Abstract

Much research has been dedicated to understanding political competition as well as to building models that explain voter turnout. One approach that has garnered attention is the ethical voter framework. In this paper, we embed an ethical voter model in a general spatial model of competition with simple assumptions around party formation and mobilization. We study the impact of political positions on voter behavior and characterize how adjustments in position by the candidates affect the size and enthusiasm levels of the parties. We find that for a sufficiently polarized electorate, the median voter theorem breaks down and candidates adopt a mixed strategy with support over moderate and extreme stances.[1]

1 Introduction

In spatial models of political competition, voters have preferences over positions on an election issue or set of issues. Candidates compete for the support of voters by taking stances on the issue(s). Hotelling (1929) is credited with the introduction of this framework; his model illustrates how competing firms choose to keep their products undifferentiated in an effort to maintain the largest market share possible. Hotelling notes the application to modelling electoral contests, but Downs (1957b) develops this idea further, positing that the distribution of the electorate on the issue space determines how the politicians behave. For an electorate that is not significantly polarized, two parties on a uni-dimensional issue space will choose to hold the same stance coinciding with the ideal point for the median voter – a finding often referred to as the *median voter theorem* (Downs, 1957b). Finally, Downs also observes that in a costly voting environment, a rational voter may abstain given the infinitesimally small probability of casting a pivotal vote (Downs, 1957a), a concept known as the *paradox of not voting.*

Much of the literature on voting models focuses on explaining the phenomena of substantial voter turnout while maintaining a rational voter framework. Ledyard's work (1981, 1984) allows for both the electorate and the candidates to be strategic agents and shows that if the positions of the candidates are not identical, there is positive turnout. However, given this positive turnout, candidates have a tendency

[1]Essay contains 5,427 words, treating a full line equation as 12 words

to converge to the position of the median voter and when both candidates occupy the same position, no one votes (Ledyard, 1981, 1984). Palfrey and Rosenthal (1985) examine the impact of strategic uncertainty on voters and demonstrate that as elections become large, the proportion of voters with positive voting costs that participate shrinks, and in the limit, only those with non-zero or negative costs participate. The concept that some voters actually derive utility from the act of voting is known as *ethical voting* and is a core assumption in the seminal work by Feddersen and Sandroni (2006). Their paper does not use a model of spatial competition; they take as given the division of the electorate and solve for the partial equilibrium in the absence of strategic candidates. However, by having a continuum of voters and a concept of group-coordinated rational voting, their model allows for intuitive statics and plausible levels of turnout.

This paper contributes to the literature of spatial political competition by studying candidate positioning behavior while taking voting behavior as given through an ethical voter model. We begin by providing an overview of Feddersen and Sandroni's seminal ethical voter model in order to familiarize the reader with the behavior of the two-party electorate. Given that their model only gives us a characterization of the partial equilibrium resulting from political parties that have already been formed, we provide two key assumptions on voter preferences and propensity for ethical voting. These assumptions will induce strategic positioning of our candidates. We will layer these assumptions sequentially, where the first illustrates how a simple description of voter preferences yields the popular result of the *median voter theorem*. The second assumption, which structurally determines the levels of ethical voting, adds complexity to the candidates' voting positions as they attempt to balance the incentive to win over moderate voters while still energizing their base. Finally, we explore equilibria through simulation and show how the density of voter opinions over the issue space is critical to whether the median voter result holds or the candidates adopt more extreme positions.

2 Overview of the Ethical Voter Model [2]

We begin by summarizing the ethical voter model by Timothy Feddersen and Alvaro Sandroni (2006). Their model hinges on a concept of ethical voting in order to explain positive turnout in elections by avoiding a pivotal voter framework.

Assume there are two candidates and a continuum of voters. Each voter can choose to vote or abstain from the election, which is decided by majority rule. By voting in the election, voters incur costs which are drawn independently from a uniform distribution on the interval $[0, \bar{c}]$ where $\bar{c}$ is the maximum voting cost. The voters are aligned to a particular candidate and have the following utility functions:

$$R_1 = wp - \phi \tag{1}$$

$$R_2 = w(1-p) - \phi \tag{2}$$

where w is the importance of the election, p is the probability of candidate 1 winning, and ϕ is the social cost to society of voting. Note that ϕ is society's collective cost and not the costs of each individual party. The size of party 1 is given by the parameter $k \in (0, \frac{1}{2}]$ and is, by default, the minority party.

In other voting models where voters make individually rational decisions regarding the expected utility of voting, equilibria often are characterized by no voter turnout. Rather then deriving utility from being the pivotal voter for one's candidate, voters instead follow a decision rule set collectively by their party. The group of voters that derive utility from following this rule are called *ethical voters*, and their proportion of their party is given by q_i, a random variable distributed uniformly over the interval $[0, 1]$. Ethical voters receive sufficient utility from participating in the election that outweighs the costs of voting[3]. The other portion of the party are *abstainers* and choose not to vote.

The rule σ_i specifies the threshold cost at which ethical voters of party i should turn out to vote. Therefore, the expected voter turnout for party 1 would be $kE[q_1]\sigma_1$.

[2]This entire section summarizes the model by Feddersen and Sandroni (2006) to provide sufficient context for the novel work that follows. For a more thorough reading, please refer to their paper.

[3]In their paper, ethical voters receive utility D to vote, which can be strictly less than $\bar{c}$. For the purposes of this extension, we assume $D > \bar{c}$.

While a higher σ_i leads to larger turnout, it also increases the social cost of voting which can be written as:

$$\phi(\sigma_1, \sigma_2) \equiv \overbrace{kE[q_1]\int_0^{\sigma_1} \bar{c}x dx}^{\text{party 1 turnout costs}} + \underbrace{(1-k)E[q_2]\int_0^{\sigma_2} \bar{c}x dx}_{\text{party 2 turnout costs}} \tag{3}$$
$$= \frac{\bar{c}}{4}\left(k\sigma_1^2 + (1-k)\sigma_2^2\right)$$

where the second line is given by the fact that $E[q_i] = \frac{1}{2}$. Note that the minority party wins under the event $kq_1\sigma_1 > (1-k)q_2\sigma_2$ with probability described by the ratio distribution:

$$\frac{q_2}{q_1} \sim F\left(\frac{k\sigma_1}{(1-k)\sigma_2}\right) = p(\sigma_1, \sigma_2) \tag{4}$$

See Appendix A.1 for the form of the ratio distribution. The maximization problems faced by party 1 and party 2 are now given by:

$$R_1(\sigma_1, \sigma_2) = wp(\sigma_1, \sigma_2) - \phi(\sigma_1, \sigma_2) \tag{5}$$

$$R_2(\sigma_1, \sigma_2) = w(1 - p(\sigma_1, \sigma_2)) - \phi(\sigma_1, \sigma_2) \tag{6}$$

The optimal voting rules for both parties are then derived through the first order conditions:

$$wf\left(\bar{k}\frac{\sigma_1}{\sigma_2}\right)\frac{\bar{k}}{\sigma_2} - \bar{c}kE(\tilde{q}_1)\sigma_1 = \begin{cases} = 0 & \text{if } \sigma_1 \in (0,1) \\ > 0 & \text{if } \sigma_1 = 1 \end{cases} \tag{7}$$

$$wf\left(\bar{k}\frac{\sigma_1}{\sigma_2}\right)\frac{\bar{k}\sigma_1}{(\sigma_2)^2} - \bar{c}(1-k)E(\tilde{q}_2)\sigma_2 = \begin{cases} = 0 & \text{if } \sigma_2 \in (0,1) \\ > 0 & \text{if } \sigma_2 = 1 \end{cases} \tag{8}$$

where $\bar{k} = \frac{k}{1-k}$.

These two first order equations in conjunction with the form of the ratio distribution for $F\left(\bar{k}\frac{\sigma_1}{\sigma_2}\right)$ fully characterize the optimal voting rules for both parties. A full treatment of the properties of this model are given in their paper. To summarize some key findings, from the first order conditions we find that in equilibrium[4]:

$$\sigma_1^* = \sqrt{\frac{w}{c}} \frac{1}{\sqrt[4]{k(1-k)}} \tag{9}$$

$$\frac{\sigma_2^*}{\sigma_1^*} = \sqrt{\frac{k}{1-k}} \tag{10}$$

where k can also be thought of as a level of disagreement. When k is low, the majority party has an overwhelming majority and therefore does not need to exert as much effort in getting its members to vote relative to the minority party. As k gets larger, the electorate is more divided and the size advantage for the majority party becomes slimmer, hence the increase in the majority party's voting rule as seen in Figure 1. At the extreme, when the two parties evenly split the electorate, both exert the same turnout. Notice that by definition, since $k \leq \frac{1}{2}$, $\frac{\sigma_2^*}{\sigma_1^*} \leq 1$. As a result, the majority party turns out a smaller portion of their party, but, in expectation, more candidate 2 voters show up to vote given the difference in party size.

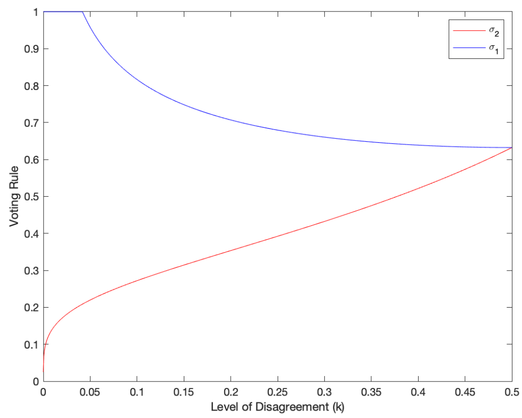

Figure 1: Turnout Fraction with Level of Disagreement

It is important to notice that in this model, several parameters are exogeneous (i.e., k, w, $\bar{c}$, etc.), and the proportion of ethical voters in each party are also given by a standard uniform distribution. Our goal now is to close parts of the model by endogenizing key parameters to allow for the organic development of political parties and candidate positioning.

[4]Results only displayed for the interior solution. For some parameterizations, $\sigma_1, \sigma_2 = 1$.

3 Political Positioning with Endogenous Party Formation

Suppose now the continuum of voters is concerned with a single issue in the election and let the set of stances be given by the interval $\Omega = [0, 1]$. Let $G(\omega)$ be the cumulative distribution function describing the electorate's stances on this issue. Assume that the distribution is continuous.

Now introduce two candidates to the election. Each candidate simultaneously announces their own stance on the spectrum Ω. Let ω_1 and ω_2 be the stances of candidate 1 and candidate 2 respectively. We assume that the candidates can neither lie about their stance nor refrain from announcing a stance. These stances in turn induce the formation of political parties as voters' preferences for candidates are based solely on this one issue.

Assumption 1. *Assume that the electorate can precisely determine where the candidates' stances lie in Ω and align themselves to each party by choosing the candidate that is closest to the voter's stance on the issue. In the event that the candidates occupy the same position, voters to the "left" vote for candidate 1, and voters to the "right" vote for candidate 2.*

Therefore, we can describe the size of each party. Without loss of generality, let the stance of candidate 1 be to the "left" of candidate 2: $\omega_1 \leq \omega_2$. Note this assumption is simply for labeling convenience. Then,

$$k = G(\bar{\omega}) = \int_0^{\frac{\omega_1+\omega_2}{2}} g(x)dx \tag{11}$$

Here we have endogenized the size of each party based on the preferences of the electorate over the candidates' stances. Without any further adjustments to the model, we can derive a simple proposition regarding how candidates choose their positions:

Proposition 1. *The equilibrium strategy for both candidates is to set stance at the median of the electorate's distribution (i.e., $\omega_1 = \omega_2 = \omega^*$ where $G(\omega^*) = \frac{1}{2}$).*

Proof **:** We first show that candidate 1 has an incentive to move ω_1 to the "right" towards ω_2. Both candidates attempt to maximize the probability of winning the election by strategically choosing ω_i, taking ω_j as given. For candidate 1, the maximization problem is

$$\max_{\omega_1} F\left(\bar{k}\frac{\sigma_1^*}{\sigma_2^*}\right)$$

The marginal increase in probability of winning with respect to ω_1 is:

$$\frac{dF}{d\omega_1} = f\left(\bar{k}\frac{\sigma_1^*}{\sigma_2^*}\right)\left(\frac{d\bar{k}}{d\omega_1}\frac{\sigma_1^*}{\sigma_2^*} + \bar{k}\frac{d\left(\sigma_1^*/\sigma_2^*\right)}{d\omega_1}\right) \tag{12}$$

where $\frac{d\bar{k}}{d\omega_1}$ can be written as:

$$\begin{aligned}\frac{d\bar{k}}{d\omega_1} &= \frac{(1-k)\frac{dk}{d\omega_1} - k\frac{d(1-k)}{d\omega_1}}{(1-k)^2} \\ &= \frac{g(\bar{w})}{2(1-G(\bar{\omega}))^2}\end{aligned} \tag{13}$$

where $\bar{\omega} = \frac{\omega_1+\omega_2}{2}$. Furthermore, recall that in equilibrium $\frac{\sigma_2*}{\sigma_1*} = \sqrt{\bar{k}}$. Then we can write $\frac{d\left(\sigma_1^*/\sigma_2^*\right)}{d\omega_1}$ as:

$$\frac{d\left(\sigma_1^*/\sigma_2^*\right)}{d\omega_1} = -\frac{1}{2}\bar{k}^{-\frac{3}{2}}\frac{d\bar{k}}{d\omega_1} \tag{14}$$

Finally, $\bar{k}\frac{\sigma_2^*}{\sigma_1^*} = \sqrt{\bar{k}} \leq 1 \implies f(\bar{k}\frac{\sigma_2^*}{\sigma_1^*}) = \frac{1}{2}$. Thus, $\frac{dF}{d\omega_1}$ is:

$$\begin{aligned}\frac{dF}{d\omega_1} &= \frac{1}{2}\left(\frac{d\bar{k}}{d\omega_1}\bar{k}^{-\frac{1}{2}} - \frac{1}{2}\bar{k}^{-\frac{1}{2}}\frac{d\bar{k}}{d\omega_1}\right) \\ &= \frac{1}{8}\frac{g(\bar{\omega})}{(1-G(\bar{\omega}))^{\frac{3}{2}}G(\bar{\omega})^{\frac{1}{2}}} > 0\end{aligned} \tag{15}$$

Therefore, the minority candidate has an incentive to move their stance towards the stance of candidate 2 in order to gain more voters caught between ω_1 and ω_2, which could be thought of generally as *moderate voters*. Candidate 2 faces the symmetric problem:

$$\max_{\omega_2} 1 - F\left(\bar{k}\frac{\sigma_1}{\sigma_2}\right)$$

where there is an incentive to move to the left.

Therefore in equilibrium both candidates align to the same point on the spectrum Ω. We claim additionally that this point must be at the median of the electorate's distribution (i.e., ω' where $G(w') = \frac{1}{2}$).

Suppose to the contrary that $\omega_1 = \omega_2 \neq \omega'$. Without loss of generality, suppose both candidates take the stance $\omega < \omega'$. Then, by deviating from ω_1 to $\omega_1 + \epsilon$ for some sufficiently small ϵ, candidate 1 would claim the majority boosting the probability of winning since $\int_{\omega_1+\frac{\epsilon}{2}}^{1} g(\omega) d\omega > \frac{1}{2}$ if $\omega_1 + \epsilon < \omega'$. $\square$

By just endogenizing the sizes of the majority and minority parties, we find that candidates align to the median of the electorate's distribution over the election issue. This is an intuitive result given that the candidates' stances only have an impact on the probability of winning through gaining a larger share of the electorate. Furthermore, we can think of this simple proposition as an analog of the *median voter theorem* in the context of an ethical voter model. Downs (1957b) notes that this result holds for an electorate that is not significantly polarized on the issue space; for more polarized distributions, the candidates may choose to adopt more extreme stances. However, his argument relies on an assumption of elasticity in voting wherein for some voters significantly distant from either platform there will be abstentions. In this paper, we do not make this assumption as abstentions are naturally captured by the ethical voter framework. However, in reality we observe that candidates do not always take the most moderate stance possible; there appear to be strategic benefits to taking a more polarizing stance on an issue. To explore this further, we next provide some additional structure to our framework to demonstrate how levels of ethical voting could allow for the rise of polarized stances.

4 Political Positioning with Endogeneous Party Mobilization

We now maintain the way in which voters align to their respective parties and offer another modification that will allow for the endogeneous determination of ethical voter populations.

Assumption 2. *Suppose the expected proportion of ethical voters is proportional to* *the level of enthusiasm in the party for the party platform.*

We now provide some more structure around this core assumption, first by illustrating how enthusiasm is affected by the composition of the electorate. Then we will close the model and demonstrate how this assumption impacts the calculus of our politicians.

4.1 Modeling Party Enthusiasm and and Intra-Party Dissonance

Recall that the proportion of ethical voters of the minority and majority party are given respectively by q_1 and q_2, which are uniform random variables with support over the interval $[0, 1]$. Maintaining this distribution and the upper bound on the support, the proportions of ethical voters take on the following distributions:

$$q_1 \sim U[a(\omega_1, \omega_2), 1] \quad q_2 \sim U[b(\omega_1, \omega_2), 1]$$

Here, $a(\omega_1, \omega_2)$ and $b(\omega_1, \omega_2)$ can be be interpreted as measures of enthusiasm to participate in the electoral process. For example, if the minority candidate's electorate is particularly excited to vote (perhaps because they strongly agree with their policy stances) then $a(\omega_1, \omega_2)$ will be higher, leading to a larger expected turnout of the electorate. It is important to note that this also guarantees a minimum level of support in the election.

Suppose the level of enthusiasm is also inversely proportional to the distance between a voter's individual stance on the election issue and their candidate's stance on the issue. Therefore, a party that has a more consolidated electorate that shares the same or close to the same view as their candidate will turn out in greater numbers for that candidate. This leads us to a definition of the described consolidation (or lack thereof). Define *intra-party dissonance* as:

$$V_1(\omega_1, \omega_2) = \int_0^{\frac{\omega_1+\omega_2}{2}} (x - \omega_1)^2 g(x) dx \tag{16}$$

$$V_2(\omega_1, \omega_2) = \int_{\frac{\omega_1+\omega_2}{2}}^{1} (x - \omega_2)^2 g(x) dx \tag{17}$$

These quantities capture the level of disagreement a party has with its candidate. For example, if the distribution placed two masses of individuals on two stances in the issue space, a candidate could choose ω_i to be exactly on that mass, thereby making $V_i = 0$. Likewise, if the distribution is uniform and there is a wider range of beliefs held by the electorate, the candidates can decrease V_i by moving their stance towards the center of their party on the issue space.

One important aspect of these quantities is that intra-party dissonance is impacted by the candidate's stance in two ways. First, if a candidate moves closer to the average belief of the party, it lowers the discord within the party through the effect on the integrand. Second, by moving towards the center of the party (i.e., to left for candidate 1), the party also shrinks as those moderate voters at the margin flip and join the other party.

Returning to our measures of party enthusiasm, let:

$$a(\omega_1, \omega_2) = \bar{a}e^{-\alpha V_1(\omega_1, \omega_2)} \tag{18}$$

$$b(\omega_1, \omega_2) = \bar{b}e^{-\beta V_2(\omega_1, \omega_2)} \tag{19}$$

where $\bar{a}, \bar{b} \in [0, 1)$ are maximum enthusiasm levels. These functional forms have the property that enthusiasm approaches 0 as $V_i \to \infty$ and approaches $\bar{a}$ or $\bar{b}$ as $V_i \to 0$. The parameters α and β describe how sharply enthusiasm shifts with respect to intra-party dissonance.

Before integrating this framework into the broader maximization problem faced by the candidates in choosing ω_i, we consider how changes in ω_i affect enthusiasm through changes in intra-party dissonance. Take ω_2 as given. Then the change in enthusiasm with respect to a change in candidate 1's stance is given by:

$$\frac{\partial a}{\partial \omega_1} = -\alpha\bar{a}e^{-\alpha V_1(\omega_1, \omega_2)}\frac{\partial V_1}{\partial \omega_1} \tag{20}$$

It is clear that the sign of the relationship between $\frac{\partial a}{\partial \omega_1}$ depends on $\frac{\partial V_1}{\partial \omega_1}$:

$$
\begin{aligned}
\frac{\partial V_1}{\partial \omega_1} &= \frac{\partial}{\partial \omega_1}\left[\int_0^{\bar{\omega}} (x-\omega_1)^2 g(x)dx\right] \\
&= \underbrace{\frac{g(\bar{\omega})(\bar{\omega}-\omega_1)^2}{2}}_{\geq 0} + \underbrace{2\int_0^{\bar{\omega}} G(x)dx}_{\geq 0} + \underbrace{2G(\bar{\omega})}_{\geq 0}\underbrace{(\omega_1-\bar{\omega})}_{\leq 0}
\end{aligned} \tag{21}
$$

Therefore, we have

$$
\frac{\partial V_1}{\partial \omega_1} > 0 \iff \frac{g(\bar{\omega})(\bar{\omega}-\omega_1)^2}{2} + 2\int_0^{\bar{\omega}} G(x)dx > 2G(\bar{\omega})(\bar{\omega}-\omega_1) \tag{22}
$$

Remark 1. *When candidates share the same stance, intra-party dissonance is strictly increasing.*

This can be seen simply by letting $\omega_1 = \bar{\omega} = \omega_2 \implies \frac{\partial V_1}{\partial \omega_1} = 2\int_0^{\bar{\omega}} G(x)dx > 0$. Therefore, no matter the shape of the electorate's distribution, intra-party dissonance is increasing when candidate 1 aligns their self with candidate 2. To understand when a shift to the middle for candidate 1 could lead to a decrease in intra-party dissonance, let the electorate's stance on the issue be uniform ($g(x) = 1$), let $\omega_1 = 0$, and $\omega_2 = 1$. Here, $\frac{\partial V_1}{\partial \omega_1} = \frac{-\omega_2^2}{8}$. Therefore, from a hyper-polarized stance, candidate 1 could reduce intra-party dissonance by shifting towards the middle, thereby increasing enthusiasm for the candidate. See Figure 2 which illustrates these relationships with a particular parameterization[5]. Note that these values are generated by holding the stance of candidate 2 steady at 0.5 and varying ω_1 between 0 and 0.5. We choose to examine the situation where $\omega_2 = 0.5$ because this is the equilibrium strategy[6] when only party sizes are determined by the positions of the candidates.

Now, given that the ethical voter turnout is dependent on the stances of the two candidates, the optimal decision rules solved by the electorate's maximization problems will also be functions of the stances of the candidates. Recall that the expected social cost of voting is given by:

$$
\phi(\sigma_1, \sigma_2) = \frac{\bar{c}}{2}\left(kE(q_1)(\sigma_1)^2 + (1-k)E(q_2)(\sigma_2)^2\right) \tag{23}
$$

where now, $E(q_1) = \frac{1+a}{2}$ and $E(q_2) = \frac{1+b}{2}$.

[5] Figure 2 valid for a uniform issue distribution, $\alpha = 500$.

[6] True assuming the median of the distribution is at 0.5.

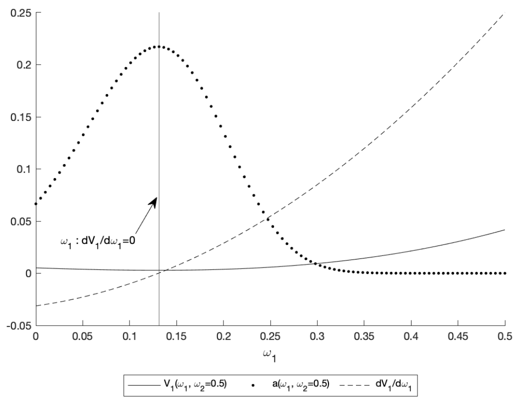

Figure 2: Intra-party Dissonance and Enthusiasm

Remark 2. *The decision rule set by each party is inversely proportional to the party's level of enthusiasm.*

The first order conditions that characterize the solution to the electorate's maximization problems are given by:

$$wf(\bar{k}\frac{\sigma_1}{\sigma_2})\frac{\bar{k}}{\sigma_2} - \bar{c}k\left(\frac{1+a}{2}\right)\sigma_1 = \begin{cases} = 0 & \text{if } \sigma_1 \in (0,1) \\ > 0 & \text{if } \sigma_1 = 1 \end{cases} \tag{24}$$

$$wf(\bar{k}\frac{\sigma_1}{\sigma_2})\frac{\bar{k}\sigma_1}{\sigma_2^2} - \bar{c}(1-k)\left(\frac{1+b}{2}\right)\sigma_2 = \begin{cases} = 0 & \text{if } \sigma_2 \in (0,1) \\ > 0 & \text{if } \sigma_2 = 1 \end{cases} \tag{25}$$

If we restrict our attention to interior solutions for both parties, we can find the following relationships:

$$\begin{aligned} \sigma_1^* &= \sqrt{\frac{w}{\bar{c}}}\sqrt{f(\bar{k}\frac{\sigma_1}{\sigma_2})}\frac{1}{\sqrt[4]{k(1-k)}}\frac{\sqrt[4]{E[q_2]}}{(E[q_1])^{\frac{3}{4}}} \\ &= \sqrt{\frac{w}{\bar{c}}}\sqrt{f(\bar{k}\frac{\sigma_1}{\sigma_2})}\frac{\sqrt{2}}{\sqrt[4]{k(1-k)}}\frac{\sqrt[4]{1+b}}{(1+a)^{\frac{3}{4}}} \end{aligned} \tag{26}$$

$$\frac{\sigma_2^*}{\sigma_1^*} = \sqrt{\frac{1+a}{1+b}\bar{k}} \tag{27}$$

Notice also that we can recover the functions from the original ethical voter model by fixing a and b equal to 0. The ratio distribution when $a, b \neq 0$ is given in Appendix A.2.

Recall that the decision rules are the parties' optimal responses to the parameters of the model and are set to compel the proportion of ethical voters to participate that balances the (expected) benefits and costs of voting. Allowing the minimum threshold of ethical voting to vary through a and b induces a response similar to that of a change in $\bar{k}$. Holding all else constant, an increase in a (an increase in enthusiasm by party 1 for candidate 1) induces party 1 to relax their voting rule relative to that of party 2 as it now requires a lower threshold to generate the same turnout. One can see this dynamic in Figure 3 where approaching the stance that generates peak enthusiasm, the minority party shifts to compelling a smaller portion of the party to vote[7].

Remark 3. *The majority party will set their voting decision rule higher than that of the minority party if* $(1+a)(k) > (1+b)(1-k)$.

In the original model, because $\bar{k} \leq 1$, the majority party would never need to encourage a strictly higher proportion of their electorate to go out and vote. In this adapted model, the voting rule for the majority party can be higher as the ability to have a more enthusiastic electorate could place the minority party at a comparative turnout advantage.

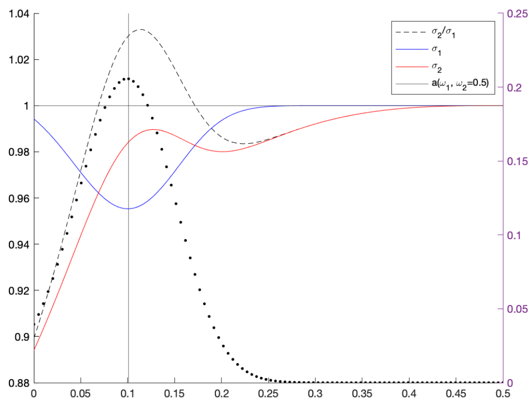

Figure 3: Voting Rule Statics with Endogeneous Ethical Voter Populations

Also, notice in Figure 3 that the peak in $\frac{\sigma_2^*}{\sigma_1^*}$ slightly lags the peak in enthusiasm. By shifting ω_1 slightly towards the right, the candidate-party variance increases slightly but is outweighed by the gain in moderate voters which leads to an increase in $\bar{k}$.

[7]Valid for the U-quadratic distribution, $\alpha = 500$.

4.2 Analysis of Positioning Incentives

We now return to the maximization problem of the minority candidate, which can now be written as:

$$\max_{\omega_1} F\left(\bar{k}\frac{\sigma_1^*}{\sigma_2^*}; a, b\right)$$

Suppose $\sigma_1 < 1$ and $\sigma_2 < 1$. Then we can substitute $\bar{k}\frac{\sigma_1^*}{\sigma_2^*}$ with $\sqrt{\frac{1+b}{1+a}\bar{k}}$. For notational cleanliness, let $\sqrt{\frac{1+b}{1+a}\bar{k}} = z$. Then taking ω_2 as given, the derivative with respect to ω_1 becomes:

$$\frac{\partial F}{\partial \omega_1} = \overbrace{f(z; a, b)\frac{\partial z}{\partial \omega_1}}^{\textit{competition effect}} + \underbrace{\frac{\partial F}{\partial a}\frac{\partial a}{\partial \omega_1} + \frac{\partial F}{\partial b}\frac{\partial b}{\partial \omega_1}}_{\textit{enthusiasm effect}} \tag{28}$$

The first term $f(z; a, b)\frac{\partial z}{\partial \omega_1}$ can be thought of as the *competition effect* on the probability that the minority candidate wins. $F(z; a, b)$ is monotonically increasing in z, and z is increasing in both $\bar{k}$ and $\frac{\sigma_1^*}{\sigma_2^*}$. Therefore, if a small increase in ω_1 leads to winning over voters equidistant between the two candidates (boosting $\bar{k}$) or leads to an increase in party 1's effort relative to party 2's effort (boosting $\frac{\sigma_1^*}{\sigma_2^*}$), then the effect is magnified by $f(z; a, b)$.

Since $z = \sqrt{\frac{1+b}{1+a}\bar{k}}$, we can expand the expression for $\frac{\partial z}{\partial \omega_1}$:

$$\begin{aligned}
\frac{\partial z}{\partial \omega_1} &= \frac{\partial\sqrt{\frac{1+b}{1+a}\bar{k}}}{\partial \omega_1} \\
&= \left[\frac{1}{2}\left(\frac{1+b}{1+a}\bar{k}\right)^{-\frac{1}{2}}\right] \cdot \left[\frac{\partial \bar{k}}{\partial \omega_1}\frac{1+b}{1+a} + \bar{k}\frac{(1+a)\frac{\partial b}{\partial \omega_1} - (1+b)\frac{\partial a}{\partial \omega_1}}{(1+a)^2}\right] \\
&= \left[\frac{1}{2}\left(\frac{1+b}{1+a}\bar{k}\right)^{-\frac{1}{2}}\right] \cdot \left[\frac{g(\bar{\omega})}{2(1-G(\bar{\omega}))^2}\frac{1+b}{1+a} + \frac{G(\bar{\omega})}{1-G(\bar{\omega})}\frac{(1+a)\frac{\partial b}{\partial \omega_1} - (1+b)\frac{\partial a}{\partial \omega_1}}{(1+a)^2}\right]
\end{aligned} \tag{29}$$

Notice that a sufficient, though not necessary condition, for $\frac{\partial z}{\partial \omega_1} > 0$ is for $(1+a)\frac{\partial b}{\partial \omega_1} - (1+b)\frac{\partial a}{\partial \omega_1} \geq 0$, which holds true if a movement to the center by candidate 1 *increases* party 1's intra-party dissonance, thereby decreasing enthusiasm. The decrease in enthusiasm leads to an increase in $\frac{\sigma_1^*}{\sigma_2^*}$ as a decrease in enthusiasm will force the minority party to increase its voting rule relative to that of the majority. This is the incentive for the candidate to actually decrease enthusiasm in an effort to compel the party to vote more aggressively. Therefore, with respect to the competition effect, a stance shift that increases $\frac{\sigma_1^*}{\sigma_2*}$ is attractive.

The second and third terms can be thought of as *enthusiasm effects* whereby a change in the stance of candidate 1 leads to a change in enthusiasm of either party. We have shown that the sign of $\frac{\partial a}{\partial \omega_1}$ depends on whether or not an increase in ω_1 leads to a decrease in the intra-party dissonance. Likewise, an increase in ω_1 impacts b only through marginally shrinking the size of party 2 and thereby decreasing party 2's intra-party dissonance, which implies that $\frac{\partial b}{\partial \omega_1} > 0$. Finally, we can also see that $\frac{\partial F}{\partial a} \geq 0$ and $\frac{\partial F}{\partial b} \leq 0$. An increase in a (decrease in b) has the effect of increasing the proportion of events that lead to the minority candidate winning. Said in another way, increasing the minimum number of possible ethical voters that turnout in an election can only improve the chances that the party wins the vote.

Therefore, given some belief about ω_2, candidate 1 seeks to position themselves so as to maximize their probability of winning, either by taking advantage of the enthusiasm effect or by maximizing the competition effect.

5 Equilibrium Characterization through Simulation

To better understand how the distribution of opinions impacts the minority candidate's choice of ω_1, we return to our fixed parameterization where $\omega_2 = 0.5$. In Figure 4, we outline four different, symmetric distributions with varying levels of disagreement regarding the particular campaign issue. *Consensus* would describe an issue that very few would disagree on and the most common stance is moderate. *Highly polarized* reflects the distribution about an issue where their is significant separation and a large portion of the electorate sticks to more extreme stances. For all of these distributions, the mean and median are the same; the only difference is the level of discord amongst the voters. Recall that we set $\omega_2 = 0.5$ since in the model where only k is an endogeneous function of the candidates' stances, $\omega_1 = \omega_2 = x$ such that $G(x) = \frac{1}{2}$ is the unique equilibrium. This is our base case to understand how the strategy chosen by candidate 1 may change from the equilibrium of $\omega_1 = \frac{1}{2}$ as the underlying distribution changes.

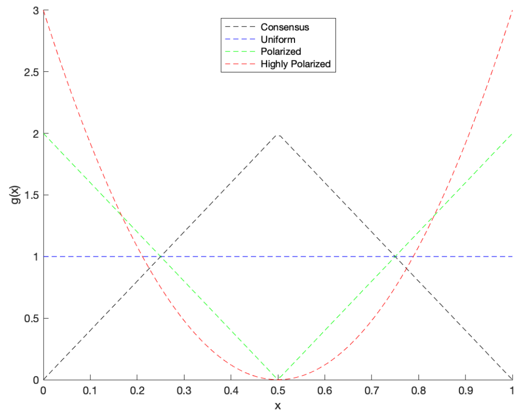

Figure 4: Electorate Stance Distributions Over Different Campaign Issues

In Figure 5, we have plotted the probability of winning for different positions of ω_1 given that candidate 2 is fixed at the middle of the distribution.

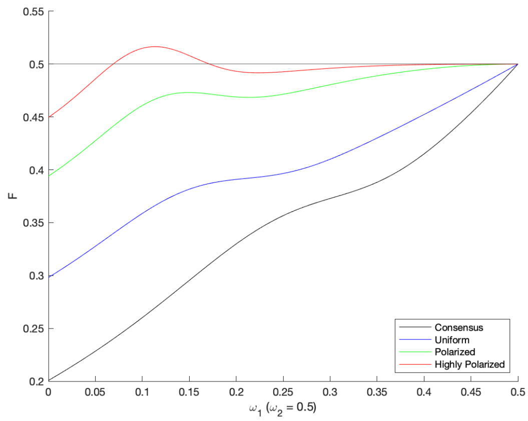

Figure 5: Probability of Candidate 1 Winning by Position for Varying Opinion Distributions

For the distributions that reflect relatively little polarization, the probability of candidate 1 winning is monotonically increasing in ω_1. Here, the enthusiasm incentive is completely outweighed by the competition incentive. While we can see that the enthusiasm effect accentuates the early rise in probability of winning as the candidate moves closer to the middle of their party, it is never large enough in either the *Consensus* or *Uniform* distribution to outweigh the incentive to win over moderate voters.

For the *Polarized* distribution, we see that for some $\omega_1 \neq \omega_2 = 0.5$, there is a stance that generates a local maximum where the enthusiasm effect is stronger than the competition effect. While globally across the entire strategy space, picking the local maximum is not candidate 1's best response to $\omega_2 = 0.5$, it illustrates how not all moderate stances are strictly preferred; there now is a second set of strategies where the enthusiasm effect is strong enough that locally, the candidate is better off maximizing enthusiasm rather than enlarging the base. In the trough between these two regions the agent is neither expressing an extreme enough view to energize base voters nor are they trying to cater to moderate voters. This concept will be interesting later on when we examine equilibria in the *Highly Polarized* distribution.

Finally, we see that for the *Highly Polarized* distribution, the enthusiasm effect is strong enough that by maximizing the enthusiasm effect we can generate a better response to the former equilibrium strategy. While candidate 1 sacrifices a portion of moderate voters, they are better off expressing an extreme view to create a larger proportion of ethical voters. This now begs the question: what is the new equilibrium in this game? We see that for the other three distributions that the unique equilibrium is still to line up at the median of the distribution. However, given that for the highly polarized distribution, candidate 1 deviates, how does candidate 2 respond?

To better understand the equilibria in this situation, recall that for finite zero-sum games we can think of a pure-strategy equilibrium as a saddle point in the payoff grid. We now discretize the stances that each agent can take and allow both agents to freely choose from the same set of n stances on the interval $[0, 1]$. In order to fill in the probabilities of winning for all possible pairs of stances, we take advantage of the symmetry of the distributions and the values of F calculated for the pairs when $\omega_1 \leq \omega_2$ and $G(\bar{\omega}) \leq 1$:

$$F(\omega_1, \omega_2) = \left\{ \begin{array}{ll} 1 - F(\omega_2, \omega_1) & \text{if } (\omega_1 > \omega_2) \wedge (G(\bar{\omega}) \leq \frac{1}{2}) \\ F(1 - \omega_2, 1 - \omega_1) & \text{if } (\omega_1 \leq \omega_2) \wedge (G(\bar{\omega}) > \frac{1}{2}) \\ F(1 - \omega_1, 1 - \omega_2) & \text{if } (\omega_1 > \omega_2) \wedge (G(\bar{\omega}) > \frac{1}{2}) \end{array} \right\}$$

This gives us a simple normal form game to analyze and visualize.

In Figure 6, we see that the surface describing the probability of candidate 1 winning is smooth since in the *Consensus* distribution the competition effect is relatively dominant, hence the smooth increase in win probability as each candidate gains a larger share of the electorate.

The sharp drop off along the $\omega_1 = \omega_2$ line is due to the fact that the electorate is perfectly discerning. Therefore, if both agents are co-located on the same position

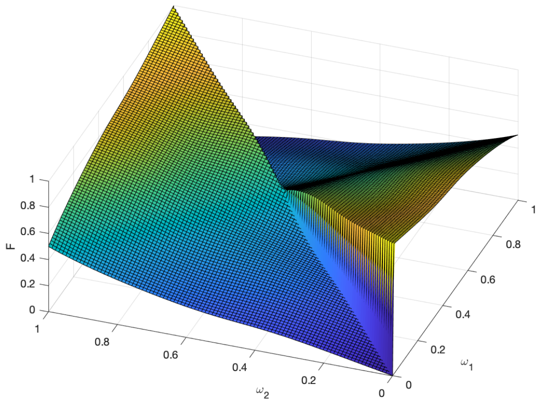

Figure 6: Probability of Candidate 1 Winning Under the Consensus Distribution

at an extreme end of the distribution, a small deviation by one agent will have a very large impact on the probability for winning as they now command virtually the entirety of the electorate. Intuitively, this drop off should decrease as the candidates are closer to the median distribution which is clear in the image below. As we illustrated in Figure 5, there is a clear equilibrium at the median of the distribution (here shown as the saddle point). The *Uniform* and *Polarized* distributions yield similar probability distributions, but are less smooth given the more pronounced enthusiasm effects. The surface for the *Polarized* distribution also has local maxima and minima away from the point $(0.5, 0.5)$ as expected.

Finally, under the *Highly Polarized* distribution in Figure 7, we see that for both candidates, the median of the distribution does not represent a strictly dominant strategy and that globally there is no saddle point. Given the lack of a pure strategy equilibrium, the only equilibrium is that of a mixed strategy.

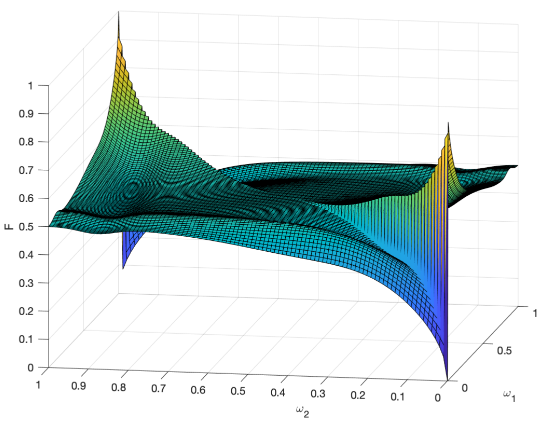

Figure 7: Probability of Candidate 1 Winning Under the Highly Polarized Distribution

In Figure 8, we observe the symmetric mixed strategies for both candidates[8]. The slight difference and reflection about the middle of the distribution can be attributed to the fact that at $(0, 0)$ we assume that candidate 1 wins with a probability of 1 while at $(1, 1)$ candidate 2 wins with a probability of 1.

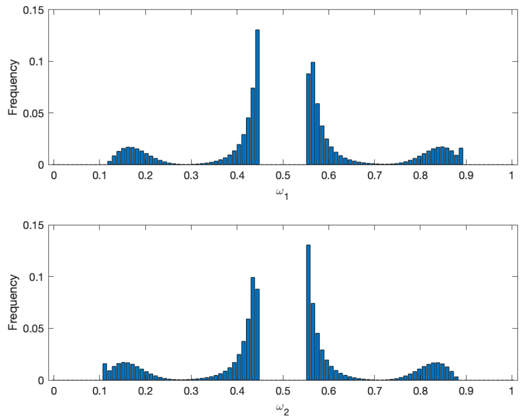

Figure 8: Mixed Strategy Under the Highly Polarized Distribution

Here there are three particular findings worth highlighting. First, as we observed in our fixed model, strategies that don't fully take advantage of the competition effect nor the enthusiasm effect are not highly valued leading to the decision-making process that a candidate should either commit to broadening their party *or* energizing the base. Second, there is no support for the prior equilibrium strategy at the median of the distribution. The candidate either adopts a slanted stance or a very extreme stance. Third, both candidates support a mixed strategy on both sides of the median. While in the context of historical party platforms, this does not seem like a very likely scenario, we have not developed a penalty for "flip-flopping" or prevented each candidate from playing certain strategies. It is in each candidate's best interest to mix over both sides of the median. Otherwise, the other candidate could place more mass on stances that effectively box out their opponent from a large portion of the electorate.

It is possible that a candidate would be penalized if historically they or their party adopted a stance on an issue that was confined to one end of the spectrum and then adopted a polar opposite stance. Either their own party would lose confidence in their candidate and/or they would not seem credible to those that hold that stance.

[8]Solved using open-source Matlab package: Bapi Chatterjee (2020). Zero Sum Game Solver (https://www.mathworks.com/matlabcentral/fileexchange/43314-zero-sum-game-solver), MATLAB Central File Exchange. Retrieved May 3, 2020.

Therefore, we take another look at the *Highly Polarized* distribution under the condition that neither candidate can take a stance past the median. Under this restriction, we see in Figure 9, that both candidates adopt an extreme, pure strategy.

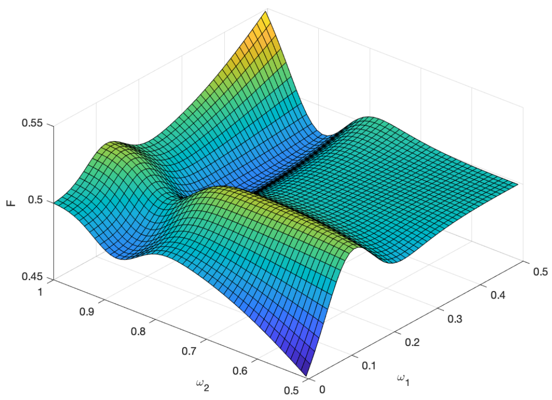

Figure 9: Probability of Candidate 1 Winning Under the Highly Polarized Distribution with Strategy Restrictions

6 Conclusion

The introduction of a single issue space in conjunction with two key assumptions around party formation and mobilization allow us to endogeneously determine key parameters of the ethical voting model and describe the political strategy of the two candidates. These assumptions give way to a strategic trade-off between appealing to a wider audience and energizing one's base. We've shown the level of polarization among the electorate is key in determining whether an agent relies on a strategy of competing for moderate voters rather than exciting its base of voters. Furthermore, the equilibrium strategies by the candidates shift abruptly when the incentive to rally the base through the enthusiasm effect overtakes the competition effect. An important avenue to test the robustness of these findings is to assume, in simulation, different functional forms for a and b. An enthusiasm effect that varies differently with intra-party dissonance could lead to a more gradual shift between the equilibrium of aligning at the median and the polarized equilibrium. Empirical work could aid in validating the assumption as well as calibrating the model.

A natural addition to this paper could be to understand how asymmetric distributions change the equilibria; in particular, can the minority party maintain a strictly higher likelihood to win an election? Also, how does expanding to a higher dimensional issue space or adding noise/cheap talk in signalling stances affect the calculus of voting? This paper provides a framework in order to test other voter behavior and modeling assumptions in a general equilibrium setting.

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A Appendix

A.1 Standard Uniform Ratio Distribution

$$F(z) = \left\{ \begin{array}{ll} \frac{z}{2} & z \leq 1 \\ 1 - \frac{1}{2z} & z > 1 \end{array} \right\}$$

$$f(z) = \left\{ \begin{array}{ll} \frac{1}{2} & z \leq 1 \\ \frac{1}{2z^2} & z > 1 \end{array} \right\}$$

A.2 Uniform Ratio Distribution when $a, b \neq 0$

$$F(z; a \leq b) = \begin{cases} 0 & z \leq b \\ \frac{1}{2(1-a)(1-b)z}(z-b)^2 & b \leq z \leq 1 \\ \frac{1}{2(1-a)z}(2z-1-b) & 1 \leq z \leq \frac{b}{a} \\ 1 - \frac{1}{2(1-a)(1-b)z}(1-az)^2 & \frac{b}{a} \leq z \leq \frac{1}{a} \\ 1 & \frac{1}{a} \leq z \end{cases}$$

$$f(z; a \leq b) = \begin{cases} 0 & z \leq b \\ \frac{1}{2(1-a)(1-b)z^2}(z^2-b^2) & b \leq z \leq 1 \\ \frac{1}{2z^2}\frac{1+b}{1-a} & 1 \leq z \leq \frac{b}{a} \\ \frac{1}{2(1-a)(1-b)z^2}(1-(az)^2) & \frac{b}{a} \leq z \leq \frac{1}{a} \\ 0 & \frac{1}{a} \leq z \end{cases}$$

$$F(z; a > b) = \begin{cases} 0 & z \leq b \\ \frac{1}{2(1-a)(1-b)z}(z-b)^2 & b \leq z \leq 1 \\ \frac{1}{2(1-a)z}(2z-1-b) & 1 \leq z \leq \frac{b}{a} \\ 1 - \frac{1}{2(1-a)(1-b)z}(1-az)^2 & \frac{b}{a} \leq z \leq \frac{1}{a} \\ 1 & \frac{1}{a} \leq z \end{cases}$$

$$f(z; a > b) = \begin{cases} 0 & z \leq b \\ \frac{1}{2(1-a)(1-b)z^2}(z^2-b^2) & b \leq z \leq 1 \\ \frac{1}{2z^2}\frac{1+b}{1-a} & 1 \leq z \leq \frac{b}{a} \\ \frac{1}{2(1-a)(1-b)z^2}(1-(az)^2) & \frac{b}{a} \leq z \leq \frac{1}{a} \\ 0 & \frac{1}{a} \leq z \end{cases}$$

These distributions collapse to the original distributions in Feddersen and Sandroni (2006) when $a, b \to 0$, taking care to use L'Hospital's rule to evaluate the boundaries of the functions.