PMT Additional Exercises (Week 4)

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October 26th, 2017

1 Discrete Structures: Equivalence Classes

In the lectures you have constructed the rational numbers $\mathbb{Q}$ from the sets $\mathbb{Z}, \mathbb{N}$. Here we construct $\mathbb{Z}$, since we only understand $\mathbb{N}$. (This is glossed over in def. 3.4.)

Let $S = \mathbb{N} \times \mathbb{N}$. Define a relation $\sim$ on S by $(m_1, n_1) \sim (m_2, n_2)$ (or $((m_1, n_1), (m_2, n_2)) \in \sim$) iff $m_1 + n_2 = m_2 + n_1$. Informally, we can think of (m, n) as $m - n$.

[construction thanks to A. Corti]

1. Show that $\sim$ is an equivalence relation.
2. Show that $(m_1, n_1) \sim (m_2, n_2)$ iff $(m_1 + k, n_1 + k) \sim (m_2 + k, n_2 + k)$ for any $k \in \mathbb{N}$.
3. Show that if $(m_1, k) \sim (m_2, k)$ then $m_1 = m_2$ for any $k \in \mathbb{N}$.
4. Similarly, show that if $(k, n_1) \sim (k, n_2)$ then $n_1 = n_2$ for any $k \in \mathbb{N}$.

   Now let $Z = S/\sim$, and define $+$ on Z by $[(m_1, n_1)] + [(m_2, n_2)] = [(m_1 + m_2, n_1 + n_2)]$.

   For convenience, represent members of Z by $\bar{z} = [(z, 0)]$ and $-\bar{z} = [(0, z)]$ for any $z \in \mathbb{N}$.
5. Show that $\bar{z} + (-\bar{z}) = [(0, 0)]$.
6. Fix any $\bar{z} \in Z$. Show that either $z \in \mathbb{N}$ or (abusing notation) $-z \in \mathbb{N}$.
7. (*) Write down an invertible function $f : \mathbb{Z} \to Z$. Show that $\forall a, b \in \mathbb{Z}, f(a + b) = f(a) + f(b)$.

2 Logic: Adequacy

[Informal definition] Say a set of connectives is *adequate* if any propositional formula of n variables can be written as some other formula only using variables $p_1, ..., p_n$ and the connectives.

For instance, $\{\neg, \vee\}, \{\neg, \rightarrow\}$ are both adequate. We've also seen that $\{\uparrow\}$ (NAND) is adequate.

1. We introduce a new connective, NOR ($\downarrow$). Show that this connective is adequate.
2. (*) Show that apart from NOR and NAND, there are no other single adequate (binary) connectives.
   [thanks to D. Evans]

3 Solutions to Equivalence Classes

1. Reflexivity. Clearly $m_1 + n_1 = m_1 + n_1$.
   Symmetry. Suppose $m_1 + n_2 = m_2 + n_1$. Then clearly $m_2 + n_1 = m_1 + n_2$.
   Transitivity. Suppose $m_1 + n_2 = m_2 + n_1$ and $m_2 + n_3 = m_3 + n_2$. Then $m_1 + n_3 = (m_2 + n_1 - n_2) + (m_3 + n_2 - m_2) = n_1 + m_3$.

2. We have $m_1 + n_2 = m_2 + n_1$. For $\Rightarrow$, verify by adding k to all four terms. For $\Leftarrow$, fix $k = 0$.

3. Suppose $m_1 + k = m_2 + k$. Subtract k from both sides (valid as each side $\geq k$).

4. Similar to previous part.

5. This is just $[(z, 0)] + [(0, z)] = [(z + 0, 0 + z)] = [(z, z)]$ as defined by $+$ on Z. Trivially we have $(k, k) \sim (0, 0) \forall k \in \mathbb{N}$ since $k + 0 = 0 + k$. So clearly $[(z, z)] = [(0, 0)])$ since they belong to the same equivalence class.

6. Write $\bar{z} = [(m, n)]$. Either $m \geq n$, then $\bar{z} = [(m, n)] = [(m - n, 0)]$, or $m \leq n$, where $\bar{z} = [(m, n)] = [(0, n - m)]$. Note that in the case $m = n$, both are valid and of course $\bar{z} \sim (0, 0)$.

7. Define $f(x) = [(x, 0)]$ when $x \geq 0$ and $f(x) = [(0, -x)]$ otherwise. Simply enumerate 4 possibilities:

   (a) $a, b \geq 0$. $f(a + b) = [(a + b, 0)] = [(a, 0)] + [(b, 0)] = f(a) + f(b)$.
   (b) $a < 0 \leq b$. $f(a + b) = f(b - (-a)) = [(b, -a)] = [(b, 0)] + [(0, -a)] = f(a) + f(b)$.
   (c) $b < 0 \leq a$. Reverse a, b above.
   (d) $a, b < 0$. $f(a + b) = f(0 - (-(a + b))) = [(0, -(a + b)] = [(0, -a)] + [(0, -b)] = f(a) = f(b)$.

   Slightly tedious.

4 Solutions to Adequacy

1. We need only use NOR to replicate a set of adequate connectives. Write down: $\neg p \equiv p \downarrow p$ and $p \wedge q \equiv (p \downarrow p) \downarrow (q \downarrow q)$.

2. Note: there are 2^4 possible binary connectives. Suppose that a binary connective $\cdot$ is adequate. We know that $\top \cdot \top \equiv \bot$ and $\bot \cdot \bot \equiv \top$, otherwise we would be unable to express negation. Only 4 possibilities remain, of which two are NOR and NAND. In the remaining two cases, $p \cdot q$ would be logically equivalent to either $\neg p$ or $\neg q$ (draw out the truth table!), which isn't adequate by itself.