Vector Network Coding Gap Sizes for the Generalized Combination Network

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Outline

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Motivation

Network coding gives a **potential gain in throughput** by communicating more information with fewer packet transmissions compared to the routing method.

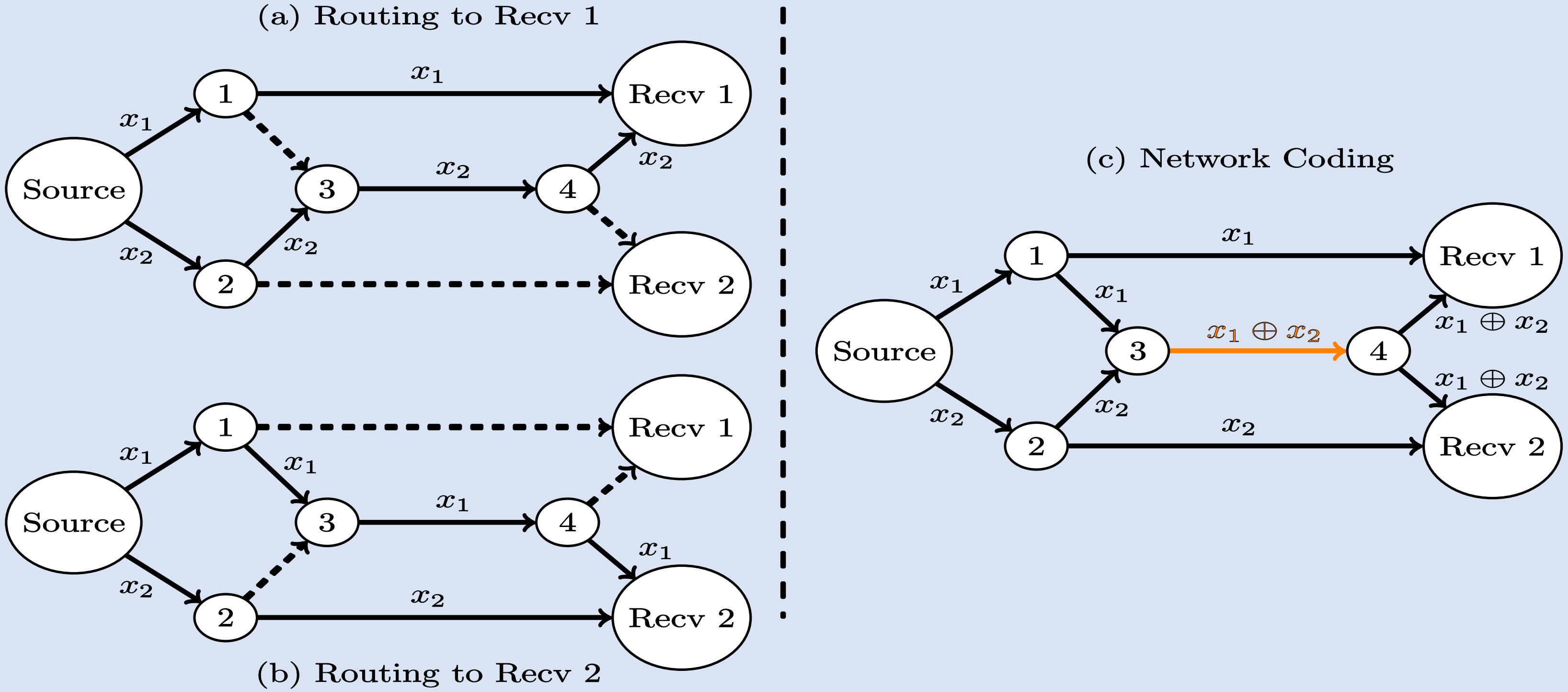


Figure 1. The butterfly network [Ahswede et al. (2000)]

Motivation (cont.)

Vector network coding (VNC) → Field size reduction [Etzion and Wachter-Zeh (2018)].

E.g. $\mathbb{F}_{2^8} \to \mathbb{F}_2^5$ (58472 middle-layer nodes)

Number of receivers is large.

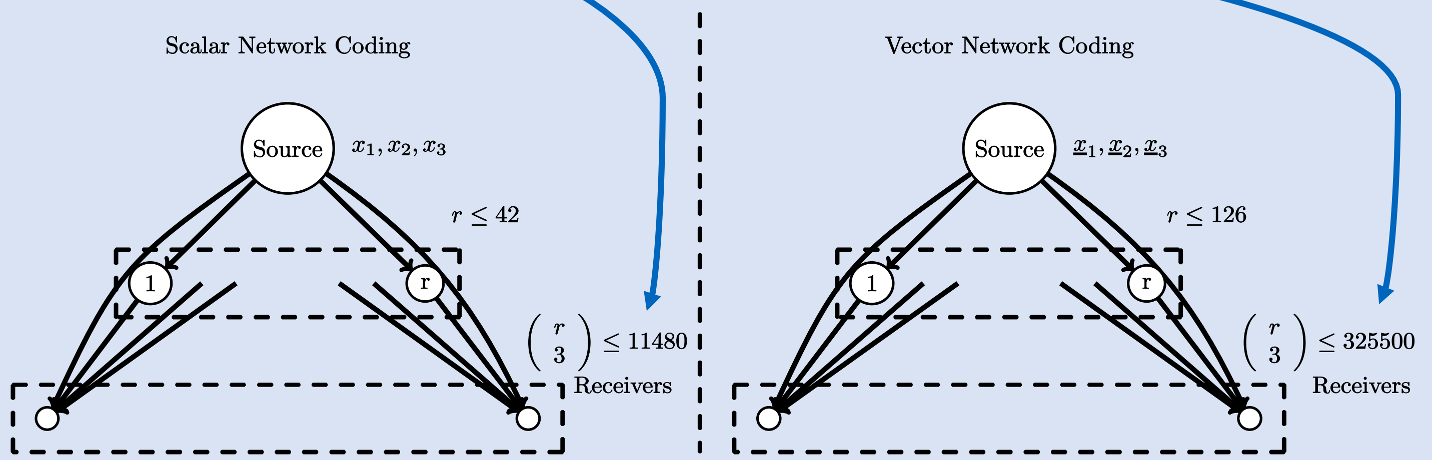


Figure 2. An example of scalar/vector network coding for a network with 3 messages [Etzion and Kurz (2019)]

Outline

1. Motivation
2. What is Network Coding?
   - Coding at a Node
   - Our Choice of Network Model
   - Network as a Matrix Channel
   - Gap size between scalar and vector solutions
3. Combinatorial Results
4. Computational Results
5. Conclusions

Coding at a node

Store-and-forward in simple routing [Yeung et al. (2006)] → each node can transmit **an arbitrary combination of its received packets** in network coding [Ahswede et al. (2000)].

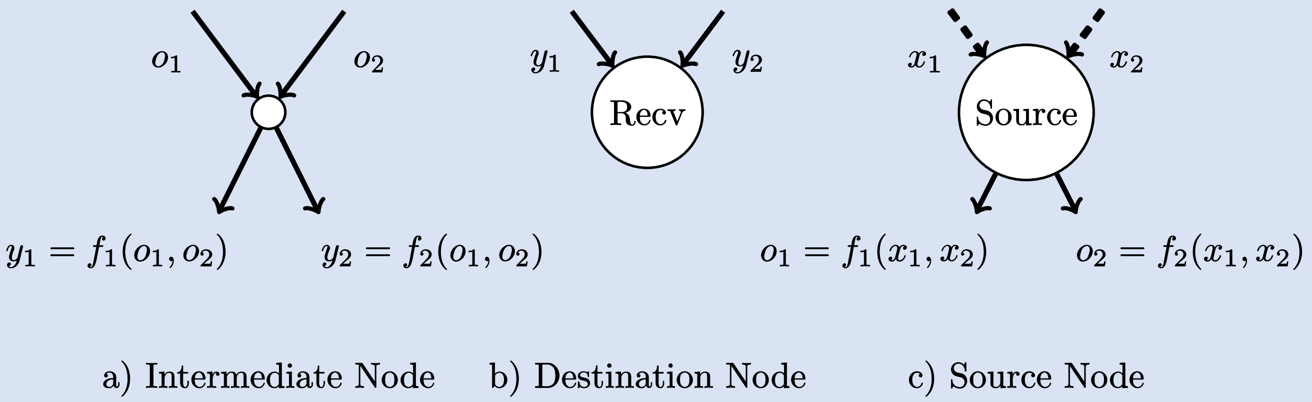


Figure 3. Incoming links and outgoing links of a node in network coding

Outline

Our Choice of Network Model

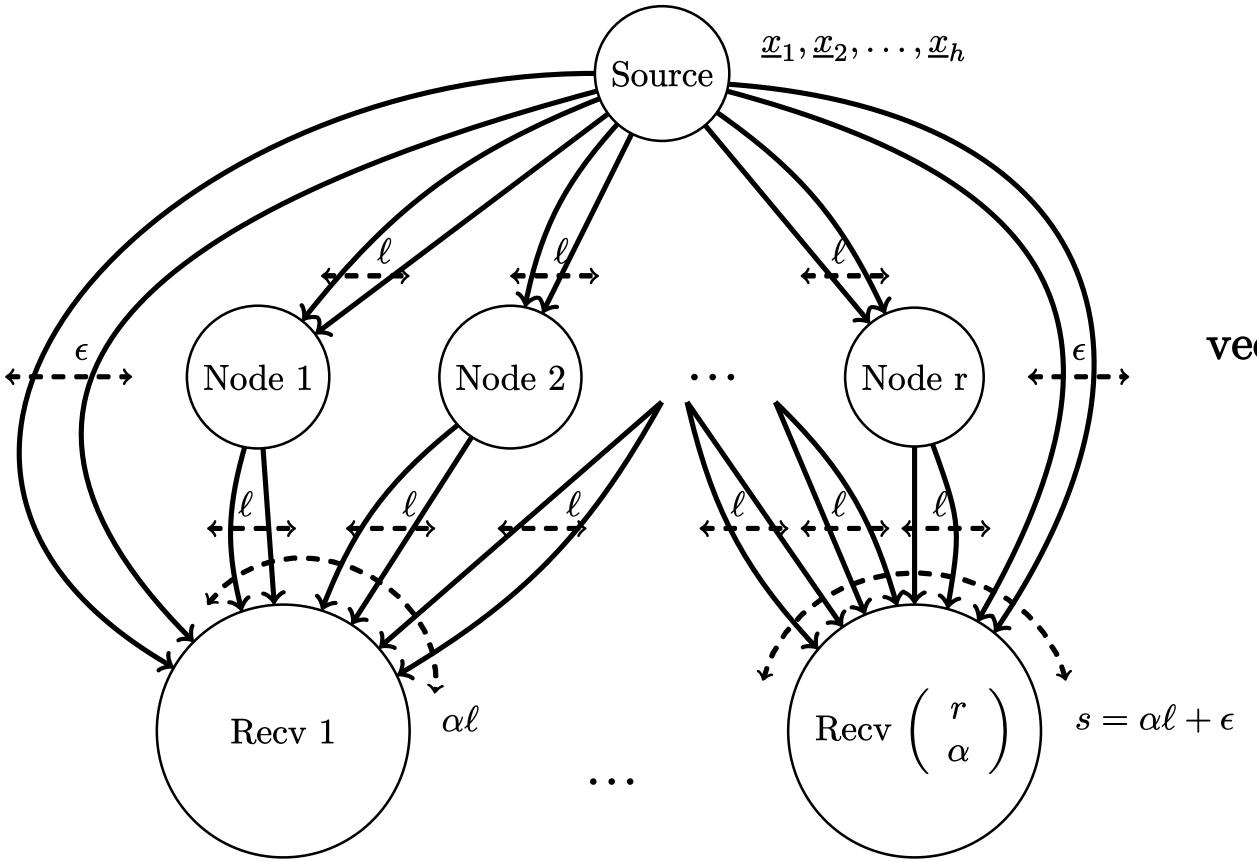


Figure 4. Generalized Combination Network (GCN)

Combination Network (CN) was introduced by Riis and Ahlswede in 2006.

A generalization of CN with ϵ direct links and ℓ multiple links was used to prove that **vector network coding outperforms scalar network coding** [Etzion and Wachter-Zeh 2018].

Able to demonstrate **networks with large capacity**.

Our Choice of Network Model (cont.)

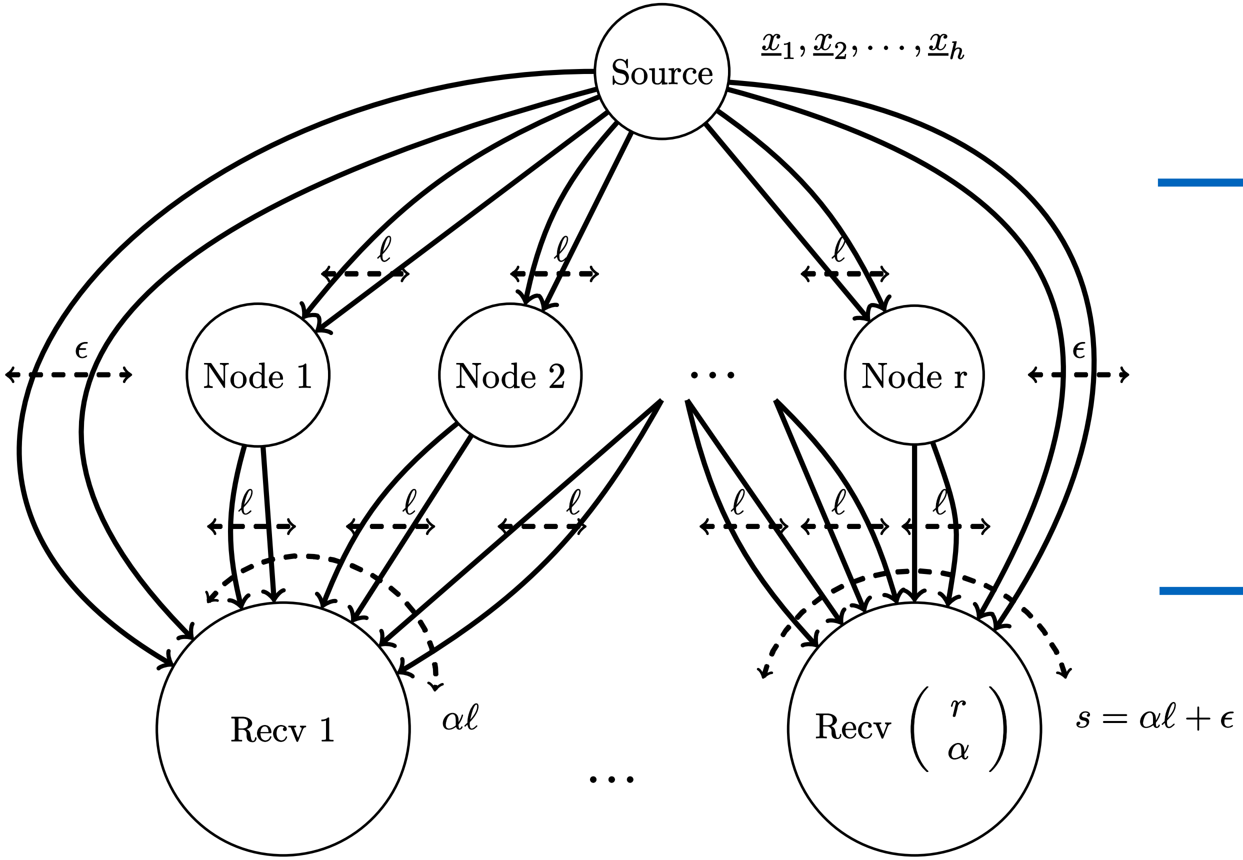


Figure 4. Generalized Combination Network (GCN)

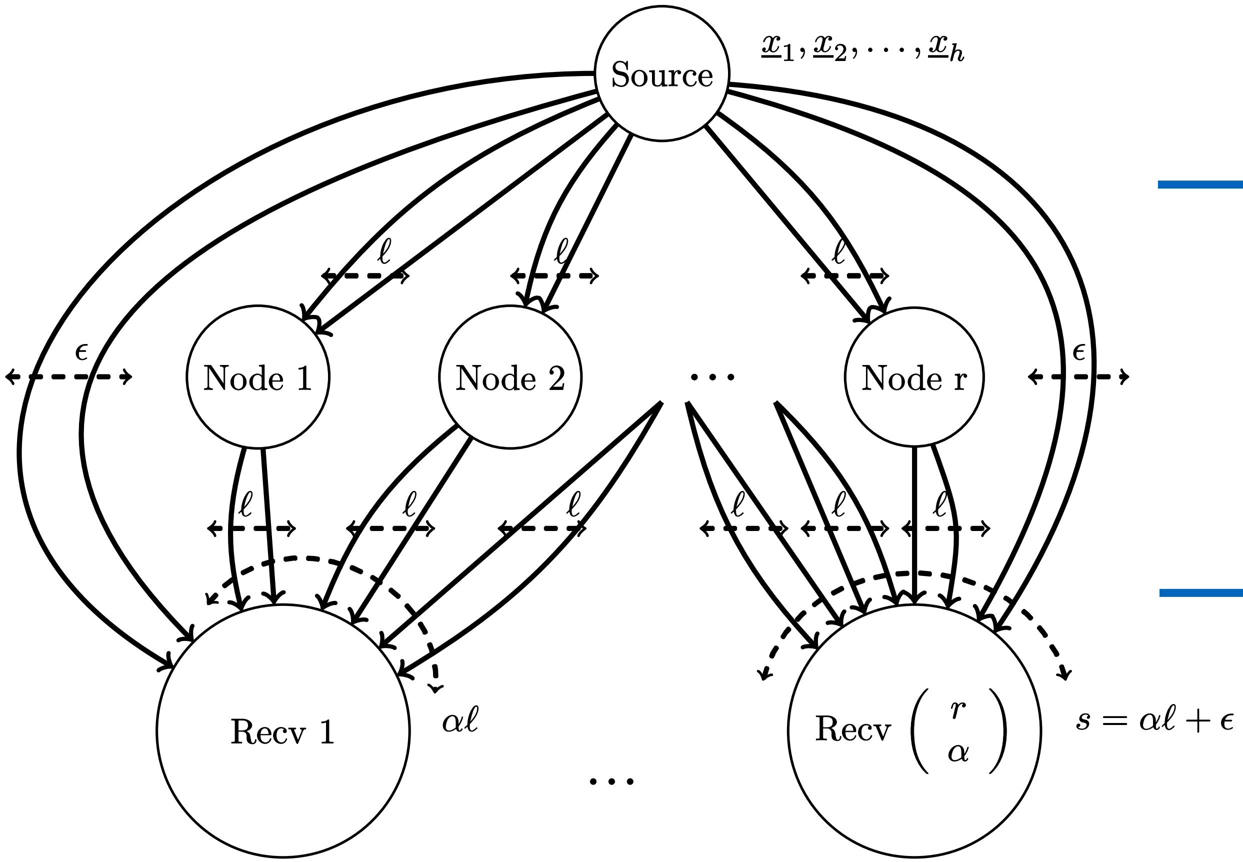
3 Layers from Top to Bottom

Source transmits $x_1, \dots, x_h \in \mathbb{F}_{q_s}$ (Scalar NC)

or

$\underline{x}_1, \dots, \underline{x}_h \in \mathbb{F}_q^t$ (Vector NC)

Our Choice of Network Model (cont.)



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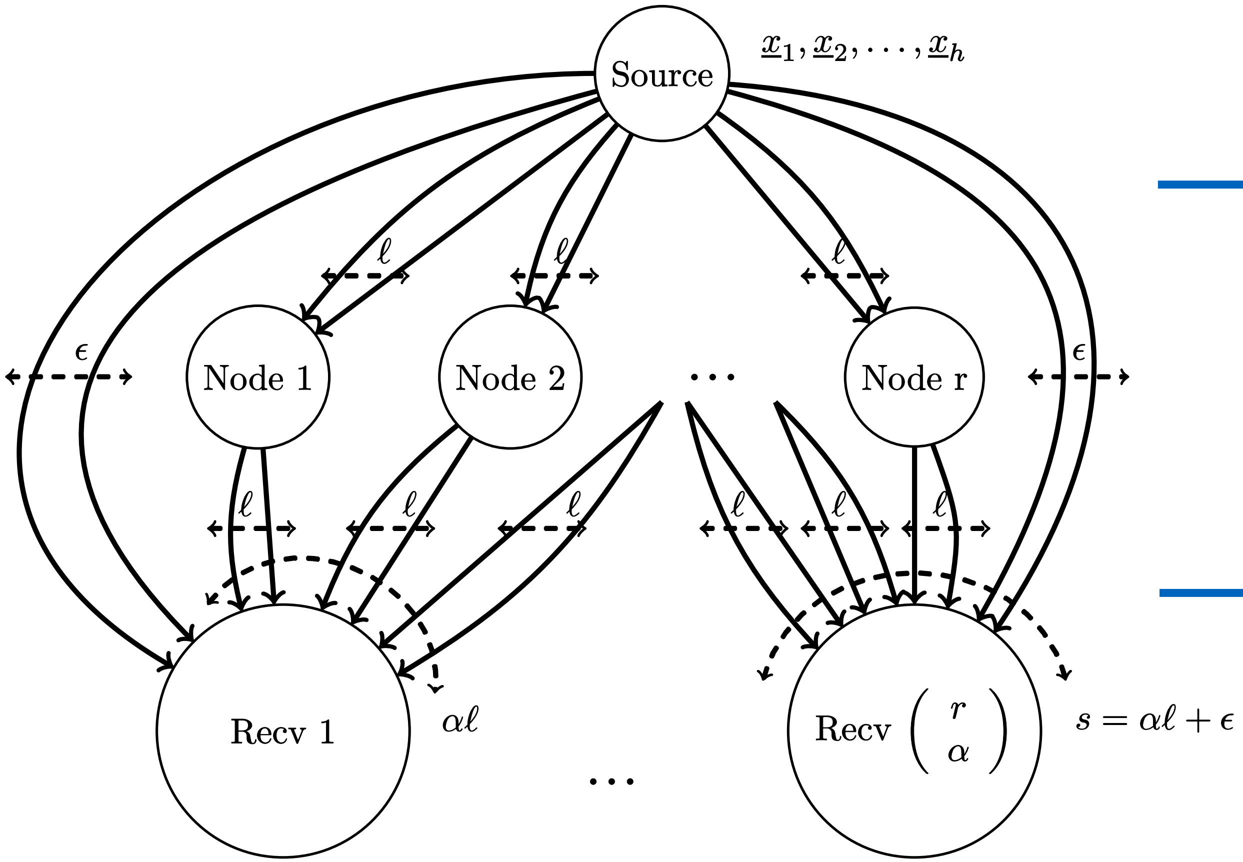
Intermediate Nodes (Nodes)

The number of nodes r is interesting to study!

Our Choice of Network Model (cont.)



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Intermediate Nodes (Nodes)

The number of nodes r is interesting to study!

Receivers

Our Choice of Network Model (cont.)

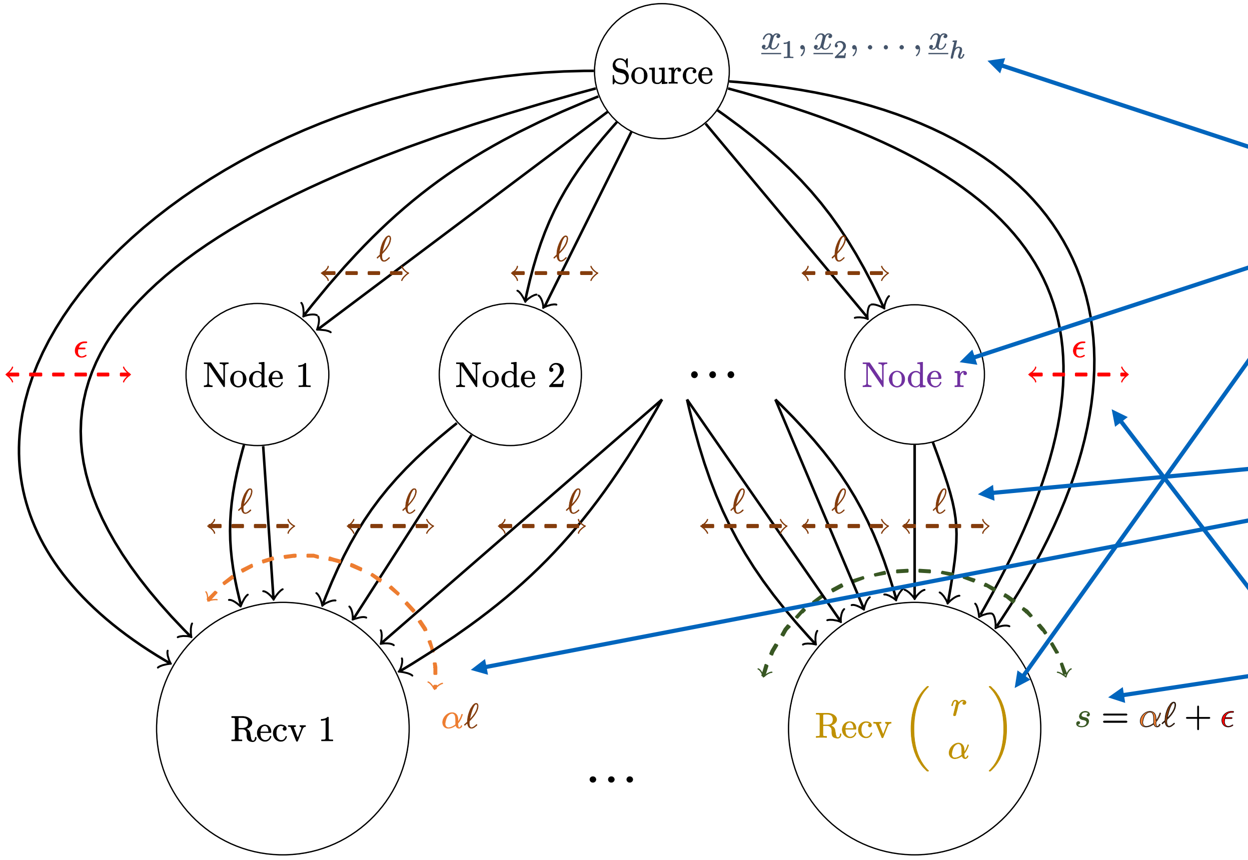


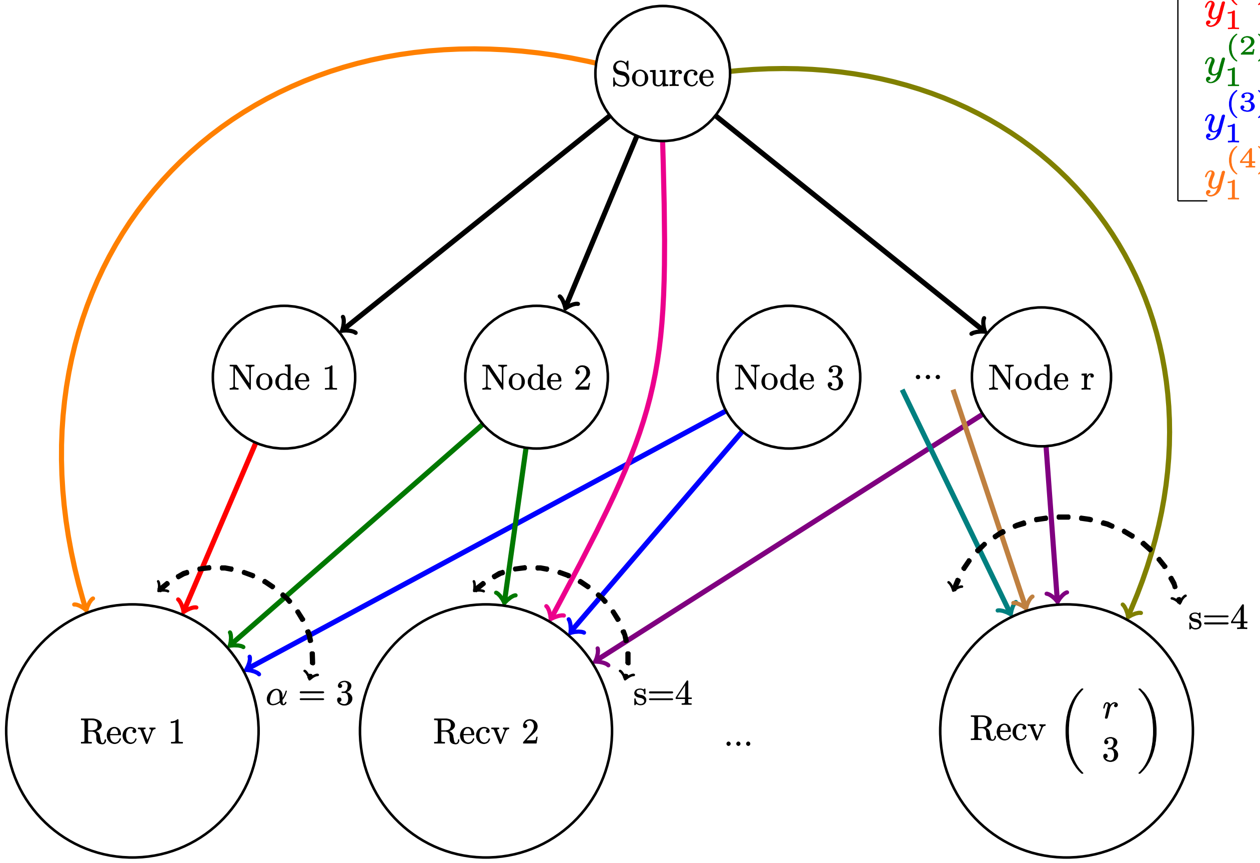
Figure 4. Generalized Combination Network (GCN)

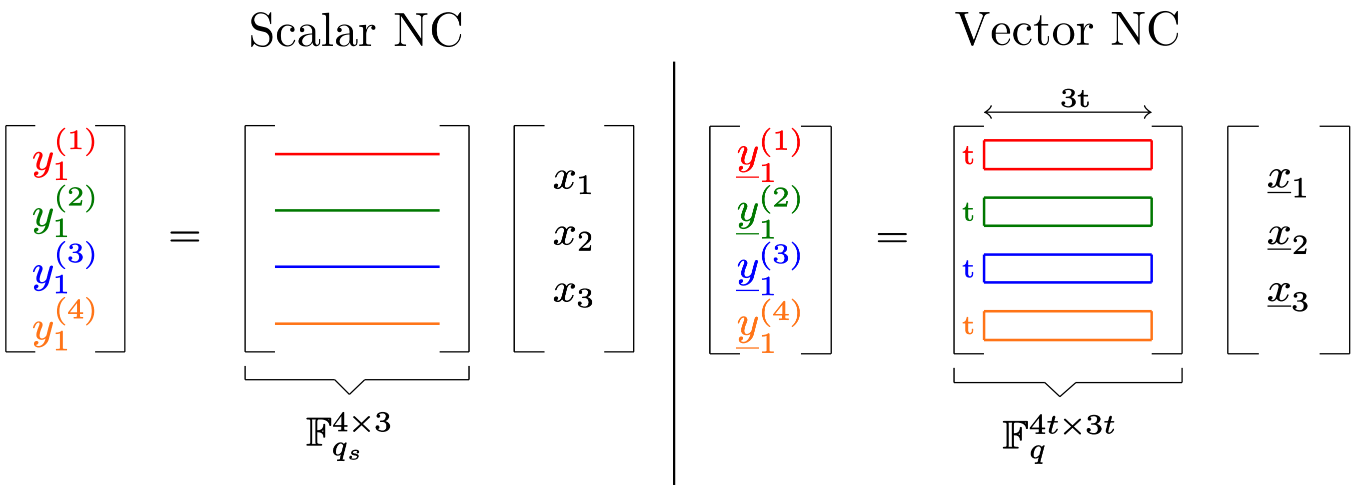
h	The number of source messages
r	The number of nodes in the middle layer
$\binom{r}{\alpha}$	The number of receivers
ℓ	Parallel links of a data unit
α	Links connected to a receiver from nodes
ϵ	Links connected to a receiver from the source
s	Total links connected to a receiver

Outline

Network as a Matrix Channel

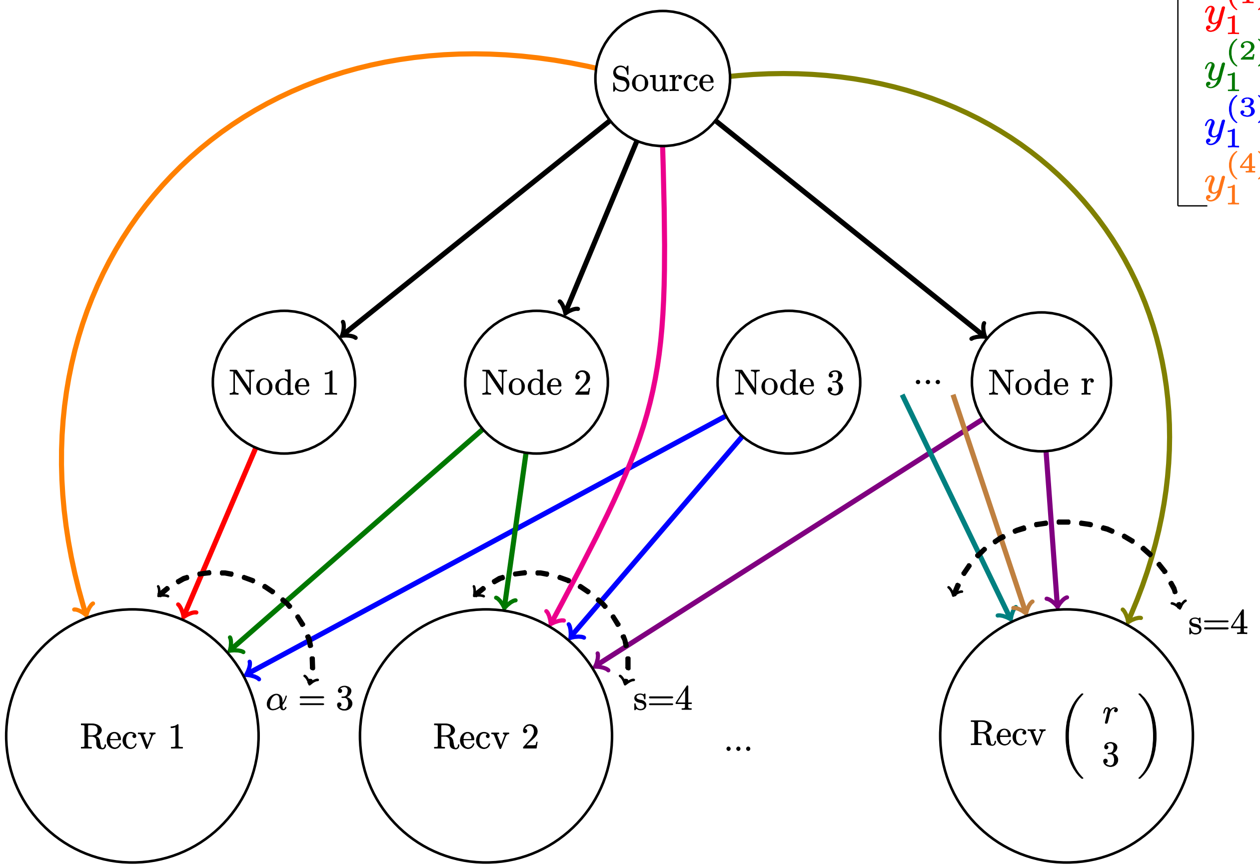
The $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network

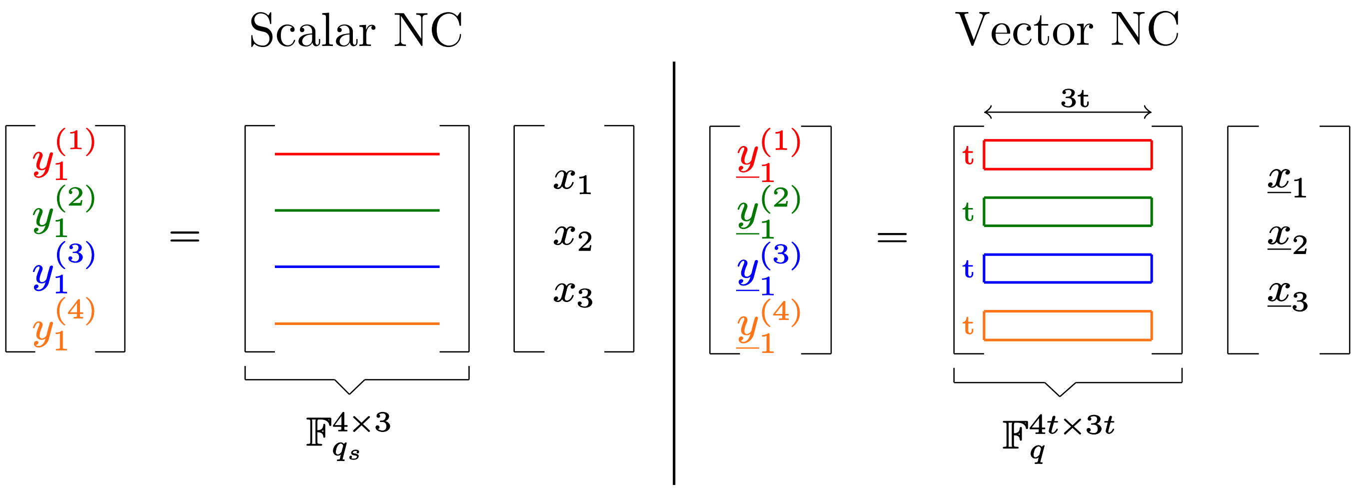




Network as a Matrix Channel (cont.)

The $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network





Scalar NC

$$\underbrace{\begin{bmatrix} y_j^{(1)} \\ \vdots \\ y_j^{(s)} \end{bmatrix}}_{\mathbb{F}_{q_s}^{s}} = \underbrace{\underline{\underline{A}}_j}_{\mathbb{F}_{q_s}^{s\times h}} \cdot \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_h \end{bmatrix}}_{\mathbb{F}_{q_s}^{h}}$$

Vector NC

$$\underbrace{\begin{bmatrix} \underline{y}_j^{(1)} \\ \vdots \\ \underline{y}_j^{(s)} \end{bmatrix}}_{\mathbb{F}_{q}^{st}} = \underbrace{\underline{\underline{A}}_j}_{\mathbb{F}_{q}^{st\times th}} \cdot \underbrace{\begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_h \end{bmatrix}}_{\mathbb{F}_{q}^{th}}$$

Network as a Matrix Channel (cont.)

Scalar NC

$$\underbrace{\begin{bmatrix} y_j^{(1)} \\ \vdots \\ y_j^{(s)} \end{bmatrix}}_{\mathbb{F}_{q_s}^{s}} = \underbrace{\underline{\underline{A}}_j}_{\mathbb{F}_{q_s}^{s \times h}} \cdot \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_h \end{bmatrix}}_{\mathbb{F}_{q_s}^{h}}$$

Vector NC

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$$\underline{\underline{A}}_j = \begin{bmatrix} \underline{a}^{(r_1)} \\ \vdots \\ \underline{a}^{(r_{\alpha l})} \\ \underline{b}^{(\epsilon(j-1)+1)} \\ \vdots \\ \underline{b}^{(\epsilon j)} \end{bmatrix}$$

Network as a Matrix Channel (cont.)

Scalar NC

$$\underbrace{\begin{bmatrix} y_j^{(1)} \\ \vdots \\ y_j^{(s)} \end{bmatrix}}_{\mathbb{F}_{q_s}^{s}} = \underbrace{\underline{\underline{A}}_j}_{\mathbb{F}_{q_s}^{s\times h}} \cdot \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_h \end{bmatrix}}_{\mathbb{F}_{q_s}^{h}}$$

Vector NC

$$\underbrace{\begin{bmatrix} \underline{y}_j^{(1)} \\ \vdots \\ \underline{y}_j^{(s)} \end{bmatrix}}_{\mathbb{F}_{q}^{st}} = \underbrace{\underline{\underline{A}}_j}_{\mathbb{F}_{q}^{st\times th}} \cdot \underbrace{\begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_h \end{bmatrix}}_{\mathbb{F}_{q}^{th}}$$

$$\underline{\underline{A}}_j = \begin{bmatrix} \underline{\underline{A}}^{(r_1)} \\ \vdots \\ \underline{\underline{A}}^{(r_{\alpha l})} \\ \underline{\underline{B}}^{(\epsilon(j-1)+1)} \\ \vdots \\ \underline{\underline{B}}^{(\epsilon j)} \end{bmatrix}$$

$$\underline{\underline{A}}_j = \begin{bmatrix} \underline{a}^{(r_1)} \\ \vdots \\ \underline{a}^{(r_{\alpha l})} \\ \underline{b}^{(\epsilon(j-1)+1)} \\ \vdots \\ \underline{b}^{(\epsilon j)} \end{bmatrix}$$

Network as a Matrix Channel (cont.)

Scalar NC | Vector NC

$$\underbrace{\begin{bmatrix} y_j^{(1)} \\ \vdots \\ y_j^{(s)} \end{bmatrix}}_{\mathbb{F}_{q^s}^{s}} = \underbrace{\underset{=j}{A}}_{\mathbb{F}_{q^s}^{s\times h}} \cdot \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_h \end{bmatrix}}_{\mathbb{F}_{q^s}^{h}} \quad \Bigg| \quad \underbrace{\begin{bmatrix} \underline{y}_j^{(1)} \\ \vdots \\ \underline{y}_j^{(s)} \end{bmatrix}}_{\mathbb{F}_{q}^{st}} = \underbrace{\underset{=j}{A}}_{\mathbb{F}_{q}^{st\times th}} \cdot \underbrace{\begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_h \end{bmatrix}}_{\mathbb{F}_{q}^{th}}$$

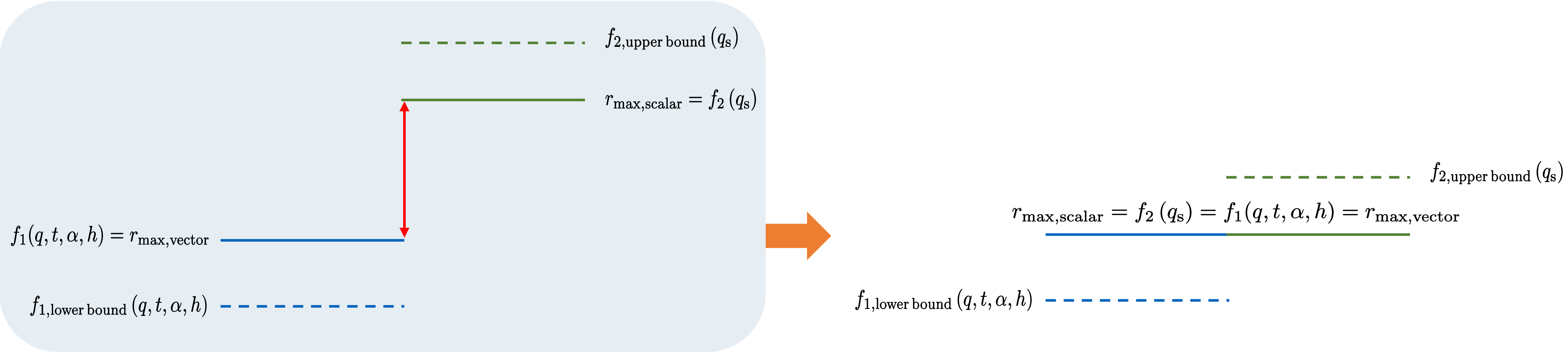
$$\underset{=j}{A} = \begin{bmatrix} \underset{=}{A}^{(r_1)} \\ \vdots \\ \underset{=}{A}^{(r_{\alpha l})} \\ \underset{=}{B}^{(\epsilon(j-1)+1)} \\ \vdots \\ \underset{=}{B}^{(\epsilon j)} \end{bmatrix}$$

$$\underset{=j}{A} = \begin{bmatrix} \underline{a}^{(r_1)} \\ \vdots \\ \underline{a}^{(r_{\alpha l})} \\ \underline{b}^{(\epsilon(j-1)+1)} \\ \vdots \\ \underline{b}^{(\epsilon j)} \end{bmatrix}$$

By using the vector coding, the upper bound number of solutions increases from q^{tsh} to q^{t^2sh}.

Outline

Gap size between scalar and vector solutions



Gap size between scalar and vector solutions (cont.)

$f_{2,\text{upper bound}}(q_s)$

$r_{\text{max,scalar}} = f_2(q_s)$

$f_1(q,t,\alpha,h) = r_{\text{max,vector}}$

$f_{1,\text{lower bound}}(q,t,\alpha,h)$

Gap [Wachter-Zeh (2018)]

$g = q_s - q_v = q_s - q^t$

$q_s(q,t,\alpha,h) = \min\{q_s : f_2(q_s) \geq f_1(q,t,\alpha,h)\}$

$f_{2,\text{upper bound}}(q_s)$

$r_{\text{max,scalar}} = f_2(q_s) = f_1(q,t,\alpha,h) = r_{\text{max,vector}}$

$f_{1,\text{lower bound}}(q,t,\alpha,h)$

$f_{1,\text{lower bound}}(q,t,\alpha,h)$ $f_{2,\text{upper bound}}(q_s)$

$q_{\text{s,min,from bound}} = \min\{q_s : f_{2,\text{upper bound}}(q_s) \geq f_{1,\text{lower bound}}(q,t,\alpha,h)\}$

$\Rightarrow g \geq g_{\text{lower bound}} = q_{\text{s,min,from bound}}(q,t,\alpha,h) - q^t$

Gap in this thesis

Gap size between scalar and vector solutions (cont.)

Network	Gaps for a specific vector solution [Etzion and Wachter-Zeh (2018)]	Lower bounds on gaps for a general vector solution [Corollary 5.4 and Corollary 5.3]
$(\epsilon = 0, \ell = 1) - \mathcal{N}_{h,r,s}$	N/A	N/A
$(\epsilon \geq 1, \ell = 1) - \mathcal{N}_{h,r,s}$	N/A	$q^{\frac{\epsilon(\alpha-h+\epsilon)}{(\alpha-1)(\alpha-h+\epsilon+1)(h-\epsilon-1)}t^2+\mathcal{O}(t)}$
$(\epsilon = 1, \ell > 1) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$	$q^{t^2/2+\mathcal{O}(t)}$	$q^{t^2/l+\mathcal{O}(t)}$
$(\epsilon = \ell - 1, \ell) - \mathcal{N}_{h=2\ell,r,s=3\ell-1}$	$q^{t^2/2+\mathcal{O}(t)}$	N/A

Table 1. Lower bounds on gaps were found in this study.

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1. Motivation
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3. Combinatorial Results
   - Proof of a gap for a network with 3 messages
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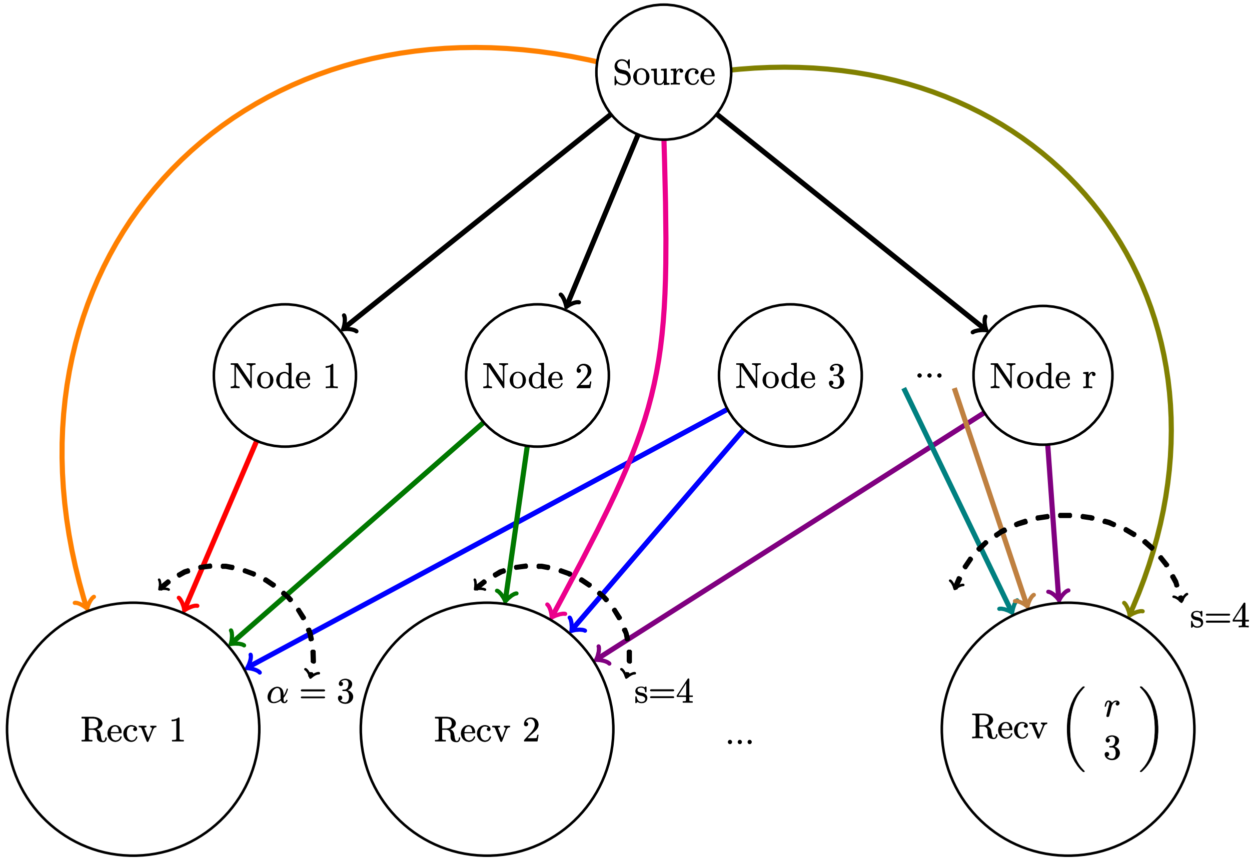
Proof of a gap for a network with 3 messages

We show the proof of the most simple case. Other cases considered in this study are similar to this proof.



Proof of a gap for a network with 3 messages (cont.)

Each receiver R_j has to solve a linear equation system of $3t$ variables with $4t$ equations to recover 3 source messages as below:

$$\begin{bmatrix} \underline{y}_j^{(1)} \\ \underline{y}_j^{(2)} \\ \underline{y}_j^{(3)} \\ \underline{y}_j^{(4)} \end{bmatrix} = \underline{\underline{A}}_j \cdot \underline{x} = \begin{bmatrix} \underline{\underline{A}}^{(r_1)} \\ \underline{\underline{A}}^{(r_2)} \\ \underline{\underline{A}}^{(r_3)} \\ \underline{\underline{B}}^{(j)} \end{bmatrix} \cdot \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \end{bmatrix},$$

with $\underline{x}_1, \dots, \underline{x}_3 \in \mathbb{F}_q^t$

$\underline{y}_j^{(1)}, \dots, \underline{y}_j^{(4)} \in \mathbb{F}_q^t$

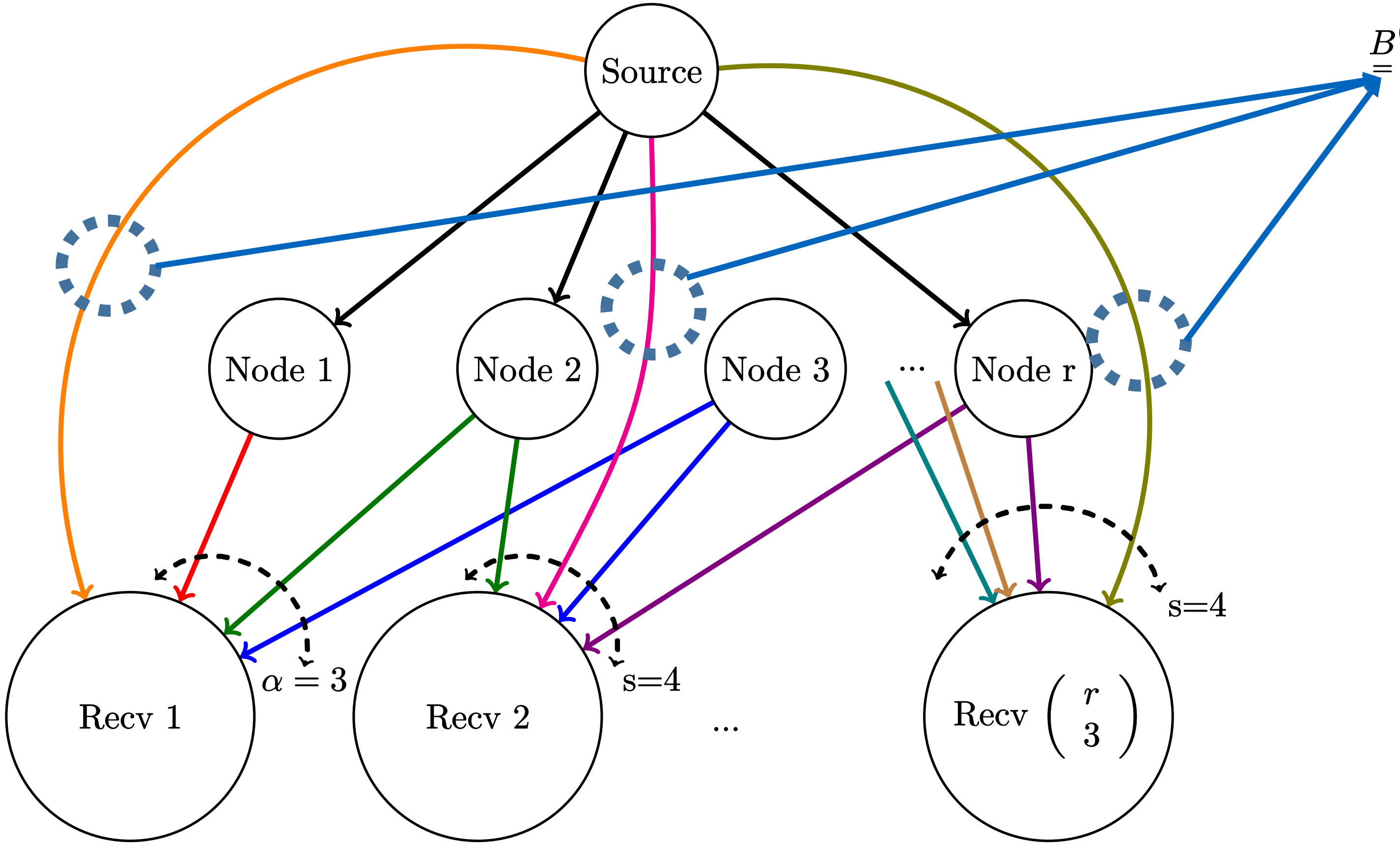
$\underline{\underline{A}}^{(r_1)}, \dots, \underline{\underline{A}}^{(r_3)} \in \mathbb{F}_q^{t\times 3t}$ for $1 \leq r_1 < r_2 < r_3 \leq r$

$\underline{\underline{B}}^{(j)} \in \mathbb{F}_q^{t\times 3t}$ for $j \in \left\{1, \dots, \binom{r}{3}\right\}$

The network is solvable if $\underline{\underline{A}}_j$ has full rank,

$$\mathrm{rk}\begin{bmatrix} \underline{\underline{A}}^{(r_1)} \\ \underline{\underline{A}}^{(r_2)} \\ \underline{\underline{A}}^{(r_3)} \\ \underline{\underline{B}}^{(j)} \end{bmatrix} \geq 3t.$$

Proof of a gap for a network with 3 messages (cont.)



$\underline{\underline{B}}^{(j)}$ can be independently chosen for any receiver R_j, we actually need only

$$\mathrm{rk}\begin{bmatrix}\underline{\underline{A}}^{(r_1)}\\ \underline{\underline{A}}^{(r_2)}\\ \underline{\underline{A}}^{(r_3)}\end{bmatrix} \geq 2t$$

$$\mathrm{rk}\begin{bmatrix}\underline{\underline{A}}^{(r_1)}\\ \underline{\underline{A}}^{(r_2)}\\ \underline{\underline{A}}^{(r_3)}\\ \underline{\underline{B}}^{(j)}\end{bmatrix} \geq 3t.$$

Proof of a gap for a network with 3 messages (cont.)

We then apply the Lovász Local lemma to calculate a gap for the network [Proposed by Schwartz (2018)].

Lemma: Symmetric Lovász Local Lemma (LLL)

A set of events $\mathcal{E}_i$, such that each event occurs with probability at most p. If each event is independent of all others except for at most d of them and $4pd \leq 1$, then:

$$\Pr\left[\bigcap_{i=1}^{n} \overline{\mathcal{E}}_i\right] > 0$$

We apply the LLL for the following events $\mathcal{E}_{r_1,r_2,r_3}$

$$\mathcal{E}_{r_1,r_2,r_3} = \left\{ \mathrm{rk} \begin{bmatrix} \underline{\underline{A}}^{(r_1)} \\ \underline{\underline{A}}^{(r_2)} \\ \underline{\underline{A}}^{(r_3)} \end{bmatrix} < 2t \right\},$$

where

$$1 \leq r_1 < r_2 < r_3 \leq r$$

$\underline{\underline{A}}^{(r_1)}, \ldots, \underline{\underline{A}}^{(r_3)} \in \mathbb{F}_q^{t \times 3t}$ are independently and uniformly random.

Proof of a gap for a network with 3 messages (cont.)

$$\Pr\left[\mathcal{E}_{r_1,r_2,r_3}\right] = \Pr\left[\operatorname{rk}\begin{bmatrix}\underline{\underline{A}}^{(r_1)}\\ \underline{\underline{A}}^{(r_2)}\\ \underline{\underline{A}}^{(r_3)}\end{bmatrix} < 2t\right] = \sum_{i=0}^{2t-1}\Pr\left[\operatorname{rk}\begin{bmatrix}\underline{\underline{A}}^{(r_1)}\\ \underline{\underline{A}}^{(r_2)}\\ \underline{\underline{A}}^{(r_3)}\end{bmatrix} = i\right]$$

$$\stackrel{1}{=} \sum_{i=0}^{2t-1}\frac{\mathrm{NM}_{i,3t,3t}}{q^{(3t)\cdot(3t)}}$$

$$= \sum_{i=0}^{2t-1}\frac{\prod_{j=0}^{i-1}\frac{\left(q^{3t}-q^{j}\right)^{2}}{q^{i}-q^{j}}}{q^{9t^2}}.$$

[1] Number of $[n \times m]$ matrices of rank i over $\mathbb{F}_q$ [Overbeck (2007)]

$$\mathrm{NM}_{i,n,m} = \prod_{j=0}^{i-1}\frac{\left(q^{m}-q^{j}\right)\left(q^{n}-q^{j}\right)}{q^{i}-q^{j}}.$$

Proof of a gap for a network with 3 messages (cont.)

$$\Pr\left[\mathcal{E}_{r_1,r_2,r_3}\right] = \Pr\left[\operatorname{rk}\begin{bmatrix}\underline{\underline{A}}^{(r_1)}\\ \underline{\underline{A}}^{(r_2)}\\ \underline{\underline{A}}^{(r_3)}\end{bmatrix} < 2t\right] = \sum_{i=0}^{2t-1}\Pr\left[\operatorname{rk}\begin{bmatrix}\underline{\underline{A}}^{(r_1)}\\ \underline{\underline{A}}^{(r_2)}\\ \underline{\underline{A}}^{(r_3)}\end{bmatrix} = i\right]$$

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[1] Number of $[n\times m]$ matrices of rank i over $\mathbb{F}_q$ [Overbeck (2007)]

$$\mathrm{NM}_{i,n,m} = \prod_{j=0}^{i-1}\frac{\left(q^m-q^j\right)\left(q^n-q^j\right)}{q^i-q^j}.$$

Lemma 5.2

$$\Pr\left[\mathcal{E}_{r_1,r_2,r_3}\right] \le p \in \Theta\left(q^{-t^2-2t-1}\right), \forall t \geq 2.$$

We study the asymptotic behaviour of LLL parameters when $q \to \infty$ by using the Bachmann-Landau notations.

$\Theta\left(f\left(q,t\right)\right)$ for a tight bound

$\Omega\left(f\left(q,t\right)\right)$ for a lower bound

$\mathcal{O}\left(f\left(q,t\right)\right)$ for an upper bound

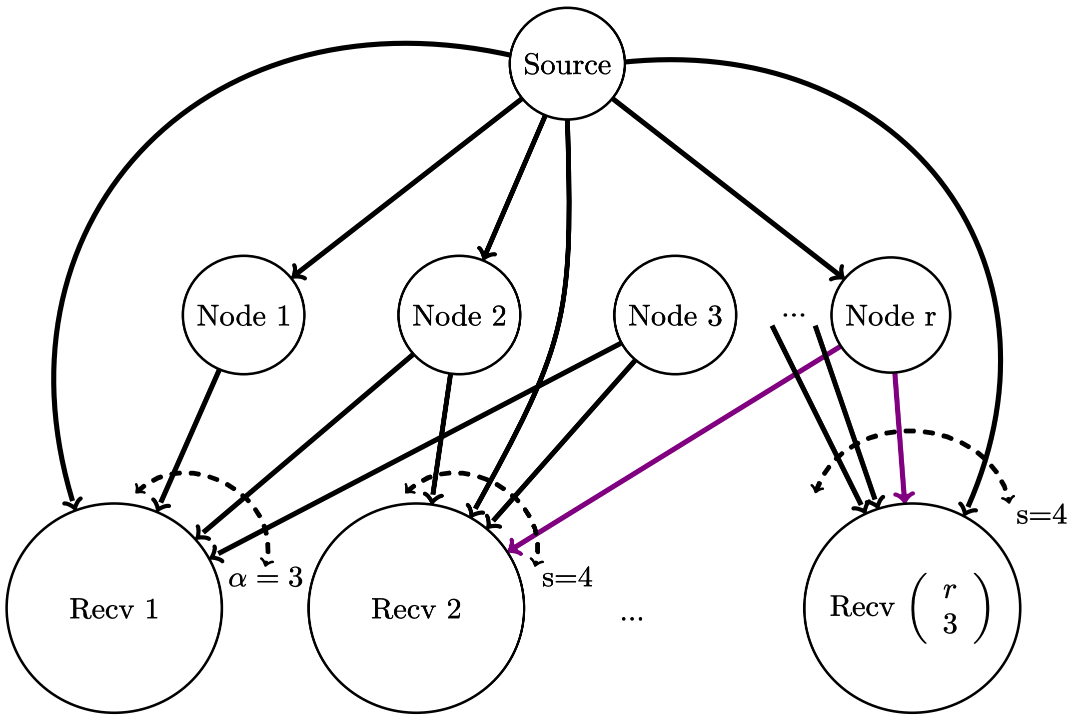
Proof of a gap for a network with 3 messages (cont.)

Lemma 5.3

Each event is dependent at most $d(r) \leq \frac{3}{2}r^2$ other events.

$\mathcal{E}_{r_1,r_2,r_3}$ is dependent on $\mathcal{E}_{r_1',r_2',r_3'}$ if and only if $\{r_1, r_2, r_3\} \cap \left\{r_1', r_2', r_3'\right\} \neq \emptyset$

$$\Rightarrow d(r) \leq 3 \cdot \binom{r-1}{2}$$



Example that "Node r" is chosen for 2 receivers.

Proof of a gap for a network with 3 messages (cont.)

Lemma 5.3

Each event is dependent at most $d(r) \leq \frac{3}{2}r^2$ other events.

Lemma 5.2

$$\Pr\left[\mathcal{E}_{r_1,r_2,r_3}\right] \leq p \in \Theta\left(q^{-t^2-2t-1}\right), \forall t \geq 2.$$

Lemma: Symmetric Lovász Local Lemma (LLL)

A set of events $\mathcal{E}_i$, such that each event occurs with probability at most p. If each event is independent of all others except for at most d of them and $4pd \leq 1$, then:

$$\Pr\left[\bigcap_{i=1}^{n} \overline{\mathcal{E}}_i\right] > 0$$

Theorem 5.1

There is an $r_{\text{max,vector}} \in \Omega\left(q^{t^2/2+\mathcal{O}(t)}\right)$ such that for any $r \leq r_{\text{max,vector}}$ there exists a vector solution for the $(\epsilon = 1, l = 1) - \mathcal{N}_{h=3,r,s=4}$ network.

Proof of a gap for a network with 3 messages (cont.)

The **optimal scalar solution** exists for $r_{\text{max,scalar}} \in \mathcal{O}\left(q_{\text{s}}^2\right)$ [Etzion and Wachter-Zeh (2018)].

Applying Theorem 5.1 on slide 32 and the formula of gap on slide 22, we have

$$\begin{aligned}
& r_{\text{max,scalar}} && = r_{\text{max,vector}} \\
\Leftrightarrow \quad & q^2_{\text{s,min,from bound}} && = q^{t^2/2+\mathcal{O}(t)} \\
\Leftrightarrow \quad & q_{\text{s,min,from bound}} && = q^{t^2/4+\mathcal{O}(t)} \\
\Rightarrow \quad & g_{\text{lower bound}} && = q_{\text{s,min,from bound}} - q_v = q^{t^2/4+\mathcal{O}(t)}
\end{aligned}$$

Corollary 5.1

The $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network has a vector solution with a gap $q^{t^2/4+\mathcal{O}(t)}$.

Similar to the $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network, we proved a gap for a **more general network** in Corollary 5.3.

Corollary 5.3

The $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h,r,s}$ network has a vector solution with a gap $q^{\frac{\alpha-h+1}{(\alpha-1)(\alpha-h+2)(h-2)}t^2+\mathcal{O}(t)}$.

Outline

1. Motivation
2. What is Network Coding?
3. Combinatorial Results
4. **Computational Results**
5. Conclusions

Computational Results

t	Scalar Solution	Vector Solution	Etzion and Kurz (2019)	Construction 1	Construction 2
2	$r_{\text{max,scalar}} = 42$	$r_{\text{max,vector}} \geq 7$	$r_{\text{max,vector}} = 121$	$r_{\text{max,vector}} = 89$	N/A
3	$r_{\text{max,scalar}} = 146$	$r_{\text{max,vector}} \geq 62$	N/A	N/A	$r_{\text{max,vector}} = 166$
4	$r_{\text{max,scalar}} = 546$	$r_{\text{max,vector}} \geq 1317$	N/A	N/A	N/A
5	$r_{\text{max,scalar}} = 2114$	$r_{\text{max,vector}} \geq 58472$	N/A	N/A	N/A
6	$r_{\text{max,scalar}} = 8322$	$r_{\text{max,vector}} > 10^6$	N/A	N/A	N/A

For $\mathbf{t} = \mathbf{2}$, Etzion and Kurz provide the best vector solution.

For $\mathbf{t} = \mathbf{3}$, this study provides the only vector solution outperforming the optimal scalar solution.

For $\mathbf{t} = \mathbf{4, 5, 6}$, the combinatorial results from this study already show large gaps.

Computational Results (cont.)

Approaches that we tried in this study:

1. Random Increasing Method

Start with a random pair [A, B], then randomly extend to [A, B, C]. Similarly, we continue to extend until no more possibility.

Computational Results (cont.)

Approaches that we tried in this study:

1. Random Increasing Method
2. Increasing Method (IM) with Extra Information

If we input a set of 4096 matrices, we know all subsets of 3 matrices satisfying the rank requirement. Then, we use this information to search for a vector solution.

$$\mathrm{rk}\begin{bmatrix} \underline{\underline{A}}^{(r_1)} \\ \underline{\underline{A}}^{(r_2)} \\ \underline{\underline{A}}^{(r_3)} \end{bmatrix} \geq 2t$$

Computational Results (cont.)

Approaches that we tried in this study:

1. Random Increasing Method

2. Increasing Method (IM) with Extra Information

3. Decreasing Method with Learning Bad Events over Each Step

Start with a set of all matrices
→ Check matrix with high frequency in events that the rank requirement is not satisfied.
→ Remove such a matrix
→ Keep doing until any 3 matrices left in the set meet the rank requirement.

$$\operatorname{rk}\begin{bmatrix} \underline{\underline{A}}^{(r_1)} \\ \underline{\underline{A}}^{(r_2)} \\ \underline{\underline{A}}^{(r_3)} \end{bmatrix} \geq 2t$$

Outline

Conclusion

Summary:

- Tool to derive general vector solutions for instances of the generalized combination network based on the Lovász Local Lemma.
- By combinatorial results, vector solutions can be generally proved to outperform scalar solutions.

Further Results: (not in presentation)

- A new bound on a three-dimensional subspace code over $\mathbb{F}_2^9$.

Open:

- Code construction of the codebook found in the computational results.
- Apply the Lovász Local Lemma to all other instances of the generalized combination network.

Conclusion (cont.)

Summary:

- Tool to derive general vector solutions for instances of the generalized combination network based on the Lovász Local Lemma.
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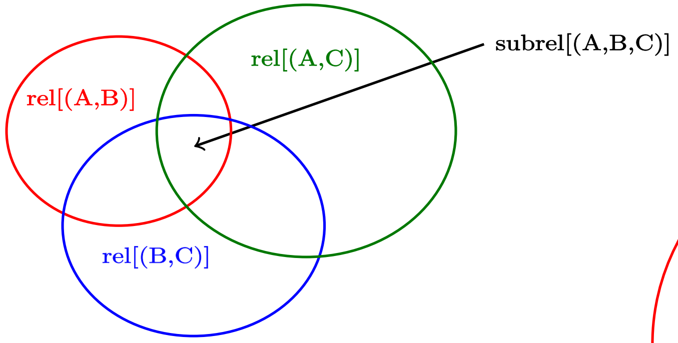
- Apply the Lovász Local Lemma to all other instances of the generalized combination network.

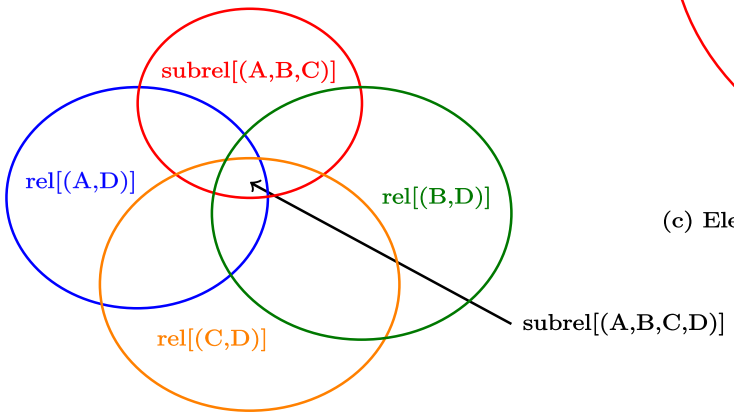
Thank you for your attention!

Computational Results

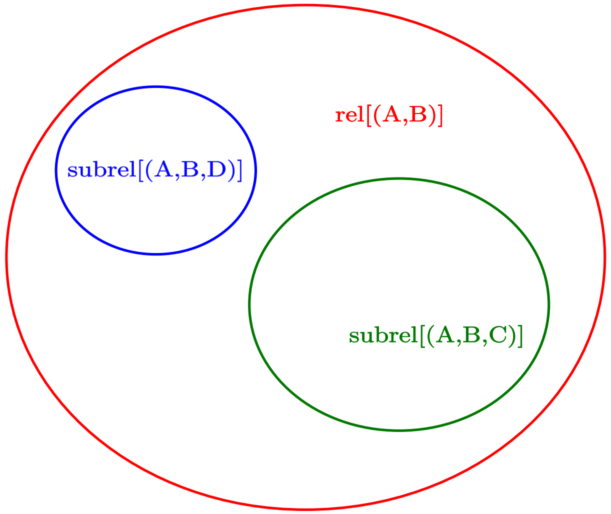
We analyze the IM for case t $=$ 3 of the $(\epsilon = 1, \ell = 1) - \mathcal{N}_{h=3,r,s=4}$ network.



(a) subrel[(A,B,C)] is a subset of rel[(A,B)], rel[(B,C)] and rel[(A,C)]



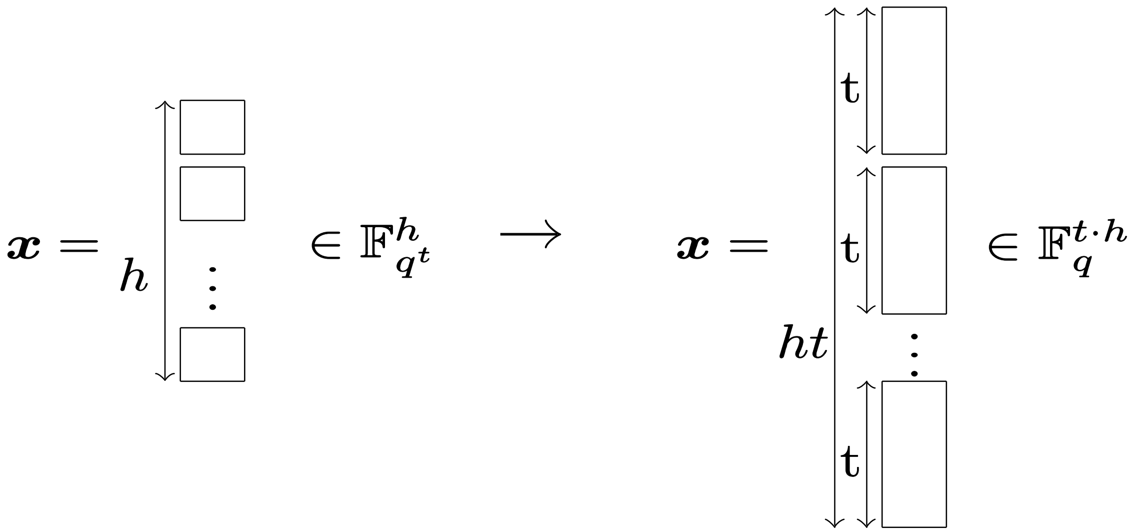
(b) subrel[(A,B,C,D)] is a subset of subrel[(A,B,C)], rel[(A,D], rel[(B,D] and rel[(C,D]



(c) Elements of subrel[.] are always contained in rel[.]

$$\mathrm{rk}\begin{bmatrix} \underset{=j}{A}^{(r_1)} \\ \underset{=j}{A}^{(r_2)} \\ \underset{=j}{A}^{(r_3)} \end{bmatrix} \geq 2t$$

Mapping



$$x_1, \dots, x_{h=3} \in \mathbb{F}_{2^2}$$

$$\boldsymbol{x}_{\text{scalar}} = (x_1, x_2, x_3) = (1, \alpha, \alpha + 1).$$

$$\boldsymbol{x}_1, \dots, \boldsymbol{x}_{h=3} \in \mathbb{F}_2^2$$

$$\boldsymbol{x}_{\text{vector}} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3).$$

$$\begin{bmatrix} x_1 = 1 \\ x_2 = \alpha \\ x_3 = \alpha + 1 \end{bmatrix} \rightarrow \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{bmatrix},$$

$$a_0 \cdot \alpha^0 + a_1 \cdot \alpha^1 + \dots + a_{t-1} \cdot \alpha^{t-1} \rightarrow \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{t-1} \end{pmatrix}.$$