a

on/for?

5.2.2. Find the Upper Bound of $r_{max,scalar}$

there is no unique upper bound

state as Theorem?

> Find $(\alpha+1)$ received vectors that span a subspace of dimension h. This implies that the α links from the middle layer carry α vectors which span a subspace of $\mathbb{F}_{q_s}^h$ whose dimension is at least $(h-1)$, with $q_s = q^t$.

Following to Theorem (4.1), we are interested in the following range: $\ell+\epsilon+1 \leq h \leq \alpha\ell+\epsilon$.

(everywhere!) and why is then $r \leq \begin{bmatrix}\alpha\\1\end{bmatrix}_{q_s}$?

For $3 \leq \alpha < h$, all α links must be distinct* $\Rightarrow r \leq \begin{bmatrix}\alpha\\1\end{bmatrix}_{q_s} \Rightarrow r \leq \mathcal{O}\left(q_s^{\alpha-1}\right)$

why? more explanation!

* do you mean lin. independent?

For $\alpha \geq h \geq 3$: to achieve $(h-1)$-subspaces of $\mathbb{F}_{q_s}^h$, no α links will contain a vector which is contained in the same $(h-2)$-subspace.

for $\epsilon > 1$, shouldn't it be $h-\epsilon$ and $h-\epsilon-1$?

what achieves this and why? This is not a sentence that makes sense...

Hence,

$$r_{max,scalar} \leq (\alpha-1)\begin{bmatrix}\alpha\\h-2\end{bmatrix}_{q_s} \Rightarrow r_{max,scalar} \in \mathcal{O}\left(q_s^{(\alpha-h+2)(h-2)t^2}\right)$$

much more details! seems to be correct

where does t come from?

5.2.3. Calculate Gap

+ a sentence

state a Th...

$$r_{max,scalar} = min\{r_{max,vector}\}$$

of them, then $d \leq 2r$.

Proof. Being similar with the previous subsections, we have:

$$d \leq \alpha \binom{r-1}{\alpha - 1} = 2\frac{(r-1)\dots(r-1)}{1!} \leq 2r$$

$\square$

Theorem 5.3. *If* $r \leq \Omega\left(q^{t^2+\mathcal{O}(t)}\right)$*, then there exists a vector solution for the* $(\epsilon = 1, \ell \geq 2)-$ $\mathcal{N}_{h=2\ell,r,s=2\ell+1}$ *network.*

Proof. As previous, we need $4 \cdot p \cdot d(r) \leq 1, \forall r \leq r_{max,vector}$ so that a vector solution exists. Following to Lemma 5.7, we have $d \leq 2r \Rightarrow 4 \cdot p \cdot 2r \leq 1 \Rightarrow r \leq \frac{1}{8p}$. We still need to maximize p to get lower on $r_{max,vector}$, and we have $p \leq \Theta\left(q^{-t^2-2t-1}\right), \forall t \geq 2$ in Lemma 5.6. Thus, $min\left\{r_{max,vector}\right\} \in \Omega\left(\frac{1}{8p}\right) = \Omega\left(q^{t^2+2t+1}\right)$.

not sure if correct number

Hence, the Local lemma in 5.1 is satisfied, when $r \leq \Omega\left(q^{t^2/2+\mathcal{O}(t)}\right)$. None of bad events occurs, so there exists a vector solution for such r. $\square$

Lemma 5.8. *A scalar solution for the* $(\epsilon = 1, \ell \geq 2) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$ *network exists, if and only if there exists a Grasmannian code* $\mathcal{G}_q\left(h = 2\ell, \ell\right)$ *such that any* $\alpha = 2$ *subspaces of the set span a subspace of dimension at least* $2\ell - 1$.

This is a subspace code!

Proof. Let's denote any 2 subspaces of $\mathcal{G}_q\left(h = 2\ell, \ell\right)$ as $\mathcal{U}$ and $\mathcal{V}$. If $\mathcal{U}$ and $\mathcal{V}$ span a subspace of dimension at least $2\ell - 1$, then we have $dim\left(\mathcal{U} + \mathcal{V}\right) = 2\ell - 1$, with $\mathcal{U} + \mathcal{V}$.

See section VII of Tuvi/Antonia's paper. There they discuss the general case $\alpha = 2$ and derive an upper bound on $r_{max,scalar}$. (all based on theory of subspace codes)

Therefore, $dim\,(\mathcal{U} \cap \mathcal{V}) = dim\,(\mathcal{U}) + dim\,(\mathcal{V}) - dim\,(\mathcal{U} + \mathcal{V}) = 1$, which leads to the subspace distance $d_s(\mathcal{U}, \mathcal{V}) = 2dim\,(\mathcal{U} + \mathcal{V}) - dim\,(\mathcal{U}) - dim\,(\mathcal{V}) = 2\ell - 2$. No 2 ℓ-dimensional subspaces of $\mathbb{F}_{q_s}^{2\ell}$ will contain a vector which is contained in the same $(2\ell - 2)$-subspace, but $(2\ell - 1)$ of such subspaces can have such vectors,

$$\Rightarrow r_{scalar} \leq (2\ell - 1) \begin{bmatrix} 2\ell \\ 2\ell - 2 \end{bmatrix}_{q_s} \Rightarrow r_{scalar} \leq \mathcal{O}\left(q_s^{l}\right)$$

lell

There are general upper bounds on the cardinality of subspace codes

Corollary 5.2. *The $(\epsilon = 1, \ell \geq 2) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$ network has a vector solution with a gap $q^{t^2/4+\mathcal{O}(t)}$.*

2ℓ

→ see Tuvis/Ad. paper & compare to yours.

Following to Section 3.4.2 and Theorem 5.3, we have the gap size

$$\begin{aligned} r_{max,scalar} &= min\left\{r_{max,vector}\right\} \\ \Leftrightarrow \quad q_s^{\ell} &= q^{t^2/2+\mathcal{O}(t)} \\ \Leftrightarrow \quad q_s &= q^{t^2/2\ell+\mathcal{O}(t)} \\ \Rightarrow \quad g &= q_s - q_v = q^{t^2/2\ell+\mathcal{O}(t)} \end{aligned} \tag{5.9}$$

This shows us that there exists a better vector solution by comparison with the gap in [EW18, Fig. 4].