Chapter 25: Joint densities

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Learning Objectives

1. Solve double integrals in our mini lesson!
2. Calculate probabilities for a pair of continuous random variables
3. Calculate a *joint and marginal* probability density function (pdf)
4. Calculate a *joint and marginal* cumulative distribution function (CDF) from a pdf

Double Integrals Mini Lesson (1/3)

Mini Lesson Example 1

Solve the following integral:

$$\int_{2}^{3}\int_{0}^{1} xydydx$$

Double Integrals Mini Lesson (2/3)

Mini Lesson Example 2

Solve the following integral:

$$\int_{2}^{3}\int_{0}^{1}(x + y)dydx$$

Double Integrals Mini Lesson (3/3)

Do this problem at home for extra practice. The solution is available in Meike's video!

Mini Lesson Example 3

Solve the following integral:

$$\int_2^3 \int_0^1 \mathrm{e}^{\mathrm{x}+\mathrm{y}}\,\mathrm{dydx}$$

How to define the joint pdf for continuous RVs?

For a single continuous RV X is a function $f_X(x)$, such that for all real values a, b with $a \leq b$,

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$$

For two continuous RVs (X and Y), we can define the **joint pdf**, $f_{X,Y}(x, y)$, such that for all real values a, b, c, d with $a \leq b$ and $c \leq d$,

$$\mathbb{P}(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x, y)dydx$$

Important properties of the joint pdf

1. Note that $f_{X,Y}(x, y) \neq \mathbb{P}(X = x, Y = y)$!!!
2. In order for $f_{X,Y}(x, y)$ to be a pdf, it needs to satisfy the properties
   - $f_{X,Y}(x, y) \geq 0$ for all x, y
   - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)dxdy = 1$

What is the joint cumulative distribution function?

Definition: Joint cumulative distribution function (Joint CDF)

The **joint cumulative distribution function (cdf)** of continuous random variables X and Y, is the function $F_{X,Y}(x, y)$, such that for all real values of x and y,

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s, t)dtds$$

Remarks:

- The definition above for $F_{X,Y}(x, y)$ is a **function** of x and y.
- The joint cdf at the point (a, b), is

$$F_{X,Y}(a, b) = \mathbb{P}(X \leq a, Y \leq b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(s, t)dtds$$

What are the marginal pdf's?

Definition: Marginal pdf's

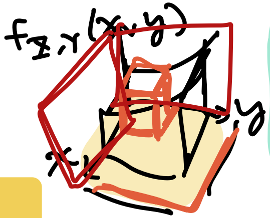
Suppose X and Y are continuous r.v.'s, with joint pdf $f_{X,Y}(x, y)$. Then the **marginal probability density functions** are

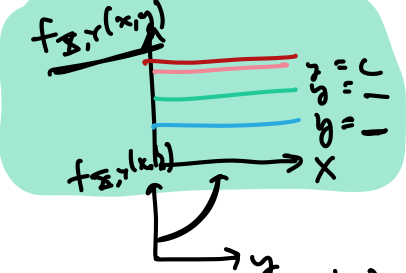
int over all y

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx$$

Example of joint pdf





Example 1.1

Let $f_{X,Y}(x,y) = \frac{3}{2}y^2$, for $0 \le x \le 2,\ 0 \le y \le 1$.

1. Find $\mathbb{P}(0 \le X \le 1, 0 \le Y \le \frac{1}{2})$.

Steps in problem

① set up domain of pdf

② shade in area of probability of interest

③ set up integral: dydx or dxdy?

domain of y and domain of x don't depend on each other & pdf is only in terms of y

$$P(0 \le X \le 1,\ 0 \le Y \le \tfrac{1}{2})$$

$$= \int_0^1 \int_0^{1/2} \frac{3}{2}y^2 \, dy\, dx$$

$$= \int_0^1 \left[\frac{1}{2} y^3 \right]_{y=0}^{y=1/2} dx$$

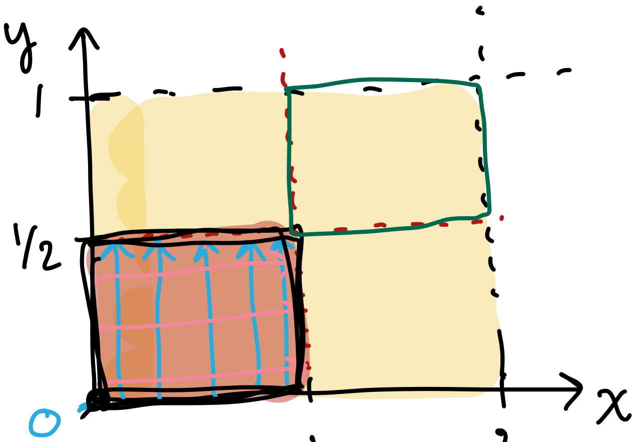
$$= \int_0^1 \left[\frac{1}{2}\left(\frac{1}{2}\right)^3 - \frac{1}{2}(0)^3 \right] dx$$

$$= \int_0^1 \frac{1}{16}\, dx = \left[\frac{1}{16} x \right]_{x=0}^{x=1}$$

$$= \frac{1}{16}(1) - \frac{1}{16}(0) = \frac{1}{16}$$

$$P(0 \le X \le 1,\ 0 \le Y \le 1/2) = 1/16$$

① & ②



$$f_X(x) = \exp\left(\frac{}{} \right)$$

Example of joint pdf

$f_X(x)$: int out y $\quad$ $f_Y(y)$: int out x

Example 1.2

Let $f_{X,Y}(x,y) = \frac{3}{2}y^2$, for $0 \le x \le 2,\ 0 \le y \le 1$.

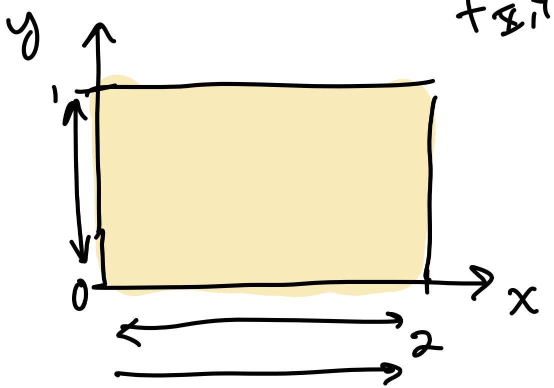
2. Find $f_X(x)$ and $f_Y(y)$.

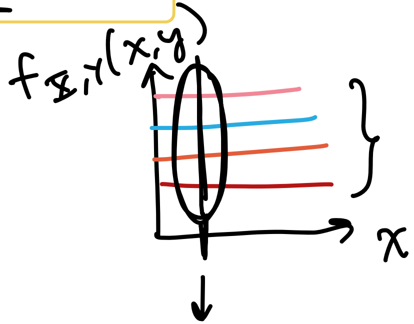
$$f_X(x) = \int_0^1 \frac{3}{2}y^2\,dy = \frac{1}{2}y^3 \Big|_{y=0}^{y=1}$$

$$= \frac{1}{2}(1)^3 - \frac{1}{2}(0)^3$$

$$= \frac{1}{2}$$

$$f_X(x) = \frac{1}{2} \quad \text{for } 0 \le x \le 2$$





$$f_Y(y) = \int_0^2 \frac{3}{2}y^2\,dx = \frac{3}{2}y^2 x \Big|_{x=0}^{x=2}$$

$$= \frac{3}{2}y^2(2) - \frac{3}{2}y^2(0)$$

$$f_Y(y) = 3y^2 \quad \text{for } 0 \le y \le 1$$

CDF

$P(y_1 \le Y \le y_2)$ $\rightarrow$ $\boxed{P(Y \le y) = \int_{-\infty}^{y} 3t^2\,dt}$ $\rightarrow$ $F_Y(y)$

Example of a *more complicated* joint pdf

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 2.1

Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$.

1. Find $f_X(x)$ and $f_Y(y)$.

Example of a *more complicated* joint pdf

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 2.2

Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$.

2. Find $\mathbb{P}(Y < 3)$.

Let's complicate this even more!

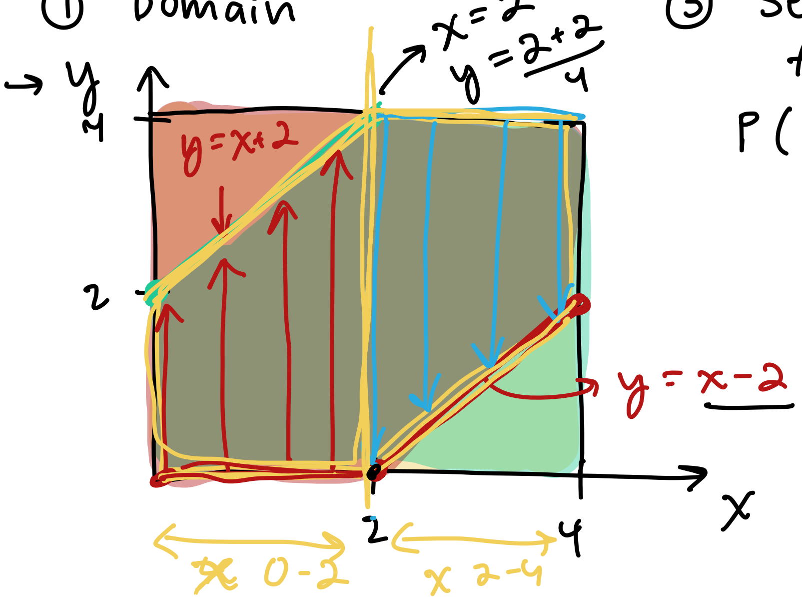
$f_{X,Y}(x,y) = \frac{1}{16}$ for $0 \le X \le 4,\ 0 \le Y \le 4$

Example 3.1

Let X and Y have constant density on the square $0 \le X \le 4, 0 \le Y \le 4.$

1. Find $\mathbb{P}(|X - Y| < 2)$

① Domain



② shade in prob $P(|X-Y|<2)$

$$-2 < x-y < 2$$

$$-2 < x - y \qquad\qquad x - y < 2$$

$$\begin{array}{ccc} +y & & +2\ +y \\ +2 & & \end{array} \qquad\qquad \begin{array}{ccc} & +y & +y \\ -2 & & -2 \end{array}$$

$$y < x+2 \qquad\qquad x-2 < y$$

$$4 \times 4 \cdot b = 1$$

$$\int_0^4 \int_0^4 b \, dy\, dx = 1$$

$$\frac{1}{4^2} \quad b = \frac{1}{16}$$

③ set up integral for probability

$$P(|X-Y|<2)$$

$$= \int_0^2 \int_0^{x+2} \frac{1}{16}\, dy\, dx + \int_2^4 \int_{x-2}^{4} \frac{1}{16}\, dy\, dx$$

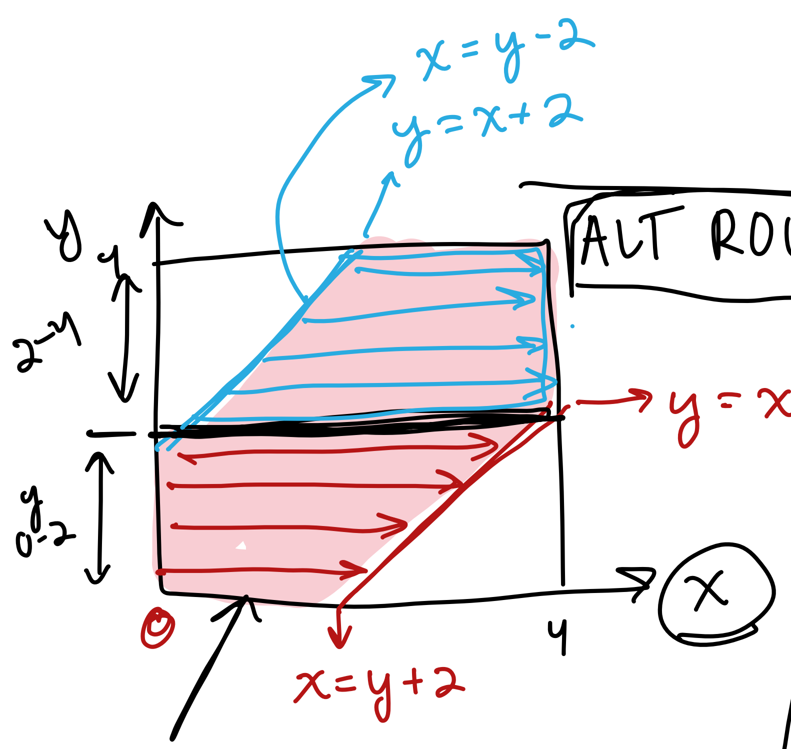
$$= \int_0^2 \left[\frac{1}{16} y \right]_{y=0}^{y=x+2} dx + \int_2^4 \left[\frac{1}{16} y \right]_{y=x-2}^{y=4} dx$$

$$= \int_0^2 \frac{1}{16}(x+2)\, dx + \int_2^4 \left[\frac{1}{16} 4 - \frac{1}{16}(x-2) \right] dx$$

$$= \int_0^2 \frac{1}{16}(x+2)\,dx + \int_2^4 \left(\frac{1}{4} - \frac{1}{16}(x-2)\right)dx$$

$$= \left[\frac{1}{32}x^2 + \frac{1}{8}x\right]_{x=0}^{x=2} + \left[\frac{1}{4}x - \frac{1}{32}x^2 - \frac{1}{8}x\right]_{x=2}^{x=4}$$

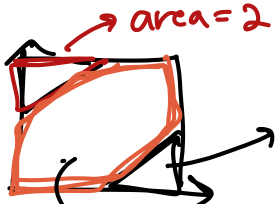
solve this @ home



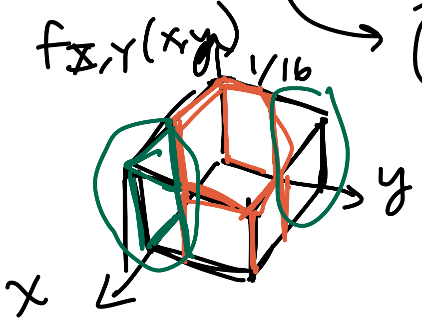
ALT ROUTE

$$\int_0^2 \int_0^{y+2} \frac{1}{16}\,dx\,dy + \int_2^4 \int_{y-2}^{4} \frac{1}{16}\,dx\,dy$$

ALT ALT ROUTE



$$\frac{1}{2}(2)(2) = 2$$



$$(4 \cdot 4) - 2 - 2 = 12$$

$$P(|X - Y| < 2) = \frac{12}{16} = 3/4$$

Let's complicate this even more!

$Z = X + Y$ $\quad$ $1+2$, $2+1$

- M is a transformation of X & Y
- CDF method: start by finding the CDF (which includes probabilities) & then find pdf from CDF:

$$F_M(m) = \int f_M(m)\,dm \rightarrow f_M(m) = \frac{d}{dm} F_M(m)$$

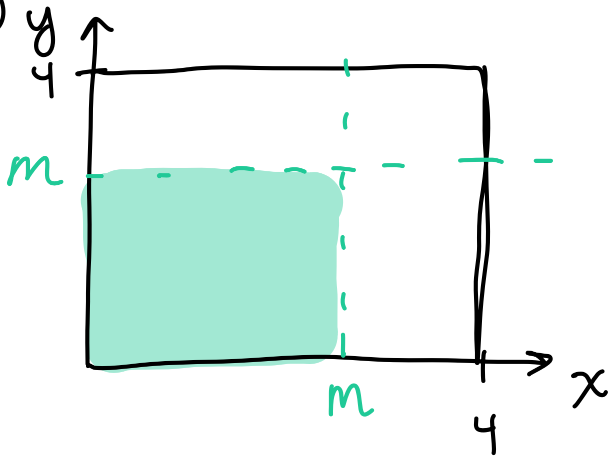
Example 3.1

Let X and Y have constant density on the square $0 \leq X \leq 4, 0 \leq Y \leq 4$.

2. Let $M = \max(X, Y)$. Find the pdf for M, that is $f_M(m)$.

$$F_M(m) = P(M \leq m) = P(\max(X, Y) \leq m)$$
$$= P(X \leq m, Y \leq m)$$

(1)



Let's complicate this even more!

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 3.3

Let X and Y have constant density on the square $0 \leq X \leq 4, 0 \leq Y \leq 4$.

3. Let $Z = \min(X, Y)$. Find the pdf for Z, that is $f_Z(z)$.

Let's complicate this even further!

Example 4

Let X and Y have joint density $f_{X,Y}(x, y) = \frac{8}{5}(x + y)$ in the region $0 < x < 1,\ \frac{1}{2} < y < 1$. Find the pdf of the r.v. Z, where $Z = XY$.