
Policy Gradients Beyond Expectations: Conditional Value-at-Risk

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Abstract

Conditional Value at Risk (CVaR) is a prominent risk measure that is being used extensively in various domains such as finance. In this work we present a new formula for the gradient of the CVaR in the form of a conditional expectation. Our result is similar to policy gradients in the reinforcement learning literature. Based on this formula, we propose novel sampling-based estimators for the CVaR gradient, and a corresponding gradient descent procedure for CVaR optimization. We evaluate our approach in learning a risk-sensitive controller for the game of Tetris, and propose an importance sampling procedure that is suitable for such domains.

1. Introduction

In simulation based optimization (Rubinstein & Kroese, 2011), a stochastic system with a set of tunable parameters θ is considered, and a performance measure $J(\theta)$ assigns a deterministic value to the system under each parameter setting. Gradient estimation methods (Fu, 2006) use Monte-Carlo (MC) sampling of system realizations to estimate the performance gradient $\frac{\partial J(\theta)}{\partial \theta}$, which is important both for parameter optimization, using stochastic gradient descent methods, and for sensitivity analysis.

Among the gradient estimation methods, the Likelihood-Ratio (LR; Glynn 1990; also known as the *score function*) is particularly popular, as it is applicable to diverse domains such as queueing systems, inventory management, and financial engineering; see Fu (2006) for a survey. The LR method is also a prominent approach in reinforcement

learning (RL; Sutton & Barto 1998), where it is commonly known as the *policy gradient* method (Baxter & Bartlett, 2001; Marbach & Tsitsiklis, 1998), and has been successfully applied to many domains such as robotics (Peters & Schaal, 2008).

Gradient estimation methods, and the LR method in particular, however, require that the performance measure be represented as an expectation of some outcome of the system, such as the waiting time in queues, or the long-term discounted return in RL. This is a serious drawback for *risk-sensitive* optimization, in which additional statistics of the system outcome, such as its variance or percentile, are taken into account (Luenberger, 1998).

Indeed, in this work we focus on risk-sensitive optimization, and specifically on the *Conditional Value at Risk* (CVaR; Rockafellar & Uryasev 2000) risk measure. The CVaR¹ at level $\alpha\%$ of some random variable is defined as its expected value in the worst $\alpha\%$ of the outcomes. The CVaR has been used and studied extensively, in particular in financial domains, and is known to be both a *coherent* (Artzner et al., 1999) and a *spectral* (Acerbi, 2002) risk measure. In other domains, the CVaR is gaining popularity due to its intuitive notion of risk on the one hand, and favorable mathematical properties on the other (Rockafellar & Uryasev, 2000; Ruszczyński & Shapiro, 2006).

In this paper we extend the LR method for gradient estimation of the CVaR performance measure. Our main contribution is a new formula for the CVaR gradient, which is in the form of a conditional expectation, similar to the standard LR derivation. This formula leads to a MC sampling-based estimator, and a stochastic gradient descent optimization procedure. Our second contribution is an application of our method to the RL domain, where we optimize a risk-sensitive policy for the Tetris game. To our knowledge, this

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¹Also known as expected shortfall or expected tail loss.

is the first CVaR optimization method that is applicable to such a challenging RL domain. In addition, we propose an importance-sampling (IS) procedure to reduce the variance in the gradient estimate, and suggest a heuristic family of IS distributions that are suitable for optimizing the CVaR in RL.

Closely related to our work are the studies by [Hong & Liu \(2009\)](#) and [Scaillet \(2004\)](#), who proposed perturbation analysis (PA) style estimators for the gradient of the CVaR. In PA, as opposed to LR, it is assumed that the tunable parameters θ do not affect the distribution of the stochastic system, but only affect the value that is assigned to each statistical outcome. This assumption makes for a simpler gradient formula, but limits the range of applications. In particular, the methods of [Hong & Liu \(2009\)](#) and [Scaillet \(2004\)](#) are not applicable to RL. The CVaR gradient formula proposed in this paper is indeed different than the formulae of [Hong & Liu \(2009\)](#) and [Scaillet \(2004\)](#), and applies to a much broader range of applications.

LR gradient estimators for other risk measures have been proposed by [Borkar \(2001\)](#) for exponential utility functions, and by [Tamar et al. \(2012\)](#) for mean-variance. These measures however, consider a very different notion of risk than the CVaR. For example, the mean-variance measure is known to underestimate the risk of rare, but catastrophic events ([Agarwal & Naik, 2004](#)).

CVaR optimization in RL is, to our knowledge, quite a new subject. [Petrik & Subramanian \(2012\)](#) propose an impressive method based on stochastic dual dynamic programming for optimizing the CVaR in large scale Markov decision processes. Their method, however, is limited to linearly controllable problems. Our method does not require this condition, which does not hold, for example, in the Tetris domain we present. [Morimura et al. \(2010\)](#) optimize the expected return, but use a CVaR based risk-sensitive policy for guiding the exploration while learning. Their method estimates the entire return distribution from every state of the system, and therefore does not scale to large problems. Finally, an approach to optimize the CVaR by dynamic programming was proposed by [Boda & Filar \(2006\)](#), but applies only in limited, and small, problems.

2. Background

We describe some preliminaries about the conditional value-at-risk.

Let Z denote a random variable with a cumulative distribution function (C.D.F.) $F_Z(z) = \Pr(Z \leq z)$. For convenience, we assume that Z is a continuous random variable, meaning that $F_Z(z)$ is everywhere continuous. We also assume that Z is bounded by \bar{Z} .

Given a confidence level $\alpha \in (0, 1)$, the α -Value-at-Risk, (or α -quantile) of Z is denoted $\nu_\alpha(Z)$, and given by

$$\nu_\alpha(Z) = F_Z^{-1}(\alpha) \doteq \inf \{z : F_Z(z) \geq \alpha\}. \quad (1)$$

The α -Conditional-Value-at-Risk of Z is denoted by $\Phi_\alpha(Z)$ and defined as the expectation of the α fraction of the worst outcomes of Z

$$\Phi_\alpha(Z) = \mathbb{E}[Z | Z \leq \nu_\alpha(Z)]. \quad (2)$$

In the next section, we present a formula for the sensitivity of $\Phi_\alpha(Z)$ to changes in the distribution of Z .

3. CVaR Gradient Estimation

In this section we present our new formula for the gradient of the CVaR.

Consider again a random variable Z , but now let its probability density function $f_Z(z; \theta)$ be parameterized by a vector $\theta \in \mathbb{R}^k$. We let $\nu_\alpha(Z; \theta)$ and $\Phi_\alpha(Z; \theta)$ denote the VaR and CVaR of Z as defined in Eq. (1) and (2) when the parameter is θ .

In this section we are interested in the sensitivity of the CVaR to the parameter vector, as expressed by the gradient $\frac{\partial}{\partial \theta_j} \Phi_\alpha(Z; \theta)$.

For technical convenience, we make the following assumption on Z :

Assumption 1. Z is a continuous random variable, and bounded in $[-b, b]$ for all θ .

We also make the following smoothness assumption on $\nu_\alpha(Z; \theta)$ and $\Phi_\alpha(Z; \theta)$

Assumption 2. For all θ and $1 \leq j \leq k$, the gradients $\frac{\partial}{\partial \theta_j} \nu_\alpha(Z; \theta)$ and $\frac{\partial}{\partial \theta_j} \Phi_\alpha(Z; \theta)$ are well defined and bounded.

Since Z is continuous, Assumption 2 is satisfied when $\frac{\partial}{\partial \theta_j} f_Z(z; \theta)$ is bounded, thus it is not a strong assumption.

The next assumption is standard in LR gradient estimates

Assumption 3. For all θ , z , and $1 \leq j \leq k$, we have that $\frac{\partial}{\partial \theta_j} f_Z(z; \theta) / f_Z(z; \theta)$ is well defined and bounded.

The next proposition gives a LR-style sensitivity formula for $\Phi_\alpha(Z; \theta)$.

Proposition 1. Let Assumptions 1, 2, and 3 hold. Then

$$\begin{aligned} \frac{\partial}{\partial \theta_j} \Phi_\alpha(Z; \theta) &= \\ &= \mathbb{E}^\theta \left[\frac{\partial}{\partial \theta_j} \log f_Z(Z; \theta) (Z - \nu_\alpha(Z; \theta)) \middle| Z \leq \nu_\alpha(Z; \theta) \right]. \end{aligned}$$

Proof. Define the set D_θ as

$$D_\theta = \{z \in [-b, b] : z \leq \nu_\alpha(Z; \theta)\}.$$

By definition, $D_\theta \equiv [-b, \nu_\alpha(Z; \theta)]$, and

$$\int_{z \in D_\theta} f_Z(z; \theta) dz = \alpha. \quad (3)$$

Also, by definition (2) we have

$$\Phi_\alpha(Z; \theta) = \int_{z \in D_\theta} \frac{f_Z(z; \theta) z}{\alpha} dz = \int_{-b}^{\nu_\alpha(Z; \theta)} \frac{f_Z(z; \theta) z}{\alpha} dz.$$

Now, using the Leibniz rule we have

$$\begin{aligned} \frac{\partial}{\partial \theta_j} \Phi_\alpha(Z; \theta) &= \frac{\partial}{\partial \theta_j} \int_{-b}^{\nu_\alpha(Z; \theta)} \frac{f_Z(z; \theta) z}{\alpha} dz \\ &= \int_{-b}^{\nu_\alpha(Z; \theta)} \frac{\frac{\partial}{\partial \theta_j} f_Z(z; \theta) z}{\alpha} dz \\ &\quad + \frac{\frac{\partial}{\partial \theta_j} \nu_\alpha(Z; \theta) f_Z(\nu_\alpha(Z; \theta); \theta) \nu_\alpha(Z; \theta)}{\alpha}. \end{aligned} \quad (4)$$

Also, from (3) we have $1 = \int_{-b}^{\nu_\alpha(Z; \theta)} \frac{f_Z(z; \theta)}{\alpha} dz$, and taking gradient we obtain

$$\begin{aligned} 0 &= \frac{\partial}{\partial \theta_j} \int_{-b}^{\nu_\alpha(Z; \theta)} \frac{f_Z(z; \theta)}{\alpha} dz \\ &= \int_{-b}^{\nu_\alpha(Z; \theta)} \frac{\frac{\partial}{\partial \theta_j} f_Z(z; \theta)}{\alpha} dz \\ &\quad + \frac{\frac{\partial}{\partial \theta_j} \nu_\alpha(Z; \theta) f_Z(\nu_\alpha(Z; \theta); \theta)}{\alpha}. \end{aligned}$$

Rearranging, and plugging in (4) we obtain

$$\begin{aligned} \frac{\partial}{\partial \theta_j} \Phi_\alpha(Z; \theta) &= \\ &\int_{-b}^{\nu_\alpha(Z; \theta)} \frac{\frac{\partial}{\partial \theta_j} f_Z(z; \theta) (z - \nu_\alpha(Z; \theta))}{\alpha} dz \end{aligned}$$

Finally, using the standard likelihood ratio trick – multiplying and dividing by $f_Z(z; \theta)$ inside the integral, which is justified due to Assumption 3, we obtain the required expectation. \square

In a typical application, Z would correspond to the performance of some system, such as the profit in portfolio optimization, or the total reward in RL. Note that in order to use Proposition 1 in a gradient estimation algorithm, one needs access to $\frac{\partial}{\partial \theta_j} \log f_Z(Z; \theta)$: the sensitivity of the performance distribution to the parameters. Typically, the

system performance is a complicated function of a high-dimensional random variable. For example, in RL and queueing systems, the performance is a function of a trajectory from a stochastic dynamical system, and calculating its probability distribution is usually intractable. The sensitivity of the trajectory distribution to the parameters, however, is often easy to calculate, since the parameters typically control how the trajectory is generated. In the following, we generalize Proposition 1 to such cases. The utility of this generalization is further exemplified in Section 5, for the RL domain.

Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ denote an n -dimensional random variable with a finite support $[-b, b]^n$, with probability density function $f_{\mathbf{X}}(\mathbf{x}; \theta)$. Let the reward function r be a bounded mapping from $[-b, b]^n$ to \mathbb{R} . We are interested in a formula for $\frac{\partial}{\partial \theta_j} \Phi_\alpha(r(\mathbf{X}); \theta)$.

We make the following assumption, similar to Assumptions 1, 2, and 3.

Assumption 4. The reward $r(\mathbf{X})$ is a continuous random variable for all θ . Furthermore, for all θ and $1 \leq j \leq k$, the gradients $\frac{\partial}{\partial \theta_j} \nu_\alpha(r(\mathbf{X}); \theta)$ and $\frac{\partial}{\partial \theta_j} \Phi_\alpha(r(\mathbf{X}); \theta)$ are well defined and bounded. In addition $\frac{\partial \log f_{\mathbf{X}}(\mathbf{x}; \theta)}{\partial \theta_j}$ is well-defined and bounded for all \mathbf{x} and θ .

Define the level-set \mathcal{D}_θ as

$$\mathcal{D}_\theta = \{\mathbf{x} \in [-b, b]^n : r(\mathbf{x}) \leq \nu_\alpha(r(\mathbf{X}); \theta)\}.$$

We require some smoothness of the function r , that is captured by the following assumption on \mathcal{D}_θ .

Assumption 5. For all θ , the set \mathcal{D}_θ may be written as a finite sum of L_θ disjoint, closed, and connected components \mathcal{D}_θ^i , each with positive measure:

$$\mathcal{D}_\theta = \sum_{i=1}^{L_\theta} \mathcal{D}_\theta^i.$$

Assumption 5 may be satisfied, for example, when r is Lipschitz.

Our main result in this paper is given in the next proposition: a sensitivity formula for $\Phi_\alpha(r(\mathbf{X}); \theta)$.

Proposition 2. Let Assumption 4 and 5 hold. Then

$$\begin{aligned} \frac{\partial}{\partial \theta_j} \Phi_\alpha(r(\mathbf{X}); \theta) &= \\ &= \mathbb{E}^\theta \left[\frac{\partial \log f_{\mathbf{X}}(\mathbf{X}; \theta)}{\partial \theta_j} (r(\mathbf{X}) - \nu_\alpha(r(\mathbf{X}); \theta)) \mathbb{1}_{r(\mathbf{X}) \leq \nu_\alpha(r(\mathbf{X}); \theta)} \right]. \end{aligned}$$

The proof of Proposition 2 is similar in spirit to the proof of Proposition 1, but involves some additional difficulties

in applying the Leibnitz rule in a multidimensional setting. It is given in the supplementary material.

The sensitivity formula in Proposition 2 suggests an immediate Monte–Carlo (MC) estimation algorithm, which we label GCVaR. Let $\mathbf{x}_1, \dots, \mathbf{x}_N$ be N samples drawn i.i.d. from $f_{\mathbf{X}}(\mathbf{x}; \theta)$. We first estimate $\nu_{\alpha}(r(\mathbf{X}); \theta)$ using the empirical α -quantile² \tilde{v}

$$\tilde{v} = \inf_z \hat{F}(z) \geq \alpha. \quad (5)$$

where $\hat{F}(z)$ is the empirical C.D.F. of $r(\mathbf{X})$: $\hat{F}(z) \doteq \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{r(\mathbf{x}_i) \leq z}$. The MC estimate of the gradient $\Delta_{j;N} \approx \frac{\partial}{\partial \theta_j} \Phi_{\alpha}(r(\mathbf{X}); \theta)$ is given by

$$\Delta_{j;N} = \frac{1}{\alpha N} \sum_{i=1}^N \frac{\partial \log f_{\mathbf{X}}(\mathbf{x}_i; \theta)}{\partial \theta_j} (r(\mathbf{x}_i) - \tilde{v}) \mathbf{1}_{r(\mathbf{x}_i) \leq \tilde{v}}. \quad (6)$$

Algorithm 1 GCVaR

1: **Given:**

- CVaR level α
- A reward function $r(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$
- A density function $f_{\mathbf{X}}(\mathbf{x}; \theta) : \mathbb{R}^n \rightarrow \mathbb{R}$
- A sequence $\mathbf{x}_1, \dots, \mathbf{x}_N \sim f_{\mathbf{X}}$, i.i.d.

2: Set $r_1^s, \dots, r_N^s = \text{Sort}(r(\mathbf{x}_1), \dots, r(\mathbf{x}_N))$

3: Set $\tilde{v} = r_{\lceil \alpha N \rceil}^s$

4: For $j = 1, \dots, K$ do

$$\Delta_{j;N} = \frac{1}{\alpha N} \sum_{i=1}^N \frac{\partial \log f_{\mathbf{X}}(\mathbf{x}_i; \theta)}{\partial \theta_j} (r(\mathbf{x}_i) - \tilde{v}) \mathbf{1}_{r(\mathbf{x}_i) \leq \tilde{v}}$$

5: **Return:** $\Delta_{1;N}, \dots, \Delta_{K;N}$

It is known that the empirical α -quantile is a biased estimator of $\nu_{\alpha}(r(\mathbf{X}); \theta)$. Therefore, $\Delta_{j;N}$ is also a biased estimator of $\frac{\partial}{\partial \theta_j} \Phi_{\alpha}(r(\mathbf{X}); \theta)$. We now show that $\Delta_{j;N}$ is a consistent estimator. The proof is similar to the proof of Theorem 4.1 in (Hong & Liu, 2009), and given in the supplementary material.

Proposition 3. *Let Assumption 4 and 5 hold. Then $\Delta_{j;N} \rightarrow \frac{\partial}{\partial \theta_j} \Phi_{\alpha}(r(\mathbf{X}); \theta)$ w.p. 1 as $N \rightarrow \infty$.*

Clearly, for very low quantiles, i.e., α close to 0, the estimator $\Delta_{j;N}$ of Eq. (6) would have a high variance, since the averaging is effectively only over αN samples. In order to mitigate this problem, we now propose an importance sampling procedure for estimating $\frac{\partial}{\partial \theta_j} \Phi_{\alpha}(r(\mathbf{X}); \theta)$.

²Algorithmically, this is equivalent to first sorting the $r(\mathbf{x}_i)$'s in ascending order, and then selecting \tilde{v} as the $\lceil \alpha N \rceil$ term in the sorted list.

4. Importance Sampling

Importance sampling (IS; Rubinstein & Kroese 2011) is a general procedure for reducing the variance of Monte–Carlo (MC) estimates. We first describe it in a general context, and then give the specific implementation for the CVaR sensitivity estimator.

4.1. Background

We wish to estimate the expectation $l = \mathbb{E}[H(X)]$ where X is a random variable with P.D.F. $f(x)$, and $H(x)$ is some function. The MC estimate is given by $\hat{l} = \frac{1}{N} \sum_{i=1}^N H(x_i)$, where $x_i \sim f$ are drawn i.i.d.

The IS method aims to reduce the variance of the MC estimator by using a different sampling distribution for the samples x_i . Assume we are given a sampling distribution $g(x)$, and that g dominates f in the sense that $g(x) = 0 \Rightarrow f(x) = 0$. We let \mathbb{E}_f and \mathbb{E}_g denote expectations w.r.t. f and g , respectively. Observe that $l = \mathbb{E}_f[H(X)] = \mathbb{E}_g\left[H(X) \frac{f(X)}{g(X)}\right]$, and we thus define the IS estimator \hat{l}_{IS} as

$$\hat{l}_{\text{IS}} = \frac{1}{N} \sum_{i=1}^N H(x_i) \frac{f(x_i)}{g(x_i)}, \quad (7)$$

where the x_i 's are drawn i.i.d., and now $x_i \sim g$. Obviously, selecting an appropriate g such that \hat{l}_{IS} indeed has a lower variance than \hat{l} is the heart of the problem. One approach is by the *variance minimization* method (Rubinstein & Kroese, 2011). Here, we are given a family of distributions $g(x; \omega)$ parameterized by ω , and we aim to find an ω that minimizes the variance $V(\omega) = \text{Var}_{x_i \sim g(\cdot; \omega)}(\hat{l}_{\text{IS}})$. A straightforward calculation shows that $V(\omega) = \mathbb{E}_f\left[H(X)^2 \frac{f(X)}{g(X; \omega)}\right] - l^2$, and since l does not depend on ω , we are left with the optimization problem

$$\min_{\omega} \mathbb{E}_f \left[H(X)^2 \frac{f(X)}{g(X; \omega)} \right],$$

which is typically solved approximately, by solving the sampled average approximation (SAA)

$$\min_{\omega} \frac{1}{N_{\text{SAA}}} \sum_{i=1}^{N_{\text{SAA}}} \left[H(x_i)^2 \frac{f(x_i)}{g(x_i; \omega)} \right], \quad (8)$$

where $x_i \sim f$ are i.i.d. Numerically, the SAA may be solved using (deterministic) gradient descent, by noting that $\frac{\partial}{\partial \omega} \left(\frac{f(x_i)}{g(x_i; \omega)} \right) = -\frac{f(x_i)}{g(x_i; \omega)} \frac{\partial}{\partial \omega} \log g(x_i; \omega)$.

Thus, in order to find an IS distribution g from a family of distributions $g(x; \omega)$, we draw N_{SAA} samples from the original distribution f , and solve the SAA (8) to obtain the optimal ω . We now describe how this procedure is applied for estimating the CVaR sensitivity $\frac{\partial}{\partial \theta_j} \Phi_{\alpha}(r(\mathbf{X}); \theta)$.

4.2. IS Estimate for CVaR Sensitivity

We recall the setting of Proposition 2, and assume that in addition to $f_{\mathbf{X}}(\mathbf{X}; \theta)$ we have access to a family of distributions $g_{\mathbf{X}}(\mathbf{X}; \theta, \omega)$ parameterized by ω . We follow the procedure outlined above and, using Proposition 2, and set

$$H_j(\mathbf{X}) = \frac{1}{\alpha} \frac{\partial \log f_{\mathbf{X}}(\mathbf{X}; \theta)}{\partial \theta_j} (r(\mathbf{X}) - \nu_{\alpha}(\mathbf{X}; \theta)) \mathbf{1}_{\mathbf{X} \leq \nu_{\alpha}(\mathbf{X}; \theta)}.$$

However, since $\nu_{\alpha}(\mathbf{X}; \theta)$ is not known in advance, we need a procedure for estimating it in order to plug it into Eq. (7). The empirical quantile \tilde{v} of Eq. (5) is not suitable since it uses samples from $f_{\mathbf{X}}(\mathbf{X}; \theta)$. Thus, we require an IS estimator for $\nu_{\alpha}(\mathbf{X}; \theta)$ as well. Such was proposed by Glynn (1996). Let $\hat{F}_{\text{IS}}(z)$ denote the IS empirical C.D.F. of $r(\mathbf{X})$

$$\hat{F}_{\text{IS}}(z) \doteq \frac{1}{N} \sum_{i=1}^N \frac{f_{\mathbf{X}}(\mathbf{x}_i; \theta)}{g_{\mathbf{X}}(\mathbf{x}_i; \theta, \omega)} \mathbf{1}_{r(\mathbf{x}_i) \leq z}.$$

Then, the IS empirical VaR is given by

$$\tilde{v}_{\text{IS}} = \inf_z \hat{F}_{\text{IS}}(z) \geq \alpha. \quad (9)$$

We also need to modify the variance minimization method, as we are not estimating a scalar function but a gradient in \mathbb{R}^k . We assume independence between the elements, and replace $H(x_i)^2$ in Eq. (8) with $\sum_{j=1}^k H_j(x_i)^2$.

Let us now state the estimation procedure explicitly. We first draw N_{SAA} i.i.d. samples from $f_{\mathbf{X}}(\cdot; \theta)$, and find a suitable ω by solving (8) with $H_j(\mathbf{X}) = \frac{1}{\alpha} \frac{\partial \log f_{\mathbf{X}}(\mathbf{X}; \theta)}{\partial \theta_j} (r(\mathbf{X}) - \tilde{v}) \mathbf{1}_{\mathbf{X} \leq \tilde{v}}$, where \tilde{v} is given in (5).

We then run the IS_GCVaR algorithm, as follows. We draw N i.i.d. samples $\mathbf{x}_1, \dots, \mathbf{x}_N$ from $g_{\mathbf{X}}(\cdot; \theta, \omega)$. The IS estimate of the CVaR gradient $\Delta_{j;N}^{\text{IS}}$ is given by

$$\Delta_{j;N}^{\text{IS}} = \frac{1}{\alpha N} \sum_{i=1}^N \frac{\partial \log f_{\mathbf{X}}(\mathbf{x}_i; \theta)}{\partial \theta_j} \frac{f_{\mathbf{X}}(\mathbf{x}_i; \theta) (r(\mathbf{x}_i) - \tilde{v}_{\text{IS}}) \mathbf{1}_{\mathbf{x}_i \leq \tilde{v}_{\text{IS}}}}{g_{\mathbf{X}}(\mathbf{x}_i; \theta, \omega)}, \quad (10)$$

where \tilde{v}_{IS} is given in (9).

Note that in our SAA program for finding ω , we estimate ν_{α} using crude Monte Carlo. In principle, IS may also be used for that estimate as well, with an additional optimization process for finding a suitable sampling distribution. However, a typical application of the CVaR gradient is in optimization of θ by stochastic gradient descent. There, one only needs to update ω intermittently, therefore a large sample size N_{SAA} is affordable and IS is not needed.

So far, we have not discussed how the parameterized distribution family $g_{\mathbf{X}}(\cdot; \theta, \omega)$ is obtained. While there are some

Algorithm 2 IS_GCVaR

1: **Given:**

- CVaR level α
- A reward function $r(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$
- A density function $f_{\mathbf{X}}(\mathbf{x}; \theta) : \mathbb{R}^n \rightarrow \mathbb{R}$
- A density function $g_{\mathbf{X}}(\mathbf{x}; \theta) : \mathbb{R}^n \rightarrow \mathbb{R}$
- A sequence $\mathbf{x}_1, \dots, \mathbf{x}_N \sim g_{\mathbf{X}}$, i.i.d.

2: Set $\mathbf{x}_1^s, \dots, \mathbf{x}_N^s = \text{Sort}(\mathbf{x}_1, \dots, \mathbf{x}_N)$ by $r(\mathbf{x})$

3: For $i = 1, \dots, N$ do

$$L(i) = \sum_{j=1}^i f_{\mathbf{X}}(\mathbf{x}_j^s; \theta) / g_{\mathbf{X}}(\mathbf{x}_j^s; \theta)$$

4: Set $l = \arg \min_i L(i) \geq \alpha$

5: Set $\tilde{v}_{\text{IS}} = r(\mathbf{x}_l^s)$

6: For $j = 1, \dots, K$ do

$$\Delta_{j;N}^{\text{IS}} = \frac{1}{\alpha N} \sum_{i=1}^N \frac{\partial \log f_{\mathbf{X}}(\mathbf{x}_i; \theta)}{\partial \theta_j} \frac{f_{\mathbf{X}}(\mathbf{x}_i; \theta) (r(\mathbf{x}_i) - \tilde{v}_{\text{IS}}) \mathbf{1}_{\mathbf{x}_i \leq \tilde{v}_{\text{IS}}}}{g_{\mathbf{X}}(\mathbf{x}_i; \theta, \omega)},$$

7: **Return:** $\Delta_{1;N}^{\text{IS}}, \dots, \Delta_{K;N}^{\text{IS}}$

standard approaches such as exponential tilting (Rubinstein & Kroese, 2011), this task typically requires some domain knowledge.

In the next section, we discuss an application of our approach to reinforcement learning. For this domain, we will present a heuristic method for selecting $g_{\mathbf{X}}(\cdot; \theta, \omega)$.

5. Application to Reinforcement Learning

In this section we show that the sensitivity formula of Section 3 may be used in RL for optimizing the CVaR of the total return, in a policy-gradient type scheme. We first describe some preliminaries and our RL setting, and then describe our algorithm.

5.1. RL Background

We consider an episodic³ Markov Decision Problem (MDP) M in discrete time with a finite state space X , an initial state distribution ζ_0 , a terminal state x^* , and a finite action space U . The transition probabilities are denoted by $\Pr(x'|x, u)$. We let π_{θ} denote a policy parameterized by $\theta \in \mathbb{R}^k$, that determines, for each $x \in X$, a distribu-

³This is also known as a stochastic shortest path problem; see Bertsekas 2012.

tion over actions $\Pr(u|x; \theta)$. Let $r(x)$ denote the reward at state x , and assume that it is random, and bounded by \bar{r} for all x . We also assume zero reward at the terminal state. We denote by x_k , u_k , and r_k the state, action, and reward, respectively, at time k , where $k = 0, 1, 2, \dots$

A policy is said to be *proper* (Bertsekas, 2012) if there is a positive probability that the terminal state x^* will be reached after at most n transitions, from any initial state. Throughout this section we make the following two assumptions

Assumption 6. *The policy π_θ is proper for all θ .*

Assumption 7. *For all $\theta \in \mathbb{R}^k$, $x \in X$, and $u \in U$, the gradient $\frac{\partial \log \Pr(u|x; \theta)}{\partial \theta_j}$ is well defined and bounded.*

Assumption 6 is common in episodic MDPs (Bertsekas, 2012), and requires that all states have a positive probability of being visited. Assumption 7 is standard in policy gradient literature, and a popular policy representation that satisfies it is softmax action selection (Sutton et al., 2000; Marbach & Tsitsiklis, 1998), described as follows. Let $\phi(x, u) \in \mathbb{R}^K$, denote a set of K features which depend on the state and action. The softmax policy is given by

$$\Pr(u|x; \theta) = \frac{\exp(\phi(x, u)^\top \theta)}{\sum_{u'} \exp(\phi(x, u')^\top \theta)}. \quad (11)$$

Let $\tau \doteq \min\{k > 0 | x_k = x^*\}$ denote the first visit time to the terminal state, and let the random variable B denote the accumulated reward along the trajectory until that time, discounted by $\gamma \in (0, 1]$

$$B \doteq \sum_{k=0}^{\tau-1} \gamma^k r(x_k).$$

In standard RL, the objective is to find the parameter θ that maximizes the expected return $V(\theta) = \mathbb{E}^\theta[B]$. Policy gradient methods (Marbach & Tsitsiklis, 1998; Peters & Schaal, 2008) use simulation to estimate $\frac{\partial}{\partial \theta_j} V(\theta)$, and then perform stochastic gradient ascent on the parameters θ . In this work we are risk-sensitive, and our goal is to maximize the CVaR of the total return $J(\theta)$

$$J(\theta) \doteq \Phi_\alpha(B; \theta).$$

In the spirit of policy gradient methods, we estimate $\frac{\partial}{\partial \theta_j} J(\theta)$ from simulation, and optimize θ by stochastic gradient ascent. The estimation of $\frac{\partial}{\partial \theta_j} J(\theta)$ however, is performed using the sensitivity formula introduced in Section 3, and detailed in the following section.

5.2. CVaR Policy Gradient

We now describe how to apply the algorithms of Sections 3 and 4 in the RL setting, and derive the CVaR policy gradient algorithm (CVaRPG).

Let the random variable \mathbf{X} correspond to a trajectory $x_0, u_0, r_0, \dots, x_\tau, u_\tau, r_\tau$ from the MDP M , and let \mathbf{x} denote an instance of it. By the Markov property of the MDP we write the P.D.F. of \mathbf{X} as

$$f_{\mathbf{X}}(\mathbf{x}; \theta) = \zeta_0(x_0) \prod_{t=0}^{\tau-1} \Pr(u_t|x_t; \theta) \Pr(r_t|x_t, u_t) \Pr(x_{t+1}|x_t, u_t), \quad (12)$$

and we have that

$$\frac{\partial \log f_{\mathbf{X}}(\mathbf{x}; \theta)}{\partial \theta_j} = \sum_{t=0}^{\tau-1} \frac{\partial \log \Pr(u_t|x_t; \theta)}{\partial \theta_j}. \quad (13)$$

Also, let $r(\mathbf{x})$ denote the total discounted reward in the trajectory

$$r(\mathbf{x}) \doteq \sum_{t=0}^{\tau-1} \gamma^t r(x_t). \quad (14)$$

Our estimation of $\frac{\partial}{\partial \theta_j} J(\theta)$ proceeds as follows. We simulate N trajectories $\mathbf{x}_1, \dots, \mathbf{x}_N$ of the MDP using policy π_θ . We then use Eq. (13) and (14) in the GCVaR algorithm to obtain the estimated gradient $\Delta_{1:N}, \dots, \Delta_{K:N}$.

The optimization of the parameters θ is finally performed using a gradient descent procedure. Let $\epsilon(t)$ denote a sequence of positive step-sizes. We update the j 'th component of θ at time t by

$$\theta_{j;t+1} = \theta_{j;t} + \epsilon(t) \Delta_{j;N}. \quad (15)$$

It is well known that the choice of step-size sequence $\epsilon(t)$ has a significant effect on the performance of gradient descent type algorithms. In theory, an $\mathcal{O}(1/t)$ decreasing step-size is typically required for guaranteeing convergence of the parameters, while in practice, a small but constant step-size ϵ typically works reasonably well, although hand-tuning of ϵ is required. A convergence analysis of GCVaR is not trivial, due to the bias of the gradient estimator $\Delta_{j;N}$, and is deferred to future work. In our experiments, however, we used a constant step size, and did not experience difficulties with the tweaking of ϵ .

5.3. CVaR Policy Gradient with Importance Sampling

As explained earlier, when dealing with small values of α , an IS scheme may help reduce the variance of the CVaR gradient estimator. In this section, we apply the IS estimator of Section 4 to the RL domain. As is typical in IS, the main difficulty is finding a suitable sampling distribution, and actually sampling from it. Observe that by the definition of the trajectory distribution in Eq. (12), a natural method for modifying the trajectory distribution is by modifying the MDP transition probabilities. We note, however, that by such our method actually requires access to a simulator of this modified MDP. In many applications a simulator of the original system is available anyway, thus modifying it should not be a problem.

Consider an MDP \hat{M} that is similar to M but with transition probabilities $\hat{\Pr}(x'|x, u; \omega)$, where ω is some controllable parameter. We will later specify $\hat{\Pr}(x'|x, u; \omega)$ explicitly, but for now, observe that the P.D.F. of a trajectory \mathbf{X} from the MDP \hat{M} is given by

$$g_{\mathbf{X}}(\mathbf{x}; \theta, \omega) = \zeta_0(x_0) \prod_{t=0}^{\tau-1} \Pr(u_t|x_t; \theta) \Pr(r_t|x_t, u_t) \hat{\Pr}(x_{t+1}|x_t, u_t; \omega),$$

and therefore

$$\frac{f_{\mathbf{X}}(\mathbf{x}; \theta)}{g_{\mathbf{X}}(\mathbf{x}; \theta, \omega)} = \prod_{t=0}^{\tau-1} \frac{\Pr(x_{t+1}|x_t, u_t)}{\hat{\Pr}(x_{t+1}|x_t, u_t; \omega)}. \quad (16)$$

Plugging Eq. (13), (14), and (16) in the `IS_GCVaR` algorithm gives the IS estimated gradient $\Delta_{j;N}^{\text{IS}}$, which may then be used instead of $\Delta_{j;N}$ in the parameter update equation (15).

We now turn to the problem of choosing the transition probabilities $\hat{\Pr}(x'|x, u; \omega)$ in the MDP \hat{M} , and propose a heuristic approach that is suitable for the RL domain. We first observe that by definition, the CVaR takes into account only the ‘worst’ trajectories for a given policy, therefore a suitable IS distribution should give more weight to such bad outcomes in some sense. The difficulty is how to modify the transition probabilities, which are defined per state, such that the whole trajectory will be ‘bad’. We note that this difficulty is in a sense opposite to the action selection problem: how to choose an action at each state such that the long-term reward is high. Action selection is a fundamental task in RL, and has a very elegant solution, which inspires our IS approach.

A standard approach to action selection is through the *value-function* $V(x)$ (Sutton & Barto, 1998), which assigns to each state x its expected long term outcome $\mathbb{E}[B|x_0 = x]$ under the current policy. Once the value function is known, the ‘greedy selection’ rule selects the action that maximizes the expected value of the next state. The intuition behind this rule is that since $V(x)$ captures the long-term return from x , states with higher values lead to better trajectories, and should be preferred.

By a similar reasoning, we expect that encouraging transitions to low-valued states will produce worse trajectories. We thus propose the following heuristic for the transition probabilities $\hat{\Pr}(x'|x, u; \omega)$. Assume that we have access to an approximate value function $\tilde{V}(x)$ for each state. We propose the following IS transitions for \hat{M}

$$\hat{\Pr}(x'|x, u; \omega) = \frac{\Pr(x'|x, u) \exp(-\omega \tilde{V}(x'))}{\sum_y \Pr(y|x, u) \exp(-\omega \tilde{V}(y))}. \quad (17)$$

Note that increasing ω encourages transitions to low value states, thus increasing the probability of ‘bad’ trajectories.

Obtaining an approximate value function for a given policy has been studied extensively in RL literature, and many efficient solutions for this task are known, such as LSTD (Boyan, 2002) and TD(λ) (Sutton & Barto, 1998). Here, we don’t restrict ourselves to a specific method.

6. Experimental Results

We examine Tetris as a test case for our algorithms. Tetris is a popular RL benchmark that has been studied extensively. The main challenge in Tetris is its large state space, which necessitates some form of approximation in the solution technique. Many approaches to learning controllers for Tetris are described in the literature, among them are approximate value iteration (Tsitsiklis & Van Roy, 1996), policy gradients (Kakade, 2001), and modified policy iteration (Gabillon et al., 2013).

The standard performance measure in Tetris is the expected number of cleared lines in the game. Here, we are interested in a risk-averse performance measure, captured by the CVaR of the total lines in the game. Our goal in this section is to compare the performance of a policy optimized for the CVaR criterion versus a policy obtained using the standard policy gradient method. As we will show, optimizing the CVaR indeed produces different policies, characterized by a risk-averse behavior. We note that at present, the best results in the literature (for the standard performance measure) were obtained using a modified policy iteration approach, and not using policy gradients. We emphasize that our goal here is not to compete with those results, but rather to study the policy gradient based approach. We do point out, however, that whether those methods could be extended to handle the CVaR criterion is currently not known. We now describe the specifics of the Tetris domain we used, and present our results. For a general description of the Tetris game we refer to, e.g., Gabillon et al. (2013).

We used the regular 10×20 Tetris board with the 7 standard shapes (a.k.a. *tetrominos*). In order to induce risk-sensitive behavior, we modified the reward function of the game as follows. The score for clearing 1, 2, 3 and 4 lines is 1, 4, 8 and 16 respectively. In addition, we limited the maximum number of steps in the game to 1000. These modifications strengthened the difference between the risk-sensitive and nominal policies, as they induce a tradeoff between clearing many ‘single’ lines with a low profit, or waiting for the more profitable, but less frequent, ‘batches’.

We used the softmax policy of Eq. (11), with the feature set of Thiery & Scherrer (Thiery & Scherrer, 2009).

Starting from a fixed policy parameter θ_0 , which was obtained by running several iterations of standard policy gradient (giving both methods a ‘warm start’), we ran both the

GCVaR and standard policy gradient⁴ for enough iteration such that both algorithms (approximately) converged. We set $\alpha = 0.05$ and $N = 1000$.

In Fig. 1A and Fig. 1B we present the average return $V(\theta)$ and CVaR of the return $J(\theta)$ for the policies of both algorithms at each iteration (evaluated on independent trajectories). Observe that for GCVaR, the average return has been compromised for a higher CVaR value. This compromise is further explained in Fig. 1C, where we display the reward distribution of the final policies. It may be observed that the left-tail distribution of the CVaR policy is significantly lower than the standard policy. For the risk-sensitive decision maker, such results are very important, especially if the left-tail contains catastrophic outcomes, as is common in many real-world domains, such as finance. To better understand the differences between the policies, we compare the final policy parameters θ in Fig. 1D. The most significant difference is in the parameter that corresponds to the Board Well feature. A *well* is a succession of unoccupied cells in a column, such that their left and right cells are both occupied. The controller trained by GCVaR has a smaller negative weight for this feature, compared to the standard controller, indicating that actions which creates deep-wells are repressed. Such wells may lead to a high reward when they get filled, but are risky as they heighten the board.

In Fig. 2 we demonstrate the importance of the IS in optimizing the CVaR when α is small. We chose $\alpha = 0.01$, and $N = 200$, and compared the naive GCVaR against IS_GCVaR. As our value function approximation, we exploited the fact that the soft-max policy uses $\phi(x, u)^\top \theta$ as a sort of state-action value function, and therefore set $\tilde{V}(x) = \max_u \phi(x, u)^\top \theta$. We chose ω using SAA, with trajectories from the initial policy θ_0 . We observe that IS_GCVaR converges significantly faster than GCVaR, due to the lower variance in gradient estimation.

7. Conclusion and Future Work

We presented a novel LR-style formula for the gradient of the CVaR performance criterion. Based on this formula, we proposed a MC gradient estimator, and a stochastic gradient descent procedure for CVaR optimization. To our knowledge, this is the first extension of the LR method to the CVaR performance criterion, which has proved important for risk-sensitive decision making in a broad range of applications. The LR method is considered to be more accessible compared to other gradient estimation methods,

⁴Standard policy gradient is similar to GCVaR when $\alpha = 1$. However, it is common to subtract a baseline from the reward in order to reduce the variance of the gradient estimate. In our experiments, we used the average return $\langle r \rangle$ as a baseline, and our gradient estimate was $\frac{1}{N} \sum_{i=1}^N \frac{\partial \log f_{\mathbf{x}}(\mathbf{x}_i; \theta)}{\partial \theta_j} (r(\mathbf{x}_i) - \langle r \rangle)$.

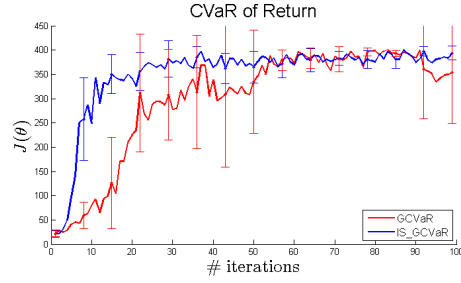


Figure 2. IS_GCVaR vs. GCVaR CVaR ($\alpha = 0.01$) of the return for IS_GCVaR and GCVaR vs. iteration.

and therefore we expect our method to be useful for risk-sensitive decision making in varied domains.

For low quantiles, it is common to couple Monte Carlo with importance sampling, and indeed we proposed an importance sampling procedure for our approach.

We evaluated our approach empirically in an RL domain: learning a risk-sensitive policy for playing Tetris. To our knowledge, such a domain is beyond the reach of existing CVaR optimization approaches. Moreover, our empirical results show that optimizing the CVaR indeed results in useful risk-sensitive policies, and motivates the use of simulation-based optimization for risk-sensitive decision making.

Currently, our approach still lacks extensive theoretical analysis. We believe that, similar to the work of Hong & Liu (2009) the bias of our gradient estimator can be shown to be $\mathcal{O}(N^{-1/2})$, although their analysis does not apply in our case. Such a result will facilitate a simple convergence proof for our stochastic gradient descent algorithm.

Finally, it would be interesting to further evaluate our method in practice. Traditional applications in finance, such as risk management for complex portfolios, are a natural candidate. However, in areas such as smart-grid control and robotics, risk-sensitive decision making is also becoming increasingly popular.

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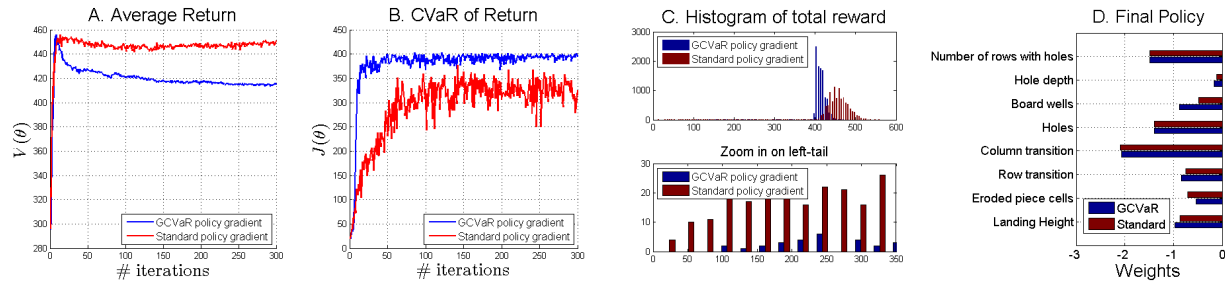


Figure 1. GCVaR vs. policy gradient. (A,B) Average return (A) and CVaR ($\alpha = 0.05$) of the return (B) for GCVaR and standard policy-gradient vs. iteration. (C) Histogram (counts from 10,000 independent runs) of the total return of the final policies. The lower plot is a zoom-in on the left-tail, and clearly shows the risk-averse behavior of the GCVaR policy. (D) Final policy parameters. Note the difference in the Board Well feature, which encourages risk taking.

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A. Proof of Proposition 2

Proof. The main difficulty in extending the proof of Proposition 1 to this case is in applying the Leibnitz rule in a multi-dimensional case. Such an extension is given by [Flanders \(1973\)](#), which we now state.

We are given an n -dimensional θ -dependent chain (field of integration) D_θ in \mathbb{R}^n . We also have an exterior differential n -form whose coefficients are time-dependent:

$$\omega = f(\mathbf{x}, \theta) dx_1 \wedge \cdots \wedge dx_n$$

The general Leibnitz rule⁵ is given by

$$\frac{\partial}{\partial \theta} \int_{D_\theta} \omega = \int_{\partial D_\theta} \mathbf{v} \lrcorner \omega + \int_{D_\theta} \frac{\partial \omega}{\partial \theta} \quad (18)$$

where \mathbf{v} denotes the vector field of velocities $\frac{\partial}{\partial \theta} \mathbf{x}$ of D_θ , and $\mathbf{v} \lrcorner \omega$ denotes the interior product between \mathbf{v} and ω (see [Flanders 1973](#) for more details).

We now write the CVaR explicitly as

$$\Phi_\alpha(r(\mathbf{X}); \theta) = \frac{1}{\alpha} \int_{\mathbf{x} \in D_\theta} f_{\mathbf{X}}(\mathbf{x}; \theta) r(\mathbf{x}) d\mathbf{x} = \frac{1}{\alpha} \sum_{i=1}^{L_\theta} \int_{\mathbf{x} \in D_\theta^i} f_{\mathbf{X}}(\mathbf{x}; \theta) r(\mathbf{x}) d\mathbf{x},$$

therefore

$$\frac{\partial}{\partial \theta_j} \Phi_\alpha(r(\mathbf{X}); \theta) = \frac{1}{\alpha} \sum_{i=1}^{L_\theta} \frac{\partial}{\partial \theta_j} \int_{\mathbf{x} \in D_\theta^i} f_{\mathbf{X}}(\mathbf{x}; \theta) r(\mathbf{x}) d\mathbf{x}. \quad (19)$$

We now treat each D_θ^i separately. Let \mathcal{X} denote the set $[-b, b]^n$ over which \mathbf{X} is defined. Obviously, $D_\theta^i \subset \mathcal{X}$.

We now make an important observation. By definition of the level-set D_θ^i , and since it is closed by Assumption 5, for every $\mathbf{x} \in \partial D_\theta^i$ we have that either

$$(a) \ r(\mathbf{x}) = \nu_\alpha(r(\mathbf{X}); \theta), \quad (20)$$

or

$$(b) \ \mathbf{x} \in \partial \mathcal{X}, \text{ and } r(\mathbf{x}) < \nu_\alpha(r(\mathbf{X}); \theta). \quad (21)$$

We write $\partial D_\theta^i = \partial D_\theta^{i,a} + \partial D_\theta^{i,b}$ where the two last terms correspond to the two possibilities in (20) and (21).

We now claim that for the boundary term $\partial D_\theta^{i,b}$, we have

$$\int_{\partial D_\theta^{i,b}} \mathbf{v} \lrcorner \omega = 0. \quad (22)$$

To see this, first note that by definition of \mathcal{X} , the boundary $\partial \mathcal{X}$ is smooth and has a unique normal vector at each point, except for a set of measure zero (the corners of \mathcal{X}). Let $\partial \tilde{D}_\theta^{i,b}$ denote the set of all points in $\partial D_\theta^{i,b}$ for which a unique normal vector exists. For each $\mathbf{x} \in \partial \tilde{D}_\theta^{i,b}$ we let \mathbf{v}_\perp and \mathbf{v}_\parallel denote the normal and tangent (with respect to $\partial \mathcal{X}$) elements of the velocity $\frac{\partial}{\partial \theta} \mathbf{x}$ at \mathbf{x} , respectively. Thus,

$$\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel.$$

For some $\epsilon > 0$ let d_ϵ denote the set $\{\mathbf{x} \in \partial D_\theta^{i,b} : r(\mathbf{x}) < \nu_\alpha(r(\mathbf{X}); \theta) - \epsilon\}$. From Assumption 4 we have that $\frac{\partial}{\partial \theta_j} \nu_\alpha(r(\mathbf{X}); \theta)$ is bounded, therefore there exists $\delta(\epsilon) > 0$ such that for all θ' that satisfy $\|\theta - \theta'\| < \delta(\epsilon)$ we have $|\nu_\alpha(r(\mathbf{X}); \theta') - \nu_\alpha(r(\mathbf{X}); \theta)| < \epsilon$, and therefore $d_\epsilon \subset \partial D_{\theta'}^{i,b}$. Since this holds for every $\epsilon > 0$, we conclude that a small change in θ does not change $\partial D_\theta^{i,b}$, and therefore we have

$$\mathbf{v}_\perp = 0, \quad \forall \mathbf{x} \in \partial \tilde{D}_\theta^{i,b}.$$

⁵The formula in ([Flanders, 1973](#)) is for a more general case where D_θ is not necessarily n -dimensional. That formula includes an additional term $\int_{D_\theta} \mathbf{v} \lrcorner d\mathbf{x} \omega$, where $d\mathbf{x}$ is the exterior derivative, which cancels in our case.

Furthermore, by definition of the interior product we have

$$\mathbf{v}_{\parallel} \lrcorner \omega = 0.$$

Therefore we have

$$\int_{\partial D_{\theta}^{i,b}} \mathbf{v} \lrcorner \omega = \int_{\partial \bar{D}_{\theta}^{i,b}} \mathbf{v} \lrcorner \omega = \int_{\partial \bar{D}_{\theta}^{i,b}} \mathbf{v}_{\parallel} \lrcorner \omega = 0,$$

and the claim follows.

Now, let $\omega = f_{\mathbf{X}}(\mathbf{x}; \theta) r(\mathbf{x}) dx_1 \wedge \dots \wedge dx_n$. Using (18), we have

$$\begin{aligned} \frac{\partial}{\partial \theta_j} \int_{\mathbf{x} \in D_{\theta}^i} \omega &= \int_{\partial D_{\theta}^i} \mathbf{v} \lrcorner \omega + \int_{D_{\theta}^i} \frac{\partial \omega}{\partial \theta} \\ &= \int_{\partial D_{\theta}^{i,a}} \mathbf{v} \lrcorner \omega + \int_{D_{\theta}^i} \frac{\partial \omega}{\partial \theta} \end{aligned} \quad (23)$$

where the last equality follows from (22) and the definition of \mathbf{v} .

Let $\tilde{\omega} = f_{\mathbf{X}}(\mathbf{x}; \theta) dx_1 \wedge \dots \wedge dx_n$. By the definition of \mathcal{D}_{θ} we have that for all θ

$$\alpha = \int_{\mathcal{D}_{\theta}} \tilde{\omega},$$

therefore, by taking a derivative, and using (22) we have

$$0 = \frac{\partial}{\partial \theta_j} \int_{\mathcal{D}_{\theta}} \tilde{\omega} = \sum_{i=1}^{L_{\theta}} \left(\int_{\partial D_{\theta}^{i,a}} \mathbf{v} \lrcorner \tilde{\omega} + \int_{D_{\theta}^i} \frac{\partial \tilde{\omega}}{\partial \theta} \right) \quad (24)$$

From (20), and linearity of the interior product we have

$$\int_{\partial D_{\theta}^{i,a}} \mathbf{v} \lrcorner \omega = \nu_{\alpha}(r(\mathbf{X}); \theta) \int_{\partial D_{\theta}^{i,a}} \mathbf{v} \lrcorner \tilde{\omega},$$

therefore, plugging in (24) we have

$$\sum_{i=1}^{L_{\theta}} \int_{\partial D_{\theta}^{i,a}} \mathbf{v} \lrcorner \omega = -\nu_{\alpha}(r(\mathbf{X}); \theta) \sum_{i=1}^{L_{\theta}} \int_{D_{\theta}^i} \frac{\partial \tilde{\omega}}{\partial \theta} \quad (25)$$

Finally, using (19), (23), and (25) we have

$$\begin{aligned} \frac{\partial}{\partial \theta_j} \Phi_{\alpha}(r(\mathbf{X}); \theta) &= \frac{1}{\alpha} \sum_{i=1}^{L_{\theta}} \left(\int_{D_{\theta}^i} \frac{\partial \omega}{\partial \theta} - \nu_{\alpha}(r(\mathbf{X}); \theta) \int_{D_{\theta}^i} \frac{\partial \tilde{\omega}}{\partial \theta} \right) \\ &= \frac{1}{\alpha} \int_{\mathcal{D}_{\theta}} \frac{\partial}{\partial \theta_j} f_{\mathbf{X}}(\mathbf{x}; \theta) (r(\mathbf{x}) - \nu_{\alpha}(r(\mathbf{X}); \theta)) d\mathbf{x}, \end{aligned}$$

and using the standard likelihood ratio trick – multiplying and dividing by $f_{\mathbf{X}}(\mathbf{x}; \theta)$ inside the integral we obtain the required expectation. \square

B. Proof of Proposition 3

Proof. Let $\nu = \nu_\alpha(r(\mathbf{X}); \theta)$. To simplify notation, we also introduce the functions $h_1(\mathbf{x}) \doteq \frac{\partial \log f_{\mathbf{X}}(\mathbf{x}; \theta)}{\partial \theta_j} r(\mathbf{x})$, and $h_2(\mathbf{x}) \doteq \frac{\partial \log f_{\mathbf{X}}(\mathbf{x}; \theta)}{\partial \theta_j}$. Thus we have

$$\begin{aligned} \Delta_{j;N} &= \frac{1}{\alpha N} \sum_{i=1}^N (h_1(\mathbf{x}_i) - h_2(\mathbf{x}_i) \tilde{\nu}) \mathbf{1}_{r(\mathbf{x}_i) \leq \tilde{\nu}} \\ &= \frac{1}{\alpha N} \sum_{i=1}^N (h_1(\mathbf{x}_i) - h_2(\mathbf{x}_i) \nu) \mathbf{1}_{r(\mathbf{x}_i) \leq \nu} \\ &\quad + \frac{1}{\alpha N} \sum_{i=1}^N (h_1(\mathbf{x}_i) - h_2(\mathbf{x}_i) \nu) (\mathbf{1}_{r(\mathbf{x}_i) \leq \tilde{\nu}} - \mathbf{1}_{r(\mathbf{x}_i) \leq \nu}) \\ &\quad + \frac{1}{\alpha N} \sum_{i=1}^N (h_1(\mathbf{x}_i) - h_2(\mathbf{x}_i)) (\tilde{\nu} - \nu) (\mathbf{1}_{r(\mathbf{x}_i) \leq \tilde{\nu}} - \mathbf{1}_{r(\mathbf{x}_i) \leq \nu}) \end{aligned} \quad (26)$$

We furthermore let $D_1(\mathbf{x}_i) \doteq (h_1(\mathbf{x}_i) - h_2(\mathbf{x}_i) \nu)$, and $D_2(\mathbf{x}_i) \doteq (h_1(\mathbf{x}_i) - h_2(\mathbf{x}_i)) (\tilde{\nu} - \nu)$. Note that by Assumption 4, D_1 and D_2 are bounded.

By Proposition 2, and the strong law of large numbers, we have that w.p. 1

$$\frac{1}{\alpha N} \sum_{i=1}^N (h_1(\mathbf{x}_i) - h_2(\mathbf{x}_i) \nu) \mathbf{1}_{r(\mathbf{x}_i) \leq \nu} \rightarrow \frac{\partial}{\partial \theta_j} \Phi_\alpha(r(\mathbf{X}); \theta). \quad (27)$$

We now show that the two additional sums in (26) vanish as $N \rightarrow \infty$. By Hölder's inequality

$$\left| \frac{1}{N} \sum_{i=1}^N D_1(\mathbf{x}_i) (\mathbf{1}_{r(\mathbf{x}_i) \leq \tilde{\nu}} - \mathbf{1}_{r(\mathbf{x}_i) \leq \nu}) \right| \leq \left(\frac{1}{N} \sum_{i=1}^N |D_1(\mathbf{x}_i)|^2 \right)^{0.5} \cdot \left(\frac{1}{N} \sum_{i=1}^N |\mathbf{1}_{r(\mathbf{x}_i) \leq \tilde{\nu}} - \mathbf{1}_{r(\mathbf{x}_i) \leq \nu}|^2 \right)^{0.5}, \quad (28)$$

and $\left(\frac{1}{N} \sum_{i=1}^N |D_1(\mathbf{x}_i)|^2 \right)^{0.5}$ is bounded. Also, note that

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N |\mathbf{1}_{r(\mathbf{x}_i) \leq \tilde{\nu}} - \mathbf{1}_{r(\mathbf{x}_i) \leq \nu}|^2 &= \frac{1}{N} \sum_{i=1}^N |\mathbf{1}_{r(\mathbf{x}_i) \leq \tilde{\nu}} - \mathbf{1}_{r(\mathbf{x}_i) \leq \nu}| \\ &= (\mathbf{1}_{\tilde{\nu} \leq \nu} - \mathbf{1}_{\nu \leq \tilde{\nu}}) \frac{1}{N} \sum_{i=1}^N (\mathbf{1}_{r(\mathbf{x}_i) \leq \tilde{\nu}} - \mathbf{1}_{r(\mathbf{x}_i) \leq \nu}) \end{aligned}$$

By Proposition 4.1 of Hong & Liu (2009), we have that w.p. 1

$$(\mathbf{1}_{\tilde{\nu} \leq \nu} - \mathbf{1}_{\nu \leq \tilde{\nu}}) \frac{1}{N} \sum_{i=1}^N (\mathbf{1}_{r(\mathbf{x}_i) \leq \tilde{\nu}} - \mathbf{1}_{r(\mathbf{x}_i) \leq \nu}) \rightarrow 0.$$

By the continuous mapping theorem, we thus have that w.p. 1

$$\left(\frac{1}{N} \sum_{i=1}^N |\mathbf{1}_{r(\mathbf{x}_i) \leq \tilde{\nu}} - \mathbf{1}_{r(\mathbf{x}_i) \leq \nu}|^2 \right)^{0.5} \rightarrow 0,$$

therefore, using Eq. (28) we have that w.p. 1

$$\frac{1}{N} \sum_{i=1}^N D_1(\mathbf{x}_i) (\mathbf{1}_{r(\mathbf{x}_i) \leq \tilde{\nu}} - \mathbf{1}_{r(\mathbf{x}_i) \leq \nu}) \rightarrow 0. \quad (29)$$

By a similar procedure we have that also

$$\frac{1}{N} \sum_{i=1}^N D_2(\mathbf{x}_i) (\mathbf{1}_{r(\mathbf{x}_i) \leq \tilde{\nu}} - \mathbf{1}_{r(\mathbf{x}_i) \leq \nu}) \rightarrow 0. \quad (30)$$

Plugging (27), (29), and (30) in (26) gives the stated result. \square