Outline

1. Motivation
2. What is Network Coding?
3. Combinatorial Results
4. Computational Results
5. Conclusions

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Motivation

Network coding gives a **potential gain in throughput** by communicating more information with fewer packet transmissions compared to the routing method.

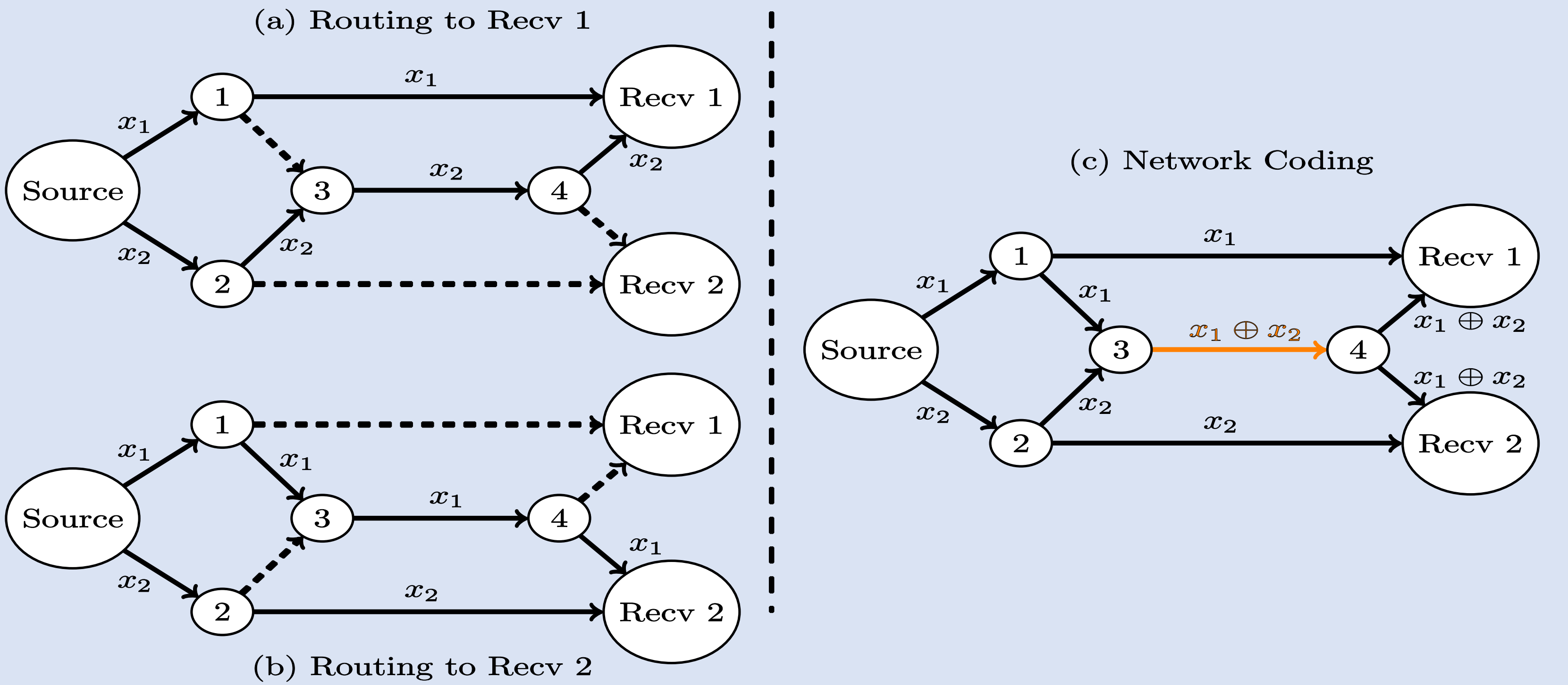

Figure 1. The butterfly network [Ahswede et al. (2000)]

Motivation (cont.)

Vector network coding solutions can **significantly reduce the required alphabet size** compared to the optimal scalar linear solution for the same network. [Ebrahimi and Fragouli (2011)]

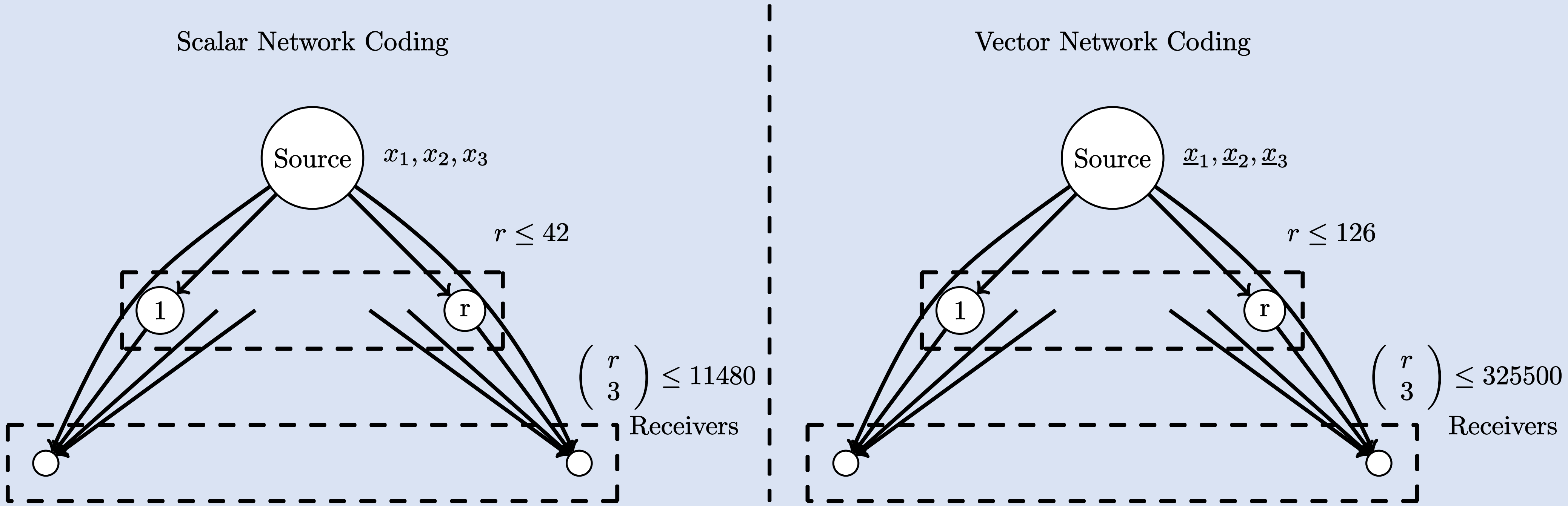

Figure 2. An example of scalar/vector network coding for a network with 3 messages

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2. What is Network Coding?
   - Coding at a Node
   - Our Choice of Network Model
   - Network as a Matrix Channel
   - Gap size between scalar and vector solutions
3. Combinatorial Results
4. Computational Results
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Coding at a node

Instead of store-and-forward in simple routing [Yeung et al. (2006)], each node can transmit **an arbitrary combination of its received packets** in network coding [Ahswede et al. (2000)].

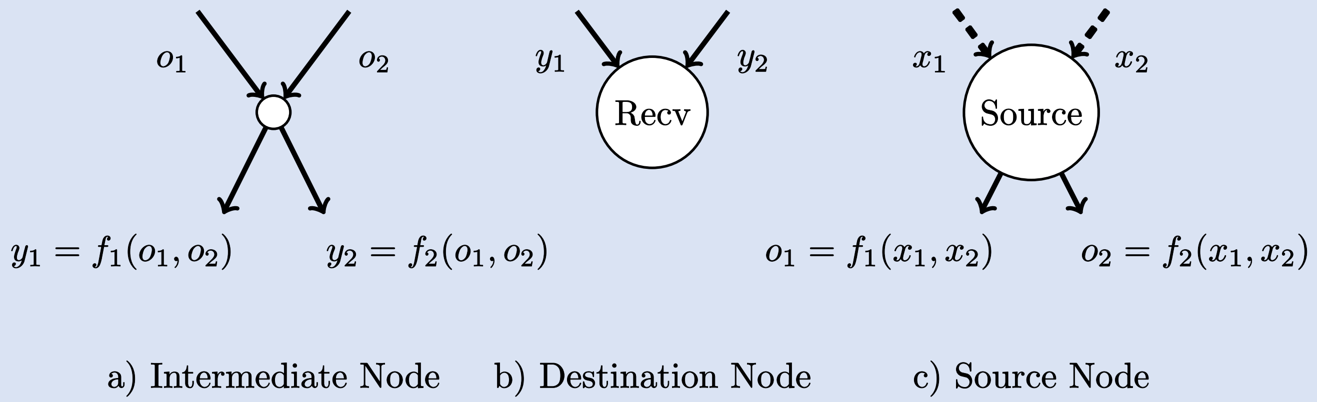


Figure 3. Incoming links and outgoing links of a node in network coding

Outline

Our Choice of Network Model

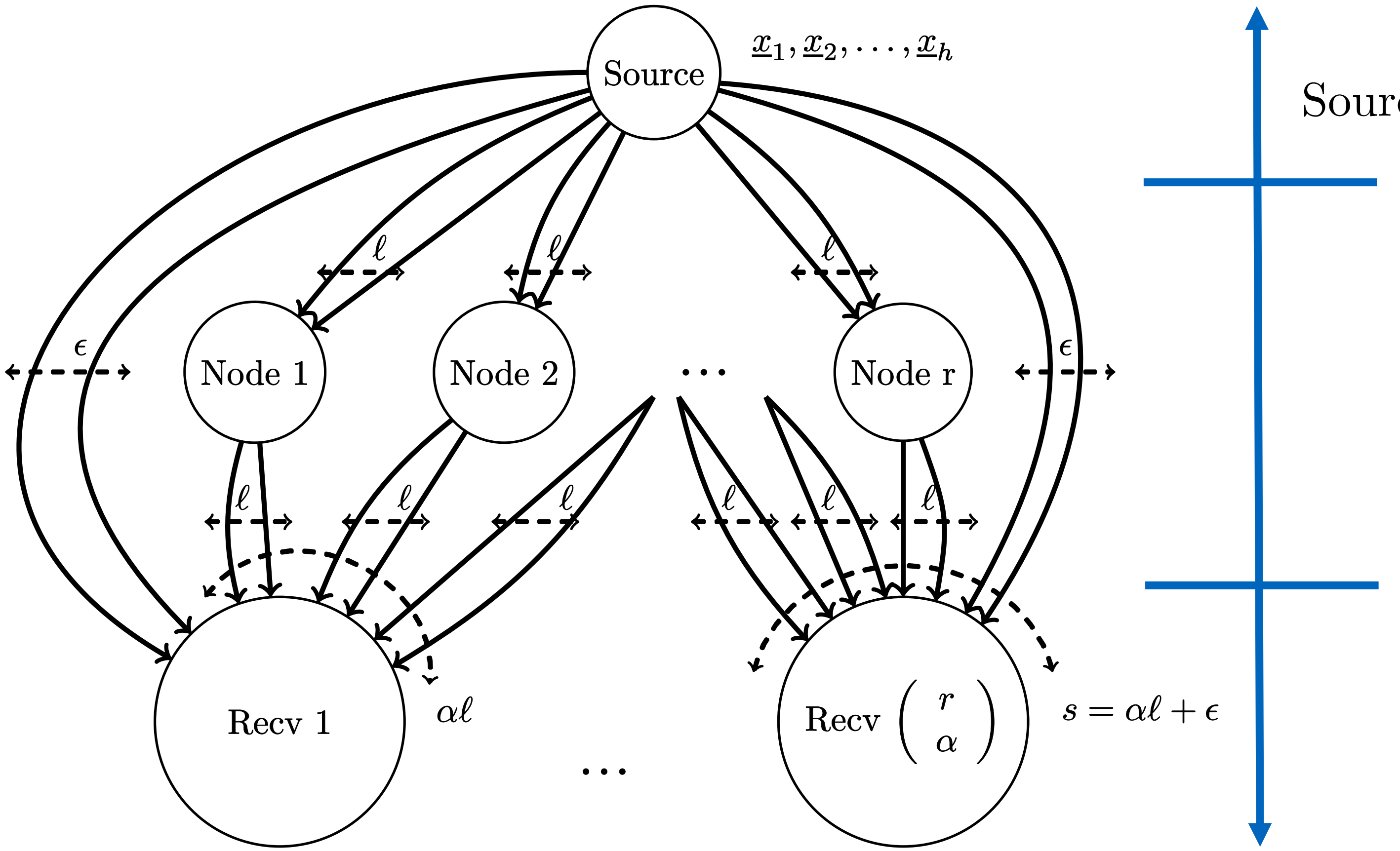


Figure 4. Generalized Combination Network (GCN)

3 Layers from Top to Bottom

Source transmits $x_1, \ldots, x_h \in \mathbb{F}_{q_s}$ (Scalar NC)

or

$\underline{x}_1, \ldots, \underline{x}_h \in \mathbb{F}_q^t$ (Vector NC)

Our Choice of Network Model (cont.)

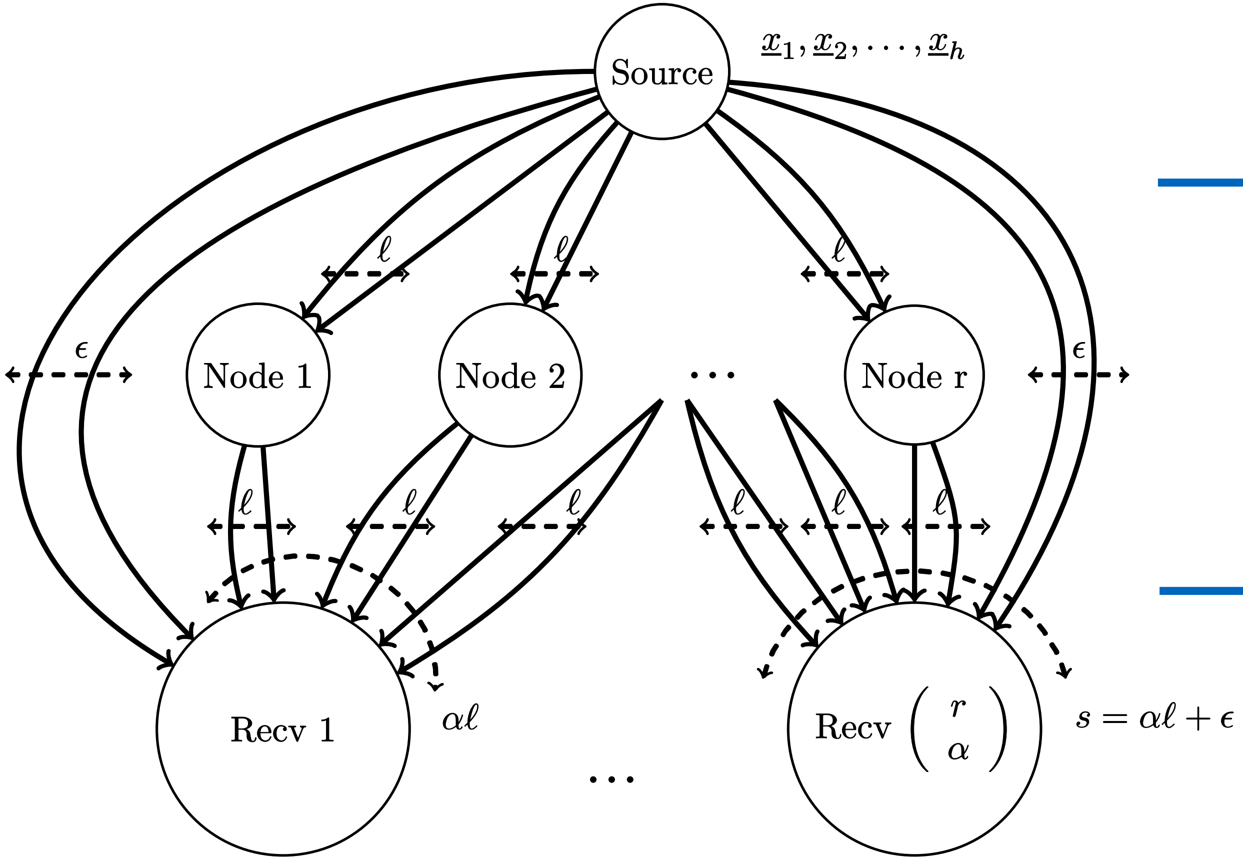


Figure 4. Generalized Combination Network (GCN)

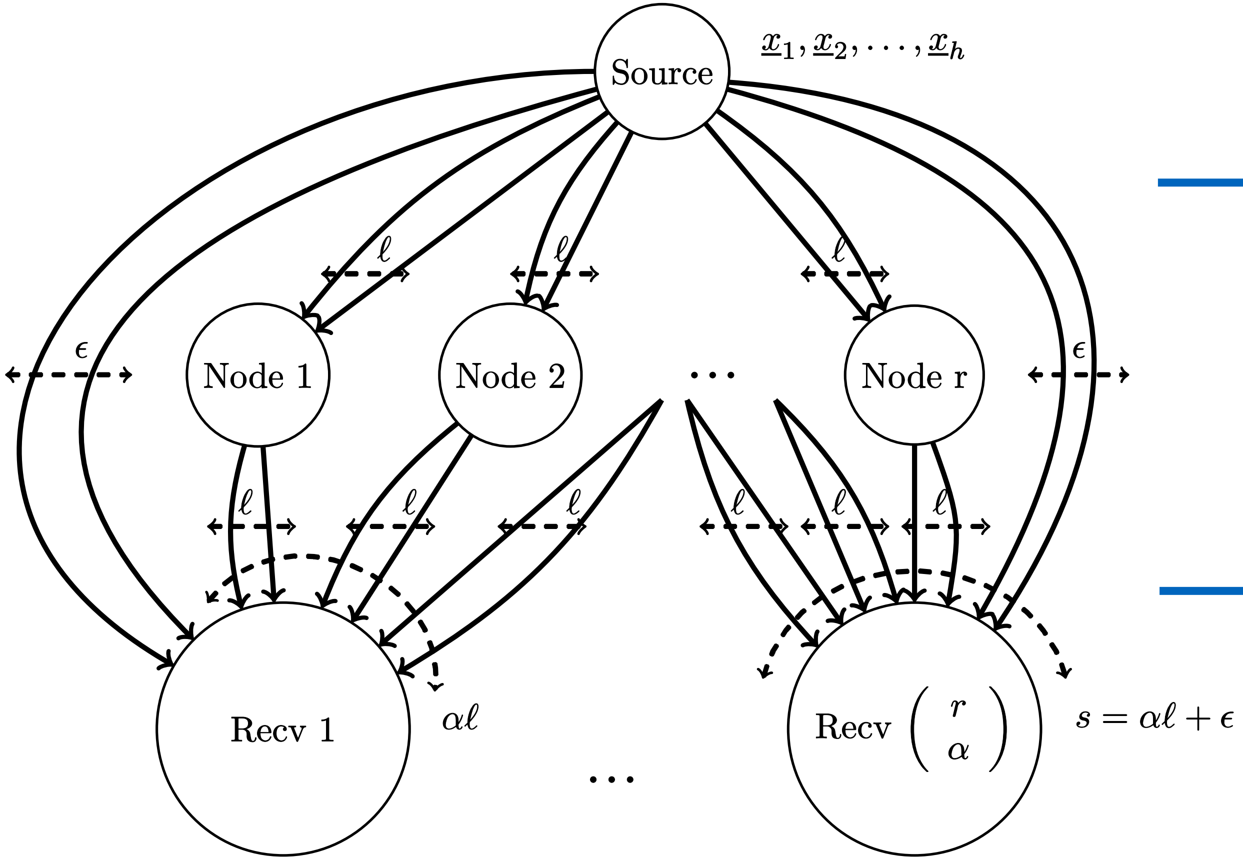
3 Layers from Top to Bottom

Source transmits $x_1, \ldots, x_h \in \mathbb{F}_{q_s}$ (Scalar NC) or $\underline{x}_1, \ldots, \underline{x}_h \in \mathbb{F}_q^t$ (Vector NC)

Intermediate Nodes (Nodes)

The number of nodes r is interesting to study!

Our Choice of Network Model (cont.)



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3 Layers from Top to Bottom

Source transmits $x_1, \dots, x_h \in \mathbb{F}_{q_s}$ (Scalar NC)
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Intermediate Nodes (Nodes)

The number of nodes r is interesting to study!

Receivers

Our Choice of Network Model (cont.)

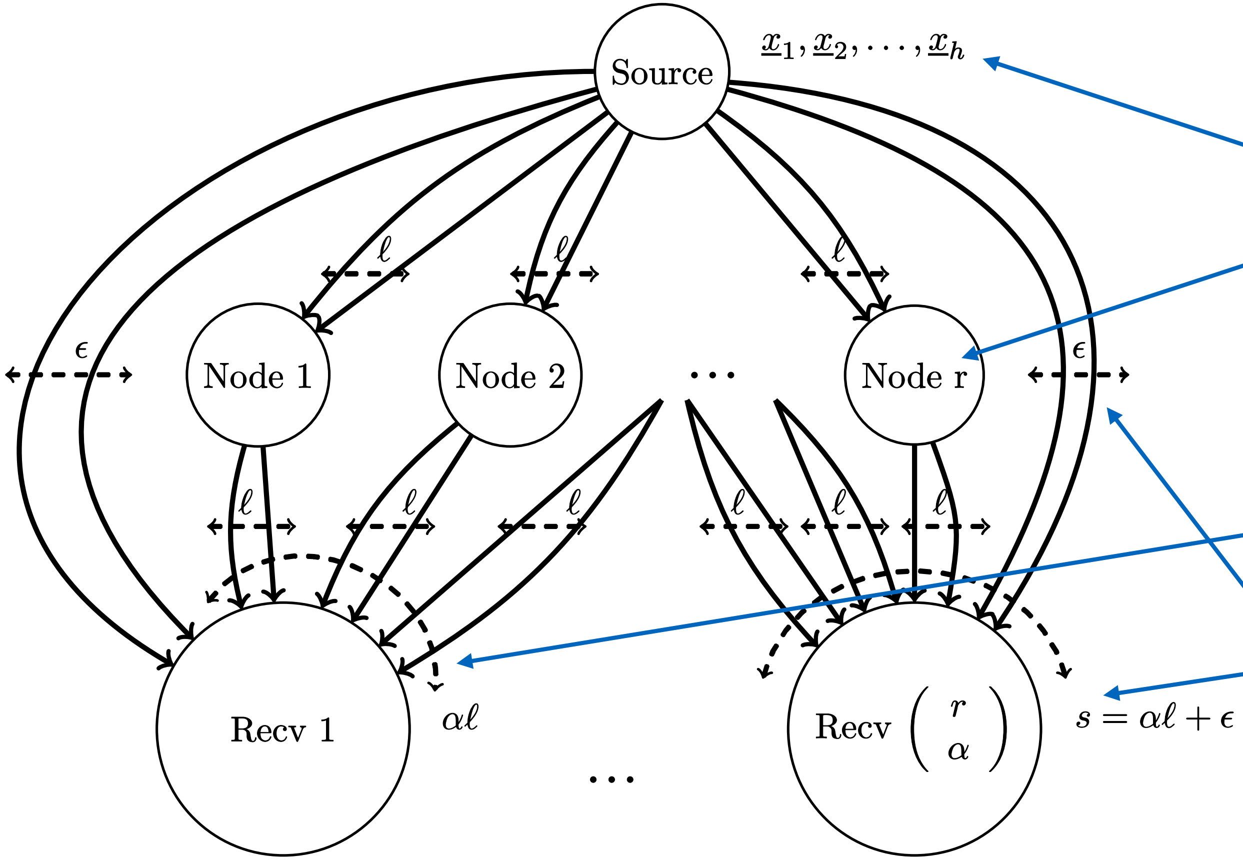


Figure 4. Generalized Combination Network (GCN)

h	The number of source messages
r	The number of nodes in the middle layer
$\binom{r}{\alpha}$	The number of receivers
ℓ	Parallel links of a data unit
α	Links connected to a receiver from nodes
ϵ	Links connected to a receiver from the source
s	Total links connected to a receiver

Our Choice of Network Model (cont.)

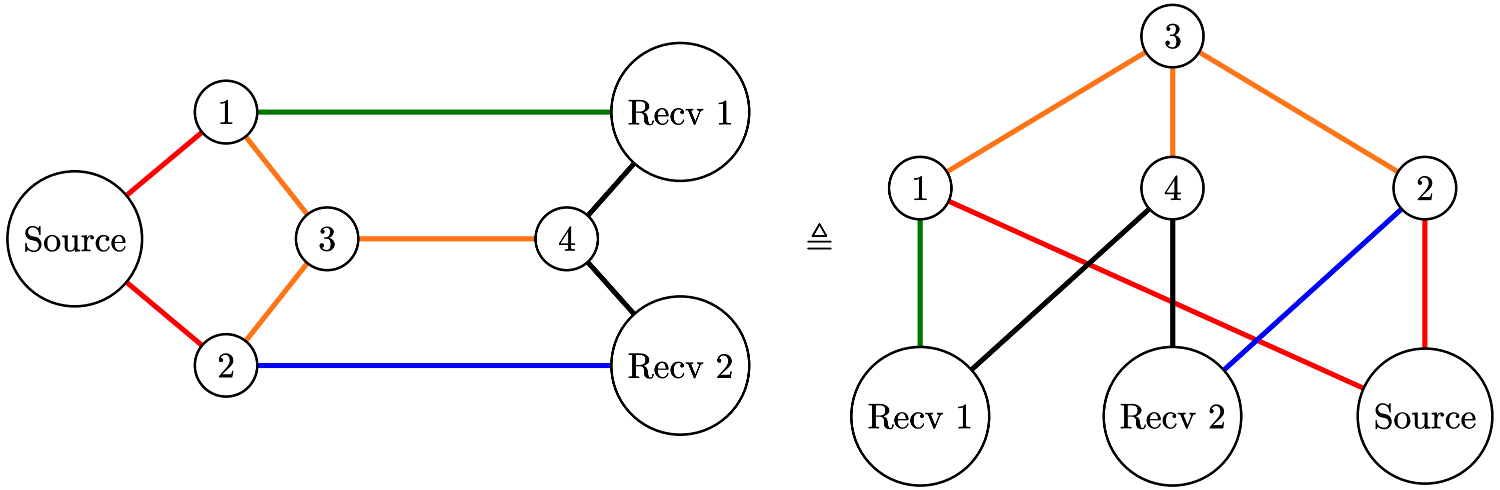


Figure 5. The butterfly network is represented as a combination network [Maheshwar et al. (2012)]

Outline

Network as a Matrix Channel

Scalar NC | Vector NC

$$\underbrace{\begin{bmatrix} y_j^{(1)} \\ \vdots \\ y_j^{(s)} \end{bmatrix}}_{\mathbb{F}_{q_s}^{s}} = \underbrace{\underline{\underline{A}}_j}_{\mathbb{F}_{q_s}^{s\times h}} \cdot \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_h \end{bmatrix}}_{\mathbb{F}_{q_s}^{h}} \qquad \underbrace{\begin{bmatrix} \underline{y}_j^{(1)} \\ \vdots \\ \underline{y}_j^{(s)} \end{bmatrix}}_{\mathbb{F}_{q}^{st}} = \underbrace{\underline{\underline{A}}_j}_{\mathbb{F}_{q}^{st\times th}} \cdot \underbrace{\begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_h \end{bmatrix}}_{\mathbb{F}_{q}^{th}}$$

$$\underline{\underline{A}}_j = \begin{bmatrix} \underline{a}^{(r_1)} \\ \vdots \\ \underline{a}^{(r_{\alpha l})} \\ \underline{b}^{(\epsilon(j-1)+1)} \\ \vdots \\ \underline{b}^{(\epsilon j)} \end{bmatrix}$$

Network as a Matrix Channel (cont.)

Scalar NC

$$\underbrace{\begin{bmatrix} y_j^{(1)} \\ \vdots \\ y_j^{(s)} \end{bmatrix}}_{\mathbb{F}_{q_s}^{s}} = \underbrace{\underline{\underline{A}}_j}_{\mathbb{F}_{q_s}^{s\times h}} \cdot \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_h \end{bmatrix}}_{\mathbb{F}_{q_s}^{h}}$$

Vector NC

$$\underbrace{\begin{bmatrix} \underline{y}_j^{(1)} \\ \vdots \\ \underline{y}_j^{(s)} \end{bmatrix}}_{\mathbb{F}_{q}^{st}} = \underbrace{\underline{\underline{A}}_j}_{\mathbb{F}_{q}^{st\times th}} \cdot \underbrace{\begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_h \end{bmatrix}}_{\mathbb{F}_{q}^{th}}$$

$$\underline{\underline{A}}_j = \begin{bmatrix} \underline{\underline{A}}^{(r_1)} \\ \vdots \\ \underline{\underline{A}}^{(r_{\alpha l})} \\ \underline{\underline{B}}^{(\epsilon(j-1)+1)} \\ \vdots \\ \underline{\underline{B}}^{(\epsilon j)} \end{bmatrix}$$

$$\underline{\underline{A}}_j = \begin{bmatrix} \underline{a}^{(r_1)} \\ \vdots \\ \underline{a}^{(r_{\alpha l})} \\ \underline{b}^{(\epsilon(j-1)+1)} \\ \vdots \\ \underline{b}^{(\epsilon j)} \end{bmatrix}$$

Network as a Matrix Channel (cont.)

Scalar NC

$$\underbrace{\begin{bmatrix} y_j^{(1)} \\ \vdots \\ y_j^{(s)} \end{bmatrix}}_{\mathbb{F}_{q^s}^{s}} = \underbrace{\underline{\underline{A}}_j}_{\mathbb{F}_{q^s}^{s\times h}} \cdot \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_h \end{bmatrix}}_{\mathbb{F}_{q^s}^{h}}$$

Vector NC

$$\underbrace{\begin{bmatrix} \underline{y}_j^{(1)} \\ \vdots \\ \underline{y}_j^{(s)} \end{bmatrix}}_{\mathbb{F}_{q}^{st}} = \underbrace{\underline{\underline{A}}_j}_{\mathbb{F}_{q}^{st\times th}} \cdot \underbrace{\begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_h \end{bmatrix}}_{\mathbb{F}_{q}^{th}}$$

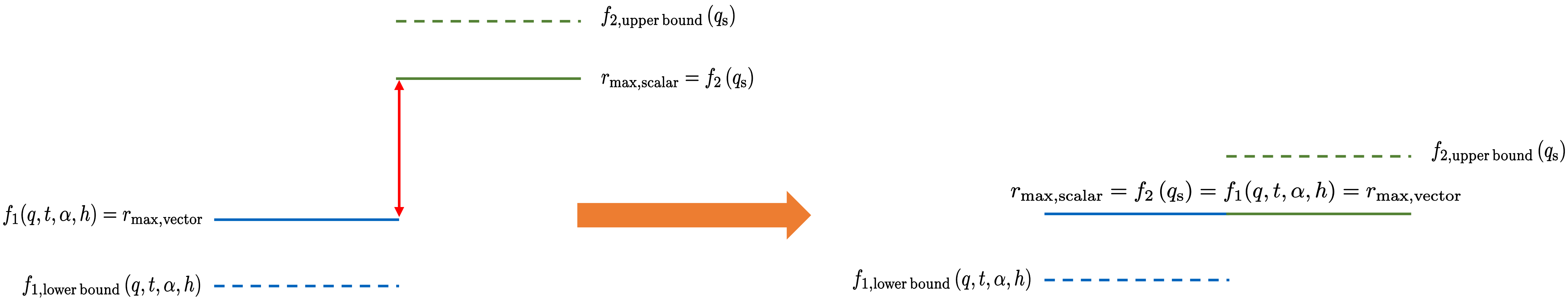
$$\underline{\underline{A}}_j = \begin{bmatrix} \underline{\underline{A}}^{(r_1)} \\ \vdots \\ \underline{\underline{A}}^{(r_{\alpha l})} \\ \underline{\underline{B}}^{(\epsilon(j-1)+1)} \\ \vdots \\ \underline{\underline{B}}^{(\epsilon j)} \end{bmatrix}$$

$$\underline{\underline{A}}_j = \begin{bmatrix} \underline{a}^{(r_1)} \\ \vdots \\ \underline{a}^{(r_{\alpha l})} \\ \underline{b}^{(\epsilon(j-1)+1)} \\ \vdots \\ \underline{b}^{(\epsilon j)} \end{bmatrix}$$

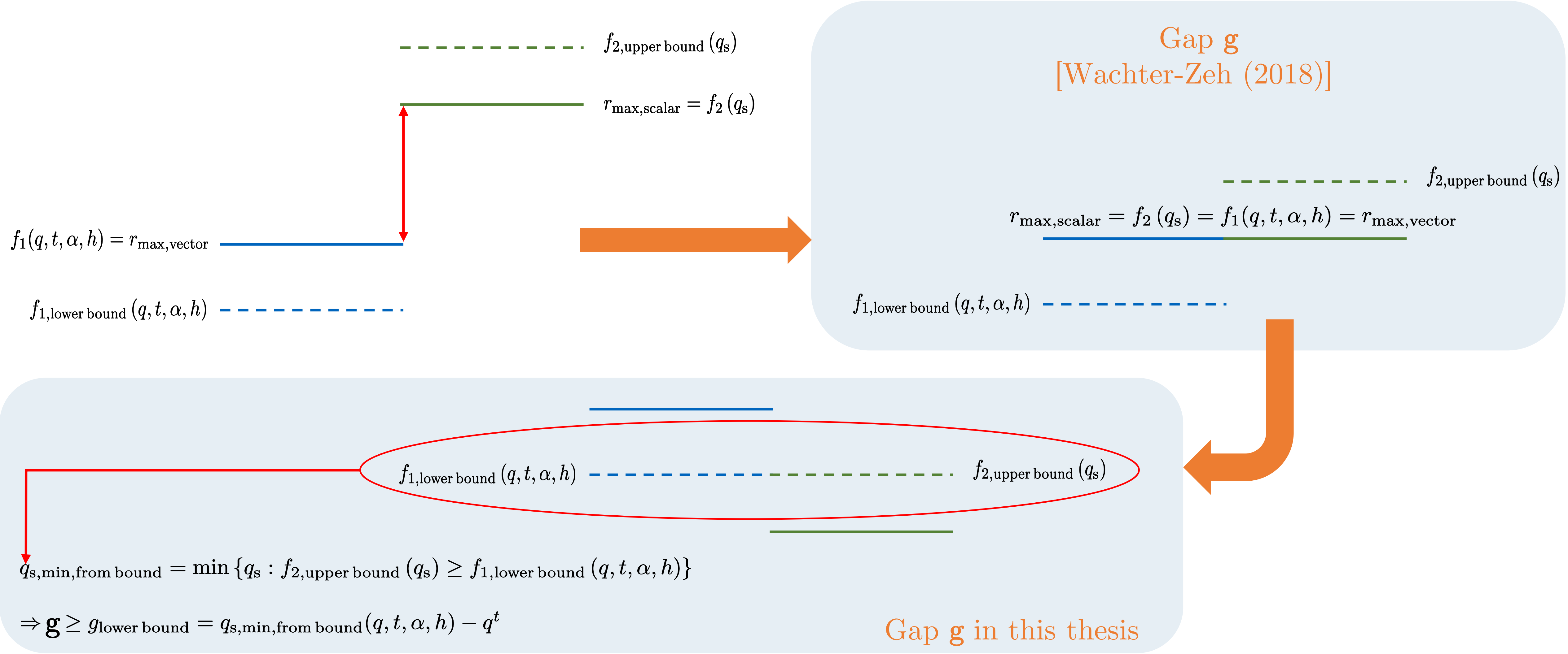
By using the vector coding, the upper bound number of solutions increases from q^{tsh} to q^{t^2sh}.

Outline

Gap size between scalar and vector solutions



Gap size between scalar and vector solutions (cont.)



Gap size between scalar and vector solutions (cont.)

Network	Gaps for a specific vector solution [Etzion and Wachter-Zeh (2018)]	Lower bounds on gaps for a general vector solution [Corollary 5.4 and Corollary 5.3]
$(\epsilon = 0, \ell = 1) - \mathcal{N}_{h,r,s}$	N/A	N/A
$(\epsilon \geq 1, \ell = 1) - \mathcal{N}_{h,r,s}$	N/A	$q^{\frac{\epsilon(\alpha-h+\epsilon)}{(\alpha-1)(\alpha-h+\epsilon+1)(h-\epsilon-1)}t^2+\mathcal{O}(t)}$
$(\epsilon = 1, \ell > 1) - \mathcal{N}_{h=2\ell,r,s=2\ell+1}$	$q^{t^2/2+\mathcal{O}(t)}$	$q^{t^2/l+\mathcal{O}(t)}$
$(\epsilon = \ell - 1, \ell) - \mathcal{N}_{h=2\ell,r,s=3\ell-1}$	$q^{t^2/2+\mathcal{O}(t)}$	N/A

Table 1. Lower bounds on gaps were found in this study.

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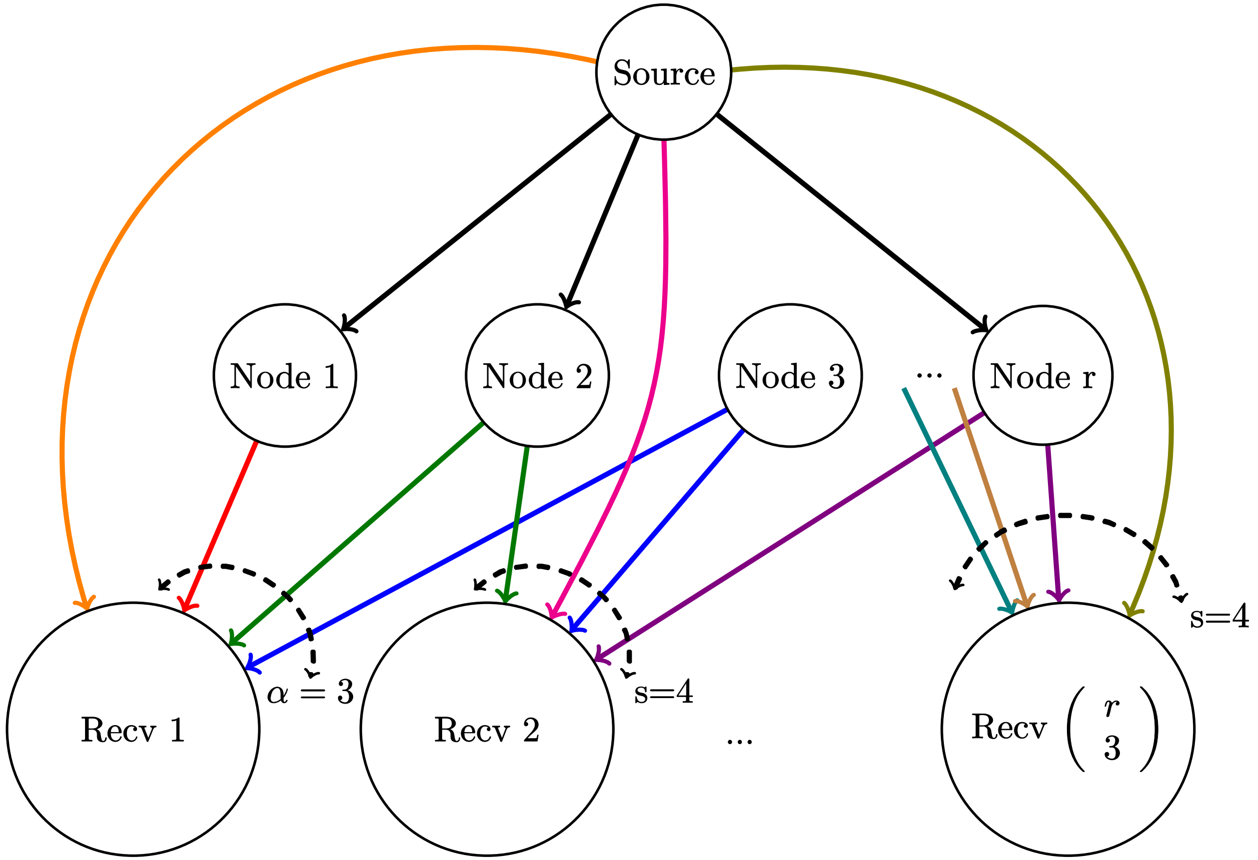
Proof of a gap for a network with 3 messages

We show the proof of the most simple case. Other cases considered in this study are similar to this proof.



Proof of a gap for a network with 3 messages (cont.)

Each receiver R_j has to solve a linear equation system of $3t$ variables with $4t$ equations to recover 3 source messages as below:

$$\begin{bmatrix} \underline{y}_j^{(1)} \\ \underline{y}_j^{(2)} \\ \underline{y}_j^{(3)} \\ \underline{y}_j^{(4)} \end{bmatrix} = \underline{\underline{A}}_j \cdot \underline{x} = \begin{bmatrix} \underline{\underline{A}}^{(r_1)} \\ \underline{\underline{A}}^{(r_2)} \\ \underline{\underline{A}}^{(r_3)} \\ \underline{\underline{B}}^{(j)} \end{bmatrix} \cdot \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \end{bmatrix},$$

with $\underline{x}_1, \ldots, \underline{x}_3 \in \mathbb{F}_q^t$

$\underline{y}_j^{(1)}, \ldots, \underline{y}_j^{(4)} \in \mathbb{F}_q^t$

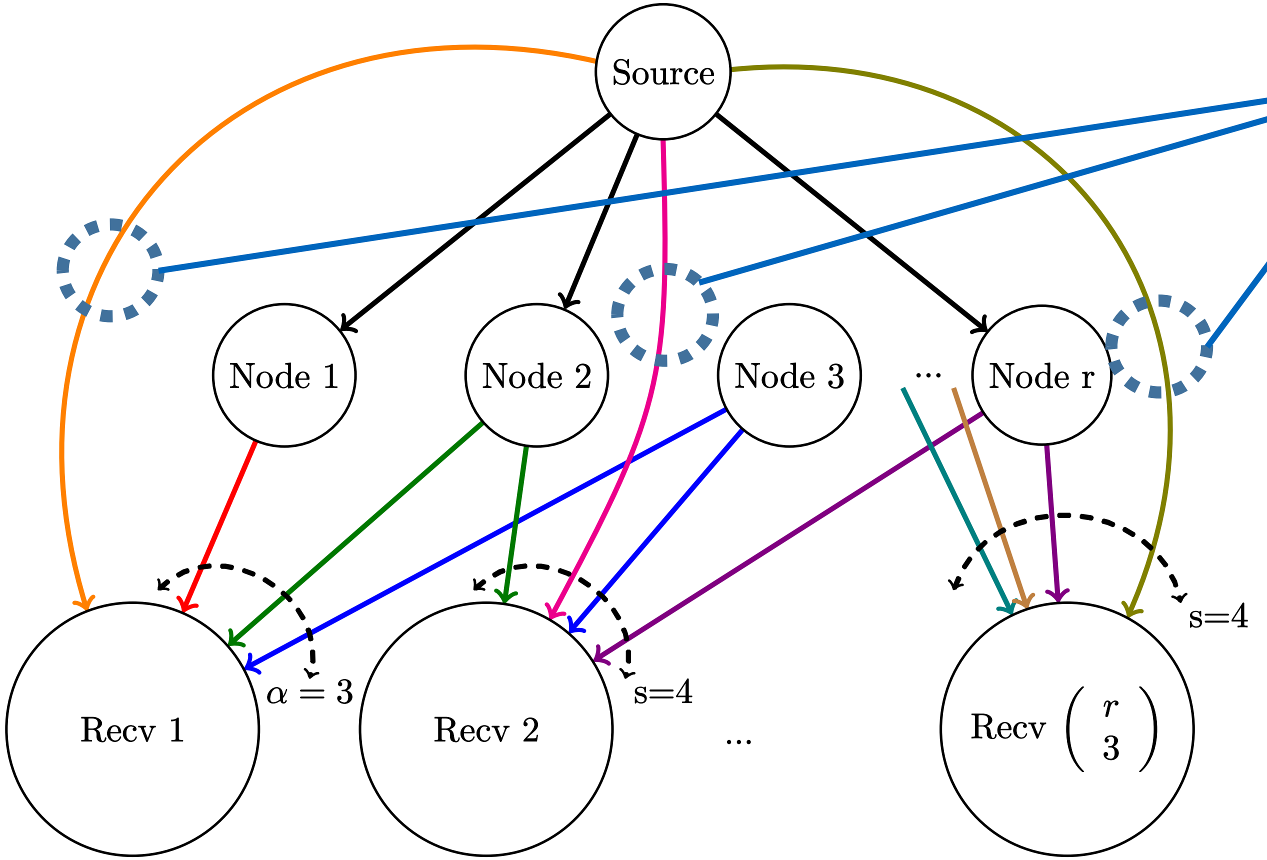
$\underline{\underline{A}}^{(r_1)}, \ldots, \underline{\underline{A}}^{(r_3)} \in \mathbb{F}_q^{t \times 3t}$ for $1 \leq r_1 < r_2 < r_3 \leq r$

$\underline{\underline{B}}^{(j)} \in \mathbb{F}_q^{t \times 3t}$ for $j \in \left\{1, \ldots, \binom{r}{3}\right\}$

The network is solvable if $\underline{\underline{A}}$ has full rank,

$$\operatorname{rk} \begin{bmatrix} \underline{\underline{A}}_j^{(r_1)} \\ \underline{\underline{A}}_j^{(r_2)} \\ \underline{\underline{A}}_j^{(r_3)} \\ \underline{\underline{B}}^{(j)} \end{bmatrix} \geq 3t.$$

Proof of a gap for a network with 3 messages (cont.)



$\underline{\underline{B}}^{(j)}$ can be independently chosen for any receiver R_j, we actually need only

$$\mathrm{rk}\begin{bmatrix} \boldsymbol{A}^{(r_1)} \\ \boldsymbol{A}^{(r_2)} \\ \boldsymbol{A}^{(r_3)} \end{bmatrix} \geq 2t$$

$$\mathrm{rk}\begin{bmatrix} \underline{\underline{A}}_j^{(r_1)} \\ \underline{\underline{A}}_j^{(r_2)} \\ \underline{\underline{A}}_j^{(r_3)} \\ \underline{\underline{B}}^{(j)} \end{bmatrix} \geq 3t.$$

Proof of a gap for a network with 3 messages (cont.)

We then apply the Local lemma to calculate a gap for the network.

Lemma: Symmetric Lovász Local Lemma (LLL) [Schwarz et al. (2013)]

A set of events $\mathcal{E}_i$, such that each event occurs with probability at most p. If each event is independent of all others except for at most of them d and $4pd \leq 1$, then:

$$\Pr\left[\bigcap_{i=1}^{n} \overline{\mathcal{E}}_i\right] > 0. \square$$