Chapter 43: Moment Generating Functions

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Learning Objectives

1. Learn the definition of a moment-generating function.
2. Find the moment-generating function of a ~~binomial~~ random variable.
3. Use a moment-generating function to find the mean and variance of a random variable.

What are moments?

Definition 1

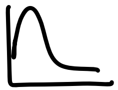
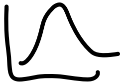
The j^{th} moment of a r.v. X is $\mathbb{E}[X^j]$

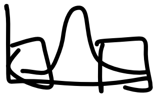
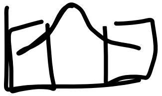
Example 1

$1^{st} - 4^{th}$ moments

① 1st moment : $E(X)$ mean / exp val ✓

② 2nd moment : $E(X^2)$ σ^2 /variance ✓

③ 3rd " " : $E(X^3)$ skewness  vs. 

④ 4th " " : $E(X^4)$ kurtosis  vs. 

Z t

What is a *moment generating function* (mgf)??

Definition 3

If X is a r.v., then the **moment generating function** (**mgf**) associated with X is:

$$M_X(t) = \mathbb{E}[e^{tX}]$$

Remarks

- For a discrete r.v., the mgf of X is

g(x)

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{all\ x} e^{tx} p_X(x)$$

- For a continuous r.v., the mgf of X is

g(x)

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$$

- The mgf $M_X(t)$ is a function of t, not of X, and it might not be defined (i.e. finite) for all values of t. We just need it to be defined for $t = 0$.

Example

Example 4

What is $M_X(t)$ for $t = 0$?

$$M_X(t) = E\left[e^{tX}\right]$$

$$M_X(t=0) = E\left(e^{0 \cdot X}\right) = E\left(e^{0}\right) = \underline{1}$$

when $t=0$, mgf is 1 for all RVs

Theorem

Theorem 5

The moment generating function uniquely specifies a probability distribution.

Theorem 6

$$\mathbb{E}[X^r] = M_X^{(r)}(0)$$

$r = 1$: $M_X'(0)$

$r = 4$: $M_X^{(4)}(0)$

(r) in this equation is the rth derivative with respect to t

- When $r = 1$, we are taking the first derivative
- When $r = 4$, we are taking the fourth derivative

Using the mgf to uniquely describe a probability distribution

Example 7

Let $X \sim Poisson(\lambda)$

1. Find the mgf of X
2. Find $\mathbb{E}[X]$
3. Find $Var(X)$

$$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

① $$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda}e^{e^t\lambda} = e^{-\lambda+\lambda e^t}$$

rule: $\sum_{n=0}^{\infty}\frac{y^n}{n!} = e^y$

$\rightarrow M_X(t) = e^{\lambda(e^t-1)}$

② $\underline{E(X)}$: $M'_X(t) = \frac{d}{dt}e^{\lambda(e^t-1)}$

$$= e^{\lambda(e^t-1)} \cdot \frac{d}{dt}(\lambda(e^t-1))$$

$$= \lambda e^t e^{\lambda(e^t-1)}$$

$$E(X) = M'_X(0) = \lambda \underbrace{e^0}_{=1} \underbrace{e^{\lambda(e^0-1)}}_{=1} \quad (e^0-1 = 0)$$

$$= \lambda$$

③ $Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = M''_X(0)$

$$= \lambda e^{0+\lambda(0)}(1+\lambda e^0)$$

$$= \lambda + \lambda^2$$

$var = \lambda + \lambda^2 - (\lambda)^2 = \lambda$

$M''_X(t) =$

$\lambda e^{t+\lambda(e^t-1)} \cdot (1+\lambda e^t)$

Theorem

Remark: Finding the mean and variance is sometimes easier with the following trick

Theorem 8

Let $R_X(t) = \ln[M_X(t)]$. Then,

$$\mu = \mathbb{E}[X] = R'_X(0), \text{ and}$$

$$\sigma^2 = Var(X) = R''_X(0)$$

Proof.

$$R'_X(t) = \frac{d}{dt} \ln(\underbrace{M_X(t)}_{g(x)}) = \underbrace{\frac{1}{M_X(t)}}_{\frac{1}{g(x)}} \cdot \underbrace{M'_X(t)}_{g'(x)}$$

$$R'_X(t=0) = \frac{1}{\underbrace{M_X(0)}_{1}} M'_X(0) = E(X)$$

Using $R_X(t)$ to uniquely describe a probability distribution

Example 9

Let $X \sim Poisson(\lambda)$.

1. Find $\mathbb{E}[X]$ using $R_X(t)$
2. Find $Var(X)$ using $R_X(t)$

① $R_X(t) = \ln(M_X(t)) = \ln\left(e^{\lambda e^t - \lambda}\right) = \lambda e^t - \lambda$

$R_X'(t) = \frac{d}{dt}(\lambda e^t - \lambda) = \underline{\lambda e^t}$

$R'_X(0) = \lambda e^0 = \lambda$

$E(X) = \lambda$

② $R_X''(t) = \frac{d}{dt}(\underbrace{\lambda e^t}_{R'_X(t)}) = \lambda e^t$

$Var(X) = R''_X(0) = \lambda e^0 = \lambda$

Using the mgf to uniquely describe the standard normal distribution

Example 10

Let Z be a standard normal random variable, i.e. $Z \sim N(0, 1)$.

1. Find the mgf of Z
2. Find $\mathbb{E}[Z]$
3. Find $Var(Z)$

① $$M_Z(t) = E[e^{tZ}] = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tz - z^2/2}\, dz$$

$$tz - z^2/2 = \underbrace{\frac{t^2 - (z-t)^2}{2}}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{t^2 - (z-t)^2}{2}}\, dz$$

$$= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2}}\, dz$$

let $u = z - t$
$du = dz$

$$= e^{t^2/2} \underbrace{\int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-u^2/2}}_{f_U(u)}\, du}_{=1} = e^{t^2/2} = M_Z(t)$$

② $E(Z) = R'_Z(0)$

$$R_Z(t) = \ln(M_Z(t)) = \ln\left(e^{t^2/2}\right) = \frac{t^2}{2}$$

$$R'_Z(t) = \frac{d}{dt}\left(t^2/2\right) = \frac{2t}{2} = t$$

$$E(Z) = R'_Z(0) = 0 \checkmark$$

vs. $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$

③ $Var(Z) = R''_Z(0)$

$$R''_Z(t) = \frac{d}{dt}(t) = 1$$

$$Var(Z) = 1 \checkmark$$

$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$

Mgf's of sums of independent RV's

Theorem 9

If X and Y are independent RV's with respective mgf's $M_X(t)$ and $M_Y(t)$, then

$$M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX}e^{tY}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t)$$

$$M'_{X+Y}(t) = M'_X(t)\, M'_Y(t)$$

Main takeaways

- Mgf's are a purely mathematically definition ✓
  - We can't really relate it to our real world analysis
- They are helpful mathematically because they are unique to a probability distribution
  - We can find the unique mgf from for a probability distribution
  - And we can find a distribution from an mgf
- Mgf's can *sometimes* make it easier to find the mean and variance of an RV
- Mgf's are most helpful when we are finding a joint distribution that is a sum or transformation of two RV's
  - Make the calculation easier!
- Mgf's are often used to prove certain distribution are sums of other ones!

More resources

- https://online.stat.psu.edu/stat414/book/export/html/676
- https://www.youtube.com/watch/ez_vq23xWrQ
- https://www.youtube.com/watch/2p9J9ChTeFI
- https://www.youtube.com/watch/A5bWU8xcQkE
- https://www.youtube.com/watch/QeUrTGFTFm4
- https://www.youtube.com/watch/HhrkwyyRtgI