Vector Network Coding Gap Sizes for the Generalized Combination Network

Ha Nguyen
ha.nguyen@tum.de

September 2, 2019

Outline

1. Motivation
2. What is Network Coding?
   - Initial Idea
   - Network as Matrix Channel
   - Generalized Combination Network
   - Introduction of Gap
3. Combinatorial Results
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h,r,s}$ Network
   - $(\epsilon = 1, \ell \geq 2) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$ Network
4. Computational Results
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network
5. Conclusions

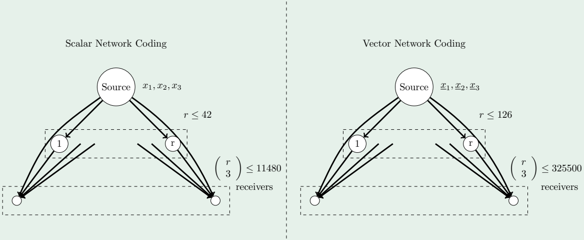
Outline

Motivation

- General vector solutions for the Generalized Combination Network was not found.

Example (A multicast network with 3 messages)

Figure 1: Vector Network Coding Outperforms Scalar Network Coding



$\Longrightarrow$ Vector Network Coding provides a higher number of receivers.

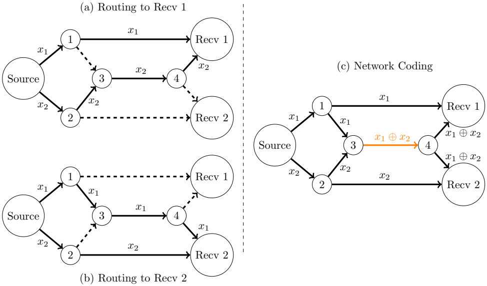
Outline

1. Motivation
2. What is Network Coding?
   - Initial Idea
   - Network as Matrix Channel
   - Generalized Combination Network
   - Introduction of Gap
3. Combinatorial Results
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h,r,s}$ Network
   - $(\epsilon = 1, \ell \geq 2) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$ Network
4. Computational Results
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network
5. Conclusions

Initial Idea

- *Network coding* was first introduced in Ahlswede et al.'s seminal paper [1] with the well-known butterfly network.

Figure 2: The butterfly network



$\Longrightarrow$ Network coding gives a potential gain in throughput by communicating more information with fewer packet transmissions compared to the routing method.

Network as Matrix Channel

- In our study, we formulate networks as matrix channels.

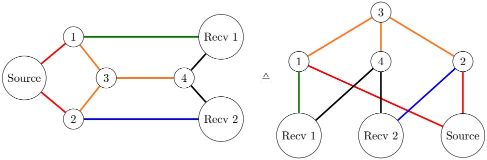
$$\underbrace{\begin{bmatrix} y_{j_1} \\ \vdots \\ y_{j_s} \end{bmatrix}}_{\mathbb{F}_{q_s}^{s}} = \underbrace{\boldsymbol{A}_j}_{\mathbb{F}_{q_s}^{s\times h}} \cdot \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_h \end{bmatrix}}_{\mathbb{F}_{q_s}^{h}} \quad (\text{Scalar}) \qquad \Bigg| \qquad \underbrace{\begin{bmatrix} \boldsymbol{y}_{j_1} \\ \vdots \\ \boldsymbol{y}_{j_s} \end{bmatrix}}_{\mathbb{F}_{q}^{st}} = \underbrace{\boldsymbol{A}_j}_{\mathbb{F}_{q}^{st\times th}} \cdot \underbrace{\begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_h \end{bmatrix}}_{\mathbb{F}_{q}^{th}} \quad (\text{Vector}) \tag{1}$$

$$\text{Scalar:}\quad \boldsymbol{A}_j = \begin{bmatrix} \boldsymbol{a}_{j_1} \\ \vdots \\ \boldsymbol{a}_{j_{\alpha\ell}} \\ \vdots \\ \boldsymbol{a}_{j_{\alpha\ell+\epsilon}} \end{bmatrix} \qquad \Bigg| \qquad \text{Vector:}\quad \boldsymbol{A}_j = \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \vdots \\ \boldsymbol{A}_{j_{\alpha l}} \\ \vdots \\ \boldsymbol{A}_{j_{\alpha l+\epsilon}} \end{bmatrix}$$

Generalized Combination Network

- We choose to use the Generalized Combination Network to analyze its network coding problems. The well-known butterfly network is isomorphic to $\mathcal{N}_{h,r=3,s=2}$, if we consider it as an undirected network [2]

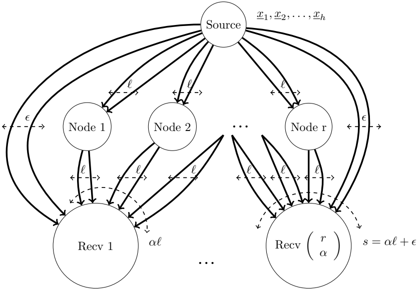
Figure 3: The butterfly network is represented as a combination network



Generalized Combination Network [cont.]

- A generalized combination network $(\epsilon, \ell) - \mathcal{N}_{h,r,s}$ consists of 3 components over 3 layers from top to bottom: "Source" in the first layer, "Intermediate Nodes" in the middle layer, and "Receiver" in the third layer.

Figure 4: The generalized network $(\epsilon, \ell) - \mathcal{N}_{h,r,s}$



Generalized Combination Network [cont.]

Table 1: Parameters of network coding

h	The number of source messages
r	The number of nodes in the middle layer
$\binom{r}{\alpha}$	The number of receivers
ℓ	The source connects to each node by ℓ parallel links, and each node also connects to one receiver by ℓ parallel links
α	A receiver is connected by any α nodes in the middle layer
ϵ	The source additionally connects to each receiver by ϵ direct parallel links
s	Each receiver is connected by s links in total, with $s = \alpha\ell + \epsilon$.

Introduction of Gap

- The gap represents the difference between the smallest field (alphabet) size for which a scalar linear solution exists and the smallest alphatbet size for which we can construct a vector solution.
- $r_{max,vector} \geq f_1(q, t, \alpha, h)$, with $f_1 : \mathbb{Z} \mapsto \mathbb{Z}$
- $r_{scalar} \leq f_2\left(q_\mathrm{s}\right)$, with $f_2 : \mathbb{Z} \mapsto \mathbb{Z}$
- $r_{max,scalar} = f_2\left(q_\mathrm{s}\right) = f_1(q, t, \alpha, h) = min\left\{r_{max,vector}\right\}$. Finally, we calculate the gap by $g = q_\mathrm{s} - q_v = q_\mathrm{s} - q^t$.

Introduction of Gap [cont.]

Table 2: New gap found in this study

Network	Gap Bounds for a specific vector solution [3]	This study proves an existence of these gaps
$(\epsilon = 0, \ell = 1) - \mathcal{N}_{h,r,s}$	N/A	N/A
$(\epsilon \geq 1, \ell = 1) - \mathcal{N}_{h,r,s}$	Unknown	$q^{\frac{\alpha-h+1}{(\alpha-1)(\alpha-h+2)(h-2)}t^2+\mathcal{O}(t)}$ (*)
$(\epsilon = 1, \ell \geq 2)-$ $\mathcal{N}_{h=2\ell,r,s=2\ell+1}$	$q^{t^2/2+\mathcal{O}(t)}$	$q^{t^2/l+\mathcal{O}(t)}$
$(\epsilon = \ell - 1, \ell) -$ $\mathcal{N}_{h=2\ell,r,s=3\ell-1}$	$q^{t^2/2+\mathcal{O}(t)}$	N/A

(*): We only consider the $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h,r,s}$ network.

Outline

1. Motivation
2. What is Network Coding?
   - Initial Idea
   - Network as Matrix Channel
   - Generalized Combination Network
   - Introduction of Gap
3. Combinatorial Results
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h,r,s}$ Network
   - $(\epsilon = 1, \ell \geq 2) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$ Network
4. Computational Results
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network
5. Conclusions

$(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network

Lemma (Symmetric Lovász local lemma (LLL) [4])

A set of events $\mathcal{E}_i$, with $i = 1, \ldots, n$, such that each event occurs with probability at most p. If each event is independent of all others except for at most d of them and $4dp \leq 1$, then:

$$Pr\left[[i = 1]n\bigcap\overline{\mathcal{E}}_i\right] > 0.$$

$(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network [cont.]

Lemma

Let $Pr\left[\mathcal{E}_i\right] = Pr\left[rk \begin{bmatrix} \boldsymbol{A}_j^{(r_1)} \\ \boldsymbol{A}_j^{(r_1)} \\ \boldsymbol{A}_j^{(r_3)} \end{bmatrix} < 2t \right] \leq p, \forall 1 \leq r_1 < r_2 < r_3 \leq r$, *and* $\boldsymbol{A}_j^{(r_1)}, \ldots, \boldsymbol{A}_j^{(r_3)} \in \mathbb{F}_q^{t \times 3t}$, *then,*

$$p \leq \Theta\left(q^{-t^2-2t-1}\right), \forall t \geq 2.$$

Lemma

Let $Pr\left[\mathcal{E}_i\right] = Pr\left[rk \begin{bmatrix} \boldsymbol{A}_j^{(r_1)} \\ \boldsymbol{A}_j^{(r_1)} \\ \boldsymbol{A}_j^{(r_3)} \end{bmatrix} < 2t \right] \leq p, \forall 1 \leq r_1 < r_2 < r_3 \leq r$ *with* $\boldsymbol{A}_j^{(r_1)}, \ldots, \boldsymbol{A}_j^{(r_3)} \in \mathbb{F}_q^{t \times 3t}$, *and each event* $\mathcal{E}_i$ *is independent of all others except for at most* d *of them, then* $d \leq \frac{3}{2} r^2$.

$(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network [cont.]

Theorem

If $r \leq \Omega\left(q^{t^2/2+\mathcal{O}(t)}\right)$, then there exists a vector solution for the $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$ network.

Corollary

The $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network has a vector solution with a gap $q^{t^2/4+\mathcal{O}(t)}$.

$(\epsilon = 1, \ell = 1) - \mathcal{N}_{h,r,s}$ Network

- Item 1
- Item 2
- Item 3

$(\epsilon = 1, \ell \geq 2) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$ Network

- Item 1
- Item 2
- Item 3

Outline

1. Motivation
2. What is Network Coding?
   - Initial Idea
   - Network as Matrix Channel
   - Generalized Combination Network
   - Introduction of Gap
3. Combinatorial Results
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h,r,s}$ Network
   - $(\epsilon = 1, \ell \geq 2) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$ Network
4. Computational Results
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network
5. Conclusions

$(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network

Table 3: r over variations of t

t	Scalar Solution	Vector Solution
1	$r_{scalar} \leq 14$	$r_{vector} \geq 3$
2	$r_{scalar} \leq 42$	$r_{vector} \geq 7$ $(67^{*}, 89^{**})$
3	$r_{scalar} \leq 146$	$r_{vector} \geq 62$ (166^{*})
4	$r_{scalar} \leq 546$	$r_{vector} \geq 1317$
5	$r_{scalar} \leq 2114$	$r_{vector} \geq 58472$
6	$r_{scalar} \leq 8322$	$r_{vector} > 10^6$

*, **: computational results in construction 1 and construction 2 respectively

Outline

1. Motivation
2. What is Network Coding?
   - Initial Idea
   - Network as Matrix Channel
   - Generalized Combination Network
   - Introduction of Gap
3. Combinatorial Results
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h,r,s}$ Network
   - $(\epsilon = 1, \ell \geq 2) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$ Network
4. Computational Results
   - $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network
5. Conclusions

Conclusions and Outlook

- If $r \leq \Omega\left(q^{t^2/2+\mathcal{O}(t)}\right)$, there always exists a vector solution for the $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network.
- The general gap $g = q^{t^2/4+\mathcal{O}(t)}$ can be achieved for the $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network
- For the $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network. Similarly we derived the gaps for the $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h,r,s}$ network and the $(\epsilon = 1, \ell > 1) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$ network, respectively with $g = q^{\frac{\alpha-h+1}{(\alpha-1)(\alpha-h+2)(h-2)}t^2+\mathcal{O}(t)}$ and $g = q^{t^2/2\ell+\mathcal{O}(t)}$.
- An improved bound $89 \leq \mathcal{A}_2(6,4,3;2) \leq 126$.
- A new bound $166 \leq \mathcal{A}_2(9,6,3;2) \leq 537$.

Thank you! Questions?

References:

R. Ahlswede, N. Cai, S.-Y. Li, and R. Yeung, "Network information flow," *IEEE Transactions on Information Theory*, vol. 46, no. 4, pp. 1204–1216, jul 2000.

S. Maheshwar, Z. Li, and B. Li, "Bounding the coding advantage of combination network coding in undirected networks," *IEEE Transactions on Information Theory*, vol. 58, no. 2, pp. 570–584, feb 2012.

T. Etzion and A. Wachter-Zeh, "Vector network coding based on subspace codes outperforms scalar linear network coding," *IEEE Transactions on Information Theory*, vol. 64, no. 4, pp. 2460–2473, April 2018.

M. Schwarz, T. S. Cubitt, and F. Verstraete, "An Information-Theoretic Proof of the Constructive Commutative Quantum Lovász Local Lemma," *arXiv e-prints*, p. arXiv:1311.6474, Nov 2013.