in \Gamma_{HN}, \quad n \geq 1, \quad (10b)$$

whenever  $|\Gamma_{HN}| > 0$ . The three field discrete problem corresponding to (1) is given by: find  $u_h^n \in \tilde{u}_D^n + V_{h,0}$ ,  $p_h^n \in M_h$ ,  $q_f^n \in \tilde{q}_N^n + X_{h,0}$  such that

$$\left( (\lambda + 2\mu) \frac{\partial u_h^n}{\partial x}, \frac{\partial \phi_h}{\partial x} \right) - \alpha \left( p_h^n, \frac{\partial \phi_h}{\partial x} \right) = \left( f(\cdot, t_n) + \bar{\rho} G \frac{\partial D}{\partial x}, \phi_h \right) + [\sigma_N \phi_h] \big|_{\Gamma_{MN}}, \quad (11a)$$

$$\forall \phi_h \in V_{h,0},$$

$$(\beta_f \phi p_h^n, \eta_h) + \alpha \left( \frac{\partial u_h^n}{\partial x}, \eta_h \right) + \tau \left( \frac{\partial q_f^n}{\partial x}, \eta_h \right) = \frac{\tau}{\rho_f} (g(\cdot, t_n), \eta_h) + (\eta_f^{n-1}, \eta_h), \quad (11b)$$

$$\forall \eta_h \in M_h,$$

$$\left( \left( \frac{\kappa}{\mu_f} \right)^{-1} q_f^n, \psi_h \right) - \left( p_h^n, \frac{\partial \psi_h}{\partial x} \right) = -[p_D(\cdot, t_n) \psi_h n] \big|_{\Gamma_{HN}} + \rho_f G \left( \frac{\partial D}{\partial x}, \psi_h \right), \quad (11c)$$

$$\forall \psi_h \in X_{h,0}.$$

Let  $U^n, P^n, Q_f^n$  the degrees of freedom of  $u_h^n, p_h^n, q_f^n$  in their bases. Then (11) can be rewritten in the matrix form

$$\begin{bmatrix} A_{uu} & -\alpha A_{pu} & 0 \\ \alpha A_{pu}^T & M_{pp} & \tau A_{qfp} \\ 0 & -A_{qfp}^T & M_{qfqf} \end{bmatrix} \begin{bmatrix} U^n \\ P^n \\ Q_f^n \end{bmatrix} = \begin{bmatrix} \mathcal{F}^n \\ \mathcal{G}^n \\ \mathcal{H}^n \end{bmatrix}, \quad n \geq 1, \quad (12)$$

where the stiffness and mass matrices  $\{A_{uu}, A_{pu}, A_{qfp}, M_{pp}, M_{qfqf}\}$  and vectors  $\{\mathcal{F}^n, \mathcal{G}, \mathcal{H}^n\}$  are described in Section 7. The linear system (12) can be further reduced to

$$\mathcal{M} \begin{bmatrix} U^n \\ P^n \end{bmatrix} = \begin{bmatrix} \mathcal{F}^n \\ -\mathcal{G}^n + \tau A_{qfp} M_{qfqf}^{-1} \mathcal{H}^n \end{bmatrix}, \quad (13a)$$

$$Q_f^n = M_{qfqf}^{-1} \left( \mathcal{H}^n + A_{qfp}^T P^n \right), \quad (13b)$$

where

$$\mathcal{M} = \begin{bmatrix} A_{uu} & -\alpha A_{pu} \\ -\alpha A_{pu}^T & -M_{pp} - \tau A_{qfp} M_{qfqf}^{-1} A_{qfp}^T \end{bmatrix}. \quad (14)$$

#### 4. ALGORITHM

The code executes the following steps:

- (1) The spatial and temporal grid are generated using the spatial domain end points,  $\{a, b\}$ , and the final time  $T_{end}$  inputted by the user.
  - (a) The spatial grid can be uniform or non-uniform. In the first case, the grid size is determined using the number of cells  $M$  as an input. For a non-uniform grid, the node positions  $\{x_{j+\frac{1}{2}}\}_j$  are inputted.

- (b) The temporal grid is uniform, and is generated from the user inputted number of time steps  $N$  as  $\tau = \frac{T_{end}}{N}$ .
- (2) The physical parameters, boundary conditions, and the initial condition is specified by the user.
- (3) The code computes the matrix  $\mathcal{M}$  given by (14) and vectors  $\{\mathcal{F}^n, \mathcal{G}^n, \mathcal{H}^n\}$  in (13).
- (4) At each  $n = 1, 2, \dots, N$ ,  $\{U^n, P^n\}$  are obtained as

$$\begin{bmatrix} U^n \\ P^n \end{bmatrix} = \mathcal{M}^{-1} \begin{bmatrix} \mathcal{F}^n \\ -\mathcal{G}^n + \tau A_{qfp} M_{qfqf}^{-1} \mathcal{H}^n \end{bmatrix}. \quad (15)$$

Then,  $Q_f^n$  is obtained from (13b).

**4.1. MATLAB implementation.** The code is run using 4 input parameters:

- (1) **el\_nodes**: This may be a scalar or a vector. If scalar, then it is the number of cells  $M$  and a uniform spatial grid is created with  $M$  cells. If it is a vector, then it contains the element nodes  $\{x_{j+\frac{1}{2}}\}_{j=0}^M$  (nodal grid), starting from  $x_{\frac{1}{2}} = a$  and including  $x_{M+\frac{1}{2}} = b$ .
- (2) **ntsteps**: This is an integer that denotes the total number of time steps  $N$ . A uniform temporal grid is generated using time step  $\tau = \frac{T_{end}}{N}$ , where  $(0, T_{end})$  is the time period of the simulation.
- (3) **BC\_flags**: This is a  $4 \times 1$  vector with each entry either 0 or 1. The entry 0 corresponds to Dirichlet boundary condition and 1 corresponds to Neumann boundary condition. The order is determined as [MMHH], i.e., the first two entries correspond to mechanical deformation at  $x = a$  and  $x = b$ , and the last two entries are for hydrological flow (pressure or flux) at  $x = a$  and  $x = b$ ; see Section 2.1.
- (4) **example**: This is an integer between 1 and 6 which toggles between particular inbuilt examples or a custom scenario: **example** = 1, 2, 3, 4 or 5 corresponds to the hard-coded examples demonstrated in Section 5 in this document. **example** = 6 corresponds to a custom scenario that needs to be set up by the user.

The code returns the following output:

- (1) **xn**: The nodal grid  $\{x_{j+\frac{1}{2}}\}_{j=0}^M$ .
- (2) **xcc**: The cell-centered grid  $\{x_j\}_{j=1}^M$ .
- (3) **t**: The temporal grid with uniform time step size  $\tau$ .
- (4) **U**: The nodal displacement profile at the final time step.
- (5) **P**: The cell-centered pressure profile at the final time step.
- (6) **Q**: The nodal flux profile at the final time step.

Example simulation runs:

```
>> [xn, xcc, t, U, P, Q] = Biot1D(10, 10, [0 0 0 0], 1);
>> [xn, xcc, t, U, P, Q] = Biot1D(10, 20, [1 0 0 1], 4);
```

For a complete demonstration of the in-built examples, see Section 5. Additional useful details about some local functions:

- (1) The spatial domain and time period may be specified in **grid**.
- (2) The physical parameters may be changed in **physical\_parameters**.
- (3) The external sources may be specified in **rhs\_f** and **rhs\_h**.

<i>Parameter</i>	<i>Value</i>	<i>Unit</i>
$\lambda$	1	[MPa]
$\mu$	1	[MPa]
$\phi$	1	[−]
$\beta_f$	1	[1/MPa]
$\kappa$	1	[m <sup>2</sup> ]
$\mu_f$	1	[MPa hr]
$\rho_f$	1	[MPa hr <sup>2</sup> /m <sup>2</sup> ]
$G$	0	[m/hr <sup>2</sup> ]

TABLE 2. Parameters used in Example 5.1, Example 5.2, and Example 5.3.

- (4) The boundary condition values may be specified by appropriately changing `exact_u`, `exact_du`, `exact_p`, and `exact_qf`.

## 5. NUMERICAL EXAMPLES

In our simulations and examples, we stick to the units of [m] for length, [hr] for time, and [MPa] for pressure.

### 5.1. Manufactured solution.

**Example 5.1.** Consider  $\Omega = (0, 1)[\text{m}]$

$$u(x, t) = -\frac{1}{\pi} \cos(\pi x) \sin\left(\frac{\pi t}{2}\right), \quad (16a)$$

$$p(x, t) = \sin(\pi x) \sin\left(\frac{\pi t}{2}\right), \quad \forall x \in (0, 1), \quad t > 0. \quad (16b)$$

The source terms are

$$f(x, t) = [-(\lambda + 2\mu)\pi + \alpha\pi] \cos(\pi x) \sin\left(\frac{\pi t}{2}\right), \quad (17a)$$

$$g(x, t) = \frac{\pi}{2}(\beta_f \phi + \alpha) \sin(\pi x) \cos\left(\frac{\pi t}{2}\right) + \frac{\kappa}{\mu} \pi^2 \sin(\pi x) \sin\left(\frac{\pi t}{2}\right). \quad (17b)$$

The physical parameters used in this example are tabulated in Table 2. We neglect gravitational effects.

The simulation is run over  $(0, 1)[\text{hr}]$  using  $M = 10$  uniform cells (corresponding to  $h = 0.1[\text{m}]$ ),  $N = 10$  time steps (corresponding to  $\tau = 0.1[\text{hr}]$ ) and all Dirichlet boundary conditions [DDDD] using the following command:

```
>> [xn, xcc, t, U, P, Q] = Biot1D(10, 10, [0 0 0 0], 1);
```

The displacement and pressure profile at the final time step is shown in Figure 1.

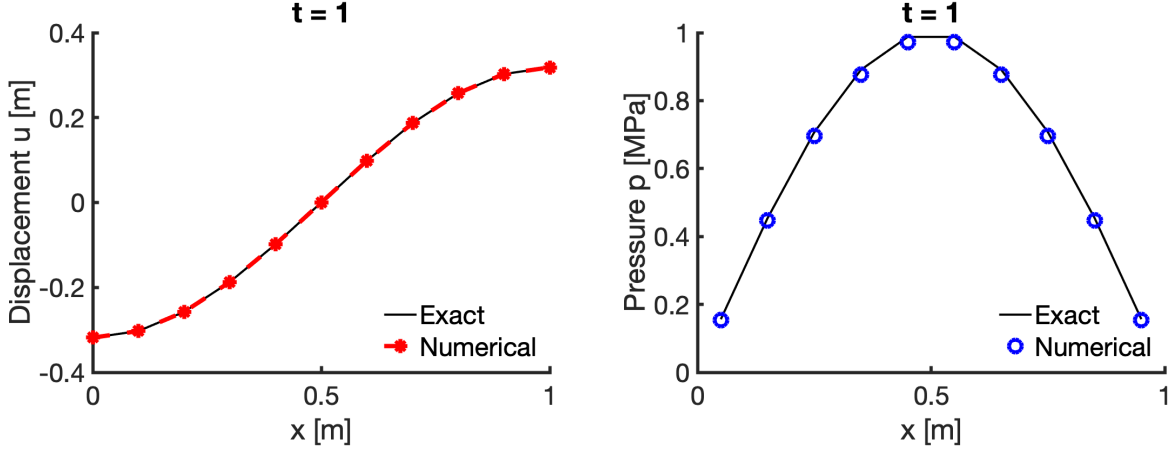


FIGURE 1. Example 5.1: Displacement and pressure profile at the final time step.

**Example 5.2.** Consider  $\Omega = (0, 1)[\text{m}]$  with

$$u(x, t) = \sin\left(\frac{\pi x}{2}\right)e^{-t}, \quad (18a)$$

$$p(x, t) = \cos\left(\frac{\pi x}{2}\right)e^{-t}, \quad \forall x \in (0, 1), \quad t > 0. \quad (18b)$$

The source terms are

$$f(x, t) = \left[ (\lambda + 2\mu)\frac{\pi^2}{4} - \alpha\frac{\pi}{2} \right] \sin\left(\frac{\pi x}{2}\right)e^{-t}, \quad (19a)$$

$$g(x, t) = \left[ -\beta_f\phi - \alpha\frac{\pi}{2} + \frac{\kappa}{\mu_f}\frac{\pi^2}{4} \right] \cos\left(\frac{\pi x}{2}\right)e^{-t}. \quad (19b)$$

We neglect gravitational effects. The physical parameters used in this example are tabulated in Table 2.

The simulation is run over  $(0, 1)[\text{hr}]$  using a non-uniform grid with  $M = 10$  cells,  $N = 10$  time steps (corresponding to  $\tau = 0.1[\text{hr}]$ ), and mixed boundary conditions [NDND] using the following commands:

```
>> el_nodes = [0; 0.05; 0.1; 0.15; 0.2; 0.4; 0.6; 0.8; 0.9; 0.95; 1.0];
>> [xn, xcc, t, U, P, Q] = Biot1D(el_nodes, 10, [1 0 1 0], 2);
```

The displacement and pressure profile at the final time step is shown in Figure 2.

**Example 5.3.** Consider  $\Omega = (0, 1)[\text{m}]$  with

$$u(x, t) = 2 - x, \quad (20a)$$

$$p(x, t) = 1 + x, \quad \forall x \in (0, 1), \quad t > 0. \quad (20b)$$

The source terms are

$$f(x, t) = \alpha, \quad (21a)$$

$$g(x, t) = 0. \quad (21b)$$

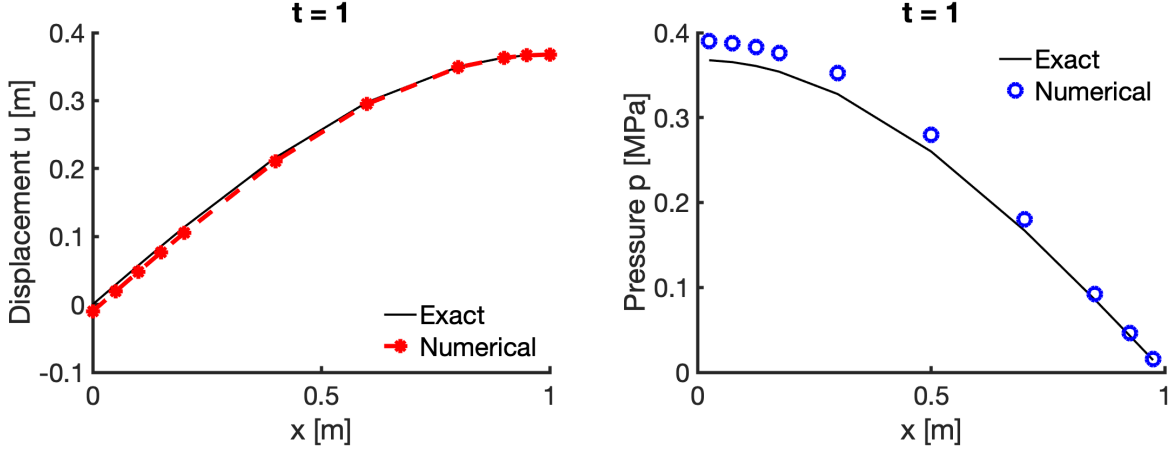


FIGURE 2. Example 5.2: Displacement and pressure profile at the final time step.

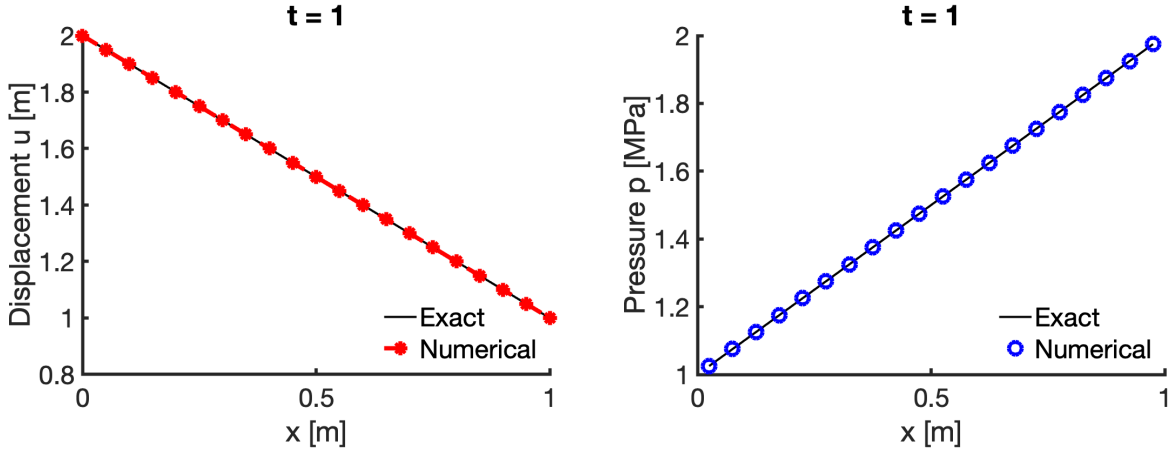


FIGURE 3. Example 5.3: Displacement and pressure profile at the final time step.

The physical parameters used in this example are tabulated in Table 2. We neglect gravitational effects.

The simulation is run over  $(0, 1)[\text{hr}]$  using a uniform grid with  $M = 20$  cells (corresponding to  $h = 0.05[\text{m}]$ ),  $N = 10$  time steps (corresponding to  $\tau = 0.1[\text{hr}]$ ), and mixed boundary conditions [DNNN] using the following commands:

```
>> [xn, xcc, t, U, P, Q] = Biot1D(20, 10, [0 1 1 1], 3);
```

The displacement and pressure profile at the final time step is shown in Figure 3.

## 5.2. Physical examples.

**Example 5.4. Consolidation test for clay:** This example is inspired from the standard consolidation tests performed at the laboratory scale [4] (Pg. 185). Consider  $\Omega = (0, 0.1)[\text{m}]$ ,



Parameter	Value	Units	Reference
$E$	20	[MPa]	[4](Pg. 406)
$\nu$	0.30	[−]	[4](Pg. 406)
$\phi$	0.50	[−]	[2](Pg. 74)
$\beta_f$	$4.16 \times 10^{-4}$	[1/MPa]	[6]
$\kappa$	$1 \times 10^{-17}$	[m <sup>2</sup> ]	[4](Pg. 373)
$\mu_f$	$2.7822 \times 10^{-13}$	[MPa hr]	[6]
$\rho_f$	998.21	[kg/m <sup>3</sup> ]	[6]
$\rho_s$	2700	[kg/m <sup>3</sup> ]	[1](Pg. 22)
$G$	$1.27290528 \times 10^8$	[m/hr <sup>2</sup> ]	[6]

TABLE 3. Physical parameters for clay used in Example 5.4.

with  $x = 0[\text{m}]$  being the top of the soil column. The source terms are

$$f(x, t) = 0, \quad (22a)$$

$$g(x, t) = 0. \quad (22b)$$

We consider the mixed boundary conditions

$$\tilde{\sigma}n = 10^{-1}, \quad x = 0, \quad (23a)$$

$$u = 0, \quad x = 0.1, \quad (23b)$$

$$p = 0, \quad x = 0, \quad (23c)$$

$$q_f = 0, \quad x = 0.1. \quad (23d)$$

We do not neglect gravitational effects. The physical parameters used in this example are tabulated in Table 3.

The simulation is run over  $(0, 24)[\text{hr}]$  using a uniform grid with  $M = 20$  cells (corresponding to  $h = 0.005[\text{m}]$ ),  $N = 10$  time steps (corresponding to  $\tau = 1.2[\text{hr}]$ ), and mixed boundary conditions [NDDN] using the following commands:

```
>> [xn, xcc, t, U, P, Q] = Biot1D(20, 10, [1 0 0 1], 4);
```

The displacement and pressure profiles at  $t = 1.2$ ,  $t = 10.8$  and  $t = 24[\text{hr}]$  are shown in Figure 4. The total settlement  $s(t) = u(0, t)$  over the simulation period  $t \in (0, 24)[\text{hr}]$  is also shown.

**Comments:** Clay is highly impermeable. Hence, the pressure decreases gradually the simulation as the water drains out of the soil. At the end of the simulation, steady state has still not been achieved. The settlement curve is qualitatively similar to [4](Pg. 186, Figure 9.26). Also, the effects of gravity are negligible compared to the applied stress of  $10^{-1}[\text{MPa}]$  when computing the total settlement: we re-run the simulation without considering the gravitational effects (i.e, with  $G = 0[\text{m/hr}^2]$ ) get a settlement of  $3.694338092420293 \times 10^{-4}[\text{m}]$ , whereas the settlement at the end of the simulation with gravitational effects is  $3.709752755037927 \times 10^{-4}[\text{m}]$ .

**Example 5.5. Consolidation test for sand and clay:** This example is to demonstrate the robustness of the solver in the presence of heterogeneity. Consider  $\Omega = (0, 1)[\text{m}]$ , with

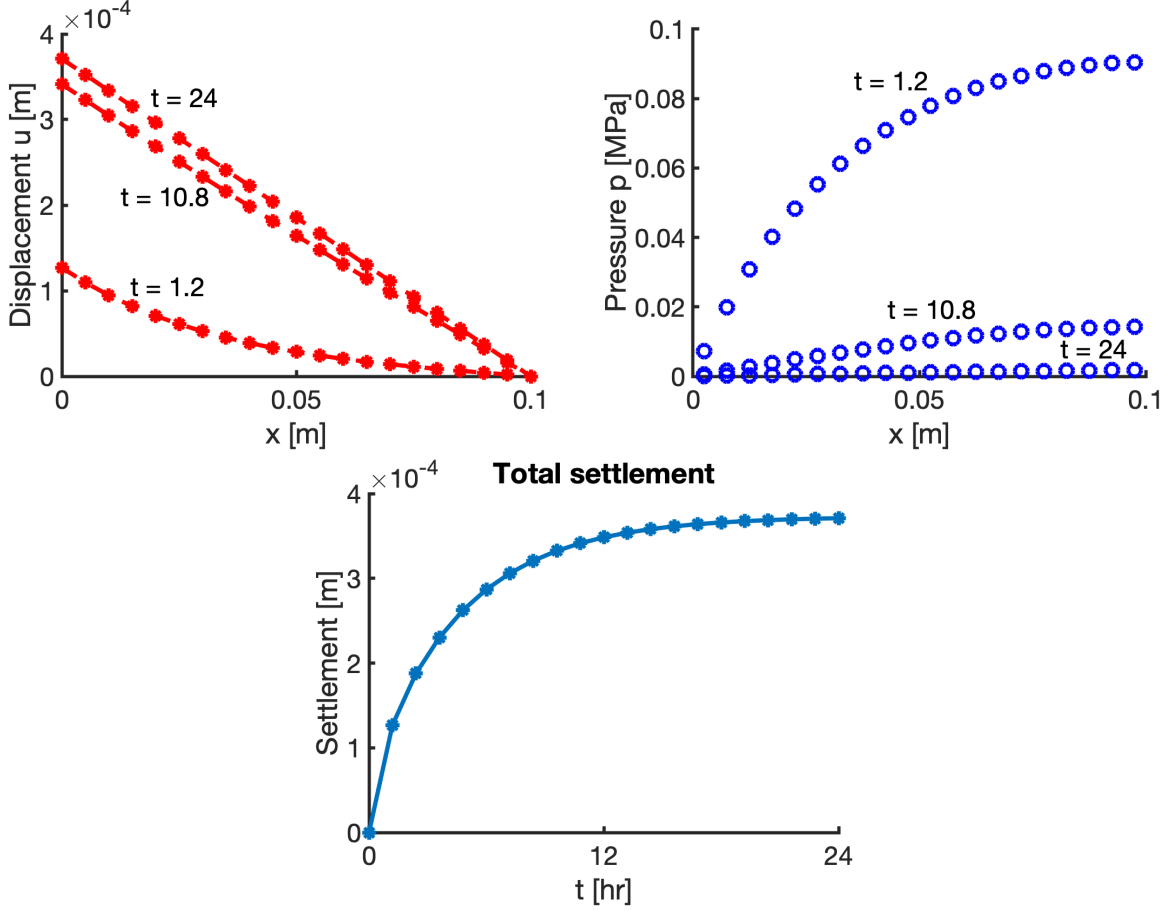


FIGURE 4. Example 5.4: Top row: Displacement and pressure profile at  $t = 1.2$ ,  $t = 10.8$  and  $t = 24$ [hr]. Bottom row: Total settlement over the simulation time period.

$x = 0$ [m] being the top of the soil column. The source terms are

$$f(x, t) = 0, \quad (24a)$$

$$g(x, t) = 0. \quad (24b)$$

We consider the mixed boundary conditions

$$\tilde{\sigma}n = 10^{-1}, \quad x = 0, \quad (25a)$$

$$u = 0, \quad x = 1, \quad (25b)$$

$$p = 0, \quad x = 0, \quad (25c)$$

$$q_f = 0, \quad x = 1. \quad (25d)$$

We do not neglect gravitational effects. To introduce heterogeneity, we consider  $\Omega = \Omega_{\text{clay}} \cup \Omega_{\text{sand}}$ , where  $\Omega_{\text{clay}} = (0, 0.5)$ [m] and  $\Omega_{\text{sand}} = (0.5, 1)$ [m]. We label this case as “Clay over sand”. We also consider the case when  $\Omega = \Omega_{\text{sand}} \cup \Omega_{\text{clay}}$  with  $\Omega_{\text{sand}} = (0, 0.5)$ [m] and  $\Omega_{\text{clay}} = (0.5, 1)$ [m], and we label this case as “Sand over clay”. The physical parameters used in this example are tabulated in Table 4.

Parameter	Value	Units	Reference
$E$	Clay: 20	[MPa]	[4](Pg. 406)
	Sand: 15	[MPa]	[4](Pg. 407)
$\nu$	Clay: 0.30	[−]	[4](Pg. 406)
	Sand: 0.25	[−]	[4](Pg. 407)
$\phi$	Clay: 0.50	[−]	[2](Pg. 74)
	Sand: 0.30	[−]	[2](Pg. 74)
$\beta_f$	$4.16 \times 10^{-4}$	[1/MPa]	[6]
$\kappa$	Clay: $1 \times 10^{-17}$	[m <sup>2</sup> ]	[4](Pg. 373)
	Sand: $1 \times 10^{-12}$	[m <sup>2</sup> ]	[4](Pg. 373)
$\mu_f$	$2.7822 \times 10^{-13}$	[MPa hr]	[6]
$\rho_f$	998.21	[kg/m <sup>3</sup> ]	[6]
$\rho_s$	Clay: 2700	[kg/m <sup>3</sup> ]	[1](Pg. 22)
	Sand: 2650	[kg/m <sup>3</sup> ]	[1](Pg. 22)
$G$	$1.27290528 \times 10^8$	[m/hr <sup>2</sup> ]	[6]

TABLE 4. Physical parameters for clay and sand used in Example 5.5.

The simulation is run over  $(0, 8760)[\text{hr}]$  (i.e., 1[year]) using a uniform grid with  $M = 20$  cells (corresponding to  $h = 0.05[\text{m}]$ ),  $N = 100$  time steps (corresponding to  $\tau = 87.6[\text{hr}]$ ), and mixed boundary conditions [NDDN] using the following commands:

```
>> [xn, xcc, t, U, P, Q] = Biot1D(20, 100, [1 0 0 1], 5);
```

The displacement and pressure profiles at the end of the simulation are shown in Figure 5. The total settlement  $s(t) = u(0, t)$  over the simulation period  $t \in (0, 24)[\text{hr}]$  is also shown for the two cases.

**Comments:** Since sand is more permeable than clay, it quickly drains out when it is over clay and the maximum settlement is reached at an early stage compared to the case of clay over sand. This is reflected by the settlement curve in Figure 5. The settlement curve is qualitatively similar to [4](Pg. 186, Figure 9.26). The settlement in the two cases is roughly the same at the end of the simulation: for “Clay over sand” the settlement is  $4.868660654353 \times 10^{-3}[\text{m}]$  and for “Sand over clay” the settlement is  $4.858040045928 \times 10^{-3}$ .

## 6. ACKNOWLEDGEMENT

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## 7. APPENDIX

Here we give the details of the matrices and vectors computed in Section 3.

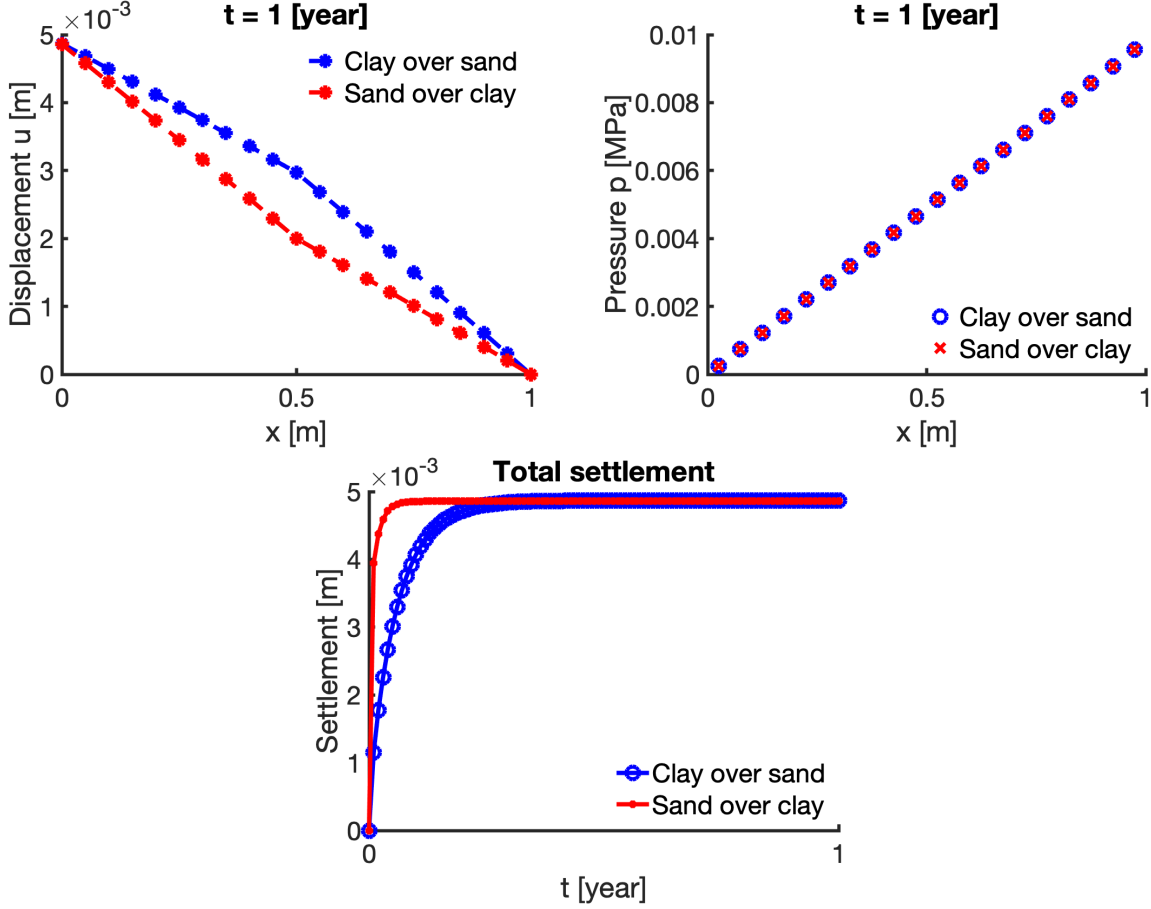


FIGURE 5. Example 5.5: Top row: Displacement and pressure profile at  $t = 1$ [year]. Bottom row: Total settlement over the simulation time period.

**7.1. Approximation spaces bases functions.** We define the basis functions  $\{\phi_{j+\frac{1}{2}}\}_{j=0}^M$ ,  $\{\eta_j\}_{j=1}^M$ , and  $\{\psi_{j+\frac{1}{2}}\}_{j=0}^M$  for  $V_h$ ,  $M_h$ , and  $X_h$ , respectively, defined in 3.1 as follows:  $\forall x \in \Omega$

$$\phi_{j+\frac{1}{2}}(x) = \begin{cases} \frac{1}{h_j} (x - x_{j-\frac{1}{2}}); & x \in \omega_j \\ \frac{1}{h_{j+1}} (x_{j+\frac{3}{2}} - x); & x \in \omega_{j+1} \\ 0; & \text{otherwise} \end{cases}, \quad \forall 1 \leq j \leq M-1, \quad (26)$$

$$\phi_{\frac{1}{2}}(x) = \begin{cases} \frac{1}{h_1} (x_{\frac{3}{2}} - x); & x \in \omega_1 \\ 0; & \text{otherwise} \end{cases}, \quad (27)$$

$$\phi_{M+\frac{1}{2}}(x) = \begin{cases} \frac{1}{h_M} (x - x_{M-\frac{1}{2}}); & x \in \omega_M \\ 0; & \text{otherwise} \end{cases}, \quad (28)$$

$$\eta_j(x) = \begin{cases} 1; & x \in \omega_j \\ 0; & \text{otherwise} \end{cases}, \quad \forall 1 \leq j \leq M, \quad (29)$$

$$\psi_{j+\frac{1}{2}} = \phi_{j+\frac{1}{2}}, \quad \forall 0 \leq j \leq M, \quad (30)$$

where (30) holds since  $H_{div}(I) = H^1(I)$ ,  $\forall I \subset \mathbb{R}$ .

**7.2. Implementation example using mixed boundary conditions [NDDN].** Consider (1) with the following mixed boundary conditions

$$\tilde{\sigma}n = \sigma_N, \quad x = a, \quad (31a)$$

$$u = u_D, \quad x = b, \quad (31b)$$

$$p = p_D, \quad x = a, \quad (31c)$$

$$q_f n = q_N, \quad x = b. \quad (31d)$$

We denote by  $U_j^n = u_h^n(x_{j-\frac{1}{2}}, t_n)$ ,  $P_j^n = p_h^n(x_j, t_n)$ , and  $Q_{fj}^n = q_{fh}^n(x_{j-\frac{1}{2}}, t_n)$ . Then we can rewrite

$$u_h^n = \sum_{j=1}^M U_j^n \phi_{j-\frac{1}{2}} + u_D(b, t_n) \phi_{M+\frac{1}{2}}, \quad (32a)$$

$$p_h^n = \sum_{j=1}^M P_j^n \eta_j, \quad (32b)$$

$$q_{fh}^n = \sum_{j=1}^M Q_{fj}^n \psi_{j-\frac{1}{2}} + q_N(b, t_n) \psi_{M+\frac{1}{2}}, \quad (32c)$$

Further denote by  $\sigma_N^n = \sigma_N(a, t_n)$ , and  $p_D^n = p_D(a, t_n)$ . The discrete system (11) can be rewritten as

$$\begin{bmatrix} A_{uu} & -\alpha A_{pu} & 0 \\ \alpha A_{pu}^T & M_{pp} & \tau A_{qfp} \\ 0 & -A_{qfp}^T & M_{qfqf} \end{bmatrix} \begin{bmatrix} U^n \\ P^n \\ Q_f^n \end{bmatrix} = \begin{bmatrix} \mathcal{F}^n \\ \mathcal{G}^n \\ \mathcal{H}^n \end{bmatrix}, \quad n \geq 1, \quad (33)$$

where the stiffness and mass matrices are

$$A_{uu} = \left[ \left( (\lambda + 2\mu) \frac{d\phi_{j-\frac{1}{2}}}{dx}, \frac{d\phi_{i-\frac{1}{2}}}{dx} \right) \right]_{1 \leq i, j \leq M}, \quad (34a)$$

$$A_{pu} = \left[ \left( \eta_j, \frac{d\phi_{i-\frac{1}{2}}}{dx} \right) \right]_{1 \leq i, j \leq M}, \quad (34b)$$

$$M_{pp} = [(\eta_j, \eta_i)]_{1 \leq i, j \leq M}, \quad (34c)$$

$$A_{qfp} = \left[ \left( \frac{d\psi_{j-\frac{1}{2}}}{dx}, \eta_i \right) \right]_{1 \leq i, j \leq M}, \quad (34d)$$

$$M_{qfqf} = \left[ \left( \left( \frac{\kappa}{\mu_f} \right)^{-1} \psi_{j-\frac{1}{2}}, \psi_{i-\frac{1}{2}} \right) \right]_{T, 1 \leq i, j \leq M}. \quad (34e)$$

The subscript  $T$  in (34e) is used to denote the use of the trapezoidal rule to evaluate the integral. This reduces  $M_{qfqf}$  to a diagonal matrix instead of a tri-diagonal system if full integration were used.

The terms on the right hand side in (33),  $\mathcal{F}^n$ ,  $\mathcal{G}^n$ , and  $\mathcal{H}^n \in \mathbb{R}^M$  are given by  $\forall 1 \leq i \leq M$ ,  $n \geq 1$ ,

$$\begin{aligned} \mathcal{F}_i^n &= \left( \left( f(\cdot, t_n) + \bar{\rho} G \frac{\partial D}{\partial x} \right), \phi_{i-\frac{1}{2}} \right) \\ &+ \begin{cases} \sigma_N(a, t_n) & i = 1 \\ 0; & 1 < i < M \\ (\lambda(x_M) + 2\mu(x_M)) \left( \frac{u_D(b, t_n)}{h_M} \right); & i = M \end{cases}, \end{aligned} \quad (35a)$$

$$\mathcal{G}_i^n = h_i \eta_{f_i}^{n-1} \eta_i - \begin{cases} 0; & 1 < i \leq M \\ \alpha u_D(b, t_n) + \tau q_N(b, t_n); & i = M \end{cases}, \quad (35b)$$

$$\mathcal{H}_i^n = \left( \rho_f G \frac{\partial D}{\partial x}, \psi_{i-\frac{1}{2}} \right) + \begin{cases} p_D(a, t_n); & i = 1 \\ 0; & 1 < i \leq M \end{cases}, \quad (35c)$$

where  $\forall n \geq 1$ ,

$$\begin{aligned} \eta_{f_i}^n &= \beta_f \phi(x_i) p_h^n(x_i) + \alpha \frac{\partial u_h^n}{\partial x}(x_i) \\ &= \begin{cases} \beta_f \phi(x_i) P_i^n + \alpha \left( \frac{U_{i+\frac{1}{2}}^n - U_{i-\frac{1}{2}}^n}{h_j} \right); & 1 \leq i < M \\ \beta_f \phi(x_i) P_i^n + \alpha \left( \frac{u_D(b, t_n) - U_{i-\frac{1}{2}}^n}{h_j} \right); & i = M \end{cases}. \end{aligned} \quad (36)$$

**Note:** In (35b), for  $n = 1$ ,  $\eta_{f_i}^0 = \eta_{f_{init}}(x_i)$  is calculated using the given initial condition (7).

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