PMT Additional Exercises (Week 3)

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1 Induction for graphs

A tree is a connected acyclic graph.

Claim: any tree with $n \in \mathbb{N}_{>0}$ vertices has $n - 1$ edges. Consider the following proof:

Base case Suppose T has exactly one vertex v. Clearly any edges must be from v to itself, as there are no other vertices. This would create a 1-cycle, so T has no edges.

Inductive step Fix a natural number k.

Inductive Hypothesis: any tree with k vertices has $k - 1$ edges.

We want to show that a tree with $k + 1$ vertices has k edges.
Consider a tree T with k vertices. It has $k - 1$ edges (I.H.).
Add a vertex v, connecting it to any of the original vertices with exactly 1 edge.
This produces a tree (as it is connected and acyclic) with $k+1$ vertices, which has $k-1+1 = k$ edges as required.

1. This proof is *wrong*. Why?
2. The claim holds. Give a valid proof.

2 Solution to flawed proof

1. A hint would be that the base case is valid *because* there is only one tree containing a single vertex.

   The trouble lies with the inductive step, as we fix a tree T with k vertices. Hence we only prove that trees of $k+1$ vertices *containing* T *as a subgraph* have k edges.

   We need to show that 'every tree can be constructed in this way', but it's in fact easier to fix our tree first and prove the result about it.

2. Use strong induction.

   Fix T with $k+1 > 1$ vertices. To prove: T has k edges.
   Remove any edge. We get exactly two subtrees (exercise) T', T'' with v', v'' vertices respectively, and $k+1 = v' + v''$.
   From the same I.H., T', T'' have a total of $v' - 1 + v'' - 1 = k + 1 - 2 = k - 1$ edges.
   Adding the original edge, T has k edges.

   [Remark: alternatively, we can conveniently pick a 'leaf' node, with only one edge, and apply the base case and inductive hypothesis, obviating the need for strong induction. However, you need to prove that such a node exists.]