Everywhere: If equations are part of a sentence, you must use ".", ",", ... as in a sentence. E.g.:

Hence, we get

$$a = b \cdot c.$$

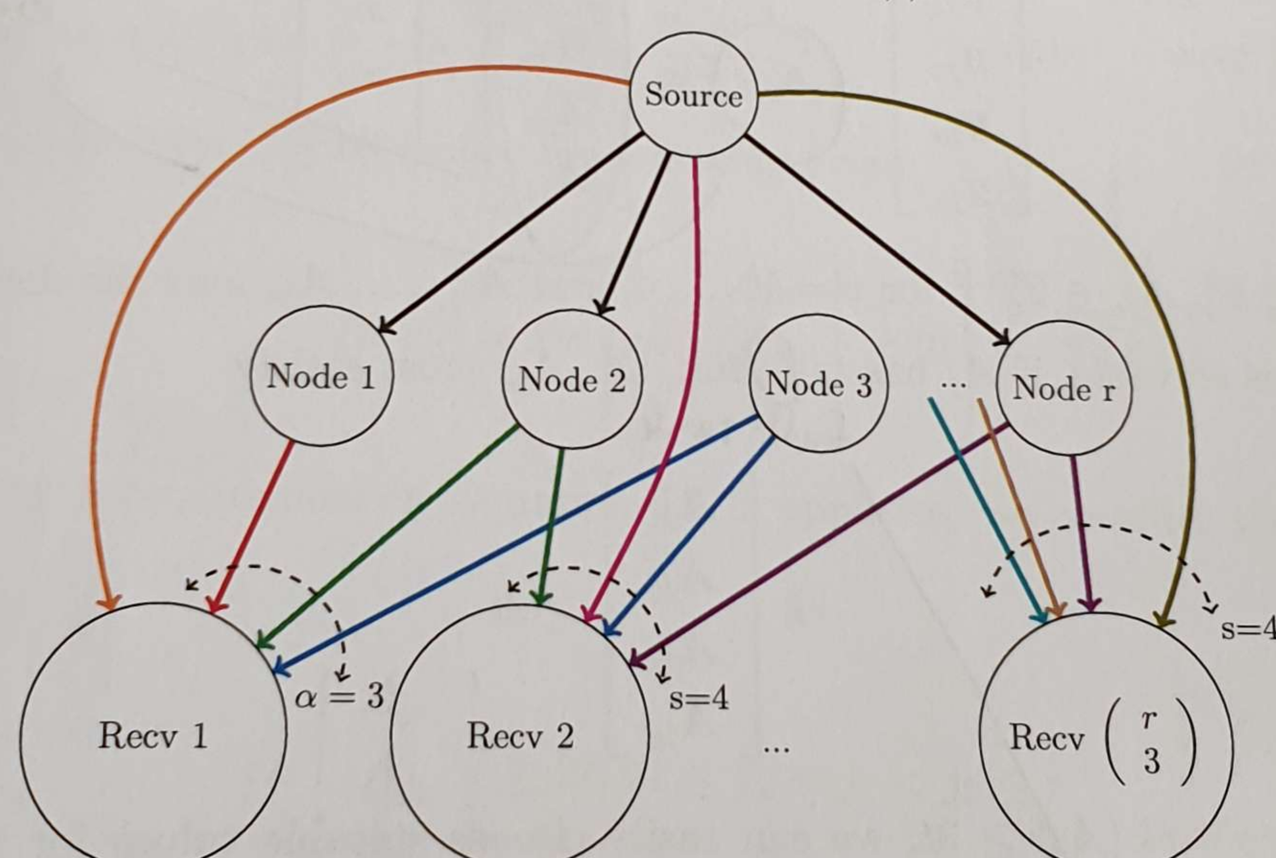
5 Combinatorial Results

outperforming scalar netw. coding

In previous studies [EW18], there was no general vector solution found for multicast networks with $h = 3$ messages. Hence, we start with a probabilistic argument to prove that there exists a vector solution outperforming the optimal linear solution for the $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network. Then we generalize the proof to the $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h,r,s}$ network.

5.1 $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network

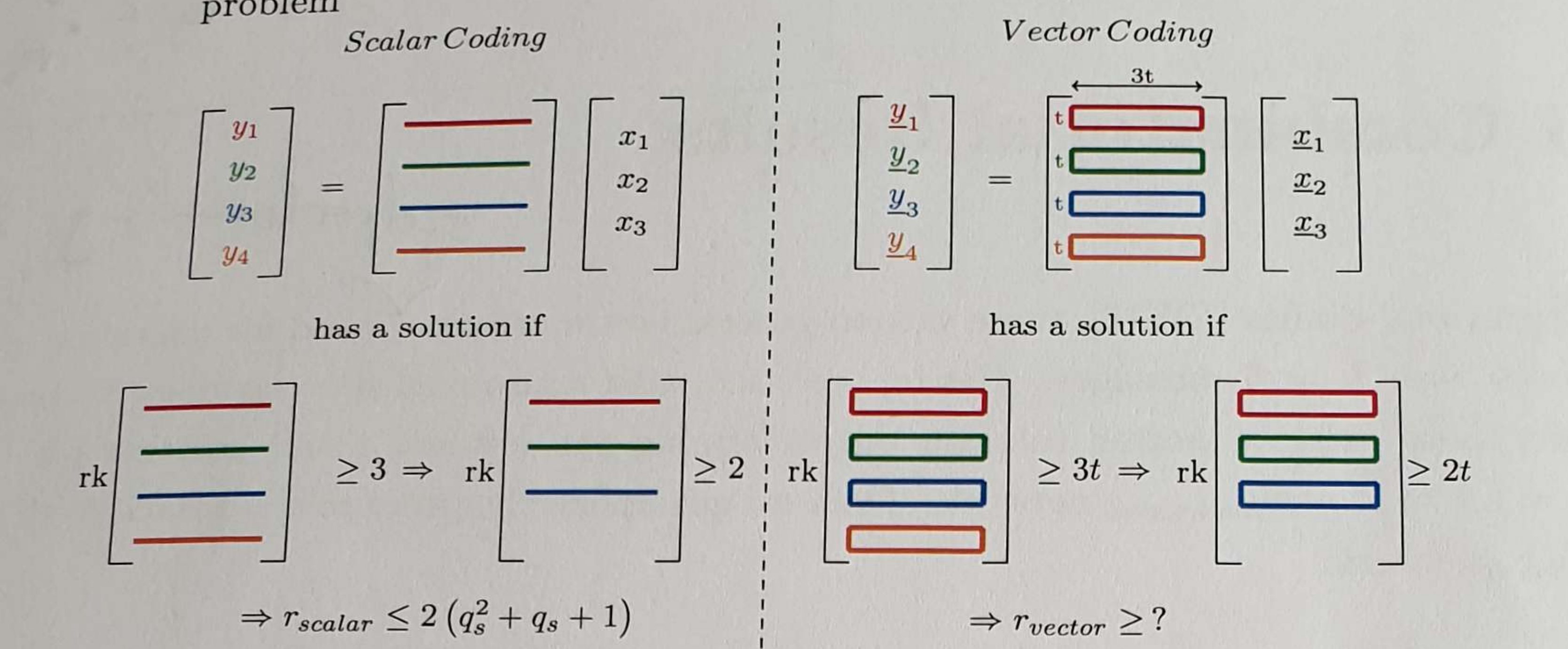
Figure 5.1: The $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network



very nice!

In this subsection, we derive a lower bound on the number of receivers for the $(\epsilon = 1, \ell = 1)-$ $\mathcal{N}_{h=3,r,s=4}$ network. Due to $\alpha = 3$, the number of receivers is $N = \binom{r_{vector}}{3}$ by definition in Section 4.1. To derive the lower bound, we introduce a rank requirement on incoming packets to each receiver.

Figure 5.2: The vector network coding of $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$ represents as a matrix problem



Following to Equation 3.1, each receiver must solve a linear equation system of 3 variables with 4 equations to recover $h = 3$ messages as below:

$$\begin{bmatrix} \boldsymbol{y}_{j_1} \\ \boldsymbol{y}_{j_2} \\ \boldsymbol{y}_{j_3} \\ \boldsymbol{y}_{j_4} \end{bmatrix} = \boldsymbol{A}_j \cdot \underline{x} = \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \\ \boldsymbol{A}_{j_4} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{bmatrix},$$

bad notation! $A_j, A_{j1} \ldots$ not distinguishable e.g. name A_j differently

with $\boldsymbol{x}_i, \boldsymbol{y}_{j_v} \in \mathbb{F}_q^t$, $\boldsymbol{A}_{j_v} \in \mathbb{F}_q^{t \times 3t}$ for $v = 1, \ldots, 4$, and $\boldsymbol{A}_{j_1}, \ldots, \boldsymbol{A}_{j_3}$ must be distinct. The network is solvable, if $\boldsymbol{A}_j$ has full-rank, i.e. $\boldsymbol{A}_{j_v}$ must satisfy:

full rank

$$rk \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \\ \boldsymbol{A}_{j_4} \end{bmatrix} \geq 3t$$

In order to satisfy $rk\,[\boldsymbol{A}_j] \geq 3t$, we can easily choose suitable values for the coefficient $\boldsymbol{A}_{j_4}$ on the direct link from the source to R_j. However, the coefficients on the links from nodes to receivers are matters. Therefore, we focus on the following requirement:

English

$$rk \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \end{bmatrix} \geq 2t \tag{5.1}$$

"–" only if "full-rank" is used as adjective, e.g.
adjective, refers to matrices
"the set of full-rank matrices" (with "-")
everywhere "the matrix has full rank" (without "-")
noun

+ refer to Fig. 5.2

This means that $rk \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \end{bmatrix} \geq 2t$ implies $rk\,[\boldsymbol{A}_j] \geq 3t$, or to satisfy $rk\,[\boldsymbol{A}_j] \geq 3t$, we need

$$rk \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \end{bmatrix} \geq 2t.$$

a bit vague. Clearly say that since A_{ju} can be independently chosen for any receiver, there is always an A_{ji} such that $rk \begin{bmatrix} A_{j1} \\ A_{j2} \\ A_{j3} \end{bmatrix} \geq 2t$ if and only if $rk(A_j) \geq 3t$

By this constraint, the problem is thus described as below:

$$\min_{rk[\boldsymbol{A}_j] \geq 3t} r_{vector}$$

$\boldsymbol{A}_{j_1}, \boldsymbol{A}_{j_2}, \boldsymbol{A}_{j_3}$ are matrices formed on any 3 of r links from nodes to receivers, i.e. these 3 matrices are randomly chosen from a set of r matrices. We formalize the problem by an approach with Lovász local lemma [Sch]. ~~"The idea proof ide~~

Lemma 5.1 (Symmetric Lovász local lemma (LLL) [SCV13]). *A set of events $\mathcal{E}_i$, with $i = 1, \ldots, n$, such that each event occurs with probability at most p. If each event is independent of all others except for at most d of them and $4dp \leq 1$, then:* $Pr\left[\bigcap_{i=1}^{n} \overline{\mathcal{E}}_i\right] > 0$.

Lemma 5.2. *For the network $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$, the probability that vector solution does not exist is equal to* $\frac{1}{q^{9t^2}} \sum_{i=0}^{2t-1} \prod_{j=0}^{i-1} \frac{(q^{3t} - q^j)^2}{q^i - q^j}$.

you messed up p and $P(\cap \bar{\mathcal{E}}_i)$!

Proof. For our problem, let $\mathcal{E}_i$ denote the following event:

$$Pr\left[\mathcal{E}_i\right] = Pr\left[rk \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \end{bmatrix} < 2t\right]$$

$p = P[\mathcal{E}_i]$ => The problem statement is completely wrong

Because the rank requirement in Equation 5.1 is opposite, we consider the complement event T:

$$rk \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \end{bmatrix} \geq 2t, \forall 1 \leq j_1 < j_2 < j_3 \leq r$$

Suggestion:
Lemma A: $p \leq \ldots$
Proof...
Lemma B: $d \leq \ldots$
Proof...
Theorem: If $r \leq \ldots$, then a solution exists
Proof: Combine Lemma A&B & LLL. □

By the intersection rule, we have:

$$T = \bigcap_{\mathcal{E}_i \in \mathcal{E}} \overline{\mathcal{E}}_i$$

this should be part of the proof of the main statement

The probability of event T indicates a measure quantifying the likelihood that we will be able to construct $rk\,[\boldsymbol{A}_j] \geq 3t$ with j_1, j_2, j_3 in the integer numbers between 1 and r,

it does not have anything to do with p! (*)

still
(*)

including both. We need to maximize r, and the rank requirement 5.1 must be satisfied, i.e. the probabilty of event T must be higher than 0:

$$
\begin{aligned}
& Pr\left[rk\begin{bmatrix}\boldsymbol{A}_{j_1}\\ \boldsymbol{A}_{j_2}\\ \boldsymbol{A}_{j_3}\end{bmatrix} \geq 2t, \forall 1 \leq j_1 < j_2 < j_3 \leq r\right] && > 0\\
\Leftrightarrow \quad & Pr\left[T\right] && > 0\\
\Leftrightarrow \quad & Pr\left[\bigcap_{\mathcal{E}_i \in \mathcal{E}} \overline{\mathcal{E}}_i\right] && > 0
\end{aligned}
$$

Following to LLL, each event occurst with probability at most p:

$$
Pr\left[\mathcal{E}_i\right] = Pr\left[rk\begin{bmatrix}\boldsymbol{A}_{i_1}\\ \boldsymbol{A}_{i_2}\\ \boldsymbol{A}_{i_3}\end{bmatrix} < 2t\right] \leq p \tag{5.2}
$$

Regarding to the left-hand side:

$$
\begin{aligned}
Pr\left[rk\begin{bmatrix}\boldsymbol{A}_{i_1}\\ \boldsymbol{A}_{i_2}\\ \boldsymbol{A}_{i_3}\end{bmatrix} < 2t\right] &= \sum_{i=0}^{2t-1} Pr\left[rk\begin{bmatrix}\boldsymbol{A}_{i_1}\\ \boldsymbol{A}_{i_2}\\ \boldsymbol{A}_{i_3}\end{bmatrix} = i\right]\\
&\overset{1}{=} \sum_{i=0}^{2t-1} \frac{N_{t,m,n}}{q^{m \cdot n}}\\
&= \sum_{i=0}^{2t-1} \frac{\prod_{j=0}^{i-1} \frac{\left(q^m - q^j\right)\left(q^n - q^j\right)}{q^i - q^j}}{q^{m \cdot n}}\\
&\overset{2}{=} \sum_{i=0}^{2t-1} \frac{\prod_{j=0}^{i-1} \frac{\left(q^{3t} - q^j\right)^2}{q^i - q^j}}{q^{9t^2}}
\end{aligned} \tag{5.3}
$$

or here

By varying t in Equation (5.3), we have the following table:

Table 5.1: r over variations of t

t	Scalar Solution	Vector Solution
1	$r_{scalar} \leq 14$	$r_{vector} \geq 3$
2	$r_{scalar} \leq 42$	$r_{vector} \geq 7$ (67*, 89**)
3	$r_{scalar} \leq 146$	$r_{vector} \geq 62$ (166*)
4	$r_{scalar} \leq 546$	$r_{vector} \geq 1317$
5	$r_{scalar} \leq 2114$	$r_{vector} \geq 58472$
6	$r_{scalar} \leq 8322$	$r_{vector} > 10^6$

*, **: computational results in construction 1 and construction 2 respectively

should be after main statement!

In the table (5.1), the vector solution outperforms the scalar solution when $t \geq 4$ for the network $(\epsilon = 1, l = 1) - N_{h=3,r,s=4}$. This is sufficient, later on we show computational results which vector solutions outperform scalar solutions in case of $t = 2$ and $t = 3$.

Lemma 5.3. *For the network $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$, the probability that vector solution does not exist is less than or equal to $\Theta\left(q^{-t^2-2t-1}\right)$.*

again messed up p

Proof. This lemma is a tight bound for Equation 5.2 in Lemma 5.2, i.e. we try to maximize p with an exact maximum value. We consider the nominator of Equation (5.3):

numerator (?)

why tight?

$$\prod_{j=0}^{i-1} \frac{\left(q^{3t} - q^j\right)^2}{q^i - q^j} = \frac{p_N^{(i)}(q)}{p_D^{(i)}(q)} = p^{(i)}(q)$$

Due to i-times product and large t:

$$\left.\begin{aligned} deg\left(p_N^{(i)}(q)\right) &= q^{i6t} \\ deg\left(p_D^{(i)}(q)\right) &= q^{i^2} \end{aligned}\right\} \Rightarrow p^{(i)}(q) \approx q^{i6t-i^2}$$

Then we have:

$$\sum_{i=0}^{2t-1}\prod_{j=0}^{i-1} \frac{\left(q^{3t} - q^j\right)^2}{q^i - q^j} = \sum_{i=0}^{2t-1} p^{(i)}(q) \approx \sum_{i=0}^{2t-1} q^{i6t-i^2}$$

To maximize the sum, we set derivation of to 0 and find its root:

$$\begin{aligned} & \left(i6t - i^2\right)' &= 0 \\ \Leftrightarrow \quad & 6t - 2i &= 0 \\ \Leftrightarrow \quad & i &= 3t \end{aligned}$$

However, the upper limit of sum is $(2t-1)$, which is less than $3t$, so

$$\Rightarrow max\left\{q^{i6t-i^2} : i = 0, 2 \ldots, 2t-1\right\} = q^{i6t-i^2}\Big|_{i=2t-1} = q^{8t^2-2t-1}.$$

Hence, by using the exact bound Θ, we have:

$$\sum_{i=0}^{2t-1} p^{(i)}(q) \in \Theta\left(max\left\{q^{i6t-i^2} : i = 1, 2 \ldots, 2t-1\right\}\right) = \Theta\left(q^{8t^2-2t-1}\right)$$

$$\Rightarrow \frac{\sum_{i=0}^{2t-1} p^{(i)}(q)}{q^{9t^2}} \in \Theta\left(q^{-t^2-2t-1}\right) \square$$

statement does not make a lot of sense

Lemma 5.4. *If $d \leq \frac{3}{2}r^2$, we have $r_{max,vector} \geq \Omega\left(q^{t^2/2+O(t)}\right)$.*

Proof. We proceed the other constraint of LLL in Lemma 5.1: $4dp \leq 1$. Regarding to d, we have:

$$\begin{aligned} d(r) &\leq 3 \cdot \binom{r-1}{2} = 3 \cdot \frac{(r-1)(r-2)}{2} = \frac{3}{2}\left(r^2 - 3r + 2\right) \\ &\leq \frac{3}{2}r^2 = d_{max}(r) \end{aligned}$$

Because $4pd_{max}(r) \leq 1$ implies that $4pd \leq 1$, we consider $d_{max}(r)$ directly:

$$4 \cdot p \cdot d_{max}(r) \leq 1 \Rightarrow 4 \cdot p \cdot \frac{3}{2}r^2 \leq 1 \Rightarrow r \leq \sqrt{\frac{1}{6p}} = r_{max,vector}$$

Similarly with above, d and r are propotional, so minimizing r is equivelent to maximizing p. The purpose is to achieve a strict lower bound proving vector solutions always outperform scalar solutions in a specific range of t, i.e., $r_{max,vector}$ asymptotes to a value higher than $r_{max,scalar}$.

By applying Lemma 5.3, we have:

$$r_{max,vector} \in \Omega\left(\sqrt{\frac{1}{6p}}\right) = \Omega\left(\sqrt{\frac{1}{6q^{-t^2-2t-1}}}\right) = \Omega\left(q^{t^2/2+O(t)}\right) \square$$

Theorem 5.1. *For the network $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$, the achieved gap is $q^{t^2/4+O(t)}$.*

Proof. In advance, we have: $r_{max,scalar} \in \mathcal{O}\left(q_s^2\right)$ [EW18], specifically,

? English?

$$r_{scalar} \leq 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{q_s} = 2\left(q_s^2 + q_s + 1\right) \tag{5.4}$$

Finally, following to Section 3.4.2 and Lemma 5.4, we have the gap size:

why so much space?

$$
\begin{array}{lll}
 & r_{max,scalar} & = r_{max,vector} \\
\Leftrightarrow & q_s^2 & = q^{t^2/2+\mathcal{O}(t)} \\
\Leftrightarrow & q_s & = q^{t^2/4+\mathcal{O}(t)} \\
\Rightarrow & g & = q_s - q_v = q^{t^2/4+\mathcal{O}(t)}
\end{array}
\tag{5.5}
$$

5.2 $(\epsilon = 1, l = 1) - \mathcal{N}_{h,r,s}$ Network

5.2.1 Find the lower bound of $r_{max,vector}$

Following to Theorem (4.1), we are interested in the following range: $\ell+\epsilon+1 \leq h \leq \alpha\ell+\epsilon$. As previous, $\boldsymbol{A}_{j_1}, \ldots, \boldsymbol{A}_{j_{h-\epsilon}} \in \mathbb{F}_q^{t\times ht}$ and we need to satisfy the following:

$$rk \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \vdots \\ \boldsymbol{A}_{j_{h-\epsilon}} \end{bmatrix} \geq ht - t \Leftrightarrow rk\left[\boldsymbol{A}_j\right] \geq (h-1)t$$

We can formulate it by the following coding problem in Grassmannian:

Find the largest set of subspaces from $\mathcal{G}_q(ht, t)$ such that any α subspaces of the set span a subspace of dimension at least $(h-1)t$.

Similar to $(\epsilon = 1, \ell = 1) - N_{3,r,4}$, we consider p to proceed LLL:

$$Pr\left[rk\left[\boldsymbol{A}\right] < (h-1)t\right] \leq p$$

Regarding to the left-hand side:

$$
\begin{aligned}
Pr\left[rk\left[\boldsymbol{A}\right] < (h-1)t\right] &= \sum_{i=0}^{(h-1)t-1} Pr\left[rk\left[\boldsymbol{A}\right] = i\right] \\
&\overset{1}{=} \sum_{i=0}^{(h-1)t-1} \frac{N_{i,\alpha t,ht}}{q^{(\alpha t)(ht)}} \\
&= \frac{1}{q^{(\alpha h)t^2}} \cdot \sum_{i=0}^{(h-1)t-1} \prod_{j=0}^{i-1} \frac{\left(q^{\alpha t} - q^j\right)\left(q^{ht} - q^j\right)}{q^i - q^j}
\end{aligned}
\tag{5.6}
$$

Firstly, we consider the product: