Chapter 29: Variance of Continuous Random Variables

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Learning Objectives

1. Calculate expected value of functions of RVs
2. Calculate variance of RVs

Expected value of a function of a continuous RV

$g(x)$ is some fn of x

How do we calculate the expected value of a function of a discrete RV or joint RVs?

For discrete RVs:

$$\mathbb{E}[g(X)] = \sum_{\{all\ x\}} g(x) p_X(x).$$

$$\mathbb{E}[g(X,Y)] = \sum_{\{all\ x\}} \sum_{\{all\ y\}} g(x,y) p_{X,Y}(x,y).$$

How do we calculate the expected value of a function of a continuous RV ~~or joint RV~~s?

For continuous RVs:

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

x^2 $(x-\mu)^2$

variance uses exp val of fn of x

Important properties of expected values of functions of continuous RVs

Function of RV with two constants

$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

Function of two RVs added

$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

Expected value of sum of ~~independent~~ RVs pt 1

If $X_1, X_2, \ldots X_n$ are continuous RVs and $a_1, a_2, \ldots a_n$ are constants, then

$$\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i]$$

linearity prop of exp vals

Expected value of multiplication of function of independent RVs

If X and Y are independent continuous RVs, and g and h are functions, then

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$$

Expected value of multiplication of independent RVs

If X and Y are independent continuous RVs, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

$Var(X+Y) = Var(X) + Var(Y)$ only if ind.

Variance of continuous RVs

How do we calculate the variance of a discrete RV?

For discrete RVs:

$$\begin{aligned} Var(X) &= \mathbb{E}[(X-\mu_X)^2] \\ &= \mathbb{E}[(X-\mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2]-(\mathbb{E}[X])^2 \\ &= \sum_{\{all\ x\}} (x-\mu_x)^2 p_X(x) \end{aligned}$$

How do we calculate the variance of a continuous RV?

For continuous RVs:

$$\begin{aligned} Var(X) &= E[(X-\mu_X)^2] \\ &= E[(X-E(X))^2] \\ &= E(X^2)-[E(X)]^2 \\ &= \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx \end{aligned}$$

Variance of an Uniform distribution

Example 2

Let $f_X(x) = \frac{1}{b-a}$, for $a \le x \le b$. Find $Var[X]$.

$$E(X) = \frac{a+b}{2}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

exp value of the square minus the square of the exp value

$$E(X^2) = \int_a^b x^2 \frac{1}{b-a}\,dx$$

$$= \frac{1}{3(b-a)} x^3 \Big|_{x=a}^{x=b}$$

$$= \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2+ab+a^2}{3} - \left[\frac{a+b}{2}\right]^2$$

$$= \frac{4(b^2+ab+a^2)}{4(3)} - \frac{3(a^2+2ab+b^2)}{3(4)}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

$$\boxed{var(X) = (b-a)^2/12}$$

Variance of exponential distribution

In the homework:

Example 3

Let $f_X(x) = \lambda e^{-\lambda x}$, for $x > 0$ and $\lambda > 0$. Find $Var[X]$.

$$Var(X) = E(X^2) - [E(X)]^2 \qquad E(X) = \frac{1}{\lambda}$$

$$= \int_0^\infty x^2 \lambda e^{-\lambda x} dx - \left(\frac{1}{\lambda}\right)^2$$

int by parts 2x

$\vdots$

$$= \frac{1}{\lambda^2}$$

Important properties of variances of continuous RVs

Function of RV with two constants

$$Var[aX + b] = a^2 Var[X]$$

Variance of sum of independent RVs pt 1

If $X_1, X_2, \ldots X_n$ are independent continuous RVs and $a_1, a_2, \ldots a_n$ are constants, then

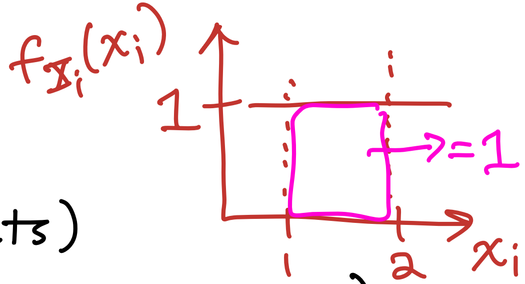
$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 Var(X_i)$$

Variance of sum of independent RVs pt 2

If $X_1, X_2, \ldots X_n$ are independent continuous RVs, then

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$$

Find the mean and sd from word problem



Example 4

A machine manufactures cubes with a side length that varies uniformly from 1 to 2 inches. Assume the sides of the base and height are equal. The cost to make a cube is 10 ¢ per cubic inch, and 5 ¢ cents for the general cost per cube. Find the mean and standard deviation of the cost to make 10 cubes.

Model cost of 10 cubes:

C = cost of 10 cubes (in cents)

C_i = cost of cube i (ith cube w/ $i = 1, \dots, 10$)

X_i = length of each side of cube i

$$C = \sum_{i=1}^{10} C_i \qquad C_i = 5 + 10\left(X_i^3\right) \qquad X_i \sim U[1,2]$$

MEAN: $E(C) = E\left[\sum_{i=1}^{10}\left(5 + 10X_i^3\right)\right]$ $E(aX+b) = aE(X)+b$

linear combos

$$= \sum_{i=1}^{10} E\left(5 + 10X_i^3\right) = \sum_{i=1}^{10}\left(5 + 10E\left(X_i^3\right)\right) \qquad \neq \left[E(X_i)\right]^3$$

$$= \sum_{i=1}^{10}\left(5 + 10\left(\frac{15}{4}\right)\right)$$

$$= \sum_{i=1}^{10} 42.5 = 425¢ = \boxed{E(C) = \$4.25}$$

$$E(\underbrace{X_i^3}_{g(X)}) = \int_1^2 \underbrace{x^3}_{g(x)}\, 1\, dx$$

$$= \frac{1}{4}x^4\Big|_{x=1}^{x=2} = \frac{1}{4}\left(2^4 - 1^4\right) = \frac{15}{4}$$

$SD = \sqrt{Var(c)}$

$Var\sum = \sum Var \rightarrow$ each cube i is independent

$$var(c) = Var\left[\sum_{i=1}^{10}(5+10X_i^3)\right] = \sum_{i=1}^{10} Var(5+10X_i^3)$$

var of const. is 0

$$= \sum_{i=1}^{10} Var(10X_i^3) = \sum_{i=1}^{10} 10^2\, Var(X_i^3)$$

$Var(aX) = a^2 Var(X)$

$X_i \sim U[1,2]$

$$Var(X_i^3) = E((X_i^3)^2) - [\underbrace{E(X_i^3)}_{15/4}]^2 = \frac{127}{7} - \left(\frac{15}{4}\right)^2 = 4.0803$$

$$\boxed{E(X_i^6) = \int_1^2 x^6(1)\,dx = \frac{1}{7}x^7\Big|_{x=1}^{x=2} = \frac{1}{7}(2^7 - 1^7) = \frac{128}{7} - \frac{1}{7} = \frac{127}{7}}$$

$$= \sum_{i=1}^{10} 10^2(4.0803) = 10 \cdot 100 \cdot 4.0803 = 4080.36¢^2$$

$$Var(c) = 4080.36 \text{¢}^2$$

$$SD(c) = \sqrt{Var(c)} = \sqrt{4080.36} = 63.8776 \text{¢}$$

$$\boxed{SD(C) = 64 \text{¢}}$$