Chapter 29: Variance of Continuous Random Variables

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Learning Objectives

1. Calculate expected value of functions of RVs
2. Calculate variance of RVs

Expected value of a function of a continuous RV

$g(x)$ is some fn of x

How do we calculate the expected value of a function of a discrete RV or joint RVs?

For discrete RVs:

$$\mathbb{E}[g(X)] = \sum_{\{\text{all } x\}} g(x)p_X(x).$$

$$\mathbb{E}[g(X,Y)] = \sum_{\{\text{all } x\}}\sum_{\{\text{all } y\}} g(x,y)p_{X,Y}(x,y).$$

How do we calculate the expected value of a function of a continuous RV or joint RVs?

For continuous RVs:

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

$$E(g(X,Y)) = \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dy\,dx$$

Expected value from a joint pdf

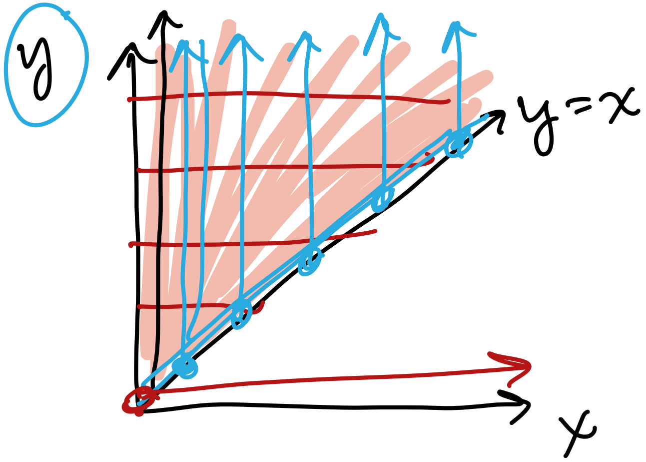
$E(X+Y) \Rightarrow g(x,y) = x+y$

$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$] in prev notes

Example 1

Let $f_{X,Y}(x,y) = 2e^{-(x+y)}$, for $0 \le x \le y$. Find $\mathbb{E}[X]$.

$g(X,Y) = x$



$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_{0}^{\infty}\int_{y=x}^{y=\infty} x\,(2e^{-x}e^{-y})\,dy\,dx$$

$$= \int_{0}^{\infty}\int_{0}^{y} x\,(2e^{-x}e^{-y})\,dx\,dy$$

double check that you get same answer

$$= \int_{0}^{\infty} x \boxed{\int_{x}^{\infty} 2e^{-x}e^{-y}\,dy}\,dx$$

$$= \dots = \frac{1}{2}$$

Remark on expected value of one RV from joint pdf

If you are given $f_{X,Y}(x, y)$ and want to calculate $\mathbb{E}[X]$, you have two options:

1. Find $f_X(x)$ and use it to calculate $\mathbb{E}[X]$.
2. Or, calculate $\mathbb{E}[X]$ using the joint density:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y}(x, y) dy dx.$$

Important properties of expected values of functions of continuous RVs

Function of RV with two constants

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

Function of two RVs added

$g(X,Y) = X+Y$

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$\int\int g(x,y) f_{XY}(x,y)$

Expected value of sum of ~~independent~~ RVs pt 1

If $X_1, X_2, \dots X_n$ are continuous RVs and $a_1, a_2, \dots a_n$ are constants, then

$$\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i]$$

Expected value of multiplication of function of independent RVs

If X and Y are independent continuous RVs, and g and h are functions, then

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$$

Expected value of multiplication of independent RVs

If X and Y are independent continuous RVs, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

$E[X_i \cdot X_i \cdot X_i]$

b/c $\not\perp$ $\neq E(X_i)E(X_i)E(X_i)$

Variance of continuous RVs

How do we calculate the variance of a discrete RV?

For discrete RVs:

$$
\begin{aligned}
Var(X) &= \mathbb{E}[(X - \mu_X)^2] \\
&= \mathbb{E}[(X - \mathbb{E}[X])^2] \\
&= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
&= \sum_{\{\text{all } x\}} (x - \mu_x)^2 p_X(x)
\end{aligned}
$$

How do we calculate the variance of a continuous RV?

For continuous RVs:

$$
\begin{aligned}
Var(X) &= E[(X - \mu_X)^2] \\
&= E[(X - E(X))^2] \\
&= E(X^2) - [E(X)]^2 \\
&= \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\,dx
\end{aligned}
$$

Variance of an Uniform distribution

Example 2

Let $f_X(x) = \frac{1}{b-a}$, for $a \leq x \leq b$. Find $Var[X]$.

Variance of exponential distribution

Example 3

Let $f_X(x) = \lambda e^{-\lambda x}$, for $x > 0$ and $\lambda > 0$. Find $Var[X]$.

Important properties of variances of continuous RVs

function of RV with two constants

$$Var[aX + b] = a^2 Var[X]$$

Variance of sum of independent RVs pt 1

If $X_1, X_2, \ldots X_n$ are independent continuous RVs and $a_1, a_2, \ldots a_n$ are constants, then

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 Var(X_i)$$

Variance of sum of independent RVs pt 2

If $X_1, X_2, \ldots X_n$ are independent continuous RVs, then

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$$

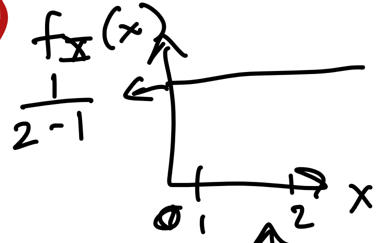
Find the mean and sd from word problem

Example 4

A machine manufactures cubes with a side length that varies uniformly from 1 to 2 inches. Assume the sides of the base and height are equal. The cost to make a cube is 10 ¢ per cubic inch, and 5 ¢ ~~cents~~ for the general cost per cube. Find the mean and standard deviation of the cost to make 10 cubes.

$[E(X_i)]^3 \to E(X_i^3)$

$\hookrightarrow X_i \not\perp X_j$



Model cost of 10 cubes:

Let C = cost of 10 cubes

C_i = cost of ith cube for $i = 1, \ldots, 10$

X_i = length cube sides for cube i

$$C = \sum_{i=1}^{10} C_i \qquad C_i = 5 + 10(X_i^3) \qquad X_i \sim U[1,2]$$

$$E(C) = E\left[\sum_{i=1}^{10} (5 + 10(X_i)^3)\right]$$

$$= \sum_{i=1}^{10} E[5 + 10X_i^3] = \sum_{i=1}^{10} (5 + 10E(X_i^3))$$

$$E(X_i^3) = \int_1^2 x^3 \frac{1}{(2-1)} dx \qquad f_X(x) = \frac{1}{b-a}$$

$$= \int_1^2 x^3 dx = \frac{1}{4} x^4 \Big|_1^2$$

$$= \frac{2^4 - 1^4}{4} = 15/4$$

$$= \sum_{i=1}^{10} (5 + 10(15/4)) = \sum_{i=1}^{10} 42.5$$

$$= 10 \cdot 42.5 = 425¢ = \$4.25$$