# PMT Revision (Week 10)

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## 1 Logic

[See handout: C140 2016 Q1c]

## 2 Relations

1. *[C142 2015 Q1d.v]* Let $R \subseteq A^2$. Prove that if $R$ is symmetric, then $R \circ R$ is also symmetric.
2. *[C142 2005 Q2b.ii]* Let $R$ be a binary relation on the natural numbers defined by $xRy$ if and only if $y = 2x$. Describe the transitive closure of $R$.
3. *[C142 2005 Q2b.iii]* A binary relation $R$ is antisymmetric if and only if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$. How many binary relations on $A = \{1, 2, 3, 4\}$ are both symmetric and antisymmetric? Justify your answer.
*[Bonus: What about an arbitrary set?]*

## 3 Functions

*[C142 2002 Q1c]* Let $f : A \to B$ and $g : B \to C$ be functions. Prove the following (otherwise give a counterexample):

1. $f$ surjective implies $g \circ f$ surjective;
2. $g \circ f$ injective implies $f$ injective;
3. $g \circ f$ injective implies $g$ injective;
4. $f, g$ surjective implies $g \circ f$ surjective.

Note: these solutions are only outlines.

# 4 Solution to Alternative Elimination

1. I would argue that (2-4) are sufficient for $A \to B$. From here, we can use EM to get $A \vee \neg A$, and $\vee E$ to get $B$.
2. First use $\vee I$ to get $A \vee B \vee C$. Assume $A$ to get $C$ then $B \vee C$ (using $\vee I$). Now use $Alt \vee E$ to get $B \vee C$. Use $Alt \vee E$ again to get $C$.
3. Observe that $\vee E$ essentially follows from the previous part. (Say: if we have ..., then get $\vdash^a C$)
4. $\vdash^a$ is complete if every valid statement expressed in propositional logic can be proven using only the rules of $\vdash^a$. It is sound if every statement that can be proven under the system is valid.
5. We know from the lectures that $\vdash$ is sound and complete. Since $Alt \vee E$ is a derived rule of $\vdash$, soundness is preserved (we can't prove anything new, hence nothing invalid). Since $\vee E$ is a derived rule of $\vdash^a$, completeness is preserved (we can prove at least the same things that $\vdash$ can).

# 5 Solutions to Relations

1. Suppose that $R$ is symmetric, i.e. $(a,b) \in R \Rightarrow (b,a) \in R$. Recall that $R \circ R = \{(a,c) : (\exists b)\,((a,b) \in R \wedge (b,c) \in R)\}$.
   Fix $(a,c) \in R \circ R$. To show: $(c,a) \in R \circ R$. But this is clear as $\exists b (a,b), (b,c) \in R$, so $(b,a), (c,b) \in R$ by symmetry, and $(c,a) \in R \circ R$.
2. $x\bar{R}y$ if and only if $\exists n \in \mathbb{N}_{>0} : y = 2^n x$.
3. $R$ is symmetric if and only if $(b,a) \in R$ whenever $(a,b) \in R$. Let $R$ be both symmetric and antisymmetric. Suppose $(a,b) \in R$. Then $(b,a) \in R$, so $a = b$.
   We easily see that the possible relations $R$ are those which only contain pairs of the form $(a,a)$. So 16.

# 6 Solutions to Functions

1. False, let $C = \{0,1\}$ and $g(x) = 0$... (write a surjective $f$ and sets)
2. We show the contrapositive. Suppose $f$ not injective. Then $\exists a_1, a_2 \in A : f(a_1) = f(a_2) \in B$. Clearly $g(f(a_1)) = g(f(a_2))$ (as $g$ is a well-defined function), so $g \circ f$ is not injective.
3. False, let $A = \{0,1\}, B = \{0,1,2\}, C = \{0,1\}, f = \mathrm{id}, g(0) = g(2) = 0, g(1) = 1$.
4. Suppose $f, g$ surjective, i.e. $\forall b \in B \exists a \in A : f(a) = b$ and $\forall c \in C \exists b \in B : g(b) = c$.
   Fix $c \in C$. Want: $\exists a \in A : g(f(a)) = c$. We have $b \in B$ s.t. $g(b) = c$. Similarly $\exists a \in A : f(a) = b$, which is what was required.