Chapter 27: Conditional Distributions

Meike Niederhausen and Nicky Wakim

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Learning Objectives

1. Calculate the conditional probability density from a joint pdf

Conditional probabilities we've seen before

What do we know about conditional probabilities for events and discrete RVs?

For events:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

For discrete RVs:

$$p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

What does it mean for conditional densities of continuous RVs?

For continuous RVs:

Example starting from a joint pmf: first try!

Example 1.1

Let $f_{X,Y}(x, y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$.

1. Find $\mathbb{P}(2 < X < 10 | Y = 4)$

What is a conditional density?

Definition: Conditional density

The conditional density of a r.v. X given $Y = y$, is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)},$$

for $f_Y(y) > 0$

Remarks

1. It follows from the definition for the conditional density $f_{X|Y}(x|y)$, that

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y).$$

2. For a fixed value of $Y = y$, the conditional density $f_{X|Y}(x|y)$ is an actual pdf, meaning
   - $f_{X|Y}(x|y) \geq 0$ for all x and y, and
   - $\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = 1.$

Example starting from a joint pmf: second try!

Example 1.1

Let $f_{X,Y}(x, y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$.

1. Find $\mathbb{P}(2 < X < 10 | Y = 4)$

Example starting from a joint pmf

Example 1.2

Let $f_{X,Y}(x,y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$.

2. Find $\mathbb{P}(X > 20|Y = 5)$

Finding probability with conditional domain and pmf

Example 2

Randomly choose a point X from the interval $[0, 1]$, and given $X = x$, randomly choose a point Y from $[0, x]$. Find $\mathbb{P}(0 < Y < \frac{1}{4})$.

Independence and conditional distributions

Question What is $f_{X|Y}(x|y)$ if X and Y are independent?

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_y(y)} = \frac{f_X(x)f_y(y)}{f_y(y)} = f_X(x)$$

- If $f_{X|Y}(x|y)$ does not depend on y (including the bounds/domain), then X and Y are independent.