ECN 532

Microeconomics II

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OLIGOPOLY MODELS

Lecture's Objective

- In this lecture we will go over some well-known models of oligopoly
  - There are a few firms in the market and large number of consumers
  - Classic models of oligopoly are Bertrand, Cournot, and Stackelberg
- We will first cover Bertrand oligopoly model
  - Firms in the market compete by choosing prices
- We will then study Cournot oligopoly with many firms
  - Firms in the market compete by choosing quantities
- Finally, we will analyze Bertrand oligopoly with differentiated products
- In each case we study Nash equilibrium in pure strategies and its properties
- We will leave Stakelberg model for later since it is a sequential game

Bertrand Oligopoly Model

Bertrand Oligopoly

- Consider the following market for a homogeneous product:
  - There are two firms, firm 1 and firm 2
  - The market demand is given by $x(p)$, strictly decreasing, continuous, with $x(p) > 0$ for all $0 < p < \bar{p} < \infty$, and $x(p) = 0$ for $p \geq \bar{p}$
  - Firms have no fixed costs and constant returns to scale production function, and thus they have constant cost per unit and marginal cost $c > 0$
  - The competitive market quantity $x(c)$ is strictly positive and less than infinity
  - Firms simultaneously announce their prices per unit, p_1 and p_2
  - The quantity demanded from firm i, $i = 1, 2$ and $j \neq i$, is

$$x_i(p_i, p_j) = \begin{cases} x(p_i) & \text{if } p_i < p_j \\ \frac{1}{2}x(p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

  - Firms produce to order, so they incur cost only for output level that it sells
  - Firm i's profit is equal to $(p_i - c)x_i(p_i, p_j)$

Bertrand Oligopoly

- It is easy to write the strategic form of the game:
  - Two players, firm 1 and firm 2
  - Strategy sets are $S_1 = [0, \infty)$, $S_2 = [0, \infty)$ with $p_i \in S_i$, $i = 1, 2$
  - Payoffs are $(p_1 - c)x_1(p_1, p_2)$ and $(p_2 - c)x_2(p_1, p_2)$
- We will focus on pure strategies and derive the Nash equilibrium
  - There is a unique Nash equilibrium in pure strategies, given by $(p_1, p_2) = (c, c)$, and both firms make zero profits
- A remarkable feature is that despite the fact that the market has only two firms, we obtain the perfectly competitive outcome
- Intuition is that competition makes each firm face an infinitely elastic demand at the price charged by the competitor

Bertrand Oligopoly

- Proof of unique Nash equilibrium $(p_1, p_2) = (c, c)$
  - Given that firm 2 charges $p_2 = c$, then by charging $p_1 = c$ firm 1 makes 0, while if it charges more it makes zero while if it charges $p_1 < c$ makes losses. Similarly for firm 2 given that $p_1 = c$. Hence, this is indeed a NE
  - If $p_i < p_j < c$ or $p_i < c < p_j$, then firm i has incentives to raise p_i to avoid strictly negative profits. Hence, there is no NE with these prices
  - If $p_i = c < p_j$, then firm i can increase profits by setting a price between c and p_j (say $p_i = c + 0.5(p_j - c)$) and make strictly positive profits. Hence, there is no NE with these prices
  - If $c < p_i \leq p_j$, then firm j has incentives to deviate and charge $p_i - \varepsilon$ for $\varepsilon > 0$ but small and sell to the entire market. For ε small, $(p_i - \varepsilon - c)x(p_i - \varepsilon) > 0.5(p_j - c)x(p_j)$, where the right side is the largest it can make at the original prices. Hence, there is no NE with these prices
  - We have ruled out all alternative prices. Thus (c, c) is the unique NE

Bertrand Oligopoly

- Remarks:
  - Similar result holds if there are $n \geq 2$ firms
  - Assume that if k firms tie at the lowest price $\tilde{p}$, then each receives $\frac{1}{k}x(\tilde{p})$
  - Then it is easy to show that $(p_1, p_2, ..., p_n) = (c, c, ..., c)$ is a NE (other NE?)
  - Note that the equilibrium is in weakly dominated strategies: $p_i = c$ is weakly dominated by charging any $p_i > c$ (check)
  - Note that if the firms agree to behave as a monopolist, then they would each charge the price that solves $\max_p (p - c)x(p)$, with FOC $x(p) + (p - c)x'(p) = 0$ or $p = c - \frac{x(p)}{x'(p)} > c$, so firms would make strictly positive profits (each, say, half of profits, or $\frac{1}{n}$ of them)
  - If no way to enforce this agreement, each has incentives to deviate (check)
  - What if the two firms have different costs, say $c_1 < c_2$?
  - What if there are n firms with different marginal costs c_i, $i = 1, 2, ..., n$?

Cournot Oligopoly Model

Cournot Oligopoly

- Consider Cournot duopoly but with homogeneous products
  - Firm 1 and firm 2 simultaneously choose q_1 and q_2
  - The inverse demand for the product is $p(q_1 + q_2)$, where $p(\cdot) = x^{-1}(\cdot)$
  - Assume $p'(q) < 0$ at all $q = q_1 + q_2 \geq 0$
  - Both firms produce at constant marginal cost $c > 0$, with $c < p(0)$
  - We will assume there is a unique $q^0 \in (0, \infty)$ such that $p(q^0) = c$ or $x(c) = q^0$, where q^0 is the competitive market quantity
  - Each firm i chooses q_i to maximize $p(q_i + q_j)q_i - cq_i$, $i = 1, 2$, $j \neq i$
  - We will restrict attention to pure strategies

Cournot Oligopoly

- Let us show that in any NE, the market price is strictly above c
  - To see this, obtain the FOC for firms 1 and 2 (check)

$$p'(q_1 + q_2)q_1 + p(q_1 + q_2) = c$$
$$p'(q_1 + q_2)q_2 + p(q_1 + q_2) = c$$

  - One can actually show that solution is interior (check)
  - Add the two FOCs to obtain

$$p'(q_1 + q_2)\frac{q_1 + q_2}{2} + p(q_1 + q_2) = c$$

  - Since $p'(q) < 0$, we obtain

$$p(q_1 + q_2) = c - p'(q_1 + q_2)\frac{q_1 + q_2}{2} > c$$

  - In any NE firms obtain strictly positive profits, unlike Bertrand

Cournot Oligopoly

- Next, let us show that in any NE, $q_1 + q_2 > q^m$, the monopoly quantity
  - We first show that we cannot have $q^m > q_1 + q_2$
  - If so, then firm i could deviate and produce $q_i = q^m - q_j$, which would increase joint profits and reduce price
  - But this means that firm j is worse off, and thus firm i's profits are strictly higher, contradicting NE
  - We next show that we cannot have $q_1 + q_2 = q^m$
  - If so, then form addition of the FOCs we obtain

$$p'(q^m)\frac{q^m}{2} + p(q^m) = c$$

    which cannot hold since in monopoly $p'(q^m)q^m + p(q^m) = c$ (why?)
  - Thus, $q_1 + q_2 > q^m$, which implies that $p(q_1 + q_2) < p(q^m)$
  - Hence, Cournot duopoly leads to aggregate quantity strictly below the competitive one but strictly above the monopoly one

Cournot Oligopoly

- All the results extend to n firms
  - Suppose that there are n firms each with $c > 0$
  - Deriving the FOCs and then adding them yield the following equation (check)

$$p'\left(\sum_{i=1}^{n} q_i\right)\frac{\sum_{i=1}^{n} q_i}{n} + p\left(\sum_{i=1}^{n} q_i\right) = c$$

  - Note that if $n = 1$ we obtain the monopoly FOC
  - As $n \to \infty$ we obtain that $p(\sum_{i=1}^{n} q_i) \to c$, the perfectly competitive case
  - To see this, note that from above $\sum_{i=1}^{n} q_i \leq q^0$ for every n (oligopoly produces aggregate output below the perfectly competitive output)
  - But then, $0 \leq \frac{\sum_{i=1}^{n} q_i}{n} \leq \frac{q^0}{n} \to 0$, and $p'(\sum_{i=1}^{n} q_i)\frac{\sum_{i=1}^{n} q_i}{n} \to 0$ as $n \to \infty$
  - As an example, assume $p = a - bq$, where $q = \sum_{i=1}^{n} q_i$, and solve for the NE and check properties above

Cournot Oligopoly

- Remarks:
  - Intuition for strictly positive profits is that now firms do not face an infinitely elastic demand: if a firm reduces its quantity, it increases the price by approximately $-p'(q)$ (q total quantity)
  - Different marginal costs easy to accommodate: try it for $n = 2$ and $c_1 < c_2$ (what does your intuition tell you?)
  - We have an analogue of the mark-up equation we saw in the monopoly case. From the FOC of firm i, we obtain
    $$\frac{p(q) - c}{p(q)} = -\frac{p'(q)}{p(q)} q_i = -p'(q) \frac{q}{p(q)} \frac{q_i}{q} = -\frac{1}{\varepsilon(q)} s_i$$
    where $s_i = \frac{q_i}{q} = \frac{q_i}{\sum_{i=1}^{n} q_i}$ is the market share of firm i
  - If firms have same market share $\frac{1}{n}$ (as in a symmetric equilibrium) we have
    $$\frac{p(q) - c}{p(q)} = -\frac{1}{n\varepsilon(q)}$$
    which goes to zero as $n \to \infty$ (why?)

Bertrand Oligopoly with Differentiated Products

Bertrand with Differentiated Products

- If firms in oligopoly choose prices but the products are differentiated, then they make strictly positive profits in NE
- Assume the following market structure:
  - There are two firms, firm 1 and firm 2
  - Each firm has constant marginal cost $c > 0$
  - Firm i faces a demand for its product given by $x_i(p_i, p_j) = a - p_i + bp_j$, $i = 1, 2$, $j \neq i$, $a > c$, and $0 < b < 2$
  - Each firm i chooses p_i to maximize profits $(p_i - c)x_i(p_i, p_j)$
- Strategic-form of the game:
  - Players: firm 1 and firm 2
  - Strategies: $p_i \in S_i = [0, \infty)$, $i = 1, 2$
  - Payoffs: firm i's payoff $(p_i - c)x_i(p_i, p_j)$, $i = 1, 2$

Bertrand with Differentiated Products

- Let us find the unique NE of the game:
  - Firm 1 solves $\max_{p_1 \geq 0}(p_1 - c)(a - p_1 + bp_2)$, and FOC is

$$a - 2p_1 + bp_2 + c = 0 \;\Rightarrow\; p_1 = \frac{a + c + bp_2}{2}$$

  - Firm 2 solves $\max_{p_2 \geq 0}(p_2 - c)(a - p_2 + bp_2)$, and FOC is

$$a - 2p_2 + bp_1 + c = 0 \;\Rightarrow\; p_2 = \frac{a + c + bp_1}{2}$$

  - Solving the two equations in two unknowns (p_1, p_2) we obtain the NE

$$p_1 = p_2 = \frac{a + c}{2 - b}$$

  - The profit for each firm is $u_1(p_1, p_2) = u_2(p_1, p_2) = \frac{(a+(b-1)c)^2}{(2-b)^2} > 0$

- Remark:
  - Intuition is that consumers do not consider the products as perfect substitutes, so if a firm that charges more does not lose all of its sales
  - Firms benefit from product differentiation or from cost reduction technologies

Stackelberg Duopoly

Leader-Follower Market

- Consider the following market structure:
  - Two firms, firm 1 (leader) and firm 2 (follower)
  - Firms produce a homogeneous product at constant marginal cost $c > 0$
  - Firm 1 chooses its production level first, then firm 2 observes the quantity produced by firm 1, and chooses its own production level
  - Market inverse demand $p(q_1 + q_2)$ determines the price per unit
  - $p(q)$ strictly decreasing with $p'(q) < 0$, $p(0) > c$, and $p''(q) \leq 0$
- The items of the extensive representation of this game are:
  - Players: firm 1 and firm 2
  - Order of moves: firm 1 chooses first and firm 2 second
  - Actions sets of each player: $q_1 \in [0, \infty)$, $q_2 \in [0, \infty)$
  - Information: firm 2 observes q_1 before choosing q_2
  - Payoffs: $u_i(q_i, q_j) = p(q_i + q_j)q_i - cq_i$, $i = 1, 2$

Leader-Follower Market

- Note that this is a game with complete and perfect information
- Unlike examples we saw, it has a continuum of strategies
- Let us find the SPE of this game by backwards induction
  - After each q_1 there is a node at which firm 2 chooses q_2

$$\max_{q_2} p(q_1 + q_2)q_2 - cq_2 \;\Rightarrow\; p'(q_1 + q_2)q_2 + p(q_1 + q_2) = c \;\Rightarrow\; q_2(q_1)$$

  - Let us move backwards and solve the problem of firm 1, which anticipates that in the continuation of the game firm 2 responds with $q_2(q_1)$

$$\max_{q_1} p(q_1 + q_2(q_1))q_1 - cq_1 \;\Rightarrow\; p'(q_1 + q_2(q_1))(1 + q_2'(q_1))q_1 + p(q_1 + q_2(q_1) =$$

  - Solving FOC for q_1 we obtain optimal quantity firm 1 produces, q_1^*
  - Then the SPE is $(q_1^*, q_2(q_1))$, and SPE outcome is $(q_1^*, q_2(q_1^*)) = (q_1^*, q_2^*)$

Leader-Follower Market

- Let us show that the output of firm 1 is larger and that of firm 2 smaller than in the Cournot duopoly case, and firm 1's profits are strictly higher
  - FOC of firm 2 is exactly the same as in Cournot (interpretation is different)
  - FOC of firm 1 has an extra term, $p'(q_1 + q_2(q_1))q_2'(q_1)$
  - We show that $q_2'(q_1) < 0$; differentiating FOC of firm 2 with respect of q_1

    $p''(q_1+q_2(q_1))(1+q_2'(q_1))q_2(q_1)+p'(q_1+q_2(q_1))q_2'(q_1)+p'(q_1+q_2(q_1))(1+q_2'(q_1))$
  - Therefore, $q_2'(q_1)$ is equal to

$$q_2' = -\frac{p''q_2 + p'}{p''q_2 + 2p'} < 0$$

  - But then the marginal benefit of expanding q_1 is higher than in Cournot duopoly, since extra term is strictly positive, and thus firm 1 produces strictly more than in Cournot case
  - Since firm 1 could have picked the q_1 from the Cournot case but chooses something different, its profits must be strictly higher

Leader-Follower Market

- How about aggregate output produced compared to Cournot duopoly?
  - Note that aggregate quantity produced is $q_1 + q_2 = q_1 + q_2(q_1)$
  - We know that q_1^* is strictly higher than in Cournot duopoly
  - Now, the derivative of the raggregate quantity is $1 + q_2'(q_1)$, where

$$q_2' = -\frac{p'' q_2 + p'}{p'' q_2 + 2p'} < 0$$

  - But then

$$1 + q_2'(q_1) = \frac{p'' q_2 + 2p' - (p'' q_2 + p')}{p'' q_2 + 2p'} = \frac{p'}{p'' q_2 + 2p'} > 0$$

  - Thus, aggregate output strictly increases

Leader-Follower Market

- How about and profits of firm 2?
  - Since aggregate output is strictly higher in Stackelberg than in Cournot, , the price per unit decreases
  - Since price is lower, aggregate profits are lower in Stackelberg than Cournot
  - To see this, let us compute aggregate profits as a function of $q_1 + q_2$:

$$p(q)q_1 - cq_1 + p(q)q_2 - cq_2 = p(q)q - cq = \Pi(q)$$

  - Then $\Pi_q = p'q + p - c$; at Cournot quantities this is equal to zero (why?)
  - But since $\Pi_{qq} = p''q + 2p' < 0$, and since aggregate quantity in Stackelberg is strictly higher than in Cournot, it follows that $\Pi(q)$ is strictly decreasing at q equal to Stackelberg aggregate quantity
  - Thus, aggregate profits are strictly lower than in Cournot
  - And since firm 1's profits go up, it must be that firm 2's profits go down

Leader-Follower Market

- Remarks:
  - There are lots of NE that are not SPE in this game
  - Example: firm 2 produces same quantity as in Cournot no matter what firm 1 produces; then optimal response for firm 1 is to produce Cournot quantity
  - Formally, firm 2 uses the strategy $q_2(q_1)$ is constant and equal to the Cournot quantity no matter what q_1 is
  - But this is clearly not SPE since it is not optimal for firm 2 to produce the Cournot quantity if firm 1 produces a q_1 different from Cournot (check)
  - Many more (an infinite number of) NE can be constructed in a similar way
  - This application illustrates how much power SPE can have over NE
  - Easy to extend the analysis with multiple followers (try it!)

Collusion

Repeated Cournot Oligopoly

- There n firms that compete as in Cournot model
- Demand is linear and given by $p = a - bq$, $q = \sum_{i=1}^{n} q_i$
- Cost of each firm is $c(q_i) = cq_i$, $0 < c < a$
- Profits of firm i are $\pi_i(q_i, q_{-i}) = (a - b(q_i, q_{-i}))q_i - cq_i$
- We assume these firms play an infinitely repeated game, in which the stage game that is repeated is the static Cournot model
- Each firm discount future payoffs with $\delta \in (0, 1)$
- Using what we learned from repeated game, we show tacit collusion can emerge, and each firm can make more profits than in static Cournot model
- Intuitively, this will require sufficiently patient firms (large δ)

NE of Stage Game and Collusive Profits

- Recall the NE of the stage game
  - Each firm i solves

$$\max_{q_i}(a-b(q_i+q_{-i})-c)q_i \;\Rightarrow\; (a-b(2q_i+q_{-i})-c=0$$

  - Solving the system of FOCs, we obtain a symmetric NE in which

$$q_i^{ne}=\frac{a-c}{b(n+1)} \;\Rightarrow\; q^{ne}=\sum_{i=1}^{n}q_i^{ne}=\frac{n(a-c)}{b(n+1)}$$

  - Each firms makes

$$\pi_i^{ne}=(a-bq_i^{ne}-c)q_i^{ne}=\frac{(a-c)^2}{b(n+1)^2}$$

  - In short, quantities and payoffs in NE of static Cournot oligopoly are for each i

$$q_i^{ne}=\frac{a-c}{b(n+1)},\ \pi_i^{ne}=\frac{(a-c)^2}{b(n+1)^2}$$

NE of Stage Game and Collusive Profits

- If these firms behave jointly as a monopolist (collusion), then
  - They solve $\max_q (a - bq)q - cq$, FOC $a - 2bq - c = 0$ and $q^m = \frac{a-c}{2b}$
  - If they split output equally, then the collusive output is $q_i^c = \frac{a-c}{2bn}$
  - Each firm then makes

$$\pi_i^c = (a - bq^m - c)q_i^c = \frac{(a-c)^2}{4bn}$$

  - Note that $\pi_i^c > \pi_i^{ne}$, and $q^m < q^{ne}$ (and so $q_i^c < q_i^{ne}$), and thus $p^m > p^{ne}$
  - In short, quantities and payoffs under collusion are for each i

$$q_i^c = \frac{a-c}{2bn}, \;\; \pi_i^c = \frac{(a-c)^2}{4bn}$$

NE of Stage Game and Collusive Profits

- Let us calculate the payoff for firm i from deviating in the stage game to its best response given that the other firms play q_i^c
  - The best deviation solves
$$\max_{q_i} \left(a - b\left(q_i + (n-1)\left(\frac{a-c}{2bn}\right)\right) - c\right) q_i$$
  - Denote the optimal deviation by q_i^d. The FOC is
$$a - b\left(2q_i^d + (n-1)\left(\frac{a-c}{2bn}\right)\right) - c = 0 \;\Rightarrow\; q_i^d = \frac{(a-c)(n+1)}{4bn} = q_i^c \frac{n+1}{2}$$
  - The profit for firm i from deviating from the collusive quantity is then (check)
$$\begin{aligned}\pi_i^d &= \left(a - b(q_i^d + (n-1)\left(\frac{a-c}{2bn}\right) - c\right) q_i^d \\ &= \left(a - b\left(\frac{(a-c)(n+1)}{4bn} + (n-1)\left(\frac{a-c}{2bn}\right)\right) - c\right) \frac{(a-c)(n+1)}{4bn} \\ &= \frac{(a-c)^2}{2bn}\frac{(n+1)^2}{2n} = \pi_i^c \frac{(n+1)^2}{2n} > \pi_i^c\end{aligned}$$

NE of Stage Game and Collusive Profits

- Remarks:
  - Calculated NE quantity of stage game, collusive one, and deviation quantity
  - It is easy to check (do it) that for each i, $q_i^c < q_i^{ne} < q_i^d$
  - Under collusion firms restrict production to maximize joint profits, so collusive quantity is lower for each firm than in NE, but in turn the best deviation is an even larger quantity than NE
  - We have also calculated the profits in each of those cases
  - It is intuitive and easy to check (do it) that for each i, $\pi_i^{ne} < \pi_i^c < \pi_i^d$
  - Now, we know in static Cournot oligopoly there is no way to sustain collusion
  - But collusion can occur in infinitely repeated game if firms are patient enough

Collusion in Infinitely Repeated Cournot Oligopoly

- Consider the following trigger strategies: each firm i plays
  - Firm i: $s_i(h^0) = q_i^c$; at any $t \geq 1$, $s_i(h^t) = q_i^c$ if h^t consists of $t-1$ repetition of $(q_1^c, ..., q_n^c)$; for any other h^t, play $s_i(h^t) = q_i^{ne}$
  - Each firm conjectures that the other firms will play according to this strategy
  - This partitions histories into two kinds: those in which q_i^c for every i has been played in every previous period, and all the other histories
  - In all other histories, $(q_1^{ne}, ..., q_n^{ne})$ in every period, a NE in each subgame
  - Consider histories of the first kind: if player i plays q_i^c, then i's payoff is (why?)

$$\pi_i^c + \frac{\delta}{1-\delta}\pi_i^c$$

  - If instead i plays q_i^d, then payoff is

$$\pi_i^d + \frac{\delta}{1-\delta}\pi_i^{ne}$$

Collusion in Infinitely Repeated Cournot Oligopoly

- We thus have the following result:
  - Every player i will have incentives to play q_i^c if and only if (check)
  $$\delta \geq \frac{\pi_i^d - \pi_i^c}{\pi_i^d - \pi_i^{ne}}$$
  - If δ is above this threshold, then the trigger strategies are a SPE, and its outcome is q_i^c in every period for every i
  - Note that it is enough for the players to understand or believe that every other player behaves according to the above trigger strategy
  - There is no need for communication among firms: tacit collusion
  - Important real-world problem repeated games have shed light on: it is a very relevant problem for antitrust policy and for economic analysis of markets
  - There are many variations and extensions of this type of construction
  - An interesting variation is one in which firms do not observe the quantities produced by other firms, and the demand is subject to random shocks, so price is a noisy signal of quantities. Important application: OPEC