The Benefits of Coding over Routing in a Randomized Setting

Tracey Ho[*], Ralf Koetter[†], Muriel Médard[*], David R. Karger[*], and Michelle Effros[‡]

[*]Massachusetts Institute of Technology, [†]Univ. of Illinois, Urbana-Champaign, [‡]California Institute of Technology

e-mail: [*]{trace@ , medard@, karger@lcs.}mit.edu, [†]koetter@uiuc.edu, [‡]effros@caltech.edu

Abstract — **We present a novel randomized network coding approach for robust, distributed transmission and compression of information in networks, and demonstrate its advantages over routing-based approaches.**

We present a randomized network coding approach for robust, distributed transmission and compression of information in networks. Network nodes transmit on each outgoing link a linear combination of incoming signals, specified by independently and randomly chosen code coefficients from some finite field $\mathbb{F}_q$. The only information needed for decoding at the receivers is the overall linear combination of source processes present in each of their incoming signals. This information can be maintained, for each signal in the network, as a vector of coefficients for each of the source processes, and updated by each coding node applying the same linear combinations to the coefficient vectors as to the data. See [1], [3] for prior work on network coding, [4] for work on randomized routing, and [2] for supporting results used in our proofs.

We give a lower bound on the success probability of random network coding for multicast connections, based on the form of transfer matrix determinant polynomials [3], that is tighter than the Schwartz-Zippel bound for general polynomials of the same total degree. The corresponding upper bound on failure probability is on the order of the inverse of the size of the finite field, showing that it can be made arbitrarily small by coding in a sufficiently large finite field, and that it decreases exponentially with the number of codeword bits. This suggests that random codes are potentially very useful for networks with unknown or changing topologies.

We model a network as a delay-free acyclic graph with unit capacity directed links[1] and one or more discrete sources.

Theorem 1 *For a feasible multicast connection problem with independent or linearly correlated sources, and a network code in which some or all code coefficients are chosen independently and uniformly over all elements of a finite field $\mathbb{F}_q$ (some coefficients can take fixed values as long as these values preserve feasibility[2]), the probability that all the receivers can decode the source processes is at least $(1 - d/q)^{\nu}$ for $q > d$, where d is the number of receivers and ν is the maximum number of links receiving signals with independent randomized coefficients in any set of links constituting a flow solution from all sources to any receiver.* □

Randomized coding can strictly outperform routing in some distributed settings. For illustration, consider the problem of sending two processes from a source node to receiver nodes in random unknown locations on a rectangular grid network, using a distributed transmission scheme that does not involve

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[1]Our model admits parallel links.

[2]i.e. the result holds for networks where not all nodes perform random coding, or where signals add by superposition on some channels

Table 1: Success probabilities of randomized routing scheme RR and randomized coding scheme RC

Receiver position		(3,3)	(10,10)	(2,4)	(8,10)
RR	upper bound	0.688	0.667	0.563	0.667
RC	$\mathbb{F}_{2^4}$ lower bound	0.597	0.098	0.597	0.126
	$\mathbb{F}_{2^6}$ lower bound	0.882	0.567	0.882	0.604
	$\mathbb{F}_{2^8}$ lower bound	0.969	0.868	0.969	0.882

any communication between nodes or routing state. To maximize the probability that any receiver node will receive two distinct messages, the best that a node with two incoming links can do is to try to preserve message diversity by sending one incoming signal on one of its two outgoing links with equal probability, and the other signal on the remaining link. We derive combinatorially an upper bound on the routing success probability for a source-receiver pair in terms of their relative grid locations, which is surpassed by the corresponding lower bound for randomized coding in sufficiently large finite fields. These bounds are tabulated in Table 1.

Our lower bound on coding success probability applies for linearly correlated sources, for which the effect of randomized coding can be viewed as distributed compression occurring within the network rather than at the sources. For a feasible multicast connection problem and a randomized code of sufficient complexity, with high probability the information flowing across any cut will be sufficient to reconstruct the original source processes. In effect, the source information is being compressed to the capacity of any cut that it passes through. This is achieved without the need for any coordination among the source nodes, which is advantageous in distributed environments where such coordination is impossible or expensive.

Finally, we note that this approach achieves robustness in a way quite different from traditional approaches. Traditionally, compression is applied at source nodes so as to minimize required transmission rate and leave spare network capacity, and the addition of new sources may require re-routing of existing connections. Our approach fully utilizes available or allocated network capacity for maximal robustness, while retaining full flexibility to accommodate changes in network topology or addition of new sources.

References

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