ECN 532

Microeconomics II

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Second Quarter 2025

STATIC GAMES WITH COMPLETE INFORMATION

Lecture's Objective

- In this lecture we will start our coverage of game theory
- After defining game theory and what games are, we will
  - Define static games with complete information
  - Learn one way to represent games: the strategic or normal form of a game
  - Learn what a strategy is, and distinguish between pure and mixed strategies
  - Learn some solution concepts, including the fundamental Nash equilibrium

Notation

- We will use the following notation:
  - $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is a strategic-form game
  - S_i is the set of (pure) strategies of player i
  - $s_i \in S_i$ is a (pure) strategy of player i
  - $S = \times_{i=1}^{n} S_i$ Cartesian product of S_i's
  - $s = (s_1, s_2, ..., s_n) \in S$ is a (pure) strategy profile
  - $s = (s_i, s_{-i})$, $s_{-i} = (s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_n)$
  - $\Delta(S_i)$ is the set of mixed strategies of player i
  - $\sigma_i \in \Delta(S_i)$ is a mixed strategy of player i
  - $\sigma_i(s_i) \geq 0$ and $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$
  - $\times_{i=1}^{n} \Delta(S_i)$ Cartesian product of $\Delta(S_i)$'s
  - $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n) \in \times_{i=1}^{n} \Delta(S_i)$ is a mixed strategy profile
  - $\sigma = (\sigma_i, \sigma_{-i})$, $\sigma_{-i} = (\sigma_1, \sigma_2, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_n)$
  - $u_i(s_i, s_{-i})$ utility of player i under (s_i, s_{-i})
  - $u_i(\sigma_i, \sigma_{-i})$ utility of player i under (σ_i, σ_{-i})

The Ground Rules of Game Theory

Preliminaries

- What is game theory?
  - The study of models of conflict and cooperation among intelligent and rational decision makers
- Important: game theory is a mathematical tool
  - Fundamental in social sciences
  - Provides insight into many important economic applications
- What is a game?
  - Any social situation involving two or more decision makers
- Subsumes a broad class of problems across disciplines
- What is a player?
  - The individuals involved in the game under consideration

Preliminaries

- What is a rational decision maker?
  - One that makes decisions consistent with their objectives, that is, maximizes utility or, if there is uncertainty, maximizes expected utility
- What is an intelligent decision maker?
  - One that understands the situation under analysis as well as we do as analysts, and thus can make the same inferences as we do
- Remarks:
  - Example of setting with rational but not intelligent decision makers: competitive equilibrium
  - Assumptions of game theory are strong, but avoids having agents that are systematically fooled or that they make costly mistakes

Taxonomy of Games

- A useful way to classify games is whether they are static or dynamic, and whether they have complete or incomplete information
- In a static game, players simultaneously and independently choose their actions, and given the combination of actions chosen, each obtains a payoff
- In a dynamic game the strategic situation unfolds sequentially, and given the players choices, at the end of the game each receives a payoff
- To define complete and incomplete information, we need the following:
  - An event is common knowledge if everyone knows the event, if everyone knows that everyone knows the event, and so on ad-infinitum

Taxonomy of Games

- A game has complete information if the following items are common knowledge among the players:
  - The actions available to every player
  - All the possible outcomes of the game, and how combinations of actions of all the players affect the outcome that obtains
  - The preferences of every player
- In games of incomplete information at least one or more players have private information about some of these features
- We will thus study four classes of games:
  - Static games with complete information
  - Dynamic games with complete information
  - Static games with incomplete information
  - Dynamic games with incomplete information

Solution Concepts

- Once we have represented a strategic situation as a game, we want to analyze it and predict what will happen
  - For this purpose we will develop several solution concepts
- Some criteria to evaluate solution concepts:
  - Existence: we want to make sure that a solution concept will actually deliver a solution in a large class of games
  - Uniqueness: we would like a solution concept to restrict behavior, and ideally deliver a unique solution, something we will see does not happen often
  - Sensitivity: a desirable feature of a solution concept is that its predictions are robust to small changes in players' payoffs
  - Welfare: does solution concept predict outcomes that are unimprovable (Pareto optimal) for the players? We will see that often this is not the case

Strategic or Normal Representation of a Game

Strategic-Form Games

- Recall that given a set A and a set B, the Cartesian product of these sets is $A \times B$, which consists of all the pairs (a, b) such that $a \in A$ and $b \in B$
- The Cartesian product of k sets A_1, A_2,...,A_k is $\times_{i=1}^{k} A_i$
  - Positive orthant of $\mathbb{R}^2$ is $\mathbb{R}^2_+ = \mathbb{R}_+ \times \mathbb{R}_+$, set of pairs (x, y) with $x \geq 0$, $y \geq 0$
  - Positive orthant of $\mathbb{R}^k$ is $\times_{i=1}^{k} \mathbb{R}_{i+}$, set of vectors $(x_1, ..., x_k)$ with $x_i \geq 0$ all i
- A strategic-form (normal-form) game $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ consists of
  - A set of players $N = \{1, 2, ..., n\}$
  - A set of strategies S_i for each player $i \in N$, where s_i denotes elements of S_i
  - A utility function $u_i : S \to \mathbb{R}$ for each player i, where $S = \times_{i \in N} S_i = \times_{i=1}^{n} S_i$

Strategic-Form Games

- This is a very useful and general way to represent games: it only requires the analyst to specify players, strategies, and payoffs
- It can be used to represent any game with complete information (we will see how to adapt it to games with incomplete information)
- It will be useful to introduce a piece of notation:
  - Consider player i and the strategy profile is $s = (s_1, s_2, ..., s_n)$
  - To isolate the role of player i, we will often denote $s = (s_i, s_{-i})$
  - Here $s_i \in S_i$, and $s_{-i} = (s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_n) \in S_{-i} = \times_{j \neq i} S_j$
  - That is, s_{-i} consists of strategies of all players except i, and similarly for S_{-i}
  - We will use the notation "$-i$" repeatedly in the sequel

Example: Prisoner's Dilemma

- Two individuals have committed a serious crime and were apprehended
- Police only has evidence to convict them for a lesser crime
- They approach each of them separately with the following deal:
    1. If you confess and incriminate your partner, then if your partner does not confess, you go free and we convict your partner for the serious crime
    2. If you do not confess and your partner does not either, then both will be convicted for the lesser crime
    3. If you confess and your partner confesses, then both are convicted for the serious crime but with leniency for cooperation
    4. If you do not confess and your partner confesses, then you will be convicted for the serious crime and your partner goes free
- Assume each individual cares only about their own welfare
- It is clear that for each individual i, $1 \succ_i 2 \succ_i 3 \succ_i 4$

Example: Prisoner's Dilemma

- Let us denote confess by C and do not confess by M (for "to mum")
- We will assign some utility numbers that preserve the above ranking
- Strategic-form representation:
  - Players: 1 and 2
  - Strategies: for player 1, C or M; for player 2, C or M
  - Payoffs: for player 1, $u_1(C, M) = 8$, $u_1(M, M) = 5$, $u_1(C, C) = 3$, and $u_1(M, C) = 0$; for player 2, $u_2(C, M) = 0$, $u_2(M, M) = 5$, $u_2(C, C) = 3$, and $u_2(M, C) = 8$
- In matrix form

Player 2

Player 1		M	C
	M	$5, 5$	$0, 8$
	C	$8, 0$	$3, 3$

Example: Coordination Game

- Consider the following game:
  - Individuals 1 and 2 want to coordinate on where to meet tonight
  - There is no way to communicate with each other
  - Both know that they prefer to be together tonight than apart
  - They both know that there are two events that they would like to attend: opera (O), and ballet (B)
  - Each decides simultaneously and independently where to show up: O or B
  - Each obtains a payoff of 0 if they do not show up at the same event; player 1 obtains 2 and player 2 obtains 1 if they both show up at O; and player 1 obtains 1 and player 2 obtains 2 if they both show up at B

Example: Coordination Game

- Strategic-form representation:
  - Players: 1 and 2
  - Strategies: for player 1, O or B; for player 2, O or B
  - Payoffs: for player 1, $u_1(O,O) = 2$, $u_1(B,B) = 1$, $u_1(O,B) = u_1(B,O) = 0$; for player 2, $u_2(O,O) = 1$, $u_2(B,B) = 2$, $u_2(O,B) = u_2(B,O) = 0$
- In matrix form

Player 2

Player 1		O	B
	O	$2,1$	$0,0$
	B	$0,0$	$1,2$

Example: Cournot Duopoly

- There are two firms in the market, firm 1 and firm 2
- Each sells the same product as the other one
- The total cost of production of each firm is $c_i(q_i) = q_i^2$, $i = 1, 2$
- Total demand for the product is $q = 100 - p$, where $q = q_1 + q_2$
- The firms compete as follows:
  - Simultaneously and independently, each chooses a quantity to produce
  - Then the demand determines the price per unit and the profits for each firm
  - Firms seek to maximize profits

Example: Cournot Duopoly

- The strategic-form of this game is
  - Players: firm 1 and firm 2
  - Strategies: for firm 1, $S_1 = [0, \infty)$; for firm 2, $s_2 \in S_2 = [0, \infty)$ ($s_i \in S_i$ is a quantity q_i of output)
  - Payoffs: for firm 1, $u_1(s_1, s_2) = (100 - s_1 - s_2)s_1 - s_1^2$, for firm 2, $u_2(s_1, s_2) = (100 - s_1 - s_2)s_2 - s_2^2$
- We cannot put it in matrix form since each firm has an infinite strategy set

Example: Nash's Demand Game

- Two players, 1 and 2, bargain over the division of v dollars
- They simultaneously and independently announce a share they would like to have for themselves
- If the sum of shares is less than v, then each gets the share they announce
- If the sum of the shares is strictly bigger than v, then each one gets zero
- Each player cares only about their share of v

Example: Nash's Demand Game

- The strategic-form of the game is as follows:
  - Players: player 1 and player 2
  - Strategies: for player 1, $S_1 = [0, v]$; for player 2, $S_2 = [0, v]$ ($s_i \in S_i$ is a the share i announces)
  - Payoffs: for player i, $i = 1, 2$

$$u_i(s_i, s_j) = \begin{cases} s_i & \text{if } s_i + s_j \leq v \\ 0 & \text{if } s_i + s_j > v \end{cases}$$

- We cannot put it in matrix form since each player has an infinite strategy set

Pure and Mixed Strategies

- In the examples S_i was relatively simple
  - Finite set or infinite set of scalars (e.g., quantities in Cournot duopoly)
- But, in other games it can be a very complex object
  - For example, it could be a detailed contingent plan of choices, in which case elements of S_i will be complicated functions
- In the examples, there was no uncertainty, but can easily be incorporated in the payoffs of the definition of a strategic-form game
  - In the Cournot duopoly example, assume that the intercept of demand can be 100 or 200 with probabilities p and $1 - p$
  - Assume both firms share this probability assessment and choose before knowing the realization of the intercept
  - Then each firm maximizes expected profits when choosing a quantity
  - Then the payoff $u_i(s_1, s_2)$ is already the expected profit of firm i

Pure and Mixed Strategies

- So far we have assumed players choose a strategy in a non-random fashion
- These strategies are called pure strategies
  - In prisoner's dilemma each player either chooses confess or do not confess
  - In the Cournot duopoly each firm chooses a quantity
- It is convenient in game theory to expand the notion of strategy and allow players to add randomness to their choices if they want
- For example, player i can flip a coin or roll a die (or another randomizing device) to determine what s_i they end up choosing among the elements in S_i
- These strategies are called mixed strategies, usually denoted with the letter σ
  - Given S_i, let $\Delta(S_i)$ be the set of probability distributions over strategies in S_i
  - That is, $\sigma_i \in \Delta(S_i)$, then $\sigma_i(s_i) \geq 0$ for every $s_i \in S_i$ and $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$
  - We denote by σ a strategy profile $(\sigma_1, \sigma_2, ..., \sigma_n) \in \times_{i \in N} \Delta(S_i)$

Pure and Mixed Strategies

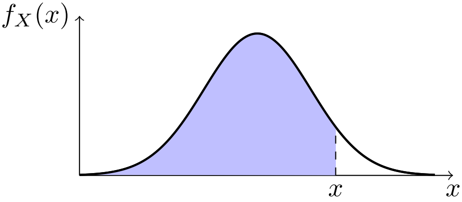
- Remarks:
  - For a simple example, in the coordination game player i can choose O with probability $\sigma_i(O) \in (0, 1)$ and B with probability $1 - \sigma_i(O)$
  - A pure strategy for i is a special case of a mixed strategy that puts probability one on a given s_i (e.g., choice of O is equivalent to choosing it with $p = 1$)
  - We will see that mixed strategies capture interesting behavior when players want to avoid being predictable
  - We will see also that mixed strategies are technically important to ensure existence of Nash equilibrium in a large class of games
  - If a player has 2 pure strategies, then a mixed strategy is summarized by one number; if 3 by two numbers (check); if k by $k - 1$ numbers (check)
  - What if i has an infinite number of pure strategies as in Cournot duopoly?

Pure and Mixed Strategies

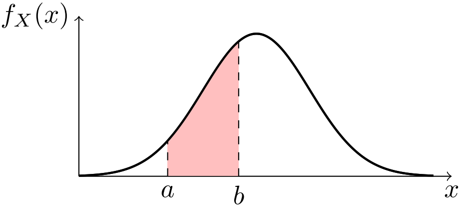
- Recall the following concepts from probability theory:
  - Assume a random variable X takes on values on an interval $[a, b]$
  - The function $F(x) = \mathbb{P}[X \leq x]$ is cumulative distribution function (cdf) of X
  - F increasing in x, with $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$
  - In fact, $F(b) = 1$ and if F is continuous at $x = a$, then $F(a) = 0$
  - If F "has a probability density function" f (pdf), then $F(x) = \int_a^x f(\tau) d\tau$; in fact, in this case $\mathbb{P}[X \in A] = \int_A f(\tau) d\tau$, where A is any "well-behaved" set
  - In this case, it is common to say that "X is distributed with density f"

Pure and Mixed Strategies

$$F(x) = P[X \leq x] = \int_0^x f_X(\tau)\, d\tau$$



$$P[a < X < b] = \int_a^b f_X(\tau)\, d\tau$$



Pure and Mixed Strategies

- Intuitively, if a player has an infinite number of pure strategies that are real numbers, we describe a mixed strategy by a cdf (or pdf if appropriate)
- That is, σ_i can be represented by $F_i(x) = \mathbb{P}[s_i \leq x]$ or by $f_i(x)$ (recall that f_i is the derivative of F_i)
  - A mixed strategy for i is a probability distribution σ_i, but we can define a random variable whose probability distribution is exactly σ_i
- Example:
  - Consider the Cournot duopoly example
  - We saw that firm i can restrict attention to $s_i \in S_i = [0, 100]$
  - An example of a mixed strategy σ_i as a cdf is (draw the pdf)

$$F_i(x) = \begin{cases} 0 & \text{if } x < 20 \\ \frac{x-20}{40} & \text{if } x \in [20, 60] \\ 1 & \text{if } x > 60 \end{cases}$$

Pure and Mixed Strategies

- Assume that players play the mixed strategy profile $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$, so each i chooses σ_i simultaneously and independently from other players
- How do we calculate player i's payoff? (Recall the notation "$-i$")
  - Recall that if X and Y are independent random variables, each with probability distribution P_X and P_Y, then joint probability distribution $P_{X,Y}$ is

$$\mathbb{P}_{X,Y}[X = x, Y = y] = \mathbb{P}_X[X = x]\mathbb{P}_Y[Y = y]$$

  - Note that in under σ, each $s = (s_1, s_2, ..., s_n)$ is chosen with probability

$$\sigma(s) = \sigma_1(s_1)\sigma_2(s_2)...\sigma_n(s_n) = \prod_{i=1}^{n} \sigma_i(s_i)$$

  - Then player i's payoff $u_i(\sigma)$ is given by expected utility

$$\begin{aligned} u_i(\sigma_i, \sigma_{-i}) &= \sum_{s \in S} \left(\prod_{i=1}^{n} \sigma_i(s_i) \right) u_i(s) \\ &= \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \sigma_i(s_i)\sigma_{-i}(s_{-i})u_i(s_i, s_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i)u_i(s_i, \sigma_{-i}) \end{aligned}$$

Pure and Mixed Strategies

- Example:
  - In the coordination game, assume each player uses

$$\sigma_i = (\sigma_i(O), \sigma_i(B)) = \left(\frac{1}{5}, \frac{4}{5}\right)$$

  - Let us calculate the expected payoff of each player $i = 1, 2$

$$u_1(\sigma_1, \sigma_2) = \frac{1}{5} \times \frac{1}{5} \times 2 + \frac{1}{5} \times \frac{4}{5} \times 0 + \frac{4}{5} \times \frac{1}{5} \times 0 + \frac{4}{5} \times \frac{4}{5} \times 1 = \frac{18}{25}$$
$$u_2(\sigma_1, \sigma_2) = \frac{1}{5} \times \frac{1}{5} \times 1 + \frac{1}{5} \times \frac{4}{5} \times 0 + \frac{4}{5} \times \frac{1}{5} \times 0 + \frac{4}{5} \times \frac{4}{5} \times 2 = \frac{33}{25}$$

Player 2

		O	B
Player 1	O	$2, 1$	$0, 0$
	B	$0, 0$	$1, 2$

Pure and Mixed Strategies

- In games with infinite number of strategies, we replace sums with integrals
- Example:
  - Consider Cournot duopoly and assume each player uses mixed strategy above
  - Then the expected profits for each firm are (check)

$$u_1(\sigma_1, \sigma_2) = \int_{20}^{60} \int_{20}^{60} \left((100 - s_1 - s_2)s_1 - s_1^2\right) \frac{1}{40} \frac{1}{40} ds_1 ds_2 = -\frac{3,200}{3}$$
$$u_2(\sigma_1, \sigma_2) = \int_{20}^{60} \int_{20}^{60} \left((100 - s_1 - s_2)s_2 - s_2^2\right) \frac{1}{40} \frac{1}{40} ds_1 ds_2 = -\frac{3,200}{3}$$

- To calculate the expected profits in this case, all you need to remember is

$$\int_a^b x^n dx = \frac{x^{n+1}}{n+1}\Big|_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

  and then first solve for the integral with respect to s_1 and then integrate the resulting expression with respect to s_2 (do it)

Dominance and Rationalizability

Introduction

- Thus far we have learned
  - Ground rules of game theory (rationality, intelligence, common knowledge)
  - The different classes of games that we will cover
  - Strategic-form (or normal-form) representation of games
- We will now turn to solution concepts
  - Predictions about what will happen in a strategic situation
- We will start with two solution concepts:
  - Iterated dominance and rationalizability
  - They study implications of rationality and common knowledge of rationality

Dominance

- Let us start with a definition:
  - Let $\sigma_i \in \Delta(S_i)$ and $s_i' \in S_i$ be strategies for player i
  - s_i' is strictly dominated by σ_i if $u_i(\sigma_i, s_{-i}) > u_i(s_i', s_{-i})$ for every $s_{-i} \in S_{-i}$
- If $\sigma_i(s_i) = 1$, then s_i' is strictly dominated by the pure strategy s_i
- A strategy is strictly dominated for i if there is an alternative strategy that yields strictly more utility to i no matter what the other players choose
- The following result follows immediately from the definition:
  - A rational player will never choose a strictly dominated strategy
- To see this, note that if a player chose a strictly dominated strategy, then they would not be maximizing their payoff
  - This is because the player has at least an alternative strategy that yields uniformly strictly higher payoffs

Dominance

- Example:
  - Consider the prisoner's dilemma
  - Note that M is strictly dominated by C for player 1
  - To see this, apply the definition: if 2 plays M, then the best for 1 is to play C, and if 2 plays C, then the best for 1 is to play C
  - No matter what 2 chooses, C yields a strictly higher payoff than M for 1
  - The same holds for player 2 (check)
  - Hence, rational players play C in this game, and the prediction is (C, C)

Player 2

Player 1		M	C
	M	$5, 5$	$0, 8$
	C	$8, 0$	$3, 3$

Dominance

- Example:
  - In the strategic-form game below, no strategy for either player is strictly dominated by another pure strategy (check)
  - But, note that player 2 will not play R if player 1 plays either A or B (check)
  - We will show that R is strictly dominated by a mixed strategy $\sigma_2 = (\sigma_2(L), \sigma_2(M), \sigma_2(R)) = (p, 1-p, 0)$
  - From definition, must show $u_2(A, \sigma_2) > u_2(A, R)$ and $u_2(B, \sigma_2) > u_2(B, R)$
  - This is the same as $3p > 1$ or $p > \frac{1}{3}$, and $4(1-p) > 1$ or $p < \frac{3}{4}$
  - Hence, R is strictly dominated by σ_2 with any p such that $\frac{1}{3} < p < \frac{3}{4}$

Player 2

Player 1	L	M	R
A	$4, 3$	$0, 0$	$2, 1$
B	$0, 0$	$3, 4$	$1, 1$

Dominance

- In the prisoner's dilemma, since there are only two pure strategies and M is strictly dominated by C, it follows that C is a dominant strategy
- Let us define this formally:
  - A strategy s_i for player i is a strictly dominant strategy if for every $s_i' \in S_i$ with $s_i' \neq s_i$, and for every $s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$
- The following result follows immediately from the definition:
  - A rational player will always choose a strictly dominant strategy
  - If every player i has a strictly dominant strategy s_i^d, then players' rationality predicts the strategy profile $(s_1^d, s_2^d, ..., s_n^d)$ as the outcome of the game
- If strictly dominant strategy profile exists, it is unique and robust to small changes in payoffs, but may not be Pareto optimal (e.g., prisoner's dilemma)
- In most economic applications players do not have strictly dominant strategies (e.g., coordination game), so this solution concept is not powerful enough

Iterated Dominance

- Rationality implies that players will not use a strictly dominated strategy
- We will now use common knowledge of rationality to develop a solution concept based on strictly dominated strategies
- The basic idea is that rationality and common knowledge of rationality allow us to remove strictly dominated strategies iteratively
- Let us explain this first with a simple example:
  - Note that in this game C is strictly dominated by R for player 2
  - Thus, rational player 2 will not play C

Player 2

		L	C	R
	U	$4, 3$	$5, 1$	$6, 2$
Player 1	M	$2, 1$	$8, 4$	$3, 6$
	D	$3, 0$	$9, 6$	$2, 8$

Iterated Dominance

- Since rationality is common knowledge, both know that 2 will not play C
- In more detail:
  - Player 2 is rational and does not play C; player 1 knows player 2 is rational and does not play C; player 2 knows that player 1 knows that player 2 is rational and does not play C
- Hence, we can eliminate C from the analysis of the game

		Player 2	
		L	R
Player 1	U	$4, 3$	$6, 2$
	M	$2, 1$	$3, 6$
	D	$3, 0$	$2, 8$

Iterated Dominance

- Now D is strictly dominated by U and M is strictly dominated by U for 1
- Hence, player 1 will not play D or M
- Since rationality is common knowledge, both know 1 will not play D or M
- Hence, we can delete these strategies from the game to obtain

		Player 2	
		L	R
Player 1	U	$4, 3$	$6, 2$

Iterated Dominance

- But now R is strictly dominated by L for player 2
- Hence, the only strategy profile that survives the iterated elimination of strictly dominated strategies is (U, L), and this is our prediction
- Remarks:
  - In most games, a unique prediction will not obtain (e.g., coordination game), but still a useful concept since often it reduces complexity of the game
  - Note that this concept just uses rationality and common knowledge of rationality (although the latter is quite strong)

Iterated Dominance

- Let us see another example:
  - No pure strategy is strictly dominated by another pure strategy for any of the two players (check)
  - But R is strictly dominated by $\sigma_2 = (\sigma_2(L), \sigma_2(C), \sigma_2(R)) = (p, 1-p, 0)$
  - To see this, note that $u_2(A, \sigma_2) = p10 > 3 = u_2(A, R)$ if and only if $p > \frac{3}{10}$
  - Also, $u_2(B, \sigma_2) = (1-p)10 > 3 = u_2(B, R)$ if and only if $p < \frac{7}{10}$
  - Hence, σ_2 strictly dominates R so long as $\frac{3}{10} < p < \frac{7}{10}$

Player 2

Player 1		L	M	R
	A	$4, 10$	$3, 0$	$1, 3$
	B	$0, 0$	$2, 10$	$10, 3$

Iterated Dominance

- Since player 2 is rational, they will not choose R
- Common knowledge of rationality implies we can eliminate R from the game
- But, now B is strictly dominated by A for player 1
- Thus, player 1 will not choose B, and common knowledge of rationality implies we can eliminate it from the game
- Now M is strictly dominated by L, and reasoning as before, we eliminate M
- Hence, the prediction is (A, L)

Player 2

Player 1		L	M
	A	$4, 10$	$3, 0$
	B	$0, 0$	$2, 10$

Iterated Dominance

- An example with an infinite number of strategies:
  - Cournot duopoly game with linear costs $c(s_i) = 10s_i$
  - Each firm solves $\max_{s_i}(100 - s_i - s_j)s_i - 10s_i$, $i = 1, 2$, $j \neq i$
  - FOC is (check) $s_1 = \frac{90-s_2}{2}$, $s_2 = \frac{90-s_1}{2}$
  - No firm will choose a quantity strictly larger than 45
  - Why? Because if $s_2 = 0$, then $s_1 = 45$, and if $s_1 = 0$, then $s_2 = 45$
  - Thus, every quantity strictly above 45 is strictly dominated for each firm, and reasoning as above, can be eliminated from the game
  - No firm will choose a quantity strictly smaller than 22.5
  - Why? Because if $s_2 = 45$, then $s_1 = 22.5$, and if $s_1 = 45$, then $s_2 = 22.5$
  - Thus, we can eliminate all quantities below 22.5 for each firm
  - After two rounds of elimination of strictly dominated strategies we know that only relevant quantities are $22.5 < s_i < 45$, $i = 1, 2$

Iterated Dominance

- After an infinite number of rounds, the only strategy profile that survives the iterated elimination of strictly dominated strategies is $(s_1, s_2) = (30, 30)$
- Suppose not, and suppose there is an interval $[s_{\min}, s_{\max}]$ that survives
- Then $s_{\min}$ and $s_{\max}$ satisfy the following equations:

$$s_{\max} = \frac{90 - s_{\min}}{2}, \text{ and } s_{\min} = \frac{90 - s_{\max}}{2}$$

- But, the unique solution to these two equations is $s_{\min} = s_{\max} = 30$
- In short, the solution concept of iterated elimination of strictly dominated strategies yields a unique prediction in Cournot duopoly game
- A picture of the two equations illustrates this process of elimination

Iterated Dominance

- How do we define iterated elimination of strictly dominated strategies?

- Suppose for each player i and each $t = 1, 2, ..., T$, there is a set of strategies X_i^t such that

  - $X_i^1 = S_i$ (we start with the set of all strategies of i)

  - X_i^{t+1} is a subset of X_i^t (at each stage we may eliminate some strategies)

  - For $t = 1, ..., T-1$ and for each player every strategy in X_i^t but not in X_i^{t+1} is strictly dominated in the game in which the set of strategies of player $j \neq i$ is X_j^t, and so we eliminated them

  - No strategy in X_i^T is strictly dominated in the game in which the set of strategies of $j \neq i$ is X_j^T

- Then the set of strategy profiles $s = (s_1, s_2, ..., s_n)$ such that $s_i \in X_i^T$ for every player survives the iterated elimination of strictly dominated strategies

Iterated Dominance

- Remarks:
  - In games with infinite strategies T can be infinity
  - The order in which strategies are eliminated is irrelevant
  - For this solution concept, existence is guaranteed, and similarly with robustness to small changes in payoffs (since dominance is strict)
  - But uniqueness does not hold (e.g., coordination game), and Pareto optimality does not hold either (e.g., prisoner's dilemma)

Rationalizability

- We used above rationality and common knowledge of rationality to
  - Eliminate the strictly dominated strategies for each player (rationality)
  - Proceed iteratively to eliminate strictly dominated strategies (common knowledge of rationality)
- The analysis focused on eliminating all those strategies that are not going to be played if players are rational and their rationality is common knowledge
- This generated a prediction: a set of strategies for each player that survive the iterated elimination of strictly dominated strategies
- In some cases, only one strategy profile survives and this is our prediction

Rationalizability

- We will now focus on a related question:
  - If players are rational and rationality is common knowledge, what are the strategy profiles may be played in a game?
- Answer: only strategies for each player that are rationalizable strategies
  - The loose idea is that in with rationalizable strategies, each player can justify their strategy choice by some conjecture they have about what other players would do, which in turn is justified by what other players think that the player under consideration would do, etc.
- One can show the following results:
  - With just two players, the set of rationalizable strategies for each player is the same as the set that survives the elimination of strictly dominated strategies
  - With more than two, set of rationalizable strategies can be a strict subset

Rationalizability

- We will need the following important concept:
  - σ_i is a best response for player i when other players choose σ_{-i} if $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$, for all $\sigma_i' \in \Delta(S_i)$
  - σ_i is never a best response if there is no σ_{-i} such that σ_i is a best response
- In words:
  - A strategy is a best response against the strategies chosen by other players if it yields as much expected utility as any other strategy of i
  - A strategy is never a best response if there is no conjecture a player can have about what others are choosing that will make it a best response
- Intuitively, strategies that are never a best response are not going to be played, whereas those that are a best response against some strategy of other players are can be part of the set of rationalizable strategies
- If a strategy is strictly dominated, then it is never a best response (check)

Rationalizability

- We can iteratively delete strategies that are never a best response
- As with strictly dominated strategies, order of elimination does not matter
- Set of strategies that survives is the rationalizable strategies of each player
  - These are strategies that a player can justify or rationalize by some conjecture about other players choosing strategies that are best response to some conjecture they have about other players that choose strategies that are best response to some conjecture...and so on
- Under weak conditions, each player has at least one rationalizable strategy
- But there could be many of them (e.g., coordination game)

Rationalizability

- Consider the following example:
  - In the first round we can eliminate B_4
  - It is strictly dominated by $\sigma_2 = (0.5, 0, 0.5, 0)$
  - A strictly dominated strategy is never a best response
- In the second round, we can eliminate A_4
  - It is strictly dominated by A_2

Player 2

Player 1		B_1	B_2	B_3	B_4
	A_1	$0,7$	$2,5$	$7,0$	$0,1$
	A_2	$5,2$	$3,3$	$5,2$	$0,1$
	A_3	$7,0$	$2,5$	$0,7$	$0,1$
	A_4	$0,0$	$0,-2$	$0,0$	$10,-1$

- No more strategies can be eliminated
- Rationalizable strategies: A_1, A_2, A_3 for 1 and B_1, B_2, B_3 for 2
  - A_1 is a best response to B_3, A_2 to B_2, and A_3 to B_1
  - B_1 is a best response to A_1, B_2 to A_2, and B_3 to A_3
  - Any of the nine strategy profiles may be played, since for each strategy a player can provide a reasonable justification for choosing it

Player 2

Player 1	B_1	B_2	B_3	B_4
A_1	$0, 7$	$2, 5$	$7, 0$	$0, 1$
A_2	$5, 2$	$3, 3$	$5, 2$	$0, 1$
A_3	$7, 0$	$2, 5$	$0, 7$	$0, 1$
A_4	$0, 0$	$0, -2$	$0, 0$	$10, -1$

Rationalizability

- Player 1 can justify choosing A_2 by the conjecture that player 2 will choose B_2, which player 1 can justify by the conjecture that player 2 thinks that 1 will choose A_2, which is reasonable if 1 thinks that 2 is thinking that player 1 thinks that 2 will play B_2, and so on ad infinitum
- Player 1 can justify choosing A_1 with the conjecture that player 2 will choose B_3, which player 1 can justify by the conjecture that player 2 thinks that player 1 will choose A_3, which is reasonable if 1 thinks that 2 is thinking that 1 things that 2 will play B_1, and so on ad infinitum

Player 2

Player 1

	B_1	B_2	B_3	B_4
A_1	$0,7$	$2,5$	$7,0$	$0,1$
A_2	$5,2$	$3,3$	$5,2$	$0,1$
A_3	$7,0$	$2,5$	$0,7$	$0,1$
A_4	$0,0$	$0,-2$	$0,0$	$10,-1$

Rationalizability

- As another example, consider the coordination game:
  - Note that no player has a never a best response strategy (why?)
  - Note that each strategy of each player is rationalizable (check)
  - Thus, any of the four strategy profiles can emerge as a prediction (how?)

Player 2

Player 1		O	B
	O	$2, 1$	$0, 0$
	B	$0, 0$	$1, 2$

Rationalizability

- Remarks:
  - This is as far as rationality and common knowledge of rationality take us
  - Sometimes (prisoner's dilemma, Cournot duopoly) we obtain single prediction
  - But in general there are multiple ones and sometimes anything goes
  - Still they are useful concepts since can reduce complexity
  - Also, for games players play for the first time rationalizability articulates the strategic uncertainty in players' minds
  - Note that existence holds, uniqueness does not (e.g., the two examples above), sensitivity does, and Pareto optimality does not (e.g., prisoner's dilemma)
  - To narrow our predictions further, we turn to solution concepts that impose equilibrium requirements

Nash Equilibrium

Nash Equilibrium in Pure Strategies

- We now turn to the most fundamental solution concept: Nash Equilibrium
- To acquire some practice, we will first define it just in terms of pure strategies, solve some examples, and then extend it to mixed strategies
  - A strategy profile $s = (s_1, s_2, ..., s_n)$ is a Nash equilibrium (in pure strategies) of $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ if for every player $i = 1, 2, ..., n$ and every strategy $s_i' \in S_i$, we have $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$
- In words:
  - A strategy profile is a Nash equilibrium (NE) if each player is choosing a best response to the strategies chosen by other players
  - Players have a conjecture about what other players are choosing, and the conjecture is correct
- Rationalizability required that each player had a reasonable conjecture about other players's choices
- NE demands that these conjectures are correct

Nash Equilibrium in Pure Strategies

- Example: prisoner's dilemma
  - Use definition to search for NE: there are 4 strategy profiles to check
  - Is (M, M) a NE? No, because at least one player has incentives to deviate (M is not a best response to M)
  - Is (M, C) a NE? No, player 1 is not playing a best response to C by player 2
  - Is (C, M) a NE? No, player 2 is not playing a best response to C by player 1
  - Is (C, C) a NE? Yes, playing C is a best response to C for each player
  - NE predicts that both players will choose C

Player 2

Player 1		M	C
	M	$5, 5$	$0, 8$
	C	$8, 0$	$3, 3$

Nash Equilibrium in Pure Strategies

- Example:
  - Find the NE in the following game

Player 2

Player 1		L	C	R
	U	$4, 3$	$5, 1$	$6, 2$
	M	$2, 1$	$8, 4$	$3, 6$
	D	$3, 0$	$9, 6$	$2, 8$

Nash Equilibrium in Pure Strategies

- Example: Cournot duopoly
  - Cournot duopoly game with linear costs $c(s_i) = 10s_i$
  - Each firm solves $\max_{s_i}(100 - s_i - s_j)s_i - 10s_i$, $i = 1, 2$, $j \neq i$
  - FOC is (check) $s_1 = \frac{90-s_2}{2}$, $s_2 = \frac{90-s_1}{2}$
  - Note what each equation means: it pins down the best response quantity for each firm given what the other firm is producing
  - To be a NE, we need both firms choosing a best response
  - But this is the same as solving both equations simultaneously
  - Hence, NE is $(s_1, s_2) = (30, 30)$

Nash Equilibrium in Pure Strategies

- Solved the examples by iterated elimination of strictly dominated strategies
- The same prediction obtains with NE solution concept
- Is this a coincidence? No
  - If a unique strategy profile survives the iterated elimination of strictly dominated strategies, then that strategy profile is a NE (proof?)
  - If each player has a strictly dominant strategy, then the resulting strategy profile is a NE (proof?)
  - If after removing iteratively all the strategies that are never a best response each player has a unique rationalizable strategy, then the resulting strategy profile is a NE (proof?)
  - Iterated dominance and rationalizability never eliminate a NE

Nash Equilibrium in Pure Strategies

- Example: Coordination Game
  - Using the definition, it is clear that there are two Nash equilibria
  - (O, O) is a Nash equilibrium: If player 1 thinks player 2 will choose O, then the best response for player 1 is to choose O; and if for player 2 thinks that player 1 will choose O, the best response for player 2 is to choose O
  - (B, B) is a Nash equilibrium: If player 1 thinks player 2 will choose B, then the best response for player 1 is to choose B, and similarly for player 2
  - Neither (O, B) nor (B, O) is a Nash equilibrium

Player 2

		O	B
Player 1	O	$2, 1$	$0, 0$
	B	$0, 0$	$1, 2$

Nash Equilibrium in Pure Strategies

- Example: Nash's demand game
  - Player 1 and player 2
  - Strategies: for player 1, $S_1 = [0, v]$; for player 2, $S_2 = [0, v]$ ($s_i \in S_i$ is a the share i announces)
  - Payoffs: for player i, $i = 1, 2$

$$u_i(s_i, s_j) = \begin{cases} s_i & \text{if } s_i + s_j \leq v \\ 0 & \text{if } s_i + s_j > v \end{cases}$$

  - We claim that any (s_1, s_2) such that $s_1 + s_2 = v$ is a Nash equilibrium
  - If player 1 conjectures that player 2 will choose $s_2 < v$, then it is a unique best response to choose $s_1 = v - s_2$ (why?), and similarly for player 2 if the conjecture is that player 1 will choose $s_1 < v$. If 1 conjectures $s_2 = v$, then a best response is $s_1 = 0$, and similarly for 2, completing the proof of claim
  - There are other Nash equilibria (check)

Nash Equilibrium in Pure Strategies

- Remarks:
  - Although there can be multiple Nash equilibria, note that "correct conjectures" significantly narrow set of predictions
  - Why would players play a Nash equilibrium?
  - If there is an obvious way to play the game, then that obvious way must be a strategy profile that is a Nash equilibrium (common conception of the game and how it is played)
  - Pre-play communication: if players can engage in pre-play discussions, then if they reach an agreement it better be that it is a Nash equilibrium of the game
  - Learning: under some conditions, if players play the same game over time, then learning process settled on a strategy profile that is a Nash equilibrium

Nash Equilibrium in Mixed Strategies

- Every example we have seen has is at least one NE in pure strategies
- But, consider the following simple example, known as matching pennies
  - (H, H) is not NE since player 1 prefers to choose T instead of H
  - (H, T) is not NE since player 2 prefers to choose H instead of H
  - (T, H) is not NE since player 1 prefers to choose T instead of H
  - (T, T) is not NE since player 2 prefers to choose H instead of T
- Hence, there is no NE in pure strategies
  - We will show below that there is a unique NE but in mixed strategies

Player 2

		H	T
Player 1	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

Nash Equilibrium in Mixed Strategies

- Let us extend the notion of NE to mixed strategies
  - $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ is a mixed strategy NE of $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ if for every $i = 1, 2, ..., n$ and every $\sigma_i' \in \Delta(S_i)$, we have $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$
- It is the same definition as before but replacing pure by mixed strategies
- Hard to check i does not have incentives to choose another mixed strategy
- Fortunately, the following result shows that it is enough to consider deviations to pure strategy
  - Let S_i^+ the set of strategies that player i plays with strictly positive probability in $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$. Then σ is a mixed strategy NE if and only if for every $i = 1, 2, ..., n$ the following conditions hold:

$$u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i}), \text{ for all } s_i, s_i' \in S_i^+$$

$$u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i}), \text{ for all } s_i \in S_i^+ \text{ and } s_i' \notin S_i^+$$

Nash Equilibrium in Mixed Strategies

- In words, this says that all the pure strategies that i plays with strictly positive probability under σ_i yield exactly the same utility against σ_{-i}, and those play with zero probability yield a weakly lower payoff that those played with strictly positive probability against σ_{-i}
- Let us prove that these conditions are necessary (check sufficiency)
  - We will show that if σ is a mixed strategy NE, then these two conditions hold
  - To see this, if either is violated, then there is $s_i \in S_i^+$ and $s_i' \in S_i$ such that $u_i(s_i', \sigma_{-i}) > u_i(s_i, \sigma_{-i})$
  - But then, player i can increase their payoff strictly against σ_{-i} by playing s_i' whenever they before played s_i
- An implication of this result is that $s = (s_1, s_2, ..., s_n)$ is a pure strategy NE if and only if it is a mixed strategy NE in which σ_i puts probability one on s_i for every player i (why?)

Nash Equilibrium in Mixed Strategies

- Example: Matching pennies
  - Let $\sigma_1 = (\sigma_1(H), \sigma_1(T)) = (p, 1-p)$ and $\sigma_2 = (\sigma_2(H), \sigma_2(T)) = (q, 1-q)$
  - We want to find values of p and q that will make (σ_1, σ_2) a mixed strategy NE
  - Use result above: to be a mixed strategy NE, it must satisfy for player 1

    $$u_1(H, \sigma_2) = u_1(T, \sigma_2) \;\Rightarrow\; q(-1) + (1-q)1 = q1 + (1-q)(-1) \;\Rightarrow\; q = 0.5$$

    and it must satisfy for player 2

    $$u_2(H, \sigma_1) = u_2(T, \sigma_1) \;\Rightarrow\; p1 + (1-p)(-1) = p(-1) + (1-p)1 \;\Rightarrow\; p = 0.5$$

  - Thus, the mixed strategy NE is $(\sigma_1, \sigma_2) = ((0.5, 0.5), (0.5, 0.5))$, and each player gets expected payoff of 0

Player 2

Player 1		H	T
	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

Nash Equilibrium in Mixed Strategies

- Example: Coordination game
  - We will see there is also a mixed strategy NE
  - Let $\sigma_1 = (\sigma_1(O), \sigma_1(B)) = (p, 1-p)$ and $\sigma_2 = (\sigma_2(O), \sigma_2(B)) = (q, 1-q)$
  - Let us use the result: for player 1 we have

$$u_1(O, \sigma_2) = u_1(B, \sigma_2) \;\Rightarrow\; q2 + (1-q)0 = q0 + (1-q)1 \;\Rightarrow\; q = \frac{1}{3}$$

    and for player 2 we have

$$u_2(O, \sigma_1) = u_2(B, \sigma_1) \;\Rightarrow\; p1 + (1-p)0 = p0 + (1-p)2 \;\Rightarrow\; p = \frac{2}{3}$$

  - Thus, the mixed strategy NE is $\sigma_1 = \left(\frac{2}{3}, \frac{1}{3}\right)$ and $\sigma_2 = \left(\frac{1}{3}, \frac{2}{3}\right)$, and each player gets expected payoff equal to $\frac{2}{3}$

Player 2

Player 1		O	B
	O	$2,1$	$0,0$
	B	$0,0$	$1,2$

Mixed Strategy Nash Equilibrium

- Example: Reporting a crime
  - A crime is observed by n people and each has to decide whether or not to report the crime to the police
  - Each person derives utility v is the crime is reported
  - But the person who reports incurs cost $c > 0$, with $c < v$, so each person prefers that someone else reports the crime
  - The strategic representation of the game is:
  - Players: n individuals
  - Strategies: $S_i = \{C, D\}$, where C is call and D is do not call
  - Payoffs: each player i's payoff is 0 if nobody reports; v is someone other than i reports; $v - c$ if i reports

Mixed Strategy Nash Equilibrium

- Continuation of example:
  - This game has n pure strategy NE in which only one player reports and the other ones do not (check)
  - There is also a mixed strategy NE that is symmetric: each player i plays

$$\sigma_i = (\sigma_i(C), \sigma_i(D)) = (p, 1-p)$$

  - To find p, by the result we have seen, it has to be the case that each i is indifferent between C and D when all other players play this strategy:

$$v - c = \mathbb{P}[\text{all other players play } D] \times 0 + \mathbb{P}[\text{at least one player } -i \text{ plays } C] \times v$$

  - What is the probability that at least one player that is not i plays C?
  - This is the same as $1 - \mathbb{P}[s_{-i} = (D, D, ..., D)] = 1 - (1-p)^{n-1}$ (check)
  - Thus, equation above is $v - c = v(1 - (1-p)^{n-1})$, which rearranges to

$$\frac{c}{v} = (1-p)^{n-1} \;\Rightarrow\; p = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}$$

Mixed Strategy Nash Equilibrium

- Continuation of the example:
  - Hence, the mixed strategy NE consists of every i playing

$$\sigma_i = (p, 1-p) = \left(1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}, \left(\frac{c}{v}\right)^{\frac{1}{n-1}}\right)$$

  - Let us examine $p = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}$
  - Note that as the group gets large (n increases), p decreases: the probability that a player calls in equilibrium decreases in the number of people around
  - How about the probability that "at least someone will call"? Does it increase or decrease in n? Fix any player i and write

$$\mathbb{P}[\text{every player plays } D] = \mathbb{P}[i \text{ plays } D]\mathbb{P}[\text{every player } -i \text{ plays } D]$$

  - But we saw that that $\mathbb{P}[i \text{ plays } D]$ increases in n (why?); also from equation above $\mathbb{P}[\text{every player } -i \text{ plays } D] = \frac{c}{v}$, which is independent of n
  - Hence, the probability that nobody calls increases in n, which is surprising

Mixed Strategy Nash Equilibrium

- Remarks:
  - NE is fundamental solution concept in games
  - We have seen many examples, some with a unique NE and some with multiple
  - In fact, in any game in which S_i has a finite number of elements for every i there exists a NE (could be in mixed strategies)
  - In games in which S_i can be uncountable, a NE in pure strategies exists if (a) S_i is nonempty, convex, compact subset of $\mathbb{R}^k$, and (b) u_i is continuous in $(s_1, s_2, ..., s_n)$ and (quasi-) concave in s_i
  - These two results cover a large class of games
  - Existence is satisfied, uniqueness is not since there can be multiple NE
  - Efficiency is not satisfied either (e.g., (C, C) in prisoner's dilemma is a NE)
  - Some NE of a given game are not robust to small changes in payoffs

Best Response Correspondence

- There is a nice way to visualize the NE in some of the examples using the concept of best response correspondence of player i, denoted BR_i:
  - BR_i of player i maps each $\sigma_{-i} \in \Delta(S_{-i})$ to a subset $BR_i(\sigma_{-i})$ of S_i, where each $\sigma_i \in BR_i(\sigma_i)$ is a best response for player i against σ_{-i}
- Why is it called "correspondence" and not "function"?
  - If a player has a unique best response to each strategy profile of the other players, then BR_i is a function (e.g., Cournot duopoly)
  - But, in some cases player i may have more than one best response when other players play σ_{-i}, hence the more general concept of correspondence
  - Note that $(\sigma_1, \sigma_2, ..., \sigma_n)$ is a NE if $\sigma_i \in BR_i(\sigma_{-i})$ for all i
  - Since a pure strategy is a special case of a mixed strategy, the same concept holds for pure strategies

Best Response Correspondence

- Example: Cournot duopoly
  - Recall that in that case FOCs are $s_1 = \frac{90-s_2}{2}$ and $s_2 = \frac{90-s_1}{2}$
  - Note what the first equation says: for each quantity s_2 that firm 2 produces, the best response for firm 1 is to produce $\frac{90-s_2}{2}$
  - Thus, $BR_1(s_2) = \frac{90-s_2}{2}$, and similarly $BR_2(s_1) = \frac{90-s_1}{2}$
  - The conditions for NE, $s_1 \in BR_1(s_2)$ and $s_2 \in BR_2(s_1)$, simply says that (s_1, s_2) must solve the two equations in two unknowns
  - Graphically, it is depicted as the intersection of the two best responses

Best Response Correspondence

- Example: Matching pennies
  - Recall $\sigma_1 = (p, 1-p)$ and $\sigma_2 = (q, 1-q)$
  - Let us denote the best response of 1 by $BR_1(q)$, since σ_2 is summarized by q
  - Facing q, payoff of H for 1 is $q(-1) + (1-q)1 = 1 - 2q$, and from T is $2q - 1$
  - Thus, if $q < 0.5$, then the best for player 1 is to choose H for sure ($p = 1$), and if $q > 0.5$ the best for 1 is to choose L for sure ($p = 0$)
  - And if $q = 0.5$, then player 1 is indifferent among all values of $p \in [0, 1]$

Player 2

		H	T
Player 1	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

Best Response Correspondence

- Continuation of the example:
  - Let us analyze $BR_2(p)$ now
  - Facing p, payoff of H for 1 is $p1 + (1-p)(-1) = 2p - 1$, and from T is $1 - 2p$
  - Thus, if $p < 0.5$, then the best for player 2 is to choose T for sure ($q = 0$), and if $p > 0.5$ the best for 2 is to choose H for sure ($q = 1$)
  - And if $p = 0.5$, then player 2 is indifferent among all values of $q \in [0, 1]$

Player 2

		H	T
Player 1	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

Best Response Correspondence

- Summarizing:

$$BR_1(q) = \begin{cases} p = 1 & \text{if } q < 0.5 \\ p \in [0,1] & \text{if } q = 0.5 \\ p = 0 & \text{if } q > 0.5 \end{cases}$$

$$BR_2(p) = \begin{cases} q = 0 & \text{if } p < 0.5 \\ q \in [0,1] & \text{if } p = 0.5 \\ q = 1 & \text{if } p > 0.5 \end{cases}$$

- The only intersection is at $p = q = 0.5$ (check), the mixed strategy NE

Best Response Correspondence

- Example: Coordination game
  - Proceeding as in the matching pennies example, we obtain

$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < \frac{1}{3} \\ p \in [0,1] & \text{if } q = \frac{1}{3} \\ p = 1 & \text{if } q > \frac{1}{3} \end{cases}$$

$$BR_2(p) = \begin{cases} q = 0 & \text{if } p < \frac{2}{3} \\ q \in [0,1] & \text{if } p = \frac{2}{3} \\ q = 1 & \text{if } p > \frac{2}{3} \end{cases}$$

  - Now there are three intersections, the three NE (check)

Player 2

Player 1		O	B
	O	$2, 1$	$0, 0$
	B	$0, 0$	$1, 2$