Chapter 29: Variance of Continuous Random Variables

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Learning Objectives

1. Calculate expected value of functions of RVs
2. Calculate variance of RVs

Expected value of a function of a continuous RV

How do we calculate the expected value of a function of a discrete RV or joint RVs?

For discrete RVs:

$$\mathbb{E}[g(X)] = \sum_{\{\text{all } x\}} g(x)p_X(x).$$

$$\mathbb{E}[g(X,Y)] = \sum_{\{\text{all } x\}} \sum_{\{\text{all } y\}} g(x,y)p_{X,Y}(x,y).$$

How do we calculate the expected value of a function of a continuous RV or joint RVs?

For continuous RVs:

Expected value from a joint pdf

Example 1

Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$. Find $\mathbb{E}[X]$.

Remark on expected value of one RV from joint pdf

If you are given $f_{X,Y}(x, y)$ and want to calculate $\mathbb{E}[X]$, you have two options:

1. Find $f_X(x)$ and use it to calculate $\mathbb{E}[X]$.
2. Or, calculate $\mathbb{E}[X]$ using the joint density:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y}(x, y) dy dx.$$

Important properties of expected values of functions of continuous RVs

Function of RV with two constants

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

Function of two RVs added

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Expected value of sum of independent RVs pt 1

If $X_1, X_2, \ldots X_n$ are continuous RVs and $a_1, a_2, \ldots a_n$ are constants, then

$$\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i]$$

Expected value of multiplication of function of independent RVs

If X and Y are independent continuous RVs, and g and h are functions, then

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$$

Expected value of multiplication of independent RVs

If X and Y are independent continuous RVs, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

Variance of continuous RVs

How do we calculate the variance of a discrete RV?

For discrete RVs:

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}[(X-\mu_X)^2] \\ &= \mathbb{E}[(X-\mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2]-(\mathbb{E}[X])^2 \\ &= \sum_{\{\text{all } x\}} (x-\mu_x)^2 p_X(x)\end{aligned}$$

How do we calculate the variance of a continuous RV?

For continuous RVs:

Variance of an Uniform distribution

Example 2

Let $f_X(x) = \frac{1}{b-a}$, for $a \leq x \leq b$. Find $Var[X]$.

Variance of exponential distribution

Example 3

Let $f_X(x) = \lambda e^{-\lambda x}$, for $x > 0$ and $\lambda > 0$. Find $\mathrm{Var}[X]$.

Important properties of variances of continuous RVs

function of RV with two constants

$$\mathrm{Var}[aX + b] = a^2 \mathrm{Var}[X]$$

Variance of sum of independent RVs pt 1

If $X_1, X_2, \ldots X_n$ are independent continuous RVs and $a_1, a_2, \ldots a_n$ are constants, then

$$\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i \right) = \sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i)$$

Variance of sum of independent RVs pt 2

If $X_1, X_2, \ldots X_n$ are independent continuous RVs, then

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

Find the mean and sd from word problem

Example 4

A machine manufactures cubes with a side length that varies uniformly from 1 to 2 inches. Assume the sides of the base and height are equal. The cost to make a cube is 10 ¢ per cubic inch, and 5 ¢ cents for the general cost per cube. Find the mean and standard deviation of the cost to make 10 cubes.