Chapter 2: Introduction to Probability

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Learning Objectives

1. Define basic axioms and propositions in probability
2. Assign probabilities to events
3. Perform manipulations on probabilities to make calculations easier

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Probability axioms
- Probability properties
- Counting
- Independence
- Conditional probability
- Bayes' Theorem
- Random Variables

Probability for discrete random variables

- Functions: pmfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Probability for continuous random variables

- Calculus
- Functions: pdfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

Probabilities of equally likely events

Probabilities of equally likely events

- "Equally likely" means the probability of any possible outcome is the same
  - Think: each side of die is equally likely or picking a card in a deck is equally likely

Pick an *equally likely* card, any *equally likely* card

Example 1

Suppose you have a regular well-shuffled deck of cards. What's the probability of drawing:

1. any heart
2. the queen of hearts
3. any queen

Let's break down this probability

If S is a finite sample space, with **equally likely outcomes**, then

$$\mathbb{P}(A) = \frac{|A|}{|S|}$$

In human speak:

- For equally likely outcomes, the probability that a certain event occurs is: the number of outcomes within the event of interest ($|A|$) **divided by** the total number of possible outcomes ($|S|$)

$$\mathbb{P}(A) = \frac{\text{total number of outcomes in event A}}{\text{total number of outcomes in sample space}}$$

- Thus, it is important to be able to count the outcomes within an event

A probability is a function...

- $\mathbb{P}(A)$ is a function with
  - **Input:** event A from the sample space S, ($A \subseteq S$)
    - $A \subseteq S$ means "A contained within S" or "A is a subset of S"
  - **Output:** a number between 0 and 1 (inclusive)

- The **probability function** maps an event (input) to value between 0 and 1 (output)
  - When we speak of the probability function, we often call the values between 0 and 1 "probabilities"
    - Example: "The probability of drawing a heart is 0.25" for $P(\text{heart}) = 0.25$

- The probability function needs to follow some specific rules!

See Probability Axioms on next slide.

Probability Axioms

Probability Axioms

Some probability properties

Some probability properties

Using the Axioms, we can prove all other probability properties! Events A, B, and C are not necessarily disjoint!

Proposition 1

For any event A, $\mathbb{P}(A) = 1 - \mathbb{P}(A^C)$

Proposition 2

$\mathbb{P}(\emptyset) = 0$

Proposition 3

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

Proposition 4

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

where A and B are not necessarily disjoint

Proposition 5

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) +& \\ \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) -& \\ \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)&\end{aligned}$$

Proposition 1 Proof

Proposition 1

For any event $A, \mathbb{P}(A) = 1 - \mathbb{P}(A^C)$

Use Axioms!

A1: $0 \leq \mathbb{P}(A) \leq 1$

A2: $\mathbb{P}(S) = 1$

A3: For disjoint A_i,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Proposition 2 Proof

Proposition 2

$\mathbb{P}(\emptyset) = 0$

Use Axioms!

A1: $0 \leq \mathbb{P}(A) \leq 1$

A2: $\mathbb{P}(S) = 1$

A3: For disjoint A_i,

$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

Proposition 3 Proof

Proposition 3

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

Use Axioms!

A1: $0 \leq \mathbb{P}(A) \leq 1$

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A3: For disjoint A_i,

$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

Proposition 4 Visual Proof

Proposition 4

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Proposition 5 Visual Proof

Proposition 5

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$$

Some final remarks on these proposition

- Notice how we spliced events into multiple **disjoint** events
  - It is often easier to work with disjoint events

- If we want to calculate the probability for one event, we may need to get creative with how we manipulate other events and the sample space
  - Helps us use any incomplete information we have

Partitions

Partitions

Definition: Partition

A set of events $\{A_i\}_{i=1}^{n}$ create a **partition** of A, if

- the A_i's are disjoint (mutually exclusive) and
- $\bigcup_{i=1}^{n} A_i = A$

Example 2

- If $A \subset B$, then $\{A, B \cap A^C\}$ is a partition of B.
- If $S = \bigcup_{i=1}^{n} A_i$, and the A_i's are disjoint, then the A_i's are a partition of the sample space.

Creating partitions is sometimes used to help calculate probabilities, since by Axiom 3 we can add the probabilities of disjoint events.

Venn Diagram Probabilities

Weekly medications

Example 3

If a subject has an

- 80% chance of taking their medication *this* week,
- 70% chance of taking their medication *next* week, and
- 10% chance of *not* taking their medication *either* week,

then find the probability of them taking their medication exactly one of the two weeks.

Hint: Draw a Venn diagram labelling each of the parts to find the probability.