ECN 532

Microeconomics II

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DYNAMIC GAMES WITH COMPLETE INFORMATION

Lecture's Objective

- In this lecture we will cover dynamic strategic situations
- We will learn the following
  - We will first introduce the extensive form representation
  - We will then focus on games with complete and perfect information and learn how to solve them by backward induction
  - Then we will turn to games with complete but imperfect information and define a more general solution concept called subgame perfect equilibrium
  - This is a refinement of Nash equilibrium
  - We will go over economic applications

Notation

- We will use the following notation:
  - Γ^e is an extensive-form game
  - Game tree is collection of nodes X with precedence relation $>$
  - H_i collection of information sets h_i of player i
  - $A_i(h_i)$ set of actions of i at h_i
  - $s_i : H_i \to A_i$, $s_i(h_i) \in A_i(h_i)$ is a pure strategy
  - σ_i is a probability distribution over pure strategies of i

Extensive-Form Games

Definition of Extensive-Form Games

- An extensive form game, denoted by Γ^e, specifies the following:

  1. Set of players $N = \{1, 2, ..., n\}$
  2. Order of moves of the players
  3. Set of actions A_i of player $i = 1, 2, ..., n$ when they can choose
  4. Knowledge players have when they choose
  5. Probability distributions over exogenous events (moves by Nature)
  6. Payoff functions for each player, u_i, as a function of outcomes
  7. All of these items are common knowledge among all the players

Definition of Extensive-Form Games

- Remarks:
  - Note that the first and last item are as in the normal form
  - The rest provides more detail about interaction (order, actions, knowledge)
  - Regarding knowledge players have, if say player i chooses after player j, this item specifies whether i observes and thus knows j's choice or not
  - When there is some exogenous uncertainty in the game (e.g., demand can be high or low with given probabilities in a game among firms), we add a player, Nature, who chooses according to a pre-specified probability distribution

Definition of Extensive-Form Games

- Example: trust game
  - One individual has to decide whether to request a service from another
  - Requesting a service means "trust" while not doing it means "no trust"
  - If she does not request, each obtains a payoff of 0
  - If she requests it and the other person provides the service well, then payoffs are 1 for each
  - If she requests it and the other provides an inferior low cost service, then payoffs are -1 for individual who requested service and 2 for the other one

Definition of Extensive-Form Games

- The extensive form game of the example is as follows:
  - Two players, 1 and 2
  - Player 1 chooses first, player 2 chooses second
  - Player 1 can choose between T or N; player 2 chooses between G or B
  - Player 2 knows when they choose that player 1 has played T
  - Payoffs are $u_1(N, G) = u_1(N, B) = 0$, $u_1(T, G) = 1$, $u_1(T, B) = -1$ for player 1; $u_2(N, G) = u_2(N, B) = 0$, $u_2(T, G) = 1$, $u_2(T, B) = 2$ for player 2

Definition of Extensive-Form Games

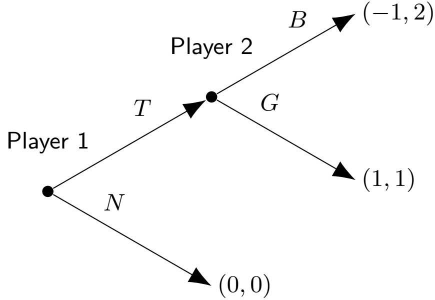
- Example: Sequential version of coordination game
  - Consider a sequential version of the coordination game in which player 1 chooses O or B first, player 2 observes and then makes a choice
- The extensive form game of the example is as follows:
  - Two players, 1 and 2
  - Player 1 chooses first, player 2 chooses second
  - Player 1 can choose between O or B; player 2 chooses between O or B
  - Player 2 knows when they choose that player 1 has played either O or B
  - Payoffs are $u_1(O,B) = u_1(B,O) = 0$, $u_1(O,O) = 2$, $u_1(B,B) = 1$ for player 1; $u_2(O,B) = u_2(B,O) = 0$, $u_2(O,O) = 1$, $u_2(B,B) = 2$ for player 2

Game Trees

- A game tree is a diagrammatic description of Γ^e
- A game tree is defined as follows:
  - It is a set of nodes $x \in X$ with a precedence relation $x > x'$ (x precedes x')
  - Every node has only one predecessor
  - $>$ is transitive, asymmetric, and incomplete (not all nodes can be ordered)
  - The root of the tree, x_0, is a node that precedes all of the other nodes
  - Nodes that precede other nodes are called terminal nodes. The set of terminal nodes is denoted by Z, which is a subset of X
  - Each terminal node has associated payoffs for each player (end of the game)
  - Every node x that is not a terminal node is assigned either to a player i with action set $A_i(x)$, or to Nature

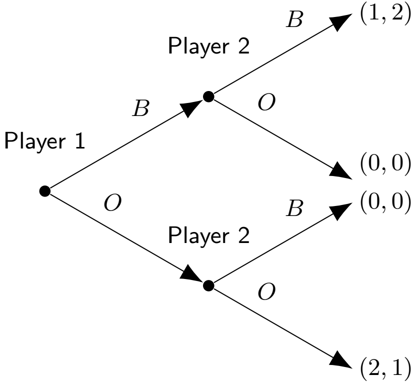
Game Trees

- Example: Game tree of trust game



Game Trees

- Example: Sequential version of coordination game



Game Trees

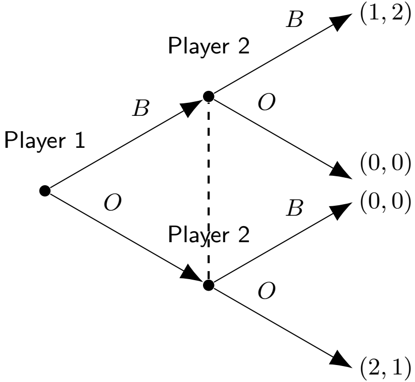
- Let us elaborate on the information a player has in a game
- Each player i has a collection of information sets $h_i \in H_i$ that partitions the nodes at which i chooses, with the following properties:
  - If h_i contains a single node x, then i knows that they are at x
  - If $x \neq x'$, $x \in h_i$ and $x' \in h_i$, then i does not know whether they are at x or x' when choosing at x
  - If $x \neq x'$ and $x \in h_i$ and $x' \in h_i$, then $A_i(x) = A_i(x')$
- Some intuition:
  - The first condition says that i knows that when the game reaches x and it is their turn to choose, i knows that they are at x
  - The second means that at h_i, both x and x' are consistent with reaching that part of the game, and i does not know at which node they are choosing
  - Finally, the last condition is for logical consistency: if the action sets at the two nodes where different, then i would know whether they are at x or x'

Perfect and Imperfect Information

- Consider a game with complete information:
  - If every information set of every player has only one node, then the game has perfect information
  - If some information sets contain several nodes, then the game has imperfect information
- Examples:
  - Coordination game has imperfect information
  - Sequential version of coordination game has perfect information
  - Trust game has perfect information
  - Game tree of prisoner's dilemma has imperfect information

Game Trees

- Example: Coordination game



Strategy in Γ^e

- A pure strategy for player i in Γ^e is $s_i : H_i \to A_i$ that assigns $s_i(h_i) \in A_i$ for every information set
  - This dovetails nicely with the idea of a strategy as a complete plan of action
  - With one caveat: the definition above requires that s_i be defined even at places that are not reached when the game is played
  - This point will become clear later
- A mixed strategy in Γ^e is a probability distribution over pure strategies s_i
- A related notion is a behavioral strategy in Γ^e:
  - It specifies for each information set h_i an independent probability distribution $\beta_i : H_i \to \Delta(A_i(h_i))$
- This says that instead of randomizing over pure strategies, player i can randomize in each h_i over the actions at that h_i
- If Γ^e has perfect recall (a player never forgets what they once knew), then mixed and behavioral strategies are equivalent

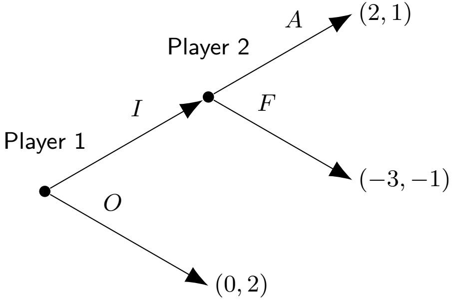
Normal-Form Game from Extensive-Form Game

- Given Γ^e, we can easily derive the associated $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$
  - Take the collection of all the pure strategies s_i defined above, and call it S_i, $i = 1, 2, ..., n$
  - For any strategy profile, define u_i as the payoff obtained in the corresponding terminal node of Γ^e
- We immediately obtain the following:
  - The set of NE of Γ^e can be obtained by deriving all the NE of the associated Γ
- Terminology:
  - Given a NE in Γ^e (in pure or mixed strategies), we say that an information set is on-the-equilibrium path if it is reached with strictly positive probability
  - Given a NE in Γ^e (in pure or mixed strategies), we say that an information set is off-the-equilibrium path if it is never reached

Backward Induction

Sequential Rationality and Credibility

- Although we could use NE as our solution concept, we will see that it is too weak in a precise sense in interactions that are sequential
- The main issue is that strategies in a NE can lack the property of being sequentially rational, or "credible"
- Let us illustrate this point with a simple example, an entry game:



Sequential Rationality and Credibility

- The strategic representation of this game is the one below:
  - Note that there are two NE: (I, A), (O, F)
  - If 1 thinks 2 will play A, the best response for 1 is I, and if 2 thinks 1 will play I, the best for 2 is A. Thus, (I, A) is a NE
  - If 1 thinks 2 will play F, the best for 1 is O, and if 2 thinks 1 will play O, F is one of the best responses. Thus, (O, F) is a NE
  - Note in passing that the second one is a NE in which player 2 plays a weakly dominated strategy, F

Player 2

Player 1		A	F
	I	$2, 1$	$-3, -1$
	O	$0, 2$	$0, 2$

Sequential Rationality and Credibility

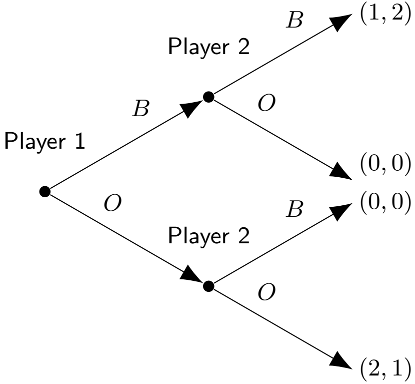
- There is something unappealing about the second NE once we look at it in the extensive form
- Loosely speaking, it relies on an empty threat, or lacks credibility
- Note that in (O, F), player 1 plays O because she believes that 2 will play F
- But, a natural conjecture should be that if player 2 is asked to make a choice at his information set, then he will play optimally having reached that point
- That is, player 1 should expect player 2 to play A instead of F, since at 2's information set, this is the optimal choice for player 2
- In other words, NE (O, F) fails to be sequentially rational, or lacks credibility
- Instead, NE (I, A) satisfies sequential rationality at each information set
- Problem: NE in an extensive form game puts no constraints on the belief a player has about how the other player would play off the equilibrium path

Backward Induction

- Let us define sequential rationality:
  - Given σ_{-i}, we say that σ_i is sequentially rational if player i is playing a best response to σ_{-i} at each of their information sets
- How do we find strategy profiles that are NE and also satisfy sequential rationality for each player?
  - For games with complete and perfect information, this is accomplished by a procedure called backward induction
  - Start at the end of the game tree and check the last information sets
  - Find at each of those information sets the player's optimal choice
  - Then move backwards to the next to last information sets, and choose the optimal action for each of the players who make a choice, assuming that in the last information sets choices will be optimal
  - Proceed inductively until reaching the root of the game tree

Backward Induction

- Example: Entry game
  - In this game backward induction leads to the prediction (I, A)
- Example: Sequential version of coordination game
  - What are the NE of this game? What is the backward induction prediction?



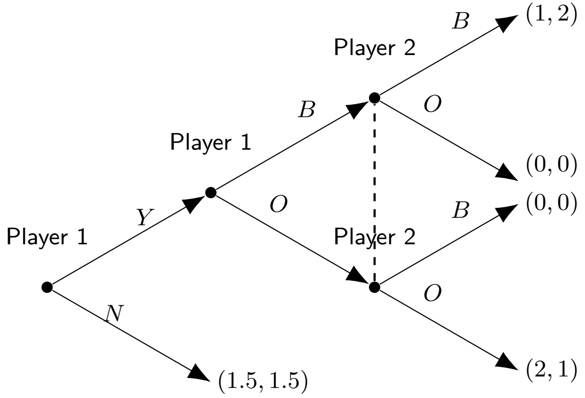
Backward Induction

- Regarding existence, one can easily prove (how?)
- In any finite game with complete and perfect information, there is a backward induction solution that is sequentially rational
- And if there no ties for a player payoffs at any of their terminal nodes, then the backward induction solution is unique
  - It follows that in any finite game with complete and perfect information, there is a NE that is sequentially rational
  - But as the examples illustrate, there could be other NE that are not credible

Subgame Perfect Equilibrium

Subgames

- We run into problems when we want to extend the backward induction procedure to games with complete but imperfect information
  - Another issue is how to analyze sequentially rational NE in games with an infinite number of stages (or periods). We will deal with this issue later
- Let us see an example with imperfect information:

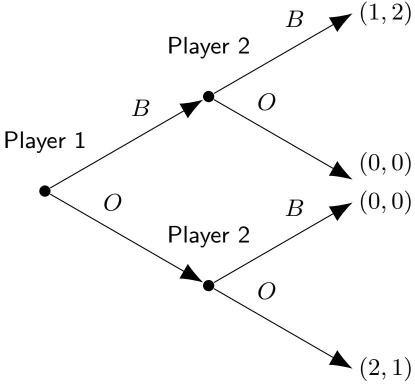


Subgames

- In this example player 1 first chooses whether to play the coordination game
- Note that it is unclear how to adapt backward induction to this case
- Player 2 at the end has an information set with two nodes, and it is not clear how player 2 will choose without having some belief about what player 1 will choose in that part of the game
- But beliefs are not part of the procedure of backward induction
- To extend the logic of backward induction, we need the notion of subgame
  - A proper subgame G of Γ^e consists of a single node and all of its successors with the property that if $x \in G$ and $x' \in h_i(x)$, then $x \in G$
  - The subgame G is itself a game tree with information sets and payoffs inherited from Γ^e

Subgames

- Examples:
  - In the previous example there are two proper subgames: the whole game, and the one that starts at the second node of player 1
  - In the sequential version of the coordination game there are three proper subgames (which ones?)

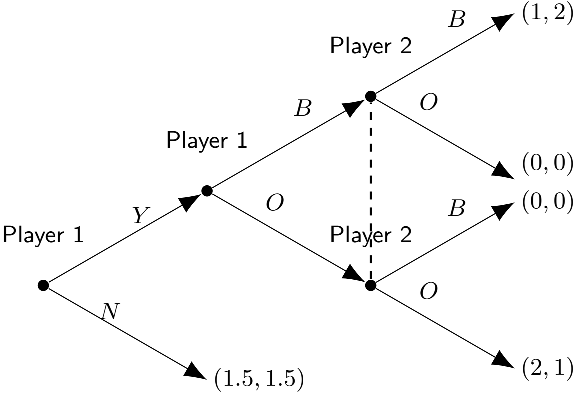


Subgame Perfect Equilibrium

- We can now define the following solution concept
  - A strategy profile $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ is a subgame perfect equilibrium (SPE) of Γ^e if for every proper subgame G of Γ^e, the restriction of σ to G is a NE
- It is also called subgame perfect Nash equilibrium (SPNE)
- Note that it is a natural generalization of the backward induction solution, and in games with perfect information SPE reduces to the NE found by backward induction
- Hence, SPE applies to both games with perfect and imperfect information
- SPE ensures that players strategies are sequentially rational in each proper subgame G (off and on equilibrium path) by requiring to be a NE in G
- In particular, this implies that a SPE is indeed a NE (the entire game is a proper subgame of itself) but in sequentially rational strategies (credible)

Subgame Perfect Equilibrium

- Example:
  - In the coordination game in which 1 can decide whether to play it or not, there are two proper subgames



Subgame Perfect Equilibrium

- Example:
  - Let us start the analysis from the back of the game: in that subgame there are two NE, (O, O) and (B, B)
  - Pick one of them, say (O, O); then moving back, player 1 anticipates that if she chooses Y the continuation is (O, O) and thus she obtains 2
  - Hence, it is optimal for her to play Y and the following is a SPE: $((Y, O), O)$
  - Pick now the NE (B, B); then moving back player 1 anticipates that if she chooses Y the continuation is (B, B) and she obtains 1
  - Hence, it is optimal for her to play N and the following is a SPE: $((N, B), B)$
  - Thus, in this game there are two SPE (check that both are NE); and there is another NE that is not SPE (which one?)

Subgame Perfect Equilibrium

- Remarks:
  - SPE was developed by Selten (1975), backward induction by Zermelo (1913)
  - SPE is a refinement of NE (it is a NE that satisfies an additional property)
  - Regarding existence, since we need to check that a SPE is a NE in each proper subgame, and there is always a NE in any finite game, it follows that a SPE exists in every finite game (with perfect or imperfect information)
  - As the previous example illustrates, it need not be unique, and one can show that it need not be efficient
  - Terminology: the strategy profile that is a SPE is a complete plan on and off the equilibrium path; the outcome of a SPE is the equilibrium path traced by the strategy profile
  - For example, $((N, B), B)$ is a SPE, whose outcome is N with payoffs $(1.5, 1.5)$

Muti-Stage Games

Definition of a Multi-Stage Games

- A particular class of extensive form games important for economic applications is the class of multi-stage games with observed actions:
  - It is divided into $T \leq \infty$ stages (or periods)
  - Within each stage, players play simultaneously and each player moves only once in each stage
  - At the end of each stage t, players know the actions taken by all the players (including moves by nature) in all previous stages
  - The payoff of player i at the end of the game depends on the entire history of play from the initial stage 0 to the final stage T
- At each stage t, the history up to that point is $h^t = (a^0, a^1, ..., a^{t-1})$, where $a^k = (a_1^k, a_2^k, ..., a_n^k)$ is vector of actions chosen by the n players in stage k
- Regarding the payoff of i (most) common specifications are:
  - The sum of payoffs of each stage $\sum_{t=0}^{T} u_i(a^t)$
  - The discounted sum of payoffs of each stage $\sum_{t=0}^{T} \delta^t u_i(a^t)$, with $\delta \in (0, 1)$

Definition of Multi-Stage Games

- Remarks:
  - One way to interpret this class of games is that players play a sequence of normal form games, which need not be the same game in each stage
  - Regarding the discounted sum of payoffs, player i receives a payoff at the end of each stage t, and so

$$\sum_{t=0}^{T} \delta^t u_i(a^t) = u_i(a^0) + \delta u_i(a^1) + \cdots + \delta^T u_i(a^T)$$

  - The discount factor δ reflects the degree of patience of player i
  - Extreme case of impatience is $\delta = 0$: only payoffs in current stage matter
  - Extreme case of patience is $\delta = 1$: receiving a given payoff today or tomorrow is valued the same way
  - Higher value of δ means that a player is more patient and thus puts higher value on future payoffs

Subgame Perfect Equilibrium

- Since a multi-stage game with observed actions is an extensive form game with complete and imperfect information
  - We use SPE as our solution concept
- Example: Prisoner's dilemma
  - Recall the prisoner's dilemma and assume that is is played twice
  - Assume payoff to each player is sum of each stage payoff
  - Find the subgame perfect equilibria of the game

Player 2

Player 1		M	C
	M	$5,5$	$0,8$
	C	$8,0$	$3,3$

Subgame Perfect Equilibrium

- The following results are now very intuitive (proof?):
  - If σ is a NE (in pure or mixed strategies) of the multi-stage game consisting of a sequence Γ^0, Γ^1,...,Γ^T normal form games, then the restriction of σ to the game in stage T must be a NE of that stage game
  - In a finite multi-stage game, if the simultaneous game played at each stage has a unique NE, then the multi-stage game has a unique SPE
- Note how the second result is based on the first one:
  - By the first result, it must be the case that in the last stage the unique NE is played after each history reaching that part
  - Moving backwards to the beginning of previous stage, the unique NE must be played since no matter what is played the continuation is the same
  - Continuing backwards we reach the root of the game tree and at that stage the unique NE is played as well
  - We have thus constructed the unique SPE of the multi-stage game

Subgame Perfect Equilibrium

- A third result is the following one:
  - In a multi-stage game, if σ^t is a NE of stage t, $t = 0, 1, ..., T$, then the resulting sequence $\sigma^0, \sigma^1, ..., \sigma^T$ is a SPE of the multi-stage game
- Regardless of history at each stage a NE is played, and thus resulting profile constitutes a SPE, since in every subgame a NE is played (check)
- Example: Variation of prisoner's dilemma
  - Assume two stages, discount factor is $\delta \in (0, 1)$
  - First stage prisoner's dilemma is played; second stage, game below is played

Player 2

Player 1		l	g
	L	$1, 1$	$-4, -1$
	G	$-1, -4$	$-2, -2$

Subgame Perfect Equilibrium

- Continuation of example:
  - In the first stage there is a unique NE, (C, C)
  - In the second stage there are two NE in pure strategies, (L, l) and (G, g) (and one in mixed strategies)
  - Applying previous result, playing (C, C) in first stage and (L, l) in second stage after every history is a SPE
  - (C, C) in first stage and (G, g) in second stage after every history is a SPE
  - (C, C) in first stage and the NE in mixed strategies in second stage after every history is a SPE

Subgame Perfect Equilibrium

- Continuation of the example:
  - We can say something much more interesting if players condition their play on the history of first stage. Consider the following strategies for each player
  - Player 1: I will play M in first stage; in the second stage, if 2 played M in first stage, I will play L; otherwise I will play G
  - Player 2: I will play M in first stage; in the second stage, if 1 played M in first stage, I will play l; otherwise I will play g
  - Let us check that these strategies are a SPE if δ is large enough
  - In the second stage, after each history a subgame starts and players play a NE
  - Consider player i, $i = 1, 2$, in the first stage
  - If player i plays M, then payoff is $5 + \delta 2$
  - If player i plays C, then payoff is $8 + \delta(-2)$
  - Thus $5 + \delta 2 \geq 8 + \delta(-2)$ if and only if $\delta \geq \frac{3}{4}$, in which case we have a SPE

Repeated Games

Definition of Repeated Game

- Repeated games are a special class of multi-stage games:
  - In this class, the same normal form game is played at every stage
  - A repeated game can have finite or infinite horizon
  - Most common player's payoff function is discounted sum of utilities, with discount factor $\delta \in (0, 1)$
- This class shows up in many relevant applications:
  - Firms competing repeatedly over time for many periods
  - A firm and its suppliers interact repeatedly over time
  - Politicians repeatedly negotiate among them in congress
  - Workers producing in teams that interact repeatedly
  - Countries engage repeatedly in bilateral or multilateral agreements
- Analysis of repeated games sheds light on logic of long-term interaction

Finitely Repeated Games

- Recall Γ is a strategic form game
- A finitely repeated game with horizon $T < \infty$ consists of
  - A stage game Γ that is played consecutively T times
  - Players' payoffs are given by a discounted sum of utilities with common discount factor $\delta \in (0, 1]$
- Remarks:
  - Note that in this case we allow $\delta = 1$ (no discounting)
  - We assume a common δ but this can easily be relaxed

Finitely Repeated Games

- Recall the following result we stated for multi-stage games:
  - In a finite multi-stage game, if the simultaneous game played at each stage has a unique NE, then the multi-stage game has a unique SPE
- This result also applies to finitely repeated games:
  - At each stage the same normal form game Γ is repeated
  - If Γ has a unique NE, then the finitely repeated game has a unique SPE
  - The outcome of that SPE is that the NE of Γ is played at every stage
- Implication:
  - In finitely repeated prisoner's dilemma, SPE outcome is (C, C) every period
  - Counterintuitive if one thinks about T very large, say 100

Finitely Repeated Games

- Things are more interesting if the stage game has more than one NE
- Example:
  - Consider the game below played twice
  - There are two NE in pure strategies in the stage game, (B, b) and (C, c)
  - There is another one in mixed strategies, but let us focus on these two
  - Note they are Pareto ranked (both players are strictly better off in (C, c))
  - By a previous result for multi-stage games, playing any sequence of these NE during the two periods constitute a SPE (check)

Player 2

Player 1		a	b	c
	A	$4, 4$	$-1, 5$	$0, 0$
	B	$5, -1$	$1, 1$	$0, 0$
	C	$0, 0$	$0, 0$	$3, 3$

Finitely Repeated Games

- Continuation of example:
  - We know in second period in each subgame a NE must be played (why?), but which one is played can be dependent on the first period
  - Suppose player 1 plays A in the first period; in second period plays C if player 2 played a in first period, otherwise plays B in second period
  - Suppose player 2 plays a in the first period; in second period plays c if player 1 played A in first period, otherwise plays b in second period
  - If $\delta \geq 0.5$, then this is a SPE, with outcome $(A, a), (C, c)$ (check)

Player 2

		a	b	c
	A	$4, 4$	$-1, 5$	$0, 0$
Player 1	B	$5, -1$	$1, 1$	$0, 0$
	C	$0, 0$	$0, 0$	$3, 3$

Finitely Repeated Games

- Remarks:
  - Note that in the second period each player is playing optimally given what the other is doing (NE logic)
  - Regarding the first period, each player by playing its part obtains 4 in that period and 3 in the second one
  - By deviating in the first period a player can obtain 5 in the first period, but given the assumed strategies, the player obtains 1 in the second period
  - By deviating a player makes 1 more in first period but 2 less in second period
  - So if second period payoffs are not discounted more than half, then each player will not deviate in first or second period, and the strategies are a SPE
  - The existence of multiple NE in stage game allows players to design strategies that incorporate "rewards and punishments"

Infinitely Repeated Games

- Now $T = \infty$, so stage game is played an infinite number of times
  - Equivalently, at each stage there is a positive probability the game continues
  - Plausible assumption in economic applications in which players do not know how many times they are going to play the stage game
  - For example, oligopolistic markets in which same set of firms compete over and over again without a foreseeable end
- Some remarks about infinite discounted sums of payoffs:
  - Suppose an individual obtains $\$100$ in perpetuity; discount factor is $\delta \in (0, 1)$
  - Let S be discounted sum $\sum_{t=0}^{\infty} \$10$; it satisfies $S = \$10 + \delta S \;\Rightarrow\; S = \frac{\$10}{1-\delta}$
  - Hence, the discounted sum of receiving an amount $\hat{u}$ of utility is $\frac{\hat{u}}{1-\delta}$
  - For a stream $(u_0, u_1, u_2, \ldots)$ we do not have a closed form in general, but as long as these utility amounts are bounded, then sum is well defined (check)
  - If $v = \sum_{i=0}^{\infty} \delta^t u_t(a^t)$, then $\bar{v} = (1-\delta) \sum_{i=0}^{\infty} \delta^t u_t(a^t)$ is the average payoff (allows to compare infinitely repeated game payoff with stage game payoffs)

Infinitely Repeated Games

- Recall that a strategy is a complete contingent plan which specifies how a player would choose at each of their information sets
- Infinitely repeated game: each player has infinite number of information sets
- How is a strategy defined in this case?
  - Key observation: each information set t identified with a unique path up to t
  - Thus, to each information set of a player, there exists a unique history that reaches that information set
  - Identify each information set with a history of play (one-to-one relationship)
- The definition of a strategy is now obvious:
  - H^t set of all t-histories $h^t \in H^t$, and $H = \cup_{t=1}^{\infty} H^t$ set of all histories
  - A pure strategy for player i in the infinitely repeated game is $s_i : H \to S_i$, where S_i is the set of actions of player i in the stage game
  - A behavioral strategy for player i in infinitely repeated game is $\sigma_i : H \to \Delta(S_i)$

Infintely Repeated Games

- Checking whether a strategy profile is SPE seems impossible
- Checking if a deviation is profitable requires to check all possible deviations (at only one information set, at two, at three,...,at an infinite number)
- A similar issue applies to any multi-stage game:
  - Checking for all possible deviations can be a very complex task
- The one-shot deviation principle (OSDP) comes to our rescue:
  - For any multi-stage game, consider σ_i, and let σ_i^{a,h_i} be the strategy that is identical to σ_i except after history h_i, where it replaces whatever σ_i plays by action a of player i
  - A strategy satisfies OSDP if, given σ_{-i}, for every a and h_i of player i, there is no deviation from σ_i to σ_i^{a,h_i} that is profitable
  - Given σ_{-i}, if σ_i satisfies OSDP, then it is optimal for player i

Infinitely Repeated Games

- This is an amazing result that drastically simplifies checking for SPE:
  - In any multi-stage game (actually in any extensive form game), one only needs to check that a strategy profile satisfies OSDP for each player
- Particularly useful in infinitely repeated games:
  - Each information set at t is associated with a history of play h^t
  - For $(\sigma_1, \sigma_2, ..., \sigma_n)$ to be SPE of infinitely repeated game, suffices to check that for each i, given σ_{-i}, the strategy σ_i satisfies OSDP
- Easy to sketch why OSDP implies optimality:
  - If σ_i satisfies OSDP but is not optimal, then some σ_i' is strictly better
  - If σ_i' involves deviations are a finite number of information sets, then at the last one of those, OSDP of σ_i implies that from that point on σ_i' is better than σ_i'
  - Moving back, same holds at next to last deviation; so on until first deviation, and it follows that σ_i has weakly higher payoff than σ_i', contradiction
  - Similar argument if σ_i' entails an infinite number of deviations

Infinitely Repeated Games

- In finitely repeated prisoner's dilemma unique SPE yields (C, C) in every t
- This need not be the case in the infinitely repeated version
- Example: Prisoner's Dilemma
  - Note there is a SPE in which each player plays $s_i(h^t) = C$ for every t
  - But if players are patient enough, there are other SPE
  - The idea once again is to structure punishments and rewards

Player 2

Player 1		M	C
	M	$5, 5$	$0, 8$
	C	$8, 0$	$3, 3$

Infinitely Repeated Games

- Continuation of example:
  - Consider the following strategies for each player $i = 1, 2$
  - Player i starts by playing M, and continues to do so at each t if history of play is (M, M), (M, M),...; if not, i plays C from that point onwards
  - Formally, for both $i = 1, 2$, $s_i(h^0) = M$, $s(h^t) = M$ if $h^t = ((M, M), (M, M), ..., (M, M))$; for any other h^t, $s_t(h^t) = C$
  - Each player starts being "nice" and continues to be nice if both have been nice in the past; otherwise, "trust" or "credibility" is lost and each switches to C
  - Called a trigger strategy: a deviation triggers a change in behavior forever

Player 2

		M	C
Player 1	M	$5, 5$	$0, 8$
	C	$8, 0$	$3, 3$

Infinitely Repeated Games

- Continuation of example:
  - Let us check that this strategy profile (s_1, s_2) is SPE of the infinitely repeated game when players are patient enough
  - Trigger strategies partition the set of all histories into two sets
  - One set consists of $h^t = ((M, M), (M, M), ..., (M, M))$ for each t
  - The other set consists of all other h^t for each t
  - In the second set, note that players are playing (C, C) all the time in each of the h^t, which is a NE in any of the subgames that starts at such h^t

Player 2

Player 1	M	C
M	$5, 5$	$0, 8$
C	$8, 0$	$3, 3$

Infinitely Repeated Games

- Continuation of the example:
  - Consider now any $h^t = ((M, M), (M, M), ..., (M, M))$
  - By OSDP, it is enough to check for a single deviation an any point in time along this history, given the strategy of the other player
  - Player i has two choices: if i plays M today then discounted payoff is (why?)

$$5 + \delta 5 + \delta^2 5 + \delta^3 5 + \cdots = \frac{5}{1-\delta} = 5 + \frac{\delta}{1-\delta} 5$$

Player 2

Player 1		M	C
	M	$5, 5$	$0, 8$
	C	$8, 0$	$3, 3$

Infinitely Repeated Games

- Continuation of example:
  - If instead player i plays C today, then discounted payoff is (why?)

$$8 + \frac{\delta}{1-\delta} 3$$

  - Thus, i's strategy is optimal if and only if (why?)

$$5 + \frac{\delta}{1-\delta} 5 \geq 8 + \frac{\delta}{1-\delta} 3 \;\Rightarrow\; \delta \geq \frac{3}{5}$$

  - Thus, if $\delta \geq 0.6$, then (s_1, s_2) is a SPE, since players are playing a NE in every proper subgame of the infinitely repeated game (check)

Player 2

Player 1	M	C
M	$5, 5$	$0, 8$
C	$8, 0$	$3, 3$

Infinitely Repeated Games

- Remarks:
  - The example illustrates a general insight from infinitely repeated games
  - Repetition without a foreseeable end can expand the predictions that we obtain from a finitely repeated game
  - In this example, the credible threat that play will switch to (C, C) forever upon a deviation from playing (M, M), makes playing (M, M) all the time an equilibrium path, which involves behavior that is not NE in the stage game
  - Same is true in a general infinitely repeated game, not just prisoner's dilemma
  - The SPE constructed is not the only one besides repetition of NE
  - For example, one can also construct a SPE in which players alternate between (M, C) in odd periods and (C, M) in even periods, again with the threat of reverting to (C, C) forever (check)
  - Indeed, the number of SPE payoffs that can be sustained in an infinitely repeated game is enormous, as exemplified by the so called "folk theorem"

Infinitely Repeated Games

- Folk theorem asserts the following:
  - Let Γ be a finite stage game that infinitely repeated
  - Let V be the convex set of payoffs constructed from payoff vectors of Γ
  - Let $u^* = (u_1^*, u_2^*, ..., u_n^*)$ be the payoff from a NE in Γ
  - Let $v = (v_1, v_2, ..., v_n)$ be a vector of payoffs in V such that $v_i > u_i^*$ for all i
  - If δ sufficiently close to 1, there is a SPE $(s_1, s_2, ..., s_n)$ in the infinitely repeated game that yields average payoffs arbitrarily close to $(v_1, v_2, ..., v_n)$
- Easy to illustrate the Folk theorem in the prisoner's dilemma (check)
- Remark:
  - There are more general versions of the folk theorem in the literature, as well versions in which the actions of players are not observable but a noisy signal is