Chapter 2: Probability

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Overview

What do you know about probabilities?

Probabilities of equally likely events

Probabilities of equally likely events (1/2)

Example 1

Suppose you have a regular well-shuffled deck of cards. What's the probability of drawing:

1. any heart
2. the queen of hearts
3. any queen

Solution:

1. any heart = 13/52 = 1/4
2. the queen of hearts = 1/52
3. any queen = 4/52 = 1/13

Probabilities of equally likely events (2/2)

If S is a finite sample space, with equally likely outcomes, then

$$\mathbb{P}(A) = \frac{|A|}{|S|}.$$

A probability is a function...

$\mathbb{P}(A)$ is a function with

- input: event A from the sample space S, ($A \subseteq S$)
- output: a number between 0 and 1 (inclusive)

$$\mathbb{P}(A) : S \to [0, 1]$$

A function that follows some specific rules though!

See Probability Axioms on next slide.

Probability Axioms

Probability Axioms

Axiom 1

For every event A, $0 \leq \mathbb{P}(A) \leq 1$.

Axiom 2

For the sample space S, $\mathbb{P}(S) = 1$.

Axiom 3

If $A_1, A_2, A_3, \ldots,$ is a collection of **disjoint** events, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Some probability properties

Some probability properties

Using the Axioms, we can prove all other probability properties!

Proposition 1

For any event A, $\mathbb{P}(A) = 1 - \mathbb{P}(A^C)$

Proposition 2

$\mathbb{P}(\emptyset) = 0$

Proposition 3

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

Proposition 4

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Proposition 5

$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$

Proposition 1 Proof

Proposition 1

For any event A, $\mathbb{P}(A) = 1 - \mathbb{P}(A^C)$

Proposition 1 Proof

Since A and A^C are disjoint, we know from Axiom 3 that $\mathbb{P}(A \cup A^C) = \mathbb{P}(A) + \mathbb{P}(A^C)$. However, $S = A \cup A^C$, and by Axiom 2, $\mathbb{P}(S) = 1$, implying $\mathbb{P}(A \cup A^C) = \mathbb{P}(S) = 1$. Thus, $\mathbb{P}(A) + \mathbb{P}(A^C) = 1$, or $\mathbb{P}(A) = 1 - \mathbb{P}(A^C)$

Proposition 2 Proof

Proposition 2

$\mathbb{P}(\emptyset) = 0$

Proposition 2 Proof (different from book)

We know $\emptyset = S^C$.

Thus by Prop 1,

$$\mathbb{P}(\emptyset) = \mathbb{P}(S^C) = 1 - \mathbb{P}(S) = 1 - 1 = 0.$$

Proposition 3 Proof

Proposition 3

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

Proposition 3 Proof

Create a partition! Make a Venn diagram.

$$\begin{aligned} B &= A \cup (B \cap A^C) \\ \mathbb{P}(B) &= \mathbb{P}(A) + \mathbb{P}(B \cap A^C) \\ \mathbb{P}(B) &\geq \mathbb{P}(A), \end{aligned}$$

since $\mathbb{P}(B \cap A^C) \geq 0$.

Partitions

Partitions

Definition: Partition

A set of events $\{A_i\}_{i=1}^{n}$ create a **partition** of A, if

- the A_i's are disjoint (mutually exclusive) and
- $\bigcup_{i=1}^{n} A_i = A$

Example 2

- If $A \subset B$, then $\{A, B \cap A^C\}$ is a partition of B.
- If $S = \bigcup_{i=1}^{n} A_i$, then the A_i's are a partition of the sample space.

Creating partitions is sometimes used to help calculate probabilities, since by Axiom 3 we can add the probabilities of disjoint events.

Venn Diagram Probabilities

Venn Diagram Probabilities

Example 3

If a subject has an

- 80% chance of taking their medication *this* week,
- 70% chance of taking their medication *next* week, and
- 10% chance of *not* taking their medication *either* week,

then find the probability of them taking their medication exactly one of the two weeks.

Hint: Draw a Venn diagram labelling each of the parts to find the probability.

Answer: $\mathbb{P}(\mathrm{A} \cap \mathrm{B}) = 0.6$.