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Vector Network Coding

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Ich versichere hiermit wahrheitsgemäß, die Arbeit bis auf die dem Aufgabensteller bereits bekannte Hilfe selbständig angefertigt, alle benutzten Hilfsmittel vollständig und genau angegeben und alles kenntlich gemacht zu haben, was aus Arbeiten anderer unverändert oder mit Abänderung entnommen wurde.

München, 18.07.2019

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Ort, Datum (Ha Nguyen)

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1 Introduction

For the traditional way of network coding, scalar solutions are used. However, it was proved in Professor Antonia's paper that vector solutions outperform scalar solutions in specific cases. It means that we can connect more devices into our network, which is especially meaningful for the IOT devices. The overview of networks where vector solutions outperform the scalar solution is described in Chapter X (Not Yet Inserted). Then we consider a case that is unable to solve by subspace codes or rank-metric codes in Chapter 2. Our computational method shows better results than the scalar solution, 67 results and 166 results respectively in case of t=2 and t=3, where scalar solution has only 42 results and 146 results. The method is described in Chapter 3.

2 Preliminaries

2.1 Definition

Definition 2.1 (Vector Space)**.** A vector space of dimension n over a finite field with q elements is denoted by $\mathbb{F}_q^n$.

Definition 2.2 (Gaussian coefficient)**.** Gaussian coefficient (also known as q-binomial), which counts the number of subspaces of dimension k in a vector space $\mathbb{F}_q^n$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^n - q^i}{q^k - q^i}$$

Definition 2.3 (Grassmannian Code)**.** A Grassmannian code is a set of all subspaces of dimension $k \leq n$ in $\mathbb{F}_q^n$, and is denoted by $\mathcal{G}_q(n,k)$. Due to being the set of all subspaces that have the same dimension k, it is also called a *constant dimension code.* The cardinality of $\mathcal{G}_q(n,k)$ is the Gaussian coefficient (also known as q-binomial), which counts the number of subspaces of dimension k in a vector space $\mathbb{F}_q^n$:

$$|\mathcal{G}_q(n,k)| = \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^n - q^i}{q^k - q^i},$$

where $q^{(n-k)k} \leq \begin{bmatrix} n \\ k \end{bmatrix}_q \leq 4q^{(n-k)k}$.

Definition 2.4 (Projective Space)**.** The *projective space of order n* is a set of all subspaces of $\mathbb{F}_q^n$, and is denoted by $\mathcal{P}_q(n)$, i.e. a union of all dimension $k = 0, \ldots n$ subspaces in $\mathbb{F}_q^n$ or $\mathcal{P}_q(n) = \bigcup_{k=0}^{n} \mathcal{G}_q(n,k)$.

Definition 2.5 (Covering Grassmannian code)**.** An $\alpha - (n,k,\delta)_q^c$ covering Grassmannian code (code in short) $\mathcal{C}$ is a subset of $\mathcal{G}_q(n,k)$ such that each subset of α codewords of $\mathcal{C}$ span a subspace whose dimension is at least $\delta + k$ in $\mathbb{F}_q^n$.

Definition 2.6 (Multiple Grassmannian code)**.** A $t - (n,k,\lambda)_q^m$ multiple Grassmannian code (code in short) $\mathcal{C}$ is a subset of $\mathcal{G}_q(n,k)$ such that each t-subspace of $\mathbb{F}_q^n$ is contained in at most λ codewords of $\mathcal{C}$. Following to [EZ19], m refers to multiplicity.

Definition 2.7. $\mathcal{A}_q(n,k,t;\lambda)$ denotes the maximum size of a $t-(n,k,\lambda)_q^m$ code, where there are no repeated codewords.

Definition 2.8 (Subspace packing)**.** A subspace packing $t-(n,k,\lambda)_q^m$ is a set $\mathcal{S}$ of k-subspaces or k-dimensional subspaces (called *blocks*), such that each t-subspace of $\mathbb{F}_q^n$ is contained in at most λ codewords of $\mathcal{C}$.

2.2 Theorem

Theorem 2.1. *If $n, k, t,$ and λ are positive integers such that $1 \leq t < k < n$ and $1 \leq \lambda \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$, then*

$$\mathcal{A}_q(n,k,t;\lambda) \leq \left\lfloor \lambda \frac{\begin{bmatrix} n \\ t \end{bmatrix}_q}{\begin{bmatrix} k \\ t \end{bmatrix}_q} \right\rfloor$$

Theorem 2.2. *If $n, k, t,$ and λ are positive integers such that $1 \leq t < k < n$ and $1 \leq \lambda \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$, then*

$$\mathcal{A}_q(n,k,t;\lambda) \leq \left\lfloor \frac{q^n-1}{q^k-1} \mathcal{A}_q(n-1,k-1,t-1;\lambda) \right\rfloor$$

3 Generalized combination Network $(\epsilon, l) - \mathcal{N}_{h,r,s}$

3.1 Description

A generalized combination network $(\epsilon, l) - \mathcal{N}_{h,r,s}$ consists of 3 components from top to bottom: “Source” in the first layer, “Node” in the middle layer, and “Receiver” in the third layer. The network has a source with h messages, r nodes, and $\left(\begin{array}{c} r \\ \alpha \end{array} \right)$ receivers, which form a single source multicast network modeled as a finite directed acyclic multigraph. The source connects to each node by l parallel links and each node also connects to a receiver by l parallel links, which are respectively called a node’s incoming and outgoing edges. Each receiver is connected by s links in total, specifically αl links from α nodes and ϵ direct links from the source, i.e. $s = \alpha l + \epsilon$. The combination network in [RA06] is the $(0, 1) - \mathcal{N}_{h,r,s}$ network and the $(1, 1) - \mathcal{N}_{h,r,s}$ network is called One-Direct Link Combination Network. Theorem 1 shows our interest of relations between the parameters h, α, ϵ and l.

Theorem 3.1. *The $(\epsilon, l) - \mathcal{N}_{h,r,s}$ network has a trivial solution if $l + \epsilon \geq h$, and it has no solution if $\alpha l + \epsilon < h$.*

Proof: Following to the network coding max-flow min-cut theorem for multicast networks, the maximum number of messages from the source to each receiver is equal to the smallest min-cut between the source and any receiver. For our considered network, s links have to be deleted to disconnect the source from the receiver, which implies that the min-cut between the source and each receiver is at least s. Hence, $h \leq s \Leftrightarrow h \leq \alpha l + \epsilon$ □
There exist at least $l + \epsilon$ disjoint links connected to each receiver. If $l + \epsilon \geq h$, each receiver can always reconstruct its requested messages on its links. Then we only need to do routing to select paths for the network. □

Remark 3.1. Following to Theorem 3.1, we are interested in networks parameters satisfying this condition: $l + \epsilon + 1 \leq h \leq \alpha l + \epsilon$.

Figure 3.1: The generalized network $(\epsilon, l) - \mathcal{N}_{h,r,s}$

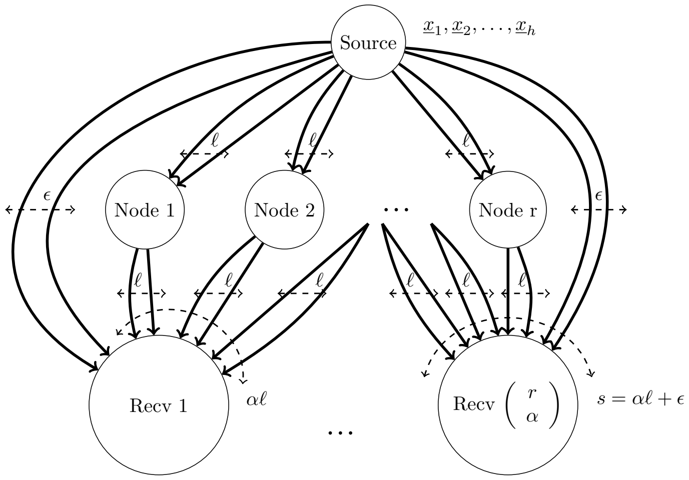


Table 3.1: Parameters of network coding

h	The number of source messages
r	The number of nodes in the middle layer
$\binom{r}{\alpha}$	The number of receivers
ℓ	The source connects to each node by ℓ parallel links, and each node also connects to one receiver by ℓ parallel links
α	A receiver is connected by any α nodes in the middle layer
ϵ	The source additionally connects to each receiver by ϵ direct parallel links
s	Each receiver is connected by s links in total, with $s = \alpha\ell + \epsilon$.

3.2 Network Coding For This Network Family

3.2.1 Scalar network coding

A message is equivalent to a symbol over $\mathbb{F}_{q_s}$. As a network of the multicast model, all receivers request the same packet of h symbols at the same time [HT13]. A packet is a 1-dimentional subspace of $\mathbb{F}_{q_s}^h$; hence, each receiver must obtain a subspace of $\mathbb{F}_{q_s}^h$, whose dimension is at least h, to be able to reconstruct the packet. Through ϵ direct links connected from the source to a receiver, the source can provide any required ϵ 1-dimensional subspaces of $\mathbb{F}_{q_s}^h$ for the corresponding receiver. Each receiver can accordingly reconstruct the packet if and only if the linear span of α l-dimensional subspaces of $\mathbb{F}_{q_s}^h$ from the nodes is at least of dimension $h-\epsilon$. When this necessary condition is satisfied, the network is said to have a *solution* or to be *solvable.*

Theorem 3.2. *The* $(0,1)-\mathcal{N}_{h,r,s}$ *network has a solution if and only if there exists an* $(r, |\mathbb{F}_{q_s}|\, h, r-\alpha+1)$ $|\mathbb{F}_{q_s}|$*-ary error correcting code. [RA06]*

Theorem 3.3. *The* $(\epsilon,l)-\mathcal{N}_{h,r,s=\alpha l+\epsilon}$*network is solvable over* $\mathbb{F}_q$ *if and only if there exists an* $\alpha-(h,l,h-l-\epsilon)_q^c$ *code with* r *codewords. [EZ19]*

3.2.2 Vector network coding

The messages are vectors of length t over $\mathbb{F}_q$. Hence, a vector solution is over field size q and dimension t. Such a vector solution has the same alphabet size as a scalar solution of field size q^t, and we denote $q_v = q^t$. A mapping from the scalar solution of field size q^t to a equivalent vector solution is represented in Example 3.1. Similarly with the scalar *linear* coding solution, each receiver can reconstruct its requested packet if and only if any α (lt)-dimensional subspaces span a subspace of dimension at least $(h-\epsilon)\,t$.

Theorem 3.4. *A vector solution for the* $(\epsilon,l)-\mathcal{N}_{h,r,s}$ *network exists if and only if there exists* $\mathcal{G}_q\,(ht,lt)$ *such that any* α *subspaces of the set span a subspace of dimension at least* $(h-\epsilon)\,t$*. [EZ19]*

Theorem 3.5. *The* $(\epsilon,l)-\mathcal{N}_{h,r,s=\alpha l+\epsilon}$*network is solvable with vectors of length* t *over* $\mathbb{F}_q$ *if and only if there exists an* $\alpha-(ht,lt,ht-lt-\epsilon t)_q^c$ *code with* r *codewords. [EZ19]*

Corollary 3.1. *The* $\alpha-(n=ht, n-k=ht-lt, \lambda=ht-lt-\epsilon t)_q^m$ *code formed from the dual subspaces of the* $\alpha-(n=ht, k=lt, \lambda=ht-lt-\epsilon t)_q^c$ *code yields the upper bound of* $\mathcal{A}_q\,(n=ht, n-k=ht-lt, \alpha;\lambda)$ *as maximum number of nodes for a vector network coding of the* $(\epsilon,l)-\mathcal{N}_{h,r,s}$ *network.*

3.2.3 Network as a matrix channel

To formulate this description, the source has a set of disjoint messages referred to packets which are either symbols from $\mathbb{F}_{q_s}$ (scalar coding) or vectors of length t over $\mathbb{F}_q$ (vector coding). Each link in the network carries functions of the packets, and a *network code* is a set of these functions. The network code is called *linear* if all the functions are linear and nonlinear otherwise. Each receiver $R_j, j\in\{1,\ldots,N\}$ requests a subset of the source's length-h messages, and this subset is called a *packet.* Through all the functions on the links from the source to each receiver, the receiver obtains several linear combinations of the h messages to form a linear system of equations for its requested packets. The coefficients of a linear combination are called *global coding vectors.* The linear equation system that any receiver R_j has to solve is as following:

$$\begin{array}{c|c} Scalar & Vector \\ \underbrace{\begin{bmatrix} y_{j_1} \\ \vdots \\ y_{j_s} \end{bmatrix}}_{\mathbb{F}_{q_s}^{s}} = \underbrace{\boldsymbol{A}_j}_{\mathbb{F}_{q_s}^{s \times h}} \cdot \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_h \end{bmatrix}}_{\mathbb{F}_{q_s}^{h}} & \underbrace{\begin{bmatrix} \underline{y}_{j_1} \\ \vdots \\ \underline{y}_{j_s} \end{bmatrix}}_{\mathbb{F}_{q}^{st}} = \underbrace{\boldsymbol{A}_j}_{\mathbb{F}_{q}^{st \times th}} \cdot \underbrace{\begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_h \end{bmatrix}}_{\mathbb{F}_{q}^{th}} \end{array} \tag{3.1}$$

The transfer matrix $\boldsymbol{A}_j$ contains the links' *global coding vectors*, which are combined by the coefficients of linear combinations on αl links from α nodes and ϵ direct-links to the corresponding receiver R_j:

$$\begin{array}{c|c} Scalar & Vector \\ \boldsymbol{A}_j = \begin{bmatrix} \underline{a}_{j_1} \\ \vdots \\ \underline{a}_{j_{\alpha l}} \\ \vdots \\ \underline{a}_{j_{\alpha l+\epsilon}} \end{bmatrix} & \boldsymbol{A}_j = \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \vdots \\ \boldsymbol{A}_{j_{\alpha l}} \\ \vdots \\ \boldsymbol{A}_{j_{\alpha l+\epsilon}} \end{bmatrix} \end{array}$$

In general, the network is represented as a matrix channel for both scalar and vector coding:

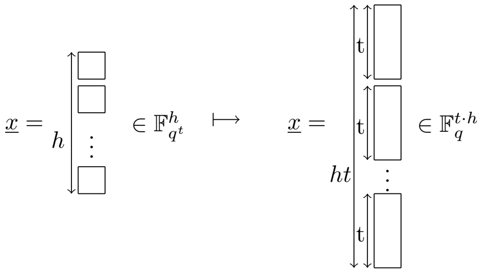
Definition 3.1. Network As Matrix Channel

The channel output can be written as: $\boldsymbol{Y}_j = \boldsymbol{A}_j \cdot \boldsymbol{X}$

Because we reconstruct $\boldsymbol{X}$ with knowing $\boldsymbol{A}_j$, i.e. the network structure is known, our network is coherent. A network is *sovable* or a network code is a *solution*, if each receiver can reconstruct its requested messages or solve the system with a unique solution for scalars $x_1, \ldots, x_h$, or vectors $\underline{x}_1, \ldots, \underline{x}_h$. Therefore, we want to find global coding vectors such that the matrix $\boldsymbol{A}_j$ has full-rank for every $j = 1, \ldots, N$, and such that q_s or q^t is minimized. In Example 3.1, we provide a vector solution of field size q and dimension t, which has the same alphabet size as a scalar solution of field size q^t

Example 3.1. Given $h = 3, q = 2, t = 2$, we consider the extension field $\mathbb{F}_{q^t=2^2}$. The example shows how mapping messages from scalar coding to vector coding.

Figure 3.2: The mapping of scalar solution over $\mathbb{F}_{q_s=q^t}$ to the equivalen vector solution



Example 3.2. We use the table of the extension field $\mathbb{F}_{2^2}$ with the primitive polynomial $f(x) = x^2 + x + 1$:

power of α	polynomial	binary vector
-	0	00
α^0	1	01
α^1	α	10
α^2	$\alpha + 1$	11

For scalar coding, the messages are $x_1, \ldots, x_{h=3} \in \mathbb{F}_{2^2}$, and for vector coding the messages are $\underline{x}_1, \ldots, \underline{x}_{h=3} \in \mathbb{F}_2^2$. From the polynomial column, let's choose arbitrarily a scalar vector $\underline{x}_{scalar} = (x_1, x_2, x_3) = (1, \alpha, \alpha + 1)$. Then, we map it to $\underline{x}_{vector} = (\underline{x}_1, \underline{x}_2, \underline{x}_3)$ by using the binary vector column as following:

$$\begin{bmatrix} x_1 = 1 \\ x_2 = \alpha \\ x_3 = \alpha + 1 \end{bmatrix} \mapsto \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{bmatrix},$$

where we use the following rule for mapping x_i individually: $a_0 \cdot \alpha^0 + a_1 \cdot \alpha^1 + \ldots + a_{t-1} \cdot \alpha^{t-1} \mapsto \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{t-1} \end{pmatrix}$.

To summarize the notations of both scalar and vector coding, we represent them in the Table 3.2:

Table 3.2: Notations of network coding

	Scalar Coding	Vector coding
Source Messages/Packets	$x_1, \dots, x_h \in \mathbb{F}_{q_s}$ $\underline{x} \in \mathbb{F}_{q_s}^h$	$\underline{x}_1, \dots, \underline{x}_h \in \mathbb{F}_q^t$ $\underline{x} \in \mathbb{F}_q^{th}$
Global Coding Vectors Of Receiver R_j	$\underline{a}_{j_1}, \dots, \underline{a}_{j_s} \in \mathbb{F}_{q_s}^h$	$\boldsymbol{A}_{j_1}, \dots, \boldsymbol{A}_{j_s} \in \mathbb{F}_q^{t \times th}$
Transfer Matrix Of Receiver R_j	$\boldsymbol{A}_j \in \mathbb{F}_{q^t}^{s \times h}$	$\boldsymbol{A}_j \in \mathbb{F}_q^{st \times th}$
Packets On Receiver R_j	$y_{j_1}, \dots, y_{j_s} \in \mathbb{F}_{q_s}$ $\underline{y} \in \mathbb{F}_{q_s}^s$	$\underline{y}_{j_1}, \dots, \underline{y}_{j_s} \in \mathbb{F}_q^t$ $\boldsymbol{Y}_j \in \mathbb{F}_q^{st}$
Number of nodes	r_{scalar}	r_{vector}

Remark 3.2. By using the vector coding, the upper bound number of solutions increases from q^{tkh} to q^{t^2kh}. Therefore, vector network coding offers more freedom in choosing the coding coefficients than does scalar linear coding for equivalent alphabet sizes, and a smaller alphabet size might be achievable [**?**]. By this advantage, we can have higher amount of receivers, i.e. higher amount of nodes, in vector network coding.

3.2.4 Comparison between scalar and vector solutions by the gap size

The *gap* represents the difference between the smallest field (alphabet) size for which a scalar linear solution exists and the smallest alphatbet size for which we can construct a vector solution. In this study, we define a solvable vector network coding over the field size $\mathbb{F}_q^t$, and we conjecture the minimum amount of nodes such vector solution can achieve, i.e. $r_{vector} \geq f_1(q)$, with $f_1 : \mathbb{Z} \mapsto \mathbb{Z}$. Meanwhile, we has a scalar solution for the same network existing if and only if: $r_{scalar} \leq f_2(q_s)$, with $f_2 : \mathbb{Z} \mapsto \mathbb{Z}$. To find the field size q_s required for a scalar solution to reach the minimum achievable vector solution's nodes in this setting, we consider $r_{max,scalar} = f_2(q_s) = f_1(q) = r_{min,vector}$. Finally, we calculate the gap by $g = q_s - q_v = q_s - q^t$. Throughout this study, we show that vectors solutions significantly reduce the required alphabet size by this gap.

3.3 Special cases of generalized combination network

3.3.1 The $(l-1)$-Direct Links and l-Parrallel Links $\mathcal{N}_{h=2l,r,s=3l-1}$

This subfamily contains the largest number of direct links from the source to the receivers. For $l \geq 2$, this network $(\epsilon = l-1, l) - \mathcal{N}_{h=2l,r,s=3l-1}$ yields the gap $q^{(l-1)t^2/l + \mathcal{O}(t)}$

between vector solutions and optimal scalar solutions. The vector solution is based on an $\mathcal{MRD}\left[lt \times lt, t\right]_q$ code. Further, the gap tends to $q^{t^2/2+\mathcal{O}(t)}$ for large l.

Lemma 3.1. *There is a scalar linear solution of field size q_s for the $(\epsilon = l-1, l) - \mathcal{N}_{h=2l,r,s=3l-1}$ network, where $l \geq 2$, if and only if $r \leq \begin{bmatrix} 2l \\ l \end{bmatrix}_{q_s}$.*

3.3.2 The 1-Direct Link and l-Parrallel Links $\mathcal{N}_{h=2l,r,s=2l+1}$

This is the smallest direct-link subfamily has an vector solution outperforming the optimal scalar solution, i.e. an vector solution outperforming the optimal scalar has not yet been found for the network $(0, l > 1) - \mathcal{N}_{h,r,s}$. Similar to the previous subfamily $(\epsilon = l-1, l) - \mathcal{N}_{h=2l,r,s=3l-1}$, when $l \geq 2$ or $h \geq 4$, this network yields the largest gap $q^{t^2/2+\mathcal{O}(t)}$ in the alphabet size by using the same approach with an $\mathcal{MRD}\left[lt \times lt, (l-1)t\right]_q$ code.

3.3.3 The ϵ-Direct Links $\mathcal{N}_{h,r,s}$

This subfamily is denoted as $(\epsilon \geq 1, l = 1) - \mathcal{N}_{h,r,s}$ and is the most focus topic on this thesis, because it motivates some interesting questions on a classic coding problem and on a new type of subspace code problem. In the chapter 3, we show our largest code set with low number of subspace codes for the network $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$.

3.3.4 The $(\epsilon = 0, l = 1) - \mathcal{N}_{h,r,s}$ Combination Network

Since the scalar solution for the combination network uses an MDS code, a vector solution based on subspace codes must go beyond the MDS bound, i.e. Singleton bound $d \leq n - k + 1$, to outperform the scalar one. In paper [EW18], it is proved that vector solutions based on subspace codes cannot outperform optimal scalar linear solutions for $h = 2$, and they conjecture it for all h. Unfortunately, a vector solution based on an $\mathcal{MRD}\left[t \times t, t\right]_q$ code is also proved that it cannot outperform the optimal scalar linear solution.

3.3.5 The 2 networks yields gap $q^{(h-3)t^2/(h-1)+\mathcal{O}(t)}$

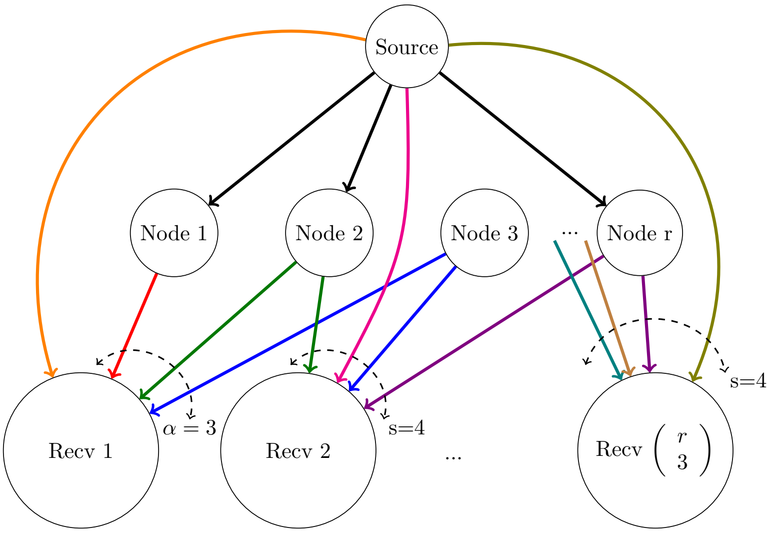
For $h = 2l - 1$: $(\epsilon = l-2, l) - \mathcal{N}_{h=2l-1,r,s=3l-2}$
For $h = 2l + 1$: $(\epsilon = l-1, l) - \mathcal{N}_{h=2l+1,r,s=3l-1}$

4 Combinatorial Results

In previous studies [EW18], there was no general vector solution found for multicast networks with $h = 3$ messages. Hence, we start with a probabilistic argument to prove that there exists a vector solution outperforming the optimal linear solution for the $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$ network. Then we generalize the proof to the $(\epsilon = 1, l = 1) - \mathcal{N}_{h,r,s}$ network.

4.1 $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$ Network

Figure 4.1: The $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$ network



Conjecture 4.1. *For the network* $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$, *the probability that vector solution does not exist is less than or equal to* $\sum_{i=0}^{2t-1} \frac{\prod_{j=0}^{i-1} \frac{\left(q^{3t}-q^j\right)^2}{q^i-q^j}}{q^{9t^2}}$.

Proof: Following to Equation 3.1, each receiver must solve a linear equation system of 3 variables with 4 equations to recover $h = 3$ messages as below:

$$\begin{bmatrix} \underline{y}_{j_1} \\ \underline{y}_{j_2} \\ \underline{y}_{j_3} \\ \underline{y}_{j_4} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \\ \boldsymbol{A}_{j_4} \end{bmatrix} \cdot \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \end{bmatrix},$$

with $\underline{x}_i, \underline{y}_{j_v} \in \mathbb{F}_q^t$, $\boldsymbol{A}_{j_v} \in \mathbb{F}_q^{t\times 3t}$ for $v = 1, \ldots, 4$, and $\boldsymbol{A}_{j_1}, \ldots, \boldsymbol{A}_{j_3}$ must be distinct.
The network is solvable, if $\boldsymbol{A}_j$ has full-rank for every $j = 1, \ldots, N$, i.e. $\boldsymbol{A}_{j_v}$ must satisfy:

$$rk \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \\ \boldsymbol{A}_{j_4} \end{bmatrix} \geq 3t$$

In order to satisfy $rk\,[\boldsymbol{A}_j] \geq 3t$, we can easily choose suitable values for the coefficient $\boldsymbol{A}_j^{(4)}$ on the direct link from the source to R_j. However, the coefficients on the links from nodes to receivers are matters. Therefore, we focus on the following requirement:

$$rk \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \end{bmatrix} \geq 2t \tag{4.1}$$

Therefore, $rk \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \end{bmatrix} \geq 2t$ implies $rk\,[\boldsymbol{A}_j] \geq 3t$, or to satisfy $rk\,[\boldsymbol{A}_j] \geq 3t$, we need $rk \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \end{bmatrix} \geq 2t$. In Section 3.1, we have $N = \begin{pmatrix} r \\ \alpha \end{pmatrix}$ and $\alpha = 3$ as a fixed values. Therefore, maximizing the number of receivers is equivalent to maximize r under the constraint 4.1. $\boldsymbol{A}_{j_1}, \boldsymbol{A}_{j_2}, \boldsymbol{A}_{j_3}$ are matrices formed on any 3 of r links from nodes to receivers, i.e. these 3 matrices are randomly chosen from a set of r matrices. We formalize the problem by an approach with Lovász local lemma [Sch].

Theorem 4.1 (Symmetric Lovász local lemma (LLL)). *[SCV13]*
A set of events $\mathcal{E}_i$, with $i = 1, \ldots, n$, such that each event occurs with probability at most p. If each event is independent of all others except for at most d of them and $4dp \leq 1$, then:

$$Pr\left[\bigcap_{i=1}^{n} \overline{\mathcal{E}}_i\right] > 0$$

For our problem, let $\mathcal{E}_i$ denote the following event:

$$Pr\left[\mathcal{E}_i\right] = Pr\left[rk\begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \end{bmatrix} < 2t\right]$$

Because the requirement on rank is opposite, we consider the complement event T:

$$rk\begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \end{bmatrix} \geq 2t, \forall 1 \leq j_1 < j_2 < j_3 \leq r$$

By the intersection rule, we have:

$$T = \bigcap_{\mathcal{E}_i \in \mathcal{E}} \overline{\mathcal{E}}_i$$

The probability of event T indicates a measure quantifying the likelihood that we will be able to construct $rk\left[\boldsymbol{A}_j\right] \geq 3t$ with j_1, j_2, j_3 in the integer numbers between 1 and r, including both. We need to maximize r, but the rank requirement is still satisfied, i.e. the probabilty must be higher than 0:

$$\begin{aligned}
& Pr\left[rk\begin{bmatrix} \boldsymbol{A}_{j_1} \\ \boldsymbol{A}_{j_2} \\ \boldsymbol{A}_{j_3} \end{bmatrix} \geq 2t, \forall 1 \leq j_1 < j_2 < j_3 \leq r\right] && > 0 \\
\Leftrightarrow \quad & Pr\left[T\right] && > 0 \\
\Leftrightarrow \quad & Pr\left[\bigcap_{\mathcal{E}_i \in \mathcal{E}} \overline{\mathcal{E}}_i\right] && > 0
\end{aligned}$$

Following to LLL, each event occurst with probability at most p:

$$Pr\left[\mathcal{E}_i\right] = Pr\left[rk\begin{bmatrix} \boldsymbol{A}_{i_1} \\ \boldsymbol{A}_{i_2} \\ \boldsymbol{A}_{i_3} \end{bmatrix} < 2t\right] \leq p$$

Regarding to the left-hand side:

$$\begin{aligned}
Pr\left[rk\begin{bmatrix}\boldsymbol{A}_{i_1}\\ \boldsymbol{A}_{i_2}\\ \boldsymbol{A}_{i_3}\end{bmatrix} < 2t\right] &= \sum_{i=0}^{2t-1} Pr\left[rk\begin{bmatrix}\boldsymbol{A}_{i_1}\\ \boldsymbol{A}_{i_2}\\ \boldsymbol{A}_{i_3}\end{bmatrix} = i\right]\\
&\overset{1}{=} \sum_{i=0}^{2t-1} \frac{N_{t,m,n}}{q^{m\cdot n}}\\
&= \sum_{i=0}^{2t-1} \frac{\prod_{j=0}^{i-1} \frac{\left(q^m-q^j\right)\left(q^n-q^j\right)}{q^i-q^j}}{q^{m\cdot n}}\\
&\overset{2}{=} \sum_{i=0}^{2t-1} \frac{\prod_{j=0}^{i-1} \frac{\left(q^{3t}-q^j\right)^2}{q^i-q^j}}{q^{9t^2}} \square \qquad (4.2)
\end{aligned}$$

By varying t in Equation (4.2), we have the following table:

Table 4.1: r over variations of t

t	Scalar Solution	Vector Solution
1	$r_{scalar} \leq 14$	$r_{vector} \geq 3$
2	$r_{scalar} \leq 42$	$r_{vector} \geq 7$ (67^*, 89^{**})
3	$r_{scalar} \leq 146$	$r_{vector} \geq 62$ (166^*)
4	$r_{scalar} \leq 546$	$r_{vector} \geq 1317$
5	$r_{scalar} \leq 2114$	$r_{vector} \geq 58472$
6	$r_{scalar} \leq 8322$	$r_{vector} > 10^6$

*, **: computational results in construction 1 and construction 2 respectively

In the table (4.1), the vector solution outperforms the scalar solution when $t \geq 4$ for the network $(\epsilon = 1, l = 1) - N_{h=3,r,s=4}$. This is sufficient, later on we show computational results which vector solutions outperform scalar solutions in case of $t = 2$ and $t = 3$.

Conjecture 4.2. *For the network* $(\epsilon = 1, l = 1)-\mathcal{N}_{h=3,r,s=4}$*, the achieved gap is* $q^{t^2/4+\mathcal{O}(t)}$.

Proof: We proceed the other constraint of LLL: $4dp \leq 1$. Regarding to d, we have:

$$\begin{aligned}
d(r) &\leq 3\cdot\begin{pmatrix} r-1\\ 2\end{pmatrix} = 3\cdot\frac{(r-1)(r-2)}{2} = \frac{3}{2}\left(r^2-3r+2\right) = d_1(r)\\
&\leq \frac{3}{2}r^2 = d_2(r)
\end{aligned}$$

Hence, we have below implications:

$$\begin{aligned} & 4 \cdot p \cdot d_2(r) && \leq 1 \\ \overset{1}{\Rightarrow} \quad & 4 \cdot p \cdot d_1(r) && \leq 1 \\ \overset{2}{\Rightarrow} \quad & 4 \cdot p \cdot d(r) && \leq 1 \end{aligned}$$

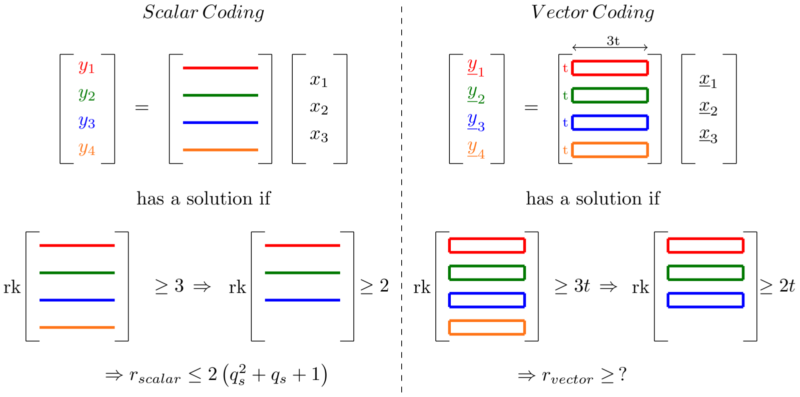
1: $d_1(r) \leq d_2(r)$

2: $d(r) \leq d_1(r)$

In consideration of $d_2(r)$:

$$4 \cdot p \cdot d_2(r) \leq 1 \Rightarrow 4 \cdot p \cdot \frac{3}{2} r^2 \leq 1 \Rightarrow r \leq \sqrt{\frac{1}{6p}} = r_{min,vector}$$

Figure 4.2: The vector network coding of $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$ represents as a matrix problem



Similarly with above, d and r are propotional, so minimizing r is equivelent to maximizing p. The purpose is to achieve a strict lower bound proving vector solutions always outperform scalar solutions in a specific range of t, i.e., r_{max}asymptotes to a value higher than r_{scalar}. Now, we begin to maximize p. We consider the nominator of Equation (4.2):

$$\prod_{j=0}^{i-1} \frac{\left(q^{3t} - q^j\right)^2}{q^i - q^j} = \frac{p_N^{(i)}(q)}{p_D^{(i)}(q)} = p^{(i)}(q)$$

Due to i-times product and large t:

$$\left.\begin{aligned} deg\left(p_N^{(i)}(q)\right) &= q^{i6t} \\ deg\left(p_D^{(i)}(q)\right) &= q^{i^2} \end{aligned}\right\} \Rightarrow p^{(i)}(q) \approx q^{i6t-i^2}$$

Then we have:

$$\sum_{i=1}^{2t-1}\prod_{j=0}^{i-1}\frac{\left(q^{3t}-q^j\right)^2}{q^i-q^j} = \sum_{i=1}^{2t-1} p^{(i)}(q) \approx \sum_{i=1}^{2t-1} q^{i6t-i^2}$$

To maximize the sum, we set derivation of to 0 and find its root:

$$\begin{aligned} & \left(i6t-i^2\right)' &= 0 \\ \Leftrightarrow \quad & 6t-2i &= 0 \\ \Leftrightarrow \quad & i &= 3t \end{aligned}$$

However, the upper limit of sum is $(2t-1)$, which is less than $3t$.

$$\Rightarrow max\left\{q^{i6t-i^2} : i = 1, 2 \ldots, 2t-1\right\} = q^{i6t-i^2}\Big|_{i=2t-1} = q^{8t^2-2t-1}$$

Hence, by using the exact bound Θ, we have:

$$\sum_{i=1}^{2t-1} p^{(i)}(q) \in \Theta\left(max\left\{q^{i6t-i^2} : i = 1, 2 \ldots, 2t-1\right\}\right) = \Theta\left(q^{8t^2-2t-1}\right)$$

$$\Rightarrow \frac{1+\sum_{i=1}^{2t-1} p^{(i)}(q)}{q^{9t^2}} \approx \frac{\sum_{i=1}^{2t-1} p^{(i)}(q)}{q^{9t^2}} \in \Theta\left(q^{-t^2-2t-1}\right)$$

Finally, we have:

$$r_{min,vector} \in \Omega\left(\sqrt{\frac{1}{6p}}\right) = \Omega\left(\sqrt{\frac{1}{6q^{-t^2-2t-1}}}\right) = \Omega\left(q^{t^2/2+\mathcal{O}(t)}\right)$$

Further we have: $r_{max,scalar} \in \mathcal{O}\left(q_s^2\right)$ [EW18], specifically,

$$r_{scalar} \leq 2\begin{bmatrix} 3 \\ 1 \end{bmatrix}_{q_s} = 2\left(q_s^2 + q_s + 1\right) \tag{4.3}$$

.

We have the gap size:

$$\begin{array}{rcl} & r_{max,scalar} & = r_{min,vector} \\ \Leftrightarrow & q_s^2 & = q^{t^2/2+\mathcal{O}(t)} \\ \Leftrightarrow & q_s & = q^{t^2/4+\mathcal{O}(t)} \\ \Rightarrow & g & = q_s - q_v = q^{t^2/4+\mathcal{O}(t)} \square \end{array}$$

4.2 $(\epsilon = 1, l = 1) - \mathcal{N}_{h,r,s}$ Network

4.2.1 Find the lower bound of $r_{max,vector}$

Following to Theorem (3.1), we are interested in the following range: $l+\epsilon+1 \leq h \leq \alpha l+\epsilon$. As previous, $\boldsymbol{A}_{j_1}, \ldots, \boldsymbol{A}_{j_{h-\epsilon}} \in \mathbb{F}_q^{t\times ht}$ and we need to satisfy the following:

$$rk \begin{bmatrix} \boldsymbol{A}_{j_1} \\ \vdots \\ \boldsymbol{A}_{j_{h-\epsilon}} \end{bmatrix} \geq ht - t \Leftrightarrow rk\left[\boldsymbol{A}_j\right] \geq (h-1)t$$

We can formulate it by the following coding problem in Grassmannian:

Find the largest set of subspaces from $\mathcal{G}_q(ht, t)$ such that any α subspaces of the set span a subspace of dimension at least $(h-1)\,t$.

Similar to $(\epsilon = 1, l = 1) - N_{3,r,4}$, we consider p to proceed LLL:

$$Pr\left[rk\left[\boldsymbol{A}\right] < (h-1)t\right] \leq p$$

Regarding to the left-hand side:

$$\begin{aligned} Pr\left[rk\left[\boldsymbol{A}\right] < (h-1)t\right] &= \sum_{i=0}^{(h-1)t-1} Pr\left[rk\left[\boldsymbol{A}\right] = i\right] \\ &\overset{1}{=} \sum_{i=0}^{(h-1)t-1} \frac{N_{i,\alpha t,ht}}{q^{(\alpha t)(ht)}} \\ &= \frac{1}{q^{(\alpha h)t^2}} \cdot \sum_{i=0}^{(h-1)t-1} \prod_{j=0}^{i-1} \frac{\left(q^{\alpha t} - q^j\right)\left(q^{ht} - q^j\right)}{q^i - q^j} \end{aligned} \tag{4.4}$$

Firstly, we consider the product:

$$\prod_{j=0}^{i-1} \frac{\left(q^{\alpha t} - q^{j}\right)\left(q^{ht} - q^{j}\right)}{q^{i} - q^{j}} = \frac{p_N^{(i)}(q)}{p_D^{(i)}(q)} = p^{(i)}(q)$$

For $t \to \infty$:

$$\left.\begin{aligned} deg\left(p_N^{(i)}(q)\right) &= q^{i(\alpha t + ht)} \\ deg\left(p_D^{(i)}(q)\right) &= q^{i^2} \end{aligned}\right\} \Rightarrow p^{(i)}(q) \approx q^{i(\alpha t+ht)-i^2}$$

Now, we exaluate $f(i) = i(\alpha t + ht) - i^2$ to find its maximum point:
$\dot{f}(i^*) = 0 \Leftrightarrow (\alpha t + ht) - 2i^* = 0 \Leftrightarrow i^* = \frac{\alpha t + ht}{2}$
We then check whether this point within the range $i = 0, \ldots, (h-1)t - 1$ as following:
$0 \leq \frac{\alpha t+ht}{2} \leq (h-1)t - 1$
With regards to the lower bound: $0 \leq \frac{\alpha t+ht}{2} \Leftrightarrow t \geq \frac{2}{\alpha+h}$, which is always true due to the given $t \geq 2$ and $\alpha, h \geq 3$.
Regarding to the upper bound: $\frac{\alpha t+ht}{2} \leq (h-1)t - 1 \Leftrightarrow t \leq \frac{-2}{\alpha+2-h}$ with $\alpha + 2 > h$ due to the given $\alpha l + \epsilon = \alpha + 1 \geq h$. This cannot happen because of $t \geq 2$, i.e. this maximum point is over then upper-range limit.

$$\begin{aligned} \Rightarrow \quad max\left\{q^{i(\alpha t+ht)-i^2} : i = 1, \ldots, (h-1)t - 1\right\} &= q^{i(\alpha t+ht)-i^2}\Big|_{i=(h-1)t-1} \\ &= q^{[(h-1)(\alpha+1)]t^2-(\alpha-h+2)t-1} \end{aligned}$$

Secondly, we apply the maximum value with the sum, we have:

$$\sum_{i=0}^{(h-1)t-1} p^{(i)}(q) \in \Theta\left(q^{[(h-1)(\alpha+1)]t^2+\mathcal{O}(t)}\right)$$

Thirdly, we consider the 3rd requirement of LLL to figure out a lower bound on r_{max}:
$d \leq \alpha \begin{pmatrix} r-1 \\ \alpha-1 \end{pmatrix} = \alpha\frac{(r-1)\ldots(r-\alpha+1)}{(\alpha-1)!} \leq \frac{\alpha}{(\alpha-1)!}r^{\alpha-1} = d_2$
We need $4dp \leq 1$, which is satisfied if $4d_2p \leq 1$. Therefore, we consider:
$\frac{\alpha}{(\alpha-1)!}r^{\alpha-1} \leq \frac{1}{4p} \Leftrightarrow r \leq \left(\frac{(\alpha-1)!}{4\alpha} \cdot \frac{1}{p}\right)^{\frac{1}{\alpha-1}}$
Finally, we have from above:

$$\begin{aligned} p &\in \Theta\left(\frac{q^{[(h-1)(\alpha+1)]t^2+\mathcal{O}(t)}}{q^{(\alpha h)t^2}}\right) \\ \Rightarrow r_{min,vector} &\in \Omega\left(q^{\frac{h-\alpha-1}{1-\alpha}t^2+\mathcal{O}(t)}\right) \end{aligned}$$

4.2.2 Find the upper bound of $r_{max,scalar}$

Find $(\alpha + 1)$ received vectors that span a subspace of dimension h. This implies that the α links from the middle layer carry α vectors which span a subspace of $\mathbb{F}_{q_s}^h$ whose dimension is at least $(h-1)$, with $q_s = q^t$.

For $3 \leq \alpha < h$: all α links must be distinct $\Rightarrow r \leq \begin{bmatrix} \alpha \\ 1 \end{bmatrix}_{q_s} \Rightarrow r \leq$

For $\alpha \geq h \geq 3$: to achieve $(h-1)$-subspaces of $\mathbb{F}_{q_s}^h$, no α links will contain a vector which is contained in the same $(h-2)$-subspace.

Hence,

$$r_{max,scalar} \leq (\alpha - 1) \begin{bmatrix} \alpha \\ h-2 \end{bmatrix}_{q_s} \Rightarrow r_{max,scalar} \in \mathcal{O}\left(q_s^{(\alpha-h+2)(h-2)t^2}\right)$$

4.2.3 Calculate the gap of size

$$\begin{aligned}
& r_{max,scalar} && = r_{min,vector} \\
\Leftrightarrow \quad & q_s^{(\alpha-h+2)(h-2)t^2} && = q^{\frac{h-\alpha-1}{1-\alpha}t^2+\mathcal{O}(t)} \\
\Leftrightarrow \quad & q_s && = q^{\frac{\alpha-h+1}{(\alpha-1)(\alpha-h+2)(h-2)}t^2+\mathcal{O}(t)} \\
\Rightarrow \quad & g && = q_s - q_v = q^{\frac{\alpha-h+1}{(\alpha-1)(\alpha-h+2)(h-2)}t^2+\mathcal{O}(t)} \square
\end{aligned}$$

5 Computational Results

In Table 4.1, our vector solutions are computed by Algorithm 1 for the $(\epsilon = 1, l = 1) - N_{3,r,4}$ network regarding to $t = 2$ and $t = 3$. Both construction 1 and 2 provide better results than scalar solutions.
Construction 1: $\boldsymbol{I}_t \mid \boldsymbol{T}$, with $\boldsymbol{T} \in \mathbb{F}_q^{t \times t(h-1)}$
Construction 2: $\boldsymbol{T} \in MatrixSpaceUrs\,(t, 3t)$
Regarding to $t = 2$, for a scalar network coding solution we need a $3 - (3, 1, 1)_4^c$ code $(q_v = 2^2)$ by Theorem 3.3. The largest such code consists of the 21 1-dimensionall subspaces of $\mathbb{F}_4^3$, each one is contained twice in the code. Therefore, the number of nodes can be at most 42 for a scalar linear coding solution, while for vector network coding 89 nodes can be used, i..e. $\mathcal{A}_{q=2}\,(n = 6, k = 4, t = 3; \lambda = 2) \geq 89$ following to Corollary 3.1. This is a new lower bound for $\mathcal{A}_2\,(6, 4, 3; 2)$ compared to a code with 51 codewords presented in [EW18]. The smallest alphabet size for a scalar solution with 89 nodes exists is $q_s = 8$. By Equation 4.3, there are 73 1-dimensional subspaces of $\mathbb{F}_8^3$, and each one can be used twice in the code; therefore, we have in total 146 possible codesword, but only 89 codewords are required.

Definition 5.1 (G3)**.** Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{F}_q^{n \times m}$. Then a set $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$ forms a subset of $g3$ if

$$rk \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \\ \boldsymbol{C} \end{bmatrix} \geq 2n$$

In other words, all $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}$ span a subspace of $\mathbb{F}_q^{2n}$ whose dimension is at least $2n$. We denote $g3_i$ as a subset of $g3$:

$$g3 = \{\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}_i\} = \{g3_i\}\,, i = 0, 1, 2, ..., |g3| - 1$$

with $g3_i = \{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\}_i$

Definition 5.2 (Relative)**.** Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{F}_q^{n \times m}$. Then $\boldsymbol{C}$ is called a relative of a tuple $(\boldsymbol{A}, \boldsymbol{B})$ if $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\} \in g3$ and denoted as following:

$$rel\,[(\boldsymbol{A},\boldsymbol{B})] = \boldsymbol{C}$$

Definition 5.3 (Sub-relative)**.** Let $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D} \in \mathbb{F}_q^{n\times m}$. Then $\boldsymbol{D}$ is called a sub-relative of a tuple $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}) \in g3$ if:

$$\begin{cases} \{\boldsymbol{A},\boldsymbol{B},\boldsymbol{D}\} \in g3 \\ \{\boldsymbol{A},\boldsymbol{C},\boldsymbol{D}\} \in g3 \\ \{\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}\} \in g3 \end{cases}$$

It is denoted as:

$$subrel\,[(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})] = \boldsymbol{D}$$

This definition is reused for a set of 5 or more matrices.

Definition 5.4 (MatrixSpace)**.** $MatrixSpace(n,m) = \left\{\boldsymbol{A} : \boldsymbol{A} \in \mathbb{F}_q^{n\times m}\right\}$

Definition 5.5 (MatrixSpace with unique row space)**.** $MatrixSpaceUrs(n,m)$ is a subspace of $\mathbb{F}_q^{n\times m}$, where any $\boldsymbol{A},\boldsymbol{B} \in \mathbb{F}_q^{n\times m}$ have their row spaces such that:

$$\mathcal{R}_q\left(\boldsymbol{A}\right) \neq \mathcal{R}_q\left(\boldsymbol{B}\right)$$

where $\mathcal{R}_q\left(.\right)$ denotes the row space of a matrix.

Algorithm 1 Increasing Method

INPUT: $g3$ of N matrices belonging to $MatrixSpace(n, m)$ or $MatrixSpaceUrs(n, m)$

1. Create a list of $rel\left[(\boldsymbol{A}, \boldsymbol{B})\right], \forall \boldsymbol{A}, \boldsymbol{B} \in MatrixSpace(n, m), \boldsymbol{A} \neq \boldsymbol{B}$

2. Choose all $\{\boldsymbol{A}, \boldsymbol{B}\}$ such that:

   $$|rel\left[(\boldsymbol{A}, \boldsymbol{B})\right]| = |rel\left[(\boldsymbol{A}, \boldsymbol{B})\right]|_{max}$$

   with an upper bound for the final result set's cardinality $|Res| \leq UB, UB = |rel\left[(\boldsymbol{A}, \boldsymbol{B})\right]|_{max}$.

3. For each found pair set of $\{\boldsymbol{A}, \boldsymbol{B}\}$, we compute the union set of the pair and its Relative, i.e., $\{\boldsymbol{A}, \boldsymbol{B}\} \cup rel\left[(\boldsymbol{A}, \boldsymbol{B})\right]$. If the union set is repeated or duplicated, we take only the first pair generating such value. We denote the chosen set as $main\_team\_and\_rel$

4. Considering $main\_team\_and\_rel_i \in main\_team\_and\_rel$ with $i = 0, 1, 2, ..., |main\_team\_and\_rel| - 1$, we have

   $$\begin{gathered} rel_j \in rel\left[main\_team\_and\_rel_i\right] \\ \forall main\_team\_and\_rel_i \in main\_team\_and\_rel \\ j = 0, 1, 2, ..., |main\_team\_and\_rel_i| - 1 \end{gathered}$$

   to compute $n\_main\_team_i$, which is combined by $\{\boldsymbol{A}, \boldsymbol{B}, rel_j\}$ if $|subrel\left[(\boldsymbol{A}, \boldsymbol{B}, rel_j)\right]|_{max}$ similarly to step 2.

5. Keep only $n\_main\_team_i$ with $|subrel\left[(n\_main\_team_i)\right]|_{max}$ with $i = 0, 1, 2, ..., |main\_team\_and\_rel| - 1$. Similar to step 3, we also avoid duplicated values here.

6. Repeat step 4, 5, 6 until $|subrel\left[(n\_main\_team_i)\right]|_{max} = 0$

OUTPUT: Get the final result set with all matrices such that:

$$Res = \left\{\boldsymbol{X}_i : \boldsymbol{X}_i \in \mathbb{F}_q^{n \times m}\right\}, i = 0, 1, ..., UB$$

with any 3 combinations of $(\boldsymbol{X}_j, \boldsymbol{X}_k, \boldsymbol{X}_t) \in g3, \forall \boldsymbol{X}_j, \boldsymbol{X}_k, \boldsymbol{X}_t \in Res, \boldsymbol{X}_j \neq \boldsymbol{X}_k \neq \boldsymbol{X}_t$ and $j \neq k \neq t$.

Example 1: Let $n = 1, m = 2, q = 2$. Then we have $N = 4$ matrices (vectors):

$$\begin{aligned} \boldsymbol{A} &= [0,0] \\ \boldsymbol{B} &= [0,1] \\ \boldsymbol{C} &= [1,0] \\ \boldsymbol{D} &= [1,1] \end{aligned}$$

Step 1: Due to, any 3 of them form a matrix with $rk \geq 2n$, we have the relative as following:

$$\begin{aligned} rel\left[(\boldsymbol{A},\boldsymbol{B})\right] &= [\boldsymbol{C},\boldsymbol{D}] \\ rel\left[(\boldsymbol{A},\boldsymbol{C})\right] &= [\boldsymbol{B},\boldsymbol{D}] \\ rel\left[(\boldsymbol{A},\boldsymbol{D})\right] &= [\boldsymbol{B},\boldsymbol{C}] \\ rel\left[(\boldsymbol{B},\boldsymbol{C})\right] &= [\boldsymbol{A},\boldsymbol{D}] \\ rel\left[(\boldsymbol{B},\boldsymbol{D})\right] &= [\boldsymbol{A},\boldsymbol{C}] \\ rel\left[(\boldsymbol{C},\boldsymbol{D})\right] &= [\boldsymbol{A},\boldsymbol{B}] \end{aligned}$$

Step 2: We get $UB = 2$ and all $\{\boldsymbol{A},\boldsymbol{B}\},\{\boldsymbol{A},\boldsymbol{C}\},\{\boldsymbol{A},\boldsymbol{D}\},\{\boldsymbol{B},\boldsymbol{C}\},\{\boldsymbol{B},\boldsymbol{D}\},\{\boldsymbol{C},\boldsymbol{D}\}$, because $\underset{\substack{\forall \boldsymbol{X},\boldsymbol{Y} \in MatrixSpace(1,2) \\ \boldsymbol{X} \neq \boldsymbol{Y}}}{max} (rel\left[(\boldsymbol{X},\boldsymbol{Y})\right]) = 2$

Step 3: Due to $\{\boldsymbol{X},\boldsymbol{Y}\} \cup rel\left[(\boldsymbol{X},\boldsymbol{Y})\right] = \{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}\}$ with $(\boldsymbol{X},\boldsymbol{Y})$ are all tuples found in Step 2. We keep only $main\_team\_and\_rel = \{(\boldsymbol{A},\boldsymbol{B}) : rel\left[(\boldsymbol{A},\boldsymbol{B})\right]\}$

Step 4: Regarding to $rel\left[(\boldsymbol{A},\boldsymbol{B})\right]$, we have $rel_0 = \boldsymbol{C}, rel_1 = \boldsymbol{D}$. Then, $|subrel\left[(\boldsymbol{A},\boldsymbol{B},rel_0)\right]| = |subrel\left[(\boldsymbol{A},\boldsymbol{B},rel_1)\right]| = 1$, so we got $n\_main\_team_0 = \{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\}$ as the only output of this step.

Step 5: Because step 4 gets only $\{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\}$, we do not need to proceed anything here.

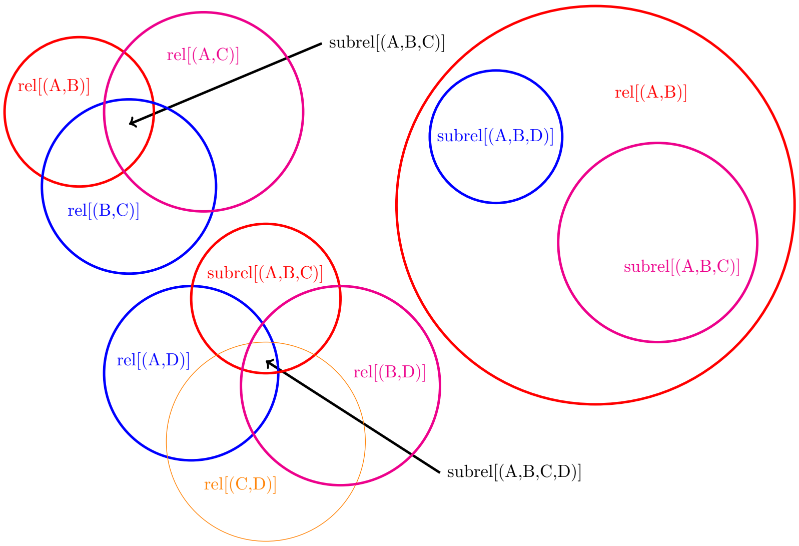
Step 6: We repeat step 4 and 5 once more and we get $Res = \{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}\}$, i.e., all the matrices can be used.

Example 2: For further understading, we use Figure 5.1 for illustration.

In the left-bottom corner, we observe that the size of relative becomes smaller when its tuple identity is larger, i.e. $|rel\left[(\boldsymbol{A},\boldsymbol{B})\right]| \geq |subrel\left[(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})\right]|$ or $|rel\left[(\boldsymbol{A},\boldsymbol{B})\right]| \geq |subrel\left[(\boldsymbol{A},\boldsymbol{B},\boldsymbol{D})\right]|$. It explains why UB is the maximum numbers of matrices that we can find in Res.

Regarding to the left-top corner and the right one, the visual explanation of $subrel$ is shown.

Figure 5.1: The vector network coding of $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$ represents as a matrix problem



6 Conclusion

7 Appendix

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