PMT Additional Exercises (Week 9)

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1 Induction

[Part of FTA] Prove (by induction) that every positive integer greater than 1 is a product of primes.

2 Discrete Structures: Selected questions

All questions are verbatim or adapted; sources indicated.

1. *[2008 exam Q3d, 9 mins]* Say that a binary relation $\sim \subset A \times A$ is *combinative* if it fulfils:

$$(\forall x, y, z) x \sim z \wedge y \sim z \Rightarrow y \sim x$$

   Prove that if R is reflexive and combinative, then R is an equivalence relation.

2. *[2007 exam Q2c, 6 mins]* Given that R, S are transitive relations on a domain A, show that $R \cap S$ is also transitive.

3. *[2015 Q1e(iii) and 2016 Q1c(iii), 3 mins]* Let $f : A \to B, g : B \to C$ be (total) functions. Prove that if $g \circ f$ is surjective, then so is g.

4. *[General, 4 mins]* For each of the following functions, write down whether they are injective or surjective individually.
   (a) $f : \mathbb{R} \to \mathbb{R}^+, f(x) = e^x$
   (b) $f : [-1, 0.5) \to [-2, 0.25], f(x) = x - x^2$

5. *[Selected M1F (A. Corti), no more than 1 min each]* True or false:
   (a) Every subset $S \subset \mathbb{N}$ has a least element (under the ordering $\leq$).
   (b) Let $f : A \to B$ be a function. $R = \{(a_1, a_2) \in A \times A : f(a_1) = f(a_2)\}$ is an equivalence relation on A.
   (c) Let $f : A \to B$ be a function. Suppose that there is a function $g : B \to A$ such that $(\forall a \in A) g \circ f(a) = a$, and another function $h : B \to A$ with $(\forall b \in B) f \circ h(b) = b$. Then $g = h$.
   (d) Let X be a finite set. The set of all functions $f : \mathbb{N} \to X$ is countable.
   (e) Let X be a finite set. The set of all functions $f : X \to \mathbb{N}$ is countable.

3 Solutions to induction

Let $P(k)$ denote that k is the product of primes. Use strong induction.

Base case: $P(2)$ holds as $2 = 2$ is the product of primes.

Inductive step: (I.H.) Suppose $P(i)$ holds for all $i < k \in \mathbb{N}$. Want to show $P(k)$.

Suppose $k \in \mathbb{N}$ is not prime (otherwise we get $P(k)$ immediately). Then $k = a \times b$ for some $a, b < k$.

But (by I.H.) $P(a)$ and $P(b)$ hold, so $a = p_1 \times \cdots \times p_i$ and $b = q_1 \times \cdots \times q_j$, where all p, q are prime.

So $k = p_1 \times \cdots \times p_i \times q_1 \times \cdots \times q_j$ and $P(k)$ holds.

So $P(n)$ holds for any $2 \leq n \in \mathbb{N}$ by the principle of induction.

4 Solutions to revision

Note: there is a distinct possibility that any of the solutions may be incorrect or insufficient.

1. To show: R is an equivalence relation. We will call the combinative property (C).
   (R) Reflexivity: given.
   (S) Symmetry: Suppose aRb. We have bRb from reflexivity. Substituting b for z, a for x in (C), we have bRa.
   (T) Transitivity: Suppose aRb, bRc. We want to show aRc. We have cRb by (S) on bRc, and aRb (given). Substituting c, a, b for x, y, z respectively in (C), we get aRc, as required.

2. R, S both subsets of $A \times A$. Transitivity means that $(\forall a, b, c \in A)\,((a, b), (b, c) \in R \Rightarrow (a, c) \in R)$, and similarly for S. Now suppose $(a, b), (b, c) \in R \cap S$. Then $(a, b), (b, c) \in R$ and $(a, b), (b, c) \in S$. By transitivity, $(a, c) \in R$ and $(a, c) \in S$, and by definition of intersection, $(a, c) \in R \cap S$. So $R \cap S$ is also transitive.

3. Suppose $g \circ f$ is surjective. Then $(\forall c \in C)(\exists a \in A) g(f(a)) = c$ (*).
   To show: g is surjective, i.e. $(\forall c \in C)(\exists b \in B) g(b) = c$. But this is clear: we can set $b = f(a)$, as existence is guaranteed by (*).

4. (a) $f : \mathbb{R} \to \mathbb{R}^+, f(x) = e^x$. Injective and surjective.
   (b) $f : [-1, 0.5) \to [-2, 0.25], f(x) = x - x^2$. Injective as monotonic increasing. Not surjective as 0.25 not mapped.

5. Informal (1 minute) 'proofs'
   (a) False. $\varnothing$.
   (b) True. Check R, S, T.
   (c) True. Notice that $|B| \geq |A|$ as $g \circ f$ is the identity map. Likewise $|A| \geq |B|$. Now satisfy yourself that $g = f^{-1} = h$.
   (d) False. Let $X = \{0, 1\}$. There is a bijection between the set of functions $f : \mathbb{N} \to X$ and the power set of $\mathbb{N}$.
   (e) True. Let $|X| = n$. Easy proof by induction:
   Base case: $n = 1$ trivial (as is $n = 0$)
   Inductive step: Assume countably many $f : \{0, 1, \cdots, k\} \to \mathbb{N}$, ordered $f_0, f_1, \cdots$. Now there are countably many choices for $f_i(k+1)$. The Cartesian product of two countable sets is also countable, and the result follows.