On page 2467 in Etzion and Wachter-Zeh, they state the gap of $(\epsilon = 1, \ell) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$. However, they did not explain the scalar linear solution. I would like you to double check for me as following.

$$\begin{bmatrix} y_{j_1} \\ \vdots \\ y_{j_s} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{\alpha\ell} \\ \vdots \\ \boldsymbol{a}_{\alpha\ell+\epsilon} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_h \end{bmatrix}$$

Notice: $h = 2\ell$ and $\alpha = \frac{s-\epsilon}{l} = 2$. This explains one of your question why Tuvi and Wachter-Zeh's gap is dependent on ℓ.

$\boldsymbol{a}_{\alpha\ell+1}, \ldots, \boldsymbol{a}_{\alpha\ell+\epsilon}$ can be independently chosen from any receiver, and there exits thus always $\boldsymbol{a}_{\alpha\ell+1}, \ldots, \boldsymbol{a}_{\alpha\ell+\epsilon}$ such that,

$$rk \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{\alpha\ell} \end{bmatrix} \geq h - \epsilon \Leftrightarrow rk \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{2\ell} \end{bmatrix} \geq 2\ell - 1,$$

if and only if $rk \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{\alpha\ell} \\ \vdots \\ \boldsymbol{a}_{\alpha\ell+\epsilon} \end{bmatrix} \geq 2\ell.$

Following to my previous report, Chapter 5 Page 31, we formulate the above problem on rank as following.

Problem: A scalar solution exists, if and only if there exists a Grasmannian code $\mathcal{G}_q\left(h = 2\ell, \ell\right)$ such that any $\alpha = 2$ subspaces of the set span a subspace of dimension at least $2\ell - 1$.

Solution:

USE A SUGGESTION YOU GAVE ME DURING OUR DISCUSSION >>> We need that no 2 ℓ-dimensional subspaces of $\mathbb{F}_{q_s}^{2\ell}$ will contain a vector which is contained in the same $(2\ell - 2)$-subspace, but $(2\ell - 1)$ of such subspaces can have such vectors.

Therefore, we have: $r_{scalar} \leq (2\ell - 1) \begin{bmatrix} 2\ell \\ 2\ell - 2 \end{bmatrix}_{q_s} \Rightarrow r_{scalar} \leq \mathcal{O}\left(q^l\right)$.

I doubt my result, because in Section IV-D, Etzion and Wachter-Zeh seem to use $r_{scalar} \leq \begin{bmatrix} 2\ell \\ \ell \end{bmatrix}_{q_s} < \mathcal{O}\left(q^{l^2}\right)$, which leads to their gap $q^{t^2/2+\mathcal{O}(t)}$ for the $(\epsilon = 1, \ell) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$ network.

To be similar with the proof that Etzion and Wachter-Zeh in Section IV-E, I re-formulate our problem. Let's denote any 2 subspaces of $\mathcal{G}_q\left(h = 2\ell, \ell\right)$ as U and V. If U and V span a subspace of dimension at least $2\ell - 1$, then we have $dim\left(U + V\right) = 2\ell - 1$. Therefore, $dim\left(U \cap V\right) = dim\left(U\right) + dim\left(V\right) - dim\left(U + V\right) = 1$, which leads to the subspace distance $d_s(U, V) = 2\ell - 2$. It shows that our above use of $\begin{bmatrix} 2\ell \\ 2\ell - 2 \end{bmatrix}_{q_s}$ is correct.