# Chapter 2: Probability

Meike Niederhausen and Nicky Wakim

2023-09-27

# Class Overview

- Probabilities of equally likely events
- Probability Axioms
- Some probability properties
- Partitions
- Venn Diagram Probabilities

# Probabilities of equally likely events

# Pick an *equally likely* card, any *equally likely* card

**Example 1**

Suppose you have a regular well-shuffled deck of cards. What's the probability of drawing:

1. any heart
2. the queen of hearts
3. any queen

sample space: 52 cards

4 suits: ♡ ◇ ♣ ♠

13 faces/#s: A 2-10 J Q K

① $P(♡) = \frac{13}{52} = \frac{1}{4} = 0.25$

probability (of)

② $P(Q♡) = \frac{1}{52}$

③ $P(Q) = \frac{4}{52} = \frac{1}{13}$

# Let's break down this probability

If $S$ is a finite sample space, with **equally likely outcomes**, then

$$\mathbb{P}(A) = \frac{|A|}{|S|}.$$

$$\frac{\text{total \# of outcomes in } A}{\text{total \# of outcome in } S}$$

$P \quad \mathbb{P} \quad \boldsymbol{P} \quad prob()$

# A probability is a function...

$\mathbb{P}(A)$ is a function with

A contained w/in S
A is a subset of S

- **Input:** event $A$ from the sample space $S$, ($A \subseteq S$)
- **Output:** a number between 0 and 1 (inclusive)

$$\mathbb{P}(A) : S \to [0, 1]$$

A function that follows some specific rules though!

*See Probability Axioms on next slide.*

# Probability Axioms

# Probability Axioms

## Axiom 1

For every event $A$, $0 \le \mathbb{P}(A) \le 1$.

## Axiom 2

$P(S) = \frac{|S|}{|S|} = 1$

For the sample space $S$, $\mathbb{P}(S) = 1$.

## Axiom 3

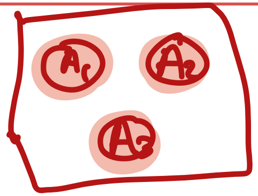
If $A_1, A_2, A_3, \ldots$, is a collection of **disjoint** events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

$P(A_1 \cup A_2 \cup A_3)$

$= P(A_1) + P(A_2) + P(A_3)$

probability of at least one $A_1 - A_n$ happening



# Some probability properties

# Some probability properties

Using the Axioms, we can prove all other probability properties!

**Proposition 1**

For any event $A$, $\mathbb{P}(A) = 1 - \mathbb{P}(A^C)$

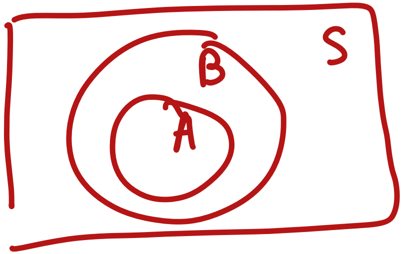
**Proposition 2**

$\mathbb{P}(\emptyset) = 0$

**Proposition 3**

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

$\subset \quad \subseteq$




**Proposition 4** A & B not necessarily disjoint

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

**Proposition 5**

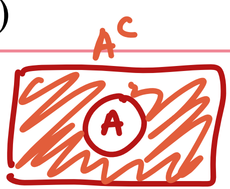
$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$

# Proposition 1 Proof

**Proposition 1**

For any event $A$, $\mathbb{P}(A) = 1 - \mathbb{P}(A^C)$

$\underline{A \cup A^c = S}$




$A$ & $A^c$ are disjoint

$$P(\underline{A \cup A^c}) = P(A) + P(A^c)$$

$$P(S) = P(A) + P(A^c)$$

$$1 = P(A) + P(A^c)$$

$$-P(A^c) \qquad\qquad -P(A^c)$$

$$P(A) = 1 - P(A^c)$$

AXIOMS

A1: $0 \le P(A) \le 1$

A2: $\underline{P(S) = 1}$

A3: $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

for disjoint $A_i$'s

# Proposition 2 Proof

**Proposition 2**

$\mathbb{P}(\emptyset) = 0$

prop 1 : $P(A) = 1 - P(A^c)$

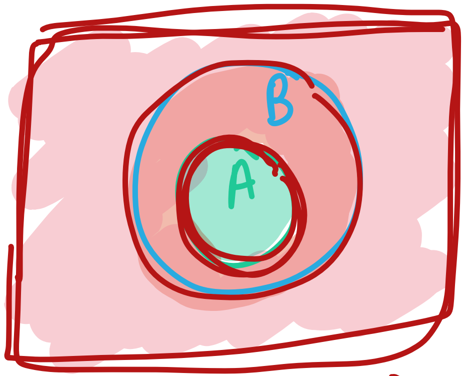
$A = \underline{\emptyset} \quad A^c = \underline{S}$

$$P(\emptyset) = 1 - P(S)$$
$$= 1 - 1$$
$$P(\emptyset) = 0$$

# Proposition 3 Proof

**Proposition 3**

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$




$\to B = A \underset{+}{\cup} (A^c \cap B)$ green + orange

disjoint events

$$P(B) = P(A \cup (A^c \cap B)) = P(A) + P(A^c \cap B)$$

$$P(B) = P(A) + \underbrace{P(A^c \cap B)}$$

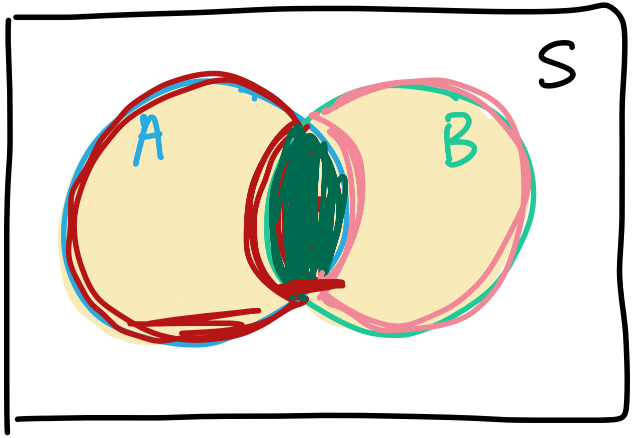
AX 1: $0 \leq P(A^c \cap B) \leq 1$

$$P(A) = P(B) - \underset{\leq 0}{P(A^c \cap B)} \leq 0 \Rightarrow P(A) \leq P(B)$$

# Proposition 4 Visual Proof

**Proposition 4** disjoint events or NOT disjoint

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$




$P(A \cup B) =$

$A \cap B^c$

$A \cap B$

$B \cap A^c$

$P(A \cap B^c) +$

$P(A \cap B) +$

$P(B \cap A^c)$

$P(A) + P(B) - P(A \cap B)$

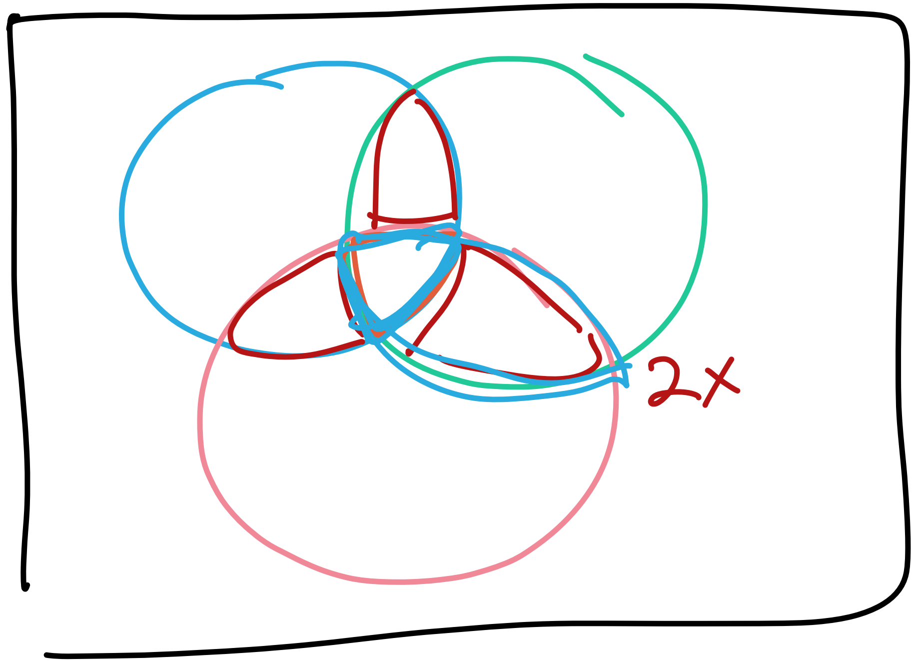
$P(A \cap B^c) + P(A \cap B)$

$P(B \cap A^c) + P(A \cap B)$

# Proposition 5 Visual Proof

**Proposition 5**

$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$



# Partitions

# Partitions

## Definition: Partition

A set of events $\{A_i\}_{i=1}^{n}$ create a **partition** of $A$, if

- the $A_i$'s are disjoint (mutually exclusive) and
- $\bigcup_{i=1}^{n} A_i = A$

## Example 2

- If $A \subset B$, then $\{A, B \cap A^C\}$ is a partition of $B$.
- If $S = \bigcup_{i=1}^{n} A_i$, and the $A_i$'s are disjoint, then the $A_i$'s are a partition of the sample space.

Creating partitions is sometimes used to help calculate probabilities, since by Axiom 3 we can add the probabilities of disjoint events.

# Venn Diagram Probabilities

# Weekly medications

**Example 3**

If a subject has an

- 80% chance of taking their medication *this* week,
- 70% chance of taking their medication *next* week, and
- 10% chance of *not* taking their medication *either* week,

then find the probability of them taking their medication exactly one of the two weeks.

*Hint: Draw a Venn diagram labelling each of the parts to find the probability.*