5. Combinatorial Results

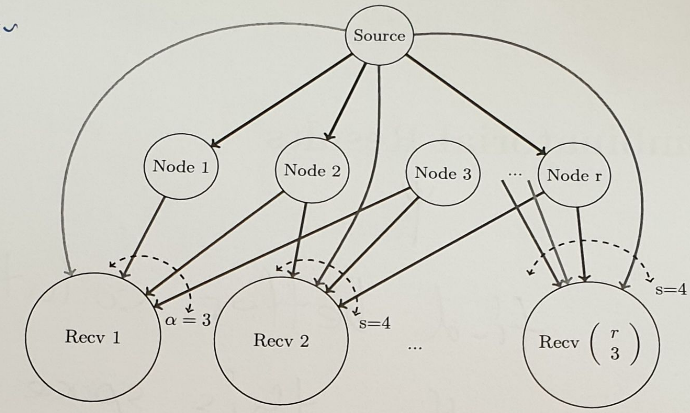
find better solution than this space (more intro?)

In previous studies [EW18], no general vector solution outperforming scalar network coding was found for multicast networks with $h = 3$ messages. Hence, we start with a probabilistic argument to prove that there exists a vector solution outperforming the optimal linear solution for the $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network. Then we generalize the proof to the $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h,r,s}$ network and the $(\epsilon = 1, \ell > 1) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$ network. As explained in Section 3.1, multiple parallel links ℓ of a data unit help us to show networks with large-capacity transmission between source and receivers. The direct links among them are not really usual in reality, i.e. a server and a client often has long-distance connection thorugh multiple intermediate nodes, it is thus interesting to study networks with $\epsilon = 1$. We formally use r_{scalar} and r_{vector} to distinguish the r parameter of GCN for scalar solutions and vector solutions to compare their gap. Because they both have the same meaning as a number of intermediate nodes in a network, we use r when we need to state a vector solution or a scalar solution exists under some conditions of r.

5.1. $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ Network

Figure 5.1.: The $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network

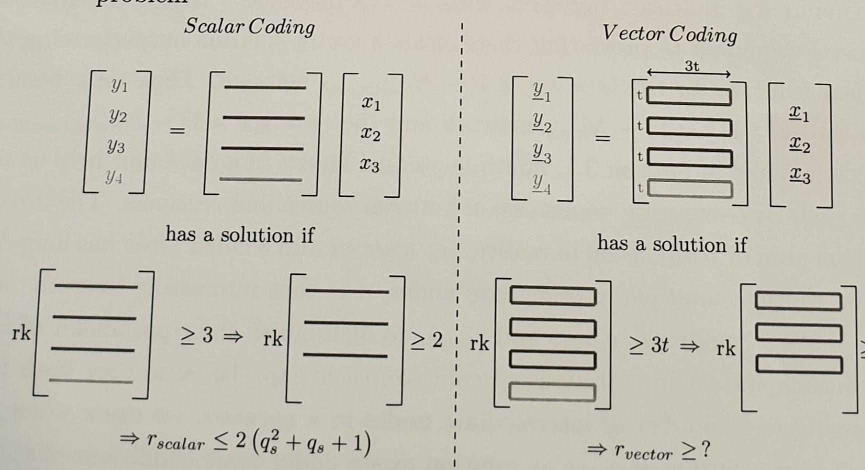
Mention both Figures in the text!



In this subsection, we derive a lower bound on the maximum number of receivers for the $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network. Due to $\alpha = 3$, the number of receivers is $N = \binom{r_{vector}}{3}$ by definition in Section 4.1. To derive the lower bound, we introduce a rank requirement on incoming packets to each receiver.

actually, on the channel matrix' rank

Figure 5.2.: The vector network coding of $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$ represents as a matrix problem



Following to Equation 3.1, each receiver R_j must solve a linear equation system of $3t$ variables with $4t$ equations to recover $h = 3$ messages as below:

$$\begin{bmatrix} \boldsymbol{y}_j^{(r_1)} \\ \boldsymbol{y}_j^{(r_2)} \\ \boldsymbol{y}_j^{(r_3)} \\ \boldsymbol{y}_j^{(r_4)} \end{bmatrix} = \boldsymbol{A}_j \cdot \underline{x} = \begin{bmatrix} \boldsymbol{A}_j^{(r_1)} \\ \boldsymbol{A}_j^{(r_2)} \\ \boldsymbol{A}_j^{(r_3)} \\ \boldsymbol{A}_j^{(r_4)} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{bmatrix},$$

with $\boldsymbol{x}_i, \boldsymbol{y}_j^{(v)} \in \mathbb{F}_q^t$, $\boldsymbol{A}_j^{(r_v)} \in \mathbb{F}_q^{t \times 3t}$ for $v = 1, \ldots, 4$, and $\boldsymbol{A}_j^{(r_1)}, \ldots, \boldsymbol{A}_j^{(r_3)}$ must be distinct. The network is solvable, if $\boldsymbol{A}_j$ has full-rank, i.e. $\boldsymbol{A}_{j_v}$ must satisfy:

$$rk \begin{bmatrix} \boldsymbol{A}_j^{(r_1)} \\ \boldsymbol{A}_j^{(r_2)} \\ \boldsymbol{A}_j^{(r_3)} \\ \boldsymbol{A}_j^{(r_4)} \end{bmatrix} \geq 3t$$

Because coding coefficients for $\boldsymbol{A}_j^{(r_4)}$ can be independently chosen for any receiver R_j, there always exists $\boldsymbol{A}_j^{(r_4)}$ such that,

$$rk \begin{bmatrix} \boldsymbol{A}_j^{(r_1)} \\ \boldsymbol{A}_j^{(r_2)} \\ \boldsymbol{A}_j^{(r_3)} \end{bmatrix} \geq 2t \tag{5.1}$$

if and only if $rk\,[\boldsymbol{A}_j] \geq 3t$.

By this constraint, the problem is thus described as below:

$$\min_{rk[\boldsymbol{A}_j] \geq 3t} r_{vector}$$

$\boldsymbol{A}_j^{(r_1)}, \boldsymbol{A}_j^{(r_2)}, \boldsymbol{A}_j^{(r_3)}$ are matrices formed on any 3 of r links from nodes to receivers, i.e. these 3 matrices are randomly chosen from a set of r matrices. We formalize the problem by an approach with Lovász local lemma, which was initially proposed by Schwartz in [Sch].

define event $\mathcal{E}_i$ here

Lemma 5.1 (Symmetric Lovász local lemma (LLL) [SCV13]). *A set of events $\mathcal{E}_i$, with $i = 1, \ldots, n$, such that each event occurs with probability at most p. If each event is independent of all others except for at most d of them and $4dp \leq 1$, then:* $Pr\left[\bigcap_{i=1}^{n} \overline{\mathcal{E}}_i\right] > 0$.

also $\mathcal{E}_{r_1,r_2,r_3}$ is a better name or $\mathcal{E}^{(r_1,r_2,r_3)}$ since where is i in $rk\begin{bmatrix} A^{(r_1)} \\ A^{(r_2)} \\ A^{(r_3)} \end{bmatrix} < 2t$? The events are indexed by r_1, r_2, r_3!

maybe use diff. letter for $\mathcal{E}_i$ (LLL is very general, but $\mathcal{E}_i$ is already the event $rk\begin{bmatrix} A^{(r_1)} \\ A^{(r_2)} \\ A^{(r_3)} \end{bmatrix} < 2t$

Lemma 5.2. *Let* $Pr[\mathcal{E}_i] = Pr\left[rk\begin{bmatrix}A_j^{(r_1)}\\ A_j^{(r_1)}\\ A_j^{(r_3)}\end{bmatrix} < 2t\right]$, ~~∀~~ $1 \le r_1 < r_2 < r_3 \le r$, *and* $A_j^{(r_1)}, \dots, A_j^{(r_3)} \in \mathbb{F}_q^{t\times 3t}$, ~~then~~, chosen independently and uniformly at random. Then,

$$\le p \in \Theta\left(q^{-t^2-2t-1}\right), \forall t \ge 2. \quad (*)$$

Proof. In Lemma 5.1, each event $\mathcal{E}_i$ is a bad event, whose occurrence is undesirable. why is this classification necessary here? Following to Equation 5.1, such a event occurs when $rk[A_j] < 3t$, and its probability is bounded by p (with $0 \le p \le 1$) as follows,

$$\cancel{Pr[\mathcal{E}_i] = Pr\left[rk\begin{bmatrix}A_j^{(r_1)}\\ A_j^{(r_1)}\\ A_j^{(r_3)}\end{bmatrix} < 2t\right] \le p.} \quad (5.2)$$

~~Regarding to the left-hand side,~~

$$\begin{aligned} Pr[\mathcal{E}_i] = Pr\left[rk\begin{bmatrix}A_j^{(r_1)}\\ A_j^{(r_2)}\\ A_j^{(r_3)}\end{bmatrix} < 2t\right] &= \sum_{i=0}^{2t-1} Pr\left[rk\begin{bmatrix}A_j^{(r_1)}\\ A_j^{(r_2)}\\ A_j^{(r_3)}\end{bmatrix} = i\right] \\ &\overset{1}{=} \sum_{i=0}^{2t-1} \frac{N_{i,m,n}}{q^{m\cdot n}} \\ &= \sum_{i=0}^{2t-1} \frac{\prod_{j=0}^{i-1}\frac{(q^m-q^j)(q^n-q^j)}{q^i-q^j}}{q^{m\cdot n}} \\ &\overset{2}{=} \sum_{i=0}^{2t-1} \frac{\prod_{j=0}^{i-1}\frac{(q^{3t}-q^j)^2}{q^i-q^j}}{q^{9t^2}} =: p. \end{aligned} \quad (5.3)$$

introduce notion before using it (e.g. explain it in one sentence above)

do not use m & n here, directly plug in actual matrix dimensions!

(1): The formula for the umber of $[m \times n]$ matrices of rank i over $\mathbb{F}_q$ was proved in [Ove07].

(2): $A_j^{(r_1)}, A_j^{(r_2)}, A_j^{(r_3)}$ vertically together form a $[3t \times 3t]$ matrix.

In the following, we ~~view~~ $\prod \dots$ as a polynomial in q

~~Sometimes~~ We consider the numerator of Equation (5.3): $\prod_{j=0}^{i-1}\frac{(q^{3t}-q^j)^2}{q^i-q^j} = \frac{p_N^{(i)}(q)}{p_D^{(i)}(q)} = p^{(i)}(q)$.

write only (5.3) (no "Equation")

Due to i-times product and large t: $\left.\begin{aligned} deg\left(p_N^{(i)}(q)\right) &= q^{i6t} \\ deg\left(p_D^{(i)}(q)\right) &= q^{i^2}\end{aligned}\right\} \Rightarrow p^{(i)}(q) \approx q^{i6t-i^2}$.

(*) it is wrong to say

"$a \le p \Rightarrow p \le \Theta(q^{-t^2-2t-1})$"

since ~~a ≤ p ...~~ also $a \le q^{t^{9999}}$ (and obviously $q^{t^{9999}} \notin \Theta(q^{-t^2-2t-1})$)

Please note the fundamental difference of "$a \le p \Rightarrow p \in \dots$" and "Let..., then $a \le p$"!

Therefore, we have: $\sum_{i=0}^{2t-1}\prod_{j=0}^{i-1} \frac{(q^{3t}-q^j)^2}{q^i-q^j} = \sum_{i=0}^{2t-1} p^{(i)}(q) \approx \sum_{i=0}^{2t-1} q^{i6t-i^2}$.
To maximize the sum, we set derivation of it to 0 and find the corresponding root:

$$\begin{aligned} & \left(i6t - i^2\right)' &= 0 \\ \Leftrightarrow\quad & 6t - 2i &= 0 \\ \Leftrightarrow\quad & i &= 3t\,. \end{aligned}$$

However, the upper limit of the sum is $(2t-1)$, which is less than $3t$ for all $t \geq 2$.

$$\Rightarrow max\left\{q^{i6t-i^2} : i = 0, 2\ldots, 2t-1\right\} = q^{i6t-i^2}\Big|_{i=2t-1} = q^{8t^2-2t-1}.$$

reconsider everywhere! (may be rewritten)

Hence, by using the exact bound ? Θ, we have: some people consider a ":" before an equation to be bad style.

$$\max_i \left[\sum_{i=0}^{2t-1} p^{(i)}(q)\right] \in \Theta\left(max\left\{q^{i6t-i^2} : i = 1, 2\ldots, 2t-1\right\}\right) = \Theta\left(q^{8t^2-2t-1}\right)$$

$$\Rightarrow \max_i \left[\frac{\sum_{i=0}^{2t-1} p^{(i)}(q)}{q^{9t^2}}\right] \in \Theta\left(q^{-t^2-2t-1}\right)$$

$$\Rightarrow p \leq \Theta\left(q^{-t^2-2t-1}\right)$$

Leave out unnecessary info!

□

why needed? Just state that $\mathcal{E}_i$ is dep. on at most $d \leq \frac{3}{2}r^2$ other events

Lemma 5.3. *Let* $Pr[\mathcal{E}_i] = Pr\left[rk\begin{bmatrix} \mathbf{A}_j^{(r_1)} \\ \mathbf{A}_j^{(r_1)} \\ \mathbf{A}_j^{(r_3)} \end{bmatrix} < 2t\right] \leq p, \forall 1 \leq r_1 < r_2 < r_3 \leq r$ *with* $\mathbf{A}_j^{(r_1)}, \ldots, \mathbf{A}_j^{(r_3)} \in \mathbb{F}_q^{t\times 3t}$*, and each event* $\mathcal{E}_i$ *is independent of all others except for at most* d *of them, then* $d \leq \frac{3}{2}r^2$*.*

Proof. Because d is a function of r in our problem, we denote d in Lemma 5.1 specifically by $d(r)$. The Local lemma 5.1 allows dependence among at most $d(r)$ events, i.e. each event considered under the Local lemma must no be independent to each other. Furthermore, $\mathbf{A}_j^{(r_1)}, \mathbf{A}_j^{(r_2)}, \mathbf{A}_j^{(r_3)}$ are matrices formed on any 3 of r links from nodes to receivers. Assume that 1 of r links is firstly chosen to form the matrix $\mathbf{A}_j^{(r_1)}$, then there are $\binom{r-1}{2}$ posibilities left to choose $\mathbf{A}_j^{(r_2)}$ and $\mathbf{A}_j^{(r_3)}$ from $r-1$ links. However, such a link can also be chosen firstly to form $\mathbf{A}_j^{(r_2)}$ or $\mathbf{A}_j^{(r_3)}$ instead. Therefore, we have a upper

This should not be part of the proof (unnecessary info). Can be written as a sentence above lemma.

If $\mathcal{E}_{r_1,r_2,r_3}$ is indexed by the r_i, then it becomes clear that $\mathcal{E}_{r_1,r_2,r_3}$ is dependent on $\mathcal{E}_{r_1',r_2',r_3'}$ if and only if $\{r_1,r_2,r_3\}\cap\{r_1',r_2',r_3'\}\neq\emptyset$

bound for $d(r)$ as following,

$$d(r) \leq 3 \cdot \binom{r-1}{2} = 3 \cdot \frac{(r-1)(r-2)}{2} = \frac{3}{2}(r^2 - 3r + 2)$$

$$\Rightarrow d(r) \leq \frac{3}{2} r^2$$

Example 5.1. $r \in \{1, 2, 3, 4\}$. Our sample space to draw any 3 matrices with orders has in total $\binom{4}{3} = 4$ samples. ◻

The context/goal is not clear!

Assume the link 1 is firstly chosen to form the matrix $A_j^{(r_1)}$, we have $\binom{3}{2} = 3$ possibilities: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$. The existence of 1 in $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ gives us some information on $rk \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ or $rk \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, which means these events $rk \begin{bmatrix} A_j^{(r_1)} \\ A_j^{(r_1)} \\ A_j^{(r_3)} \end{bmatrix} < 2t$ are dependent.

is this $\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$?

Strange word

this is only one event

We also have 3 posibilities, when we choose the link 1 to form the matrix $A_j^{(r_2)}$: $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$.

or use DeclareMathOperator for

There are again 3 posibilities, when we choose the link 1 to form the matrix $A_j^{(r_3)}$: $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$.

these are the same events

put rk(..) Pr(..) etc. in \mathrm{rk}(..) ...

Let's denote the event $rk \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} < 2t$ by $\mathcal{E}_1$. It is clearly to see that $Pr\left[rk \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} < 2t \,\middle|\, \mathcal{E}_1 \right] = 1$. The bound in Lemma 5.3 is a bit loose, because we do not neglect unsuitable matrices, e.g. $\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$. In the other words, there exists at most 9 events are dependent.

that's also why

>>> IS THIS EXAMPLE CONVINCING ENOUGH? WHY IS IT NOT $d(r) \leq 3 \cdot r$.

Everywhere!

be careful that $r_1 < r_2 < r_3$!

$\binom{r-1}{2} = ?$

To be exact it is: "There is an $r_0 \in \Omega(q^{t^2/2+O(t)})$ s.t. for any $r \leq r_0$ there exists ..."

Theorem 5.1. *If $r \leq \Omega\left(q^{t^2/2+O(t)}\right)$, then there exists a vector solution for the $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$ network.*

Proof. The Local lemma 5.1 shows that there is a positive probability that none of the? bad events occurs: $Pr\left[\bigcap_{i=1}^{n} \bar{\mathcal{E}}_i\right] > 0$. the? * if $4 \cdot P[\mathcal{E}_i] \cdot d \leq 1$?

By the intersection rule, none of bad events is equivalent to an event T that its set of outcomes are all desirable, i.e. these outcomes satisfy ~~the Equation 5.1.~~ (5.1):

$$T = \bigcap_{i=1}^{n} \bar{\mathcal{E}}_i = \left\{ rk \begin{bmatrix} \mathbf{A}_j^{(r_1)} \\ \mathbf{A}_j^{(r_2)} \\ \mathbf{A}_j^{(r_3)} \end{bmatrix} \geq 2t, \forall 1 \leq r_1 < r_2 < r_3 \leq r \right\}.$$

The probability of event T indicates a measure quantifying the likelihood that we are able to construct $rk\left[\mathbf{A}_j\right] \geq 3t$ given r links. It means that a vector solution exists if and only if the Local lemma 5.1 is satisfied such that: $4 \cdot p \cdot d(r) \leq 1, \forall r \leq r_{max,vector}$. Therefore, we must find a lower bound of $r_{max,vector}$. Furthermore, we have $d \leq \frac{3}{2}r^2$ following to Lemma 5.3, which gives: $4 \cdot p \cdot \frac{3}{2}r^2 \leq 1 \Rightarrow r \leq \sqrt{\frac{1}{6p}} = r_{max,vector}$. Thus, finding $min\left\{r_{max,vector}\right\}$ is equivalent to maximize p.

? wrong! The LLL statement is sufficient but not necessary!

use \min \max ...

By Lemma 5.2, we have $p \leq \Theta\left(q^{-t^2-2t-1}\right)$,

why min? We want to max. it!

subscript in \mathrm{...} everywhere!

(roman letters for operators!)

$$\Rightarrow min\left\{r_{max,vector}\right\} \in \Omega\left(\sqrt{\frac{1}{6p}}\right) = \Omega\left(\sqrt{\frac{1}{6q^{-t^2-2t-1}}}\right) = \Omega\left(q^{t^2/2+O(t)}\right).$$

sufficient condition of the

Hence, the Local lemma in 5.1 is satisfied, when $r \leq \Omega\left(q^{t^2/2+O(t)}\right)$ $r_0 =$. None of bad events occurs, so there exists a vector solution for such r. □

Corollary 5.1. *The $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network has a vector solution with a gap $q^{t^2/4+O(t)}$.*

Proof.

In [EW18, Sec. VIII-C], we have that $r_{max,scalar} \in O\left(q_s^2\right)$, where they proved that

$$r_{scalar} \leq 2\begin{bmatrix} 3 \\ 1 \end{bmatrix}_{q_s} = 2\left(q_s^2 + q_s + 1\right). \tag{5.4}$$

Following to Section 3.4.2 and Theorem 5.1, we have the gap size

$$
\begin{aligned}
& r_{max,scalar} && = min\left\{r_{max,vector}\right\} \\
\Leftrightarrow\ & q_s^2 && = q^{t^2/2+\mathcal{O}(t)} \\
\Leftrightarrow\ & q_s && = q^{t^2/4+\mathcal{O}(t)} \\
\Rightarrow\ & g && = q_s - q_v = q^{t^2/4+\mathcal{O}(t)}
\end{aligned}
\tag{5.5}
$$

By varying t in Equation (5.3), we have the following table:

□

Table 5.1.: r over variations of t

t	Scalar Solution	Vector Solution
1	$r_{scalar} \leq 14$	$r_{vector} \geq 3$
2	$r_{scalar} \leq 42$	$r_{vector} \geq 7$ (67*, 89**)
3	$r_{scalar} \leq 146$	$r_{vector} \geq 62$ (166*)
4	$r_{scalar} \leq 546$	$r_{vector} \geq 1317$
5	$r_{scalar} \leq 2114$	$r_{vector} \geq 58472$
6	$r_{scalar} \leq 8322$	$r_{vector} > 10^6$

*, **: Computational results in Construction 1 and Construction 2 respectively

(maybe) e.g. Constr. 1&2 not introduced

+ this also theo. shows res.

Explain everything!

Table 5.1 (capital "T" and no "(-)")

$r_{max,scalar}$ or r_{scalar} ??

In the table (5.1), the vector solution outperforms the scalar solution when $t \geq 4$ for the network $(\epsilon = 1, \ell = 1) - N_{h=3,r,s=4}$. This is sufficient, we show in Section 6 computational results which vector solutions outperform scalar solutions in case of $t = 2$ and $t = 3$.

what is sufficient?

5.2. $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h,r,s}$ Network

5.2.1. Find the lower bound of $r_{max,vector}$

say a — on/for

As previous, $\boldsymbol{A}_j^{(r_1)}, \ldots, \boldsymbol{A}_j^{(r_{h-\epsilon})} \in \mathbb{F}_q^{t \times ht}$ and we need to satisfy the following:

$$
rk \begin{bmatrix} \boldsymbol{A}_j^{(r_1)} \\ \vdots \\ \boldsymbol{A}_j^{(r_{h-\epsilon})} \end{bmatrix} \geq ht - t \Leftrightarrow rk\left[\boldsymbol{A}_j\right] \geq (h-1)t
$$

We can formulate it by the following coding problem in Grassmannian:

> Find the largest set of subspaces from $\mathcal{G}_q(ht, t)$ such that any α subspaces of the set span a subspace of dimension at least $(h-1)t$.

Similar

Same comments as in 5.2 ?

with $(\epsilon = 1, \ell = 1) - N_{3,r,4}$, we consider p to proceed the Local lemma 5.1:

$$Pr\left[rk\left[\mathbf{A}_j\right] < (h-1)t\right] \leq p \tag{5.6}$$

Lemma 5.4. *Let* $Pr\left[\mathcal{E}_i\right] = Pr\left[rk\begin{bmatrix}\mathbf{A}_j^{(r_1)}\\ \vdots \\ \mathbf{A}_j^{(r_{h-\epsilon})}\end{bmatrix} < (h-1)t\right] \leq p, \forall 1 \leq r_1 < \ldots < r_{h-\epsilon} \leq r$ *and* $\mathbf{A}_j^{(r_1)}, \ldots, \mathbf{A}_j^{(r_{h-\epsilon})} \in \mathbb{F}_q^{t\times ht}$, *then,*

$$p \leq \Theta\left(q^{(h-\alpha)t^2 + \mathcal{O}(t)}\right), \forall t \geq 2.$$

Proof. Regarding to the left-hand side in Equation 5.6:

$$\begin{aligned} Pr\left[rk\left[\mathbf{A}_j\right] < (h-1)t\right] &= \sum_{i=0}^{(h-1)t-1} Pr\left[rk\left[\mathbf{A}\right] = i\right] \\ &\overset{1}{=} \sum_{i=0}^{(h-1)t-1} \frac{N_{i,\alpha t,ht}}{q^{(\alpha t)(ht)}} \\ &= \frac{1}{q^{(\alpha h)t^2}} \cdot \sum_{i=0}^{(h-1)t-1} \prod_{j=0}^{i-1} \frac{\left(q^{\alpha t} - q^j\right)\left(q^{ht} - q^j\right)}{q^i - q^j}. \end{aligned} \tag{5.7}$$

(1): The formula for the umber of $[m \times n]$ matrices of rank i over $\mathbb{F}_q$ was proved in [Ove07].

We consider firstly the product $\prod_{j=0}^{i-1} \frac{\left(q^{\alpha t}-q^j\right)\left(q^{ht}-q^j\right)}{q^i - q^j} = \frac{p_N^{(i)}(q)}{p_D^{(i)}(q)} = p^{(i)}(q)$.

For $t \to \infty$: $\left.\begin{aligned} deg\left(p_N^{(i)}(q)\right) &= q^{i(\alpha t + ht)} \\ deg\left(p_D^{(i)}(q)\right) &= q^{i^2}\end{aligned}\right\} \Rightarrow p^{(i)}(q) \approx q^{i(\alpha t+ht) - i^2}$.

Now, we evaluate $f(i) = i(\alpha t + ht) - i^2$ to find its maximum point by its derivation :

$\dot{f}(i^*) = 0 \Leftrightarrow (\alpha t + ht) - 2i^* = 0 \Leftrightarrow i^* = \frac{\alpha t + ht}{2}$

We then check whether this point within the range $i = 0, \ldots, (h-1)t - 1$ as following: $0 \leq \frac{\alpha t+ht}{2} \leq (h-1)t - 1$.

With regards to the lower bound: $0 \leq \frac{\alpha t + ht}{2} \Leftrightarrow t \geq \frac{2}{\alpha + h}$, which is always true due to the given $t \geq 2$ and $\alpha, h \geq 3$.

Regarding to the upper bound: $\frac{\alpha t+ht}{2} \leq (h-1)t - 1 \Leftrightarrow t \leq \frac{-2}{\alpha+2-h}$ with $\alpha + 2 > h$ due to the given $\alpha l + \epsilon = \alpha + 1 \geq h$. This cannot happen because of $t \geq 2$, i.e. this maximum point is over then upper-range limit.

$$\Rightarrow\ max\left\{q^{i(\alpha t+ht)-i^2} : i=1,\dots,(h-1)t-1\right\} = q^{i(\alpha t+ht)-i^2}\Big|_{i=(h-1)t-1} = q^{[(h-1)(\alpha+1)]t^2-(\alpha-h+2)t-1}.$$

Secondly, we apply the maximum value with the sum, we have:

$$max\left\{\sum_{i=0}^{(h-1)t-1} p^{(i)}(q)\right\} \in \Theta\left(q^{[(h-1)(\alpha+1)]t^2+\mathcal{O}(t)}\right)$$

$$\Rightarrow\ max\left\{\frac{\sum_{i=0}^{(h-1)t-1} p^{(i)}(q)}{q^{(\alpha h)t^2}}\right\} \in \left(\frac{q^{[(h-1)(\alpha+1)]t^2+\mathcal{O}(t)}}{q^{(\alpha h)t^2}}\right)$$

$$\Rightarrow p \leq \Theta\left(q^{(h-\alpha)t^2+\mathcal{O}(t)}\right)$$

□

Lemma 5.5. *Let* $Pr\left[\mathcal{E}_i\right] = Pr\left[rk\left[\begin{matrix}\boldsymbol{A}_j^{(r_1)}\\ \vdots \\ \boldsymbol{A}_j^{(r_{h-\epsilon})}\end{matrix}\right] < (h-1)t\right] \leq p, \forall 1 \leq r_1 < \ldots < r_{h-\epsilon} \leq r$ *and* $\boldsymbol{A}_j^{(r_1)},\ldots,\boldsymbol{A}_j^{(r_{h-\epsilon})} \in \mathbb{F}_q^{t\times ht}$, *and each event* $\mathcal{E}_i$ *is independent of all others except for at most* d *of them, then* $d \leq \frac{\alpha}{(\alpha-1)!}r^{\alpha-1}$.

Proof. Similar with $(\epsilon=1,\ell=1)-N_{3,r,4}$, we have:

$$d \leq \alpha\binom{r-1}{\alpha-1} = \alpha\frac{(r-1)\ldots(r-\alpha+1)}{(\alpha-1)!} \leq \frac{\alpha}{(\alpha-1)!}r^{\alpha-1}$$

□

Theorem 5.2. *If* $r \leq \Omega\left(q^{\frac{h-\alpha-1}{1-\alpha}t^2+\mathcal{O}(t)}\right)$, *then there exists a vector solution for the* $(\epsilon=1,\ell=1)-N_{h,r,s}$ *network.*

Proof. As previous, satisfying LLL 5.1 is equivalent to an existence of a vector solution. We need $4dp \leq 1$, and following to Lemma 5.5: $d \leq \frac{\alpha}{(\alpha-1)!}r^{\alpha-1} \Rightarrow r \leq \left(\frac{(\alpha-1)!}{4\alpha}\cdot\frac{1}{p}\right)^{\frac{1}{\alpha-1}} = r_{max,vector}$. We thus again find a lower bound of $r_{max,vector}$.
By Lemma 5.5, we have $p \leq \Theta\left(q^{(h-\alpha)t^2+\mathcal{O}(t)}\right), \forall t \geq 2$,
$\Rightarrow min\left\{r_{max,vector}\right\} \in \Omega\left(q^{\frac{h-\alpha-1}{1-\alpha}t^2+\mathcal{O}(t)}\right)$.
Hence, the Local lemma 5.1 is satisfied, when $r \leq \Omega\left(q^{\frac{h-\alpha-1}{1-\alpha}t^2+\mathcal{O}(t)}\right)$, and a vector solution exists for such r. □

5.2.2. Find ~~the~~ a Upper Bound ~~of~~ on/for? $r_{max,scalar}$

there is no unique upper bound

> Find $(\alpha+1)$ received vectors that span a subspace of dimension h. This implies that the α links from the middle layer carry α vectors which span a subspace of $\mathbb{F}_{q_s}^h$ whose dimension is at least $(h-1)$, with $q_s = q^t$.

Following to Theorem ~~(4.1)~~, we are interested in the following range: $\ell+\epsilon+1 \leq h \leq \alpha\ell+\epsilon$. (everywhere!)

For $3 \leq \alpha < \cancel{h}$, all α links must be distinct $\Rightarrow r \leq \begin{bmatrix} \alpha \\ 1 \end{bmatrix}_{q_s} \Rightarrow r \leq \mathcal{O}\left(q_s^{\alpha-1}\right)$

For $\alpha \geq h \geq 3$: to achieve $(h-1)$-subspaces of $\mathbb{F}_{q_s}^h$, no α links will contain a vector which is contained in the same $(h-2)$-subspace.

Hence,

$$r_{max,scalar} \leq (\alpha-1)\begin{bmatrix} \alpha \\ h-2 \end{bmatrix}_{q_s} \Rightarrow r_{max,scalar} \in \mathcal{O}\left(q_s^{(\alpha-h+2)(h-2)t^2}\right)$$

state as Theorem?

5.2.3. Calculate Gap

+ a sentence

$$\begin{aligned} r_{max,scalar} &= min\{r_{max,vector}\} \\ \Leftrightarrow \quad q_s^{(\alpha-h+2)(h-2)t^2} &= q^{\frac{h-\alpha-1}{1-\alpha}t^2+\mathcal{O}(t)} \\ \Leftrightarrow \quad q_s &= q^{\frac{\alpha-h+1}{(\alpha-1)(\alpha-h+2)(h-2)}t^2+\mathcal{O}(t)} \\ \Rightarrow \quad g &= q_s - q_v = q^{\frac{\alpha-h+1}{(\alpha-1)(\alpha-h+2)(h-2)}t^2+\mathcal{O}(t)} \end{aligned}$$

state as Thm.

5.3. $(\epsilon = 1, \ell \geq 2) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$

+ Text

Lemma 5.6. *Let* $Pr\left[\mathcal{E}_i\right] = Pr\left[rk\begin{bmatrix} \boldsymbol{A}_j^{(r_1)} \\ \vdots \\ \boldsymbol{A}_j^{(r_{h-\epsilon})} \end{bmatrix} < (2\ell-1)\,t\right] \leq p, \forall 1 \leq r_1 < \ldots < r_{h-\epsilon} \leq r$ *and* $\boldsymbol{A}_j^{(r_1)}, \ldots, \boldsymbol{A}_j^{(r_{h-\epsilon})} \in \mathbb{F}_q^{t\times 2\ell t}$*, then,*

$$p \leq \Theta\left(q^{-t^2-2t-1}\right), \forall t \geq 2.$$

same comments as Sec. 5.1

Proof. A bad event in this vector network coding problem has the following probability:

$$
\begin{aligned}
Pr\left[rk\left[\begin{bmatrix} \boldsymbol{A}_j^{(r_1)} \\ \vdots \\ \boldsymbol{A}_j^{(r_{h-\epsilon})} \end{bmatrix}\right] < (2\ell-1)\,t \right] &= \sum_{i=0}^{(2\ell-1)t-1} Pr\left[rk\left[\begin{bmatrix} \boldsymbol{A}_j^{(r_1)} \\ \vdots \\ \boldsymbol{A}_j^{(r_{h-\epsilon})} \end{bmatrix}\right] = i \right] \\
&\overset{1}{=} \sum_{i=0}^{(2\ell-1)t-1} \frac{N_{i,m,n}}{q^{m\cdot n}} \\
&\overset{2}{=} \frac{1}{q^{4\ell^2 t^2}} \sum_{i=0}^{(2\ell-1)t-1} \prod_{j=0}^{i-1} \frac{\left(q^{2\ell t}-q^j\right)^2}{q^i-q^j} \qquad (5.8)
\end{aligned}
$$

(1): The formula for the umber of $[m \times n]$ matrices of rank i over $\mathbb{F}_q$ was proved in [Ove07].

(2): $s = \alpha\ell + \epsilon$ by definition in Section 4.1 $\Rightarrow \alpha = 2$, so $\boldsymbol{A}_j \in \mathbb{F}_q^{2\ell t \times 2\ell t}$ with $\boldsymbol{A}_j = \begin{bmatrix} \boldsymbol{A}_j^{(r_1)} \\ \vdots \\ \boldsymbol{A}_j^{(r_{h-\epsilon})} \end{bmatrix}$, and $\boldsymbol{A}_j$ contains(2ℓ) t-dimensional subspaces of $\mathbb{F}_q^{2\ell t}$.

We consider the product in Equation 5.8: $\prod_{j=0}^{i-1} \frac{\left(q^{2\ell t}-q^j\right)^2}{q^i-q^j} = \frac{p_N^{(i)}(q)}{p_D^{(i)}(q)} = p^{(i)}(q)$.

Due to i-times product and large t: $\left.\begin{aligned} deg\left(p_N^{(i)}(q)\right) &= q^{i4\ell t} \\ deg\left(p_D^{(i)}(q)\right) &= q^{i^2} \end{aligned}\right\} \Rightarrow p^{(i)}(q) \approx q^{i4\ell t - i^2}$.

Therefore, we have: $\sum_{i=0}^{(2\ell-1)t-1} \prod_{j=0}^{i-1} \frac{\left(q^{2\ell t}-q^j\right)^2}{q^i-q^j} = \sum_{i=0}^{(2\ell-1)t-1} p^{(i)}(q) \approx \sum_{i=0}^{(2\ell-1)t-1} q^{i4\ell t - i^2}$.

To maximize the sum, we set derivation of it to 0 and find the corresponding root:

$$
\begin{aligned}
& \left(i4\ell t - i^2\right)' &&= 0 \\
\Leftrightarrow \quad & 4\ell t - 2i &&= 0 \\
\Leftrightarrow \quad & i &&= 2\ell t
\end{aligned}
$$

However, the upper limit of the sum is $(2\ell-1)\,t-1$, which is less than $2\ell t$ for all $t \geq 2$.

$$
\Rightarrow max\left\{q^{i4\ell t - i^2} : i = 0, 2\ldots, (2\ell-1)\,t-1\right\} = \left. q^{i4\ell t - i^2}\right|_{i=(2\ell-1)t-1} = q^{4\ell^2 t^2 - t^2 - 2t - 1}
$$

Hence, by using the exact bound Θ, we have:

$$\max_i \left\{ \sum_{i=0}^{(2\ell-1)t-1} p^{(i)}(q) \right\} \in \Theta\left(q^{4\ell^2 t^2 - t^2 - 2t - 1}\right)$$

$$\Rightarrow \max_i \left\{ \frac{1}{q^{4\ell^2 t^2}} \sum_{i=0}^{(2\ell-1)t-1} p^{(i)}(q) \right\} \in \Theta\left(q^{-t^2-2t-1}\right)$$

$$\Rightarrow p \leq \Theta\left(q^{-t^2-2t-1}\right)$$

□

Lemma 5.7. *Let* $Pr[\mathcal{E}_i] = Pr\left[rk \begin{bmatrix} \boldsymbol{A}_j^{(r_1)} \\ \vdots \\ \boldsymbol{A}_j^{(r_{h-\epsilon})} \end{bmatrix} < (2\ell-1)t \right] \leq p, \forall 1 \leq r_1 < \ldots < r_{h-\epsilon} \leq r$ *and* $\boldsymbol{A}_j^{(r_1)}, \ldots, \boldsymbol{A}_j^{(r_{h-\epsilon})} \in \mathbb{F}_q^{t \times 2\ell t}$, *and each event* $\mathcal{E}_i$ *is independent of all others except for at most* d *of them, then* $d \leq 2r$.

Proof. Being similar with the previous subsections, we have:

$$d \leq \alpha \binom{r-1}{\alpha-1} = 2\frac{(r-1)\ldots(r-1)}{1!} \leq 2r$$

□

Theorem 5.3. *If* $r \leq \Omega\left(q^{t^2+\mathcal{O}(t)}\right)$, *then there exists a vector solution for the* $(\epsilon = 1, \ell \geq 2)-\mathcal{N}_{h=2\ell,r,s=2\ell+1}$ *network.*

Proof. As previous, we need $4 \cdot p \cdot d(r) \leq 1, \forall r \leq r_{max,vector}$ so that a vector solution exists. Following to Lemma 5.7, we have $d \leq 2r \Rightarrow 4 \cdot p \cdot 2r \leq 1 \Rightarrow r \leq \frac{1}{8p}$. We still need to maximize p to get lower on $r_{max,vector}$, and we have $p \leq \Theta\left(q^{-t^2-2t-1}\right), \forall t \geq 2$ in Lemma 5.6. Thus, $min\left\{r_{max,vector}\right\} \in \Omega\left(\frac{1}{8p}\right) = \Omega\left(q^{t^2+2t+1}\right)$.
Hence, the Local lemma in 5.1 is satisfied, when $r \leq \Omega\left(q^{t^2/2+\mathcal{O}(t)}\right)$. None of bad events occurs, so there exists a vector solution for such r. □

Lemma 5.8. *A scalar solution for the* $(\epsilon = 1, \ell \geq 2) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$ *network exists, if and only if there exists a Grasmannian code* $\mathcal{G}_q(h = 2\ell, \ell)$ *such that any* $\alpha = 2$ *subspaces of the set span a subspace of dimension at least* $2\ell - 1$.

Proof. Let's denote any 2 subspaces of $\mathcal{G}_q(h = 2\ell, \ell)$ as $\mathcal{U}$ and $\mathcal{V}$. If $\mathcal{U}$ and $\mathcal{V}$ span a subspace of dimension at least $2\ell - 1$, then we have $dim(\mathcal{U} + \mathcal{V}) = 2\ell - 1$, with $\mathcal{U} + \mathcal{V}$.

uu

Therefore, $dim(\mathcal{U} \cap \mathcal{V}) = dim(\mathcal{U}) + dim(\mathcal{V}) - dim(\mathcal{U} + \mathcal{V}) = 1$, which leads to the subspace distance $d_s(\mathcal{U}, \mathcal{V}) = 2dim(\mathcal{U} + \mathcal{V}) - dim(\mathcal{U}) - dim(\mathcal{V}) = 2\ell - 2$. No 2 ℓ-dimensional subspaces of $\mathbb{F}_{q_s}^{2\ell}$ will contain a vector which is contained in the same $(2\ell - 2)$-subspace, but $(2\ell - 1)$ of such subspaces can have such vectors,

$$\Rightarrow r_{scalar} \leq (2\ell - 1) \begin{bmatrix} 2\ell \\ 2\ell - 2 \end{bmatrix}_{q_s} \Rightarrow r_{scalar} \leq \mathcal{O}\left(q_s^{l}\right)$$ lell

□

Corollary 5.2. *The* $(\epsilon = 1, \ell \geq 2) - \mathcal{N}_{h=2\ell, r, s=2\ell+1}$ *network has a vector solution with a gap* $q^{t^2/4 + \mathcal{O}(t)}$.

2ℓ

Following to Section 3.4.2 and Theorem 5.3, we have the gap size

$$\begin{aligned}
& r_{max,scalar} && = min\{r_{max,vector}\} \\
\Leftrightarrow \quad & q_s^{\ell} && = q^{t^2/2 + \mathcal{O}(t)} \\
\Leftrightarrow \quad & q_s && = q^{t^2/2\ell + \mathcal{O}(t)} \\
\Rightarrow \quad & g && = q_s - q_v = q^{t^2/2\ell + \mathcal{O}(t)}
\end{aligned} \tag{5.9}$$

This shows us that there exists a better vector solution by comparison with the gap in [EW18, Fig. 4].