Chapter 11: Expected Values of Sums of Discrete RVs

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Learning Objectives

1. Calculate the mean (expected value) of *sums of* discrete random variables

linear combinations

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Probability axioms
- Probability properties
- Counting
- Independence
- Conditional probability
- Bayes' Theorem
- Random Variables

Probability for discrete random variables

- Functions: pmfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Probability for continuous random variables

- Calculus
- Functions: pdfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

Revisiting our two card draw

Example 1

Suppose you draw 2 cards from a standard deck of cards *with* replacement. Let X be the number of hearts you draw. Find $\mathbb{E}[X]$.

Recall Binomial RV with $n = 2$:

$$p_X(x) = \binom{2}{x} p^x (1-p)^{2-x} \text{ for } x = 0, 1, 2$$

$P(\heartsuit) = p = \frac{13}{52} = \frac{1}{4}$

x	x_1 0	x_2 1	x_3 2
$p_X(x)$	$\binom{2}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^2$	$\binom{2}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^1$	$\binom{2}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^0$

$$E(X) = \sum_{i=1}^{3} x_i P_X(x_i) = 0\,P_X(0) + 1\,P_X(1) + 2\,P_X(2)$$

$$= 0 \cdot (1)(1)\left(\frac{3}{4}\right)^2 + 1 \cdot \left[2\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\right] + 2\left[1 \cdot \left(\frac{1}{4}\right)^2(1)\right]$$

$$= 0 + \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

$$E(X) = np = 2 \cdot p = 2 \cdot \frac{1}{4} = \frac{1}{2}$$

Let Y = success/fail (bern) $\rightarrow X = \sum_{i=1}^{2} Y_i$

$$E(X) = E\left(\sum_{i=1}^{2} Y_i\right)$$

binom = sum of bernoullis

$E(\sum Y_i) = \sum(E(Y_i))$

What if we draw A LOT of cards?

$X = \#\ \heartsuit\text{s in } 200 \text{ draws}$

Example 2

What is the expected number of hearts in Example 1 if you draw 200 cards?

w/ replace

Recall Binomial RV with $n = 200$:

$$p_X(x) = \binom{200}{x} p^x (1-p)^{200-x}$$

for $x = 0, 1, 2, \ldots, 200$

$$E(X) = \sum_{x=0}^{200} x\, P(X = x)$$

$$= \sum_{x=0}^{200} x \binom{200}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{200-x}$$

$$X = \sum_{i=1}^{200} Y_i \quad \text{where } Y_i = \begin{cases} 1 & \text{if } \heartsuit \\ 0 & \text{if not } \heartsuit \end{cases}$$

$$E(Y_i) = p = \frac{1}{4}$$

$$E(X) = E\left[\sum_{i=1}^{200} Y_i\right]$$

$$= \sum_{i=1}^{200} E(Y_i) = \sum_{i=1}^{200} \left(\frac{1}{4}\right)$$

$$= 200\left(\frac{1}{4}\right) = 50$$

Sum of discrete RVs

Theorem 11.1: Sum of discrete RVs

For discrete r.v.'s X_i and constants $a_i, i = 1, 2, \ldots, n$,

$$\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i].$$

if $a_i = a$

expected value of sum is sum of expected value

$E(bY) = bE(Y)$ if b is a constant

Remark: The theorem holds for infinitely r.v.'s X_i as well.

- For two RVs, X and Y:
  - We can say $E[X+Y] = E[X] + E[Y]$
  - ... and constant numbers a and b, we can also say $E[aX+bY] = aE[X] + bE[Y]$
  - We can also also say $E[X-Y] = E[X] - E[Y]$, since $b = -1$

$E(X + (-1)Y)$

Corollaries from Thm 11.1

Corollary 11.1.1

For a discrete r.v. X, and constants a and b,

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b.$$

$E(\text{constant}) = \text{constant}$

Corollary 11.1.2

If $X_i, i = 1, 2, \ldots, n$, are identically distributed r.v.'s, then

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = n\mathbb{E}[X_1].$$

$$E(X) = E\left(\sum_{i=1}^{200} Y_i\right) = n\,E(Y_1)$$

identically dist

if not ident.

$$\sum_{i=1}^{200} E(Y_i)$$

p_i

Cost of hotel rooms

Example 4 not necessarily identically distributed

A tour group is planning a visit to the city of Minneapolis and needs to book 30 hotel rooms. The average price of a room is \$200. In addition, there is a 10% tourism tax for each room. What is the expected cost for the 30 hotel rooms?

Let T = cost of 30 rooms

C_i = cost of room i

$\hookrightarrow E(C_i) = 200$

$E(C_i) = E(C_j)$, $i \neq j$

$\hookrightarrow 1.10\, C_i$

can't jump to this 1st

$1.10 \cdot E(C_i)$

$$T = \sum_{i=1}^{30} 1.1 \cdot C_i$$

$E(aX) = aE(X)$

$$E(T) = E\left(\sum_{i=1}^{30} 1.1\, C_i\right) = \sum_{i=1}^{30} E[1.1\, C_i] = \sum_{i=1}^{30} 1.1\, E(C_i)$$

$$= 1.1 \sum_{i=1}^{30} E(C_i) = 1.1 \sum_{i=1}^{30} 200$$

$$= 1.1 \cdot 30 \cdot 200 = \$6,600$$