Chapter 2: Introduction to Probability

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Learning Objectives

1. Define basic axioms and propositions in probability
2. Assign probabilities to events, and perform manipulations on probabilities to make calculations easier

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Probability axioms
- Probability properties
- Counting
- Independence
- Conditional probability
- Bayes' Theorem
- Random Variables

Probability for discrete random variables

- Functions: pmfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Probability for continuous random variables

- Calculus
- Functions: pdfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

Probabilities of equally likely events

Probabilities of equally likely events

- "Equally likely" means the probability of any possible outcome is the same
  - Think: each side of die is equally likely or picking a card in a deck is equally likely

Pick an *equally likely* card, any *equally likely* card

Example 1

Suppose you have a regular well-shuffled deck of cards. What's the probability of drawing:

1. any heart
2. the queen of hearts
3. any queen

Let's break down this probability

If S is a finite sample space, with **equally likely outcomes**, then

$$\mathbb{P}(A) = \frac{|A|}{|S|}$$

In human speak:

- For equally likely outcomes, the probability that a certain event occurs is: the number of outcomes within the event of interest ($|A|$) **divided by** the total number of possible outcomes ($|S|$)
- Thus, it is important to be able to count the outcomes within an event

A probability is a function...

- $\mathbb{P}(A)$ is a function with
  - **Input:** event A from the sample space S, ($A \subseteq S$)
    - $A \subseteq S$ means "A contained within S" or "A is a subset of S"
  - **Output:** a number between 0 and 1 (inclusive)

- The probability function maps an event (or something within our sample space) to value between 0 and 1
  - When we speak of the probability function, we often call the values between 0 and 1 "probabilities"
    - Example: "The probability of drawing a heart is 0.25" for $P(\text{heart}) = 0.25$

- The probability function needs to follow some specific rules!

See Probability Axioms on next slide.

Probability Axioms

Probability Axioms

Axiom 1

For every event A, $0 \leq \mathbb{P}(A) \leq 1$. Probability is between 0 and 1.

Axiom 2

For the sample space S, $\mathbb{P}(S) = 1$.

Axiom 3

If $A_1, A_2, A_3, \ldots$, is a collection of **disjoint** events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Some probability properties

Some probability properties

Using the Axioms, we can prove all other probability properties! Events A, B, and C are not necessarily disjoint!

Proposition 1

For any event A, $\mathbb{P}(A) = 1 - \mathbb{P}(A^C)$

Proposition 2

$\mathbb{P}(\emptyset) = 0$

Proposition 3

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

Proposition 4

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Proposition 5

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) +& \\ \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) -& \\ \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)&\end{aligned}$$

Proposition 1 Proof

Proposition 1

For any event A, $\mathbb{P}(A) = 1 - \mathbb{P}(A^C)$

Use Axioms!

A1: $0 \leq \mathbb{P}(A) \leq 1$

A2: $\mathbb{P}(S) = 1$

A3: For disjoint A_i,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Proposition 2 Proof

Proposition 2

$\mathbb{P}(\emptyset) = 0$

Use Axioms!

A1: $0 \leq \mathbb{P}(A) \leq 1$

A2: $\mathbb{P}(S) = 1$

A3: For disjoint A_i,

$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

Proposition 3 Proof

Proposition 3

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

Use Axioms!

A1: $0 \leq \mathbb{P}(A) \leq 1$

A2: $\mathbb{P}(S) = 1$

A3: For disjoint A_i,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Proposition 4 Visual Proof

Proposition 4

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Proposition 5 Visual Proof

Proposition 5

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$$

Partitions

Partitions

Definition: Partition

A set of events $\{A_i\}_{i=1}^{n}$ create a **partition** of A, if

- the A_i's are disjoint (mutually exclusive) and
- $\bigcup_{i=1}^{n} A_i = A$

Example 2

- If $A \subset B$, then $\{A, B \cap A^C\}$ is a partition of B.
- If $S = \bigcup_{i=1}^{n} A_i$, and the A_i's are disjoint, then the A_i's are a partition of the sample space.

Creating partitions is sometimes used to help calculate probabilities, since by Axiom 3 we can add the probabilities of disjoint events.

Venn Diagram Probabilities

Weekly medications

Example 3

If a subject has an

- 80% chance of taking their medication *this* week,
- 70% chance of taking their medication *next* week, and
- 10% chance of *not* taking their medication *either* week,

then find the probability of them taking their medication exactly one of the two weeks.

Hint: Draw a Venn diagram labelling each of the parts to find the probability.