Lesson 6: Tests for GLMs using Likelihood function

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I’ve never explicitly said this…

- Because we are in a public health class, we are often analyzing data with sensitive outcomes

- If you ever need a moment in class because of our topic, feel free to just step out or leave and privately view the lecture

- If you need extra time on your assignments because you have an emotional response to lectures/homework/lab, just let me know! Extenuating circumstance!

Learning Objectives

1. Use the Wald test to test the significance of an estimated coefficient through confidence intervals.
2. Articulate how the Wald test, Score test, and likelihood ratio test (LRT) calculates a test statistic using the likelihood function.
3. Use the Likelihood ratio test to test the significance of estimated coefficients through formal hypothesis testing.
4. Understand when and how to use each test: Wald, Score, and Likelihood ratio

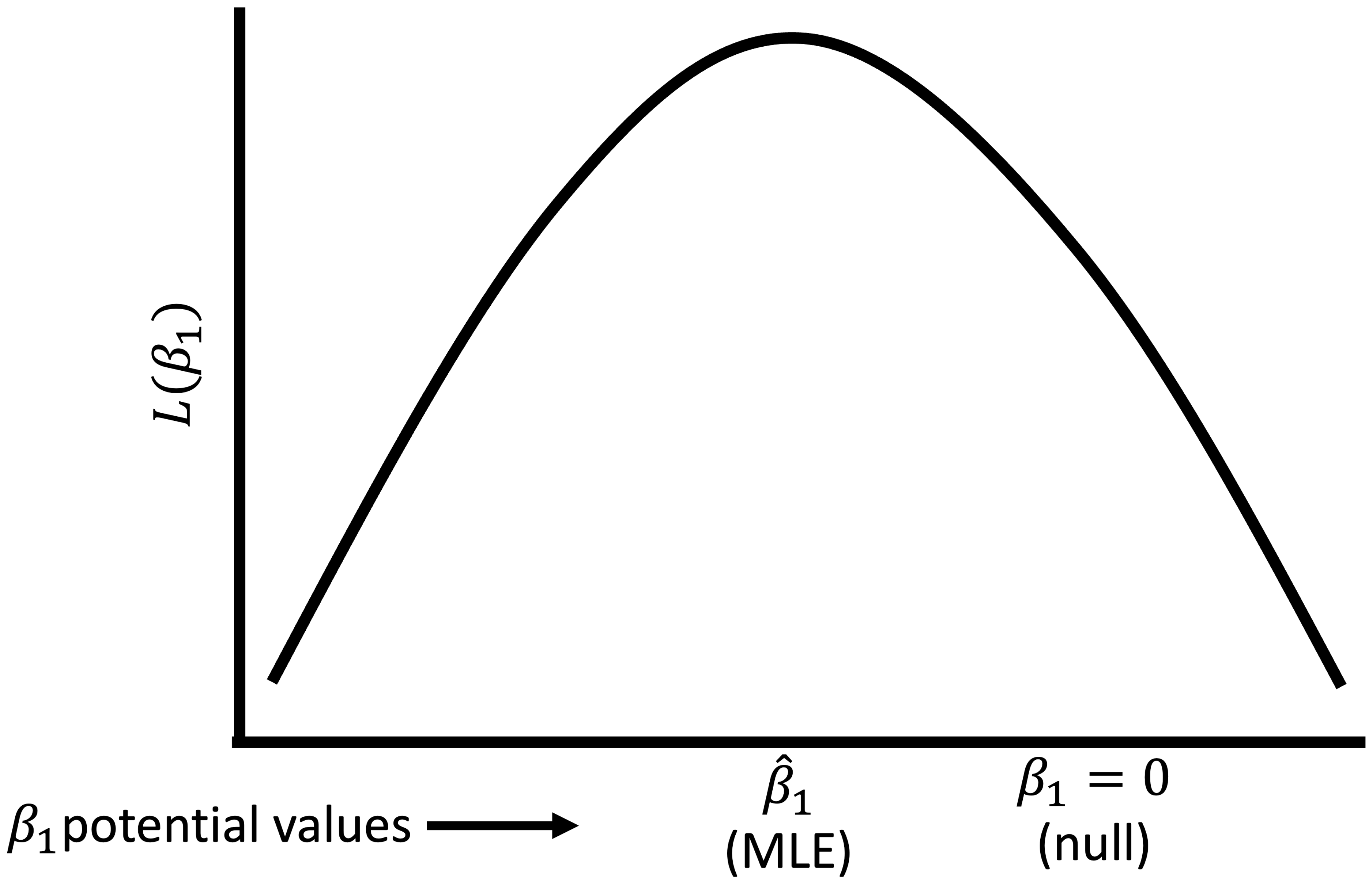
Connection between tests in linear models and GLMs

- In linear regression, we used ordinary least squares (OLS) to find the best fit model, so we could use the following tests:
  - t-test for single coefficients
  - F-test for single coefficients or groups of coefficients
- These tests **hinge on the Mean Squared Error (MSE)** which we minimized in OLS and the LINE assumptions

- In GLMs, when we use **maximum likelihood estimation (MLE)**, we **cannot** use t-tests or F-tests
  - Because we are now using likelihood to find our estimates (not OLS)
- But we have **parallel tests in MLE**!!
  - t-test $\longrightarrow$ Wald test
  - F-test $\longrightarrow$ Likelihood ratio test

Revisit the likelihood function

- **Likelihood function:** expresses the probability of the observed data as a function of the unknown parameters
  - Function that enumerates the likelihood (similar to probability) that we observe the data across the range of potential values of our coefficients
- We often compare likelihoods to see what estimates are more likely given our data
- Plot to right is a simplistic view of likelihood
  - I have flattened the likelihood that would be a function of β_0 and β_1 into a 2D plot (instead of 3D: β_0 vs. β_1 vs. $L(\beta_0, \beta_1)$)
- I use L to represent the log-likelihood and l to represent the likelihood



Introduction to three tests in GLM

- To introduce these three tests, we will work on a **single coefficient**
  - To be clear: the *Likelihood ratio test can be extended to more coefficients*
- Let's say we fit a GLM using MLE
  - We will continue to use logistic regression as our working example
- Now we want to run a hypothesis test for an individual coefficient j:
  - $H_0 : \beta_j = 0$
  - $H_1 : \beta_j \neq 0$
- Three potential tests that we use with a Likelihood function are:
  - Wald test
  - Score test
  - Likelihood ratio test (LRT)

Poll Everywhere Question 1

Revisit previous model with late stage BC diagnosis and age

- Simple logistic regression model:

$$\text{logit}(\pi(Age)) = \beta_0 + \beta_1 \cdot Age$$

Don't forget: $\pi(Age) = P(Y = 1|Age)$

```r
bc_reg = glm(Late_stage_diag ~ Age_c, data = bc, family = binomial)
summary(bc_reg)
```

```
Call:
glm(formula = Late_stage_diag ~ Age_c, family = binomial, data = bc)

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.989422   0.023205  -42.64   <2e-16 ***
Age_c        0.056965   0.003204   17.78   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 11861  on 9999  degrees of freedom
Residual deviance: 11510  on 9998  degrees of freedom
AIC: 11514

Number of Fisher Scoring iterations: 4
```

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4. Understand when and how to use each test: Wald, Score, and Likelihood ratio

Wald test (1/3)

- Very similar to a t-test!
  - But slightly different because it based in our likelihood function
- Assumes test statistic W follows a standard normal distribution under the null hypothesis
- Test statistic:

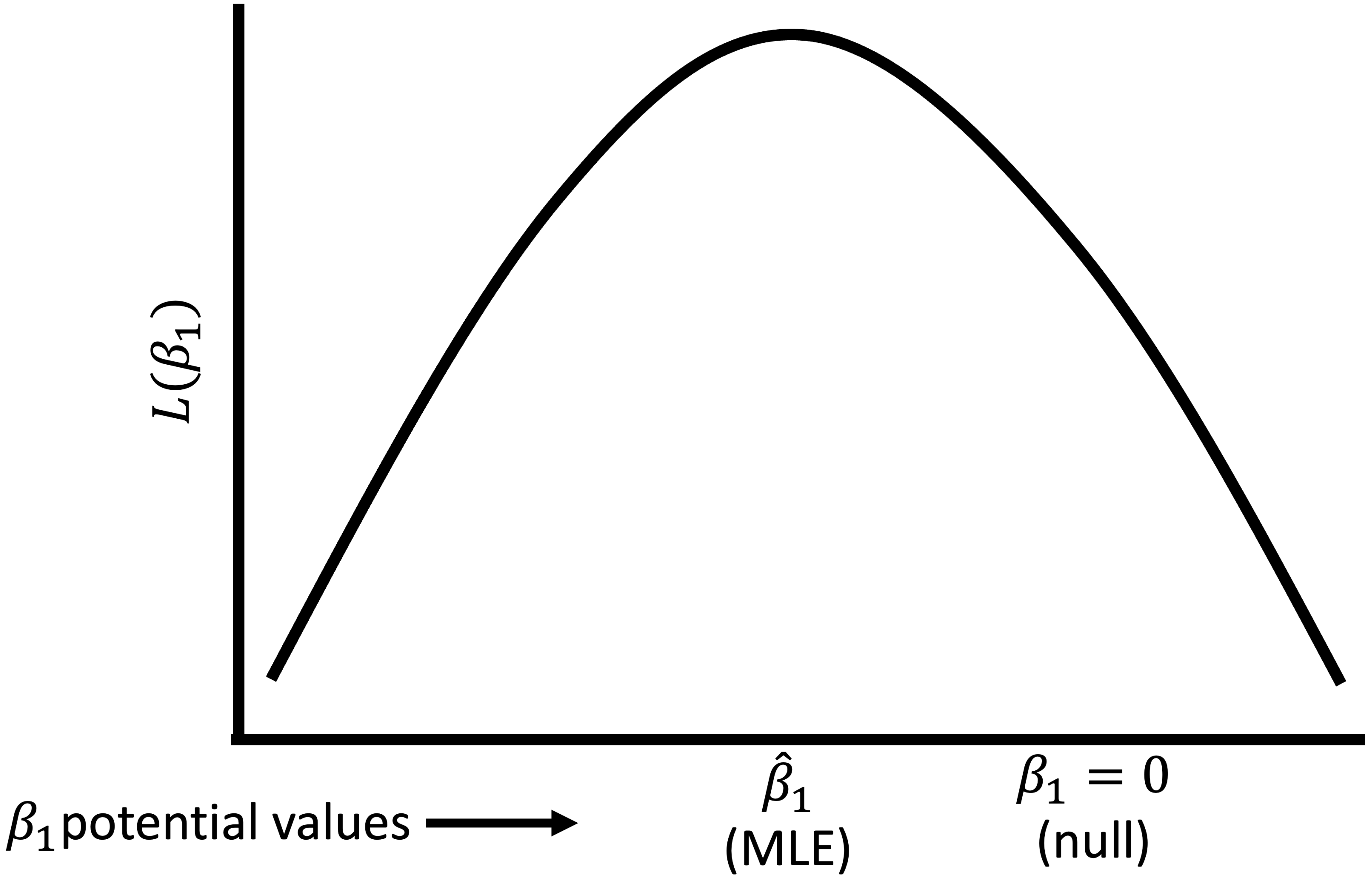
$$W = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)} \sim N(0,1)$$

  - where $\widehat{\beta}_j$ is a MLE of coefficient j
- 95% Wald confidence interval:

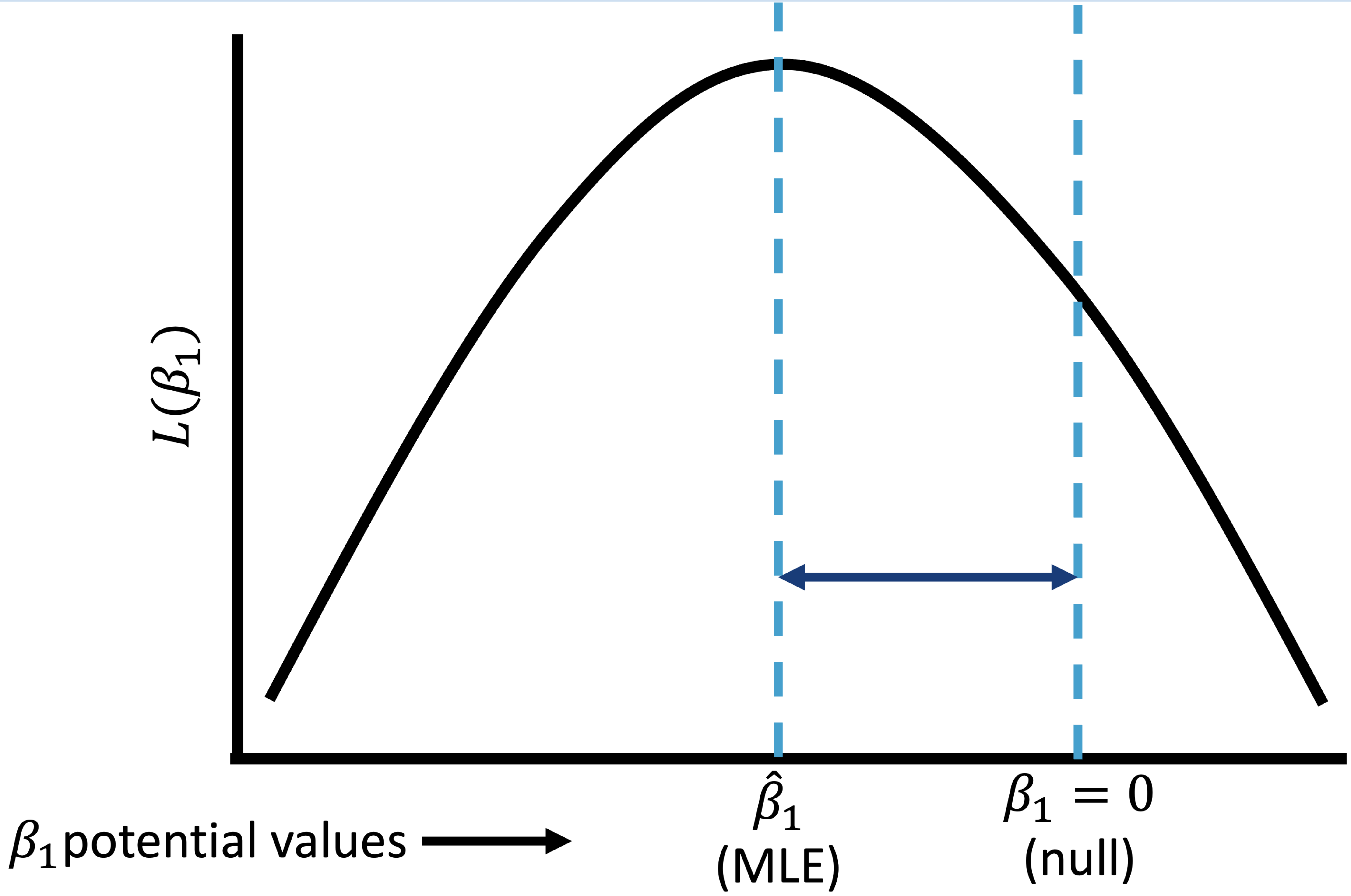
$$\hat{\beta}_1 \pm 1.96 \cdot SE_{\hat{\beta}_j}$$

- The Wald test is a routine output in R (`summary()` of `glm()` output)
  - Includes $SE_{\hat{\beta}_j}$ and can easily find confidence interval with `tidy()`

Wald test (2/3)



Wald test (3/3)



Wald test procedure with confidence intervals

1. Set the level of significance α
2. Specify the null (H_0) and alternative (H_A) hypotheses
    - In symbols
    - In words
    - Alternative: one- or two-sided?
3. Calculate the confidence interval and determine if it overlaps with null
    - Overlap with null (usually 0 for coefficient) = fail to reject null
    - No overlap with null (usually 0 for coefficient) = reject null
4. Write a conclusion to the hypothesis test
    - What is the estimate and its confidence interval?
    - Do we reject or fail to reject H_0?
    - Write a conclusion in the context of the problem

Example: BC diagnosis and age

Wald test for age coefficient

Interpret the coefficient for age in our model of late stage breast cancer diagnosis.

Needed steps:

1. Set the **level of significance** α
2. Specify the **null** (H_0) and **alternative** (H_A) **hypotheses**
3. Calculate the **confidence interval** and **determine if it overlaps with null**
4. Write a **conclusion** to the hypothesis test

Note

I don't want us to get fixated on this interpretation. This is more to introduce the process, BUT it's MUCH better to interpret the coefficient in terms of OR (next class).

Example: BC diagnosis and age

Wald test for age coefficient

Interpret the coefficient for age in our model of late stage breast cancer diagnosis.

1. Set the **level of significance** α

$$\alpha = 0.05$$

2. Specify the **null** (H_0) and **alternative** (H_A) **hypotheses**

$$\begin{aligned} H_0 &: \beta_{Age} = 0 \\ H_1 &: \beta_{Age} \neq 0 \end{aligned}$$

Example: BC diagnosis and age

Wald test for age coefficient

Interpret the coefficient for age in our model of late stage breast cancer diagnosis.

3. Calculate the **confidence interval** and **determine if it overlaps with null**

```
library(epiDisplay)
bc_reg = glm(Late_stage_diag ~ Age_c, data = bc, family = binomial)
tidy(bc_reg, conf.int=T) %>% gt() %>% tab_options(table.font.size = 35) %>%
  fmt_number(decimals = 3)
```

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	−0.989	0.023	−42.637	0.000	−1.035	−0.944
Age_c	0.057	0.003	17.780	0.000	0.051	0.063

Example: BC diagnosis and age

Wald test for age coefficient

Interpret the coefficient for age in our model of late stage breast cancer diagnosis.

4. Write a **conclusion** to the hypothesis test

For every one year increase in age, the log-odds of late stage breast cancer diagnosis increases 0.057 (95% CI: 0.051, 0.063).

There is sufficient evidence that age an breast cancer diagnosis are associated.

Note

I don't want us to get fixated on this interpretation. This is more to introduce the process, BUT it's MUCH better to interpret the coefficient in terms of OR (next class).

Poll Everywhere Question 2

Learning Objectives

1. Use the Wald test to test the significance of an estimated coefficient through confidence intervals.

2. Articulate how the Wald test, **Score test**, and likelihood ratio test (LRT) calculates a test statistic using the likelihood function.

3. Use the Likelihood ratio test to test the significance of estimated coefficients through formal hypothesis testing.

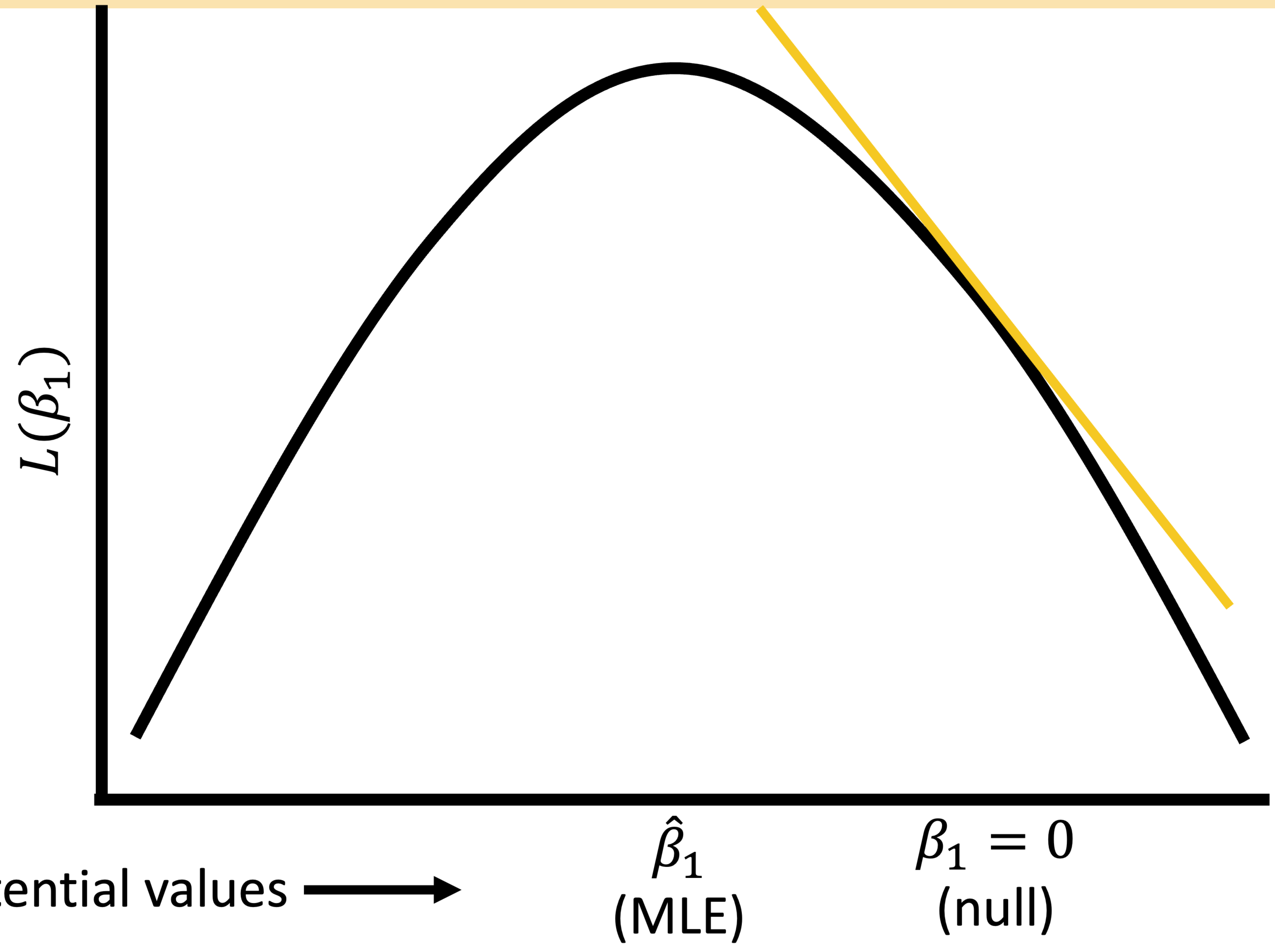
4. Understand when and how to use each test: Wald, Score, and Likelihood ratio

Score test

- Score test does not require the computation of MLE for β_1, while both likelihood test and Wald test does
  - Only need to know β_1 under the null
- Score test is based on the first and second derivatives of the log-likelihood under the null hypothesis:

$$S = \frac{\sum_{i=1}^{n} x_i(y_i - \bar{y})}{\sqrt{\bar{y}(1-\bar{y})\sum_{i=1}^{n}(x_i - \bar{x})^2}} \sim N(0,1)$$

Score test



β_1 potential values →

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Likelihood ratio test (1/3)

- Likelihood ratio test answers the question:
  - For a specific covariate, **which model tell us more about the outcome variable**: the model including the covariate or the model omitting the covariate?
  - Aka: Which model is more likely given our data: model including the covariate or the model omitting the covariate?

- Test a single coefficient by comparing different models
  - **Very similar to the F-test**

- Important: LRT can be used conduct hypothesis tests for **multiple coefficients**
  - Just like F-test, we can test a single coefficient, continuous/binary covariate, multi-level covariate, or multiple covariates

Likelihood ratio test (2/3)

- To assess the significance of a continuous/binary covariate's coefficient in the simple logistic regression, we compare the deviance (D) with and without the covariate

$$G = D\left(\text{model without } x\right) - D\left(\text{model with } x\right)$$

- For a continuous or binary variable, this is equivalent to test: $H_0 : \beta_j = 0$ vs. $H_1 : \beta_j \neq 0$
- Test statistic for LRT:

$$G = -2ln\left[\frac{\text{likelihood without } x}{\text{likelihood with } x}\right] = 2ln\left[\frac{l\left(\hat{\beta}_0, \hat{\beta}_1\right)}{l(\hat{\beta}_0)}\right]$$

Likelihood ratio test (2/3)

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LRT: what is Deviance?

- **Deviance:** quantifies the difference in likelihoods between a fitted and saturated model
  - **Fitted model:**
    - Your proposed fitted model
  - **Saturated model:**
    - A model that contains as many parameters as there are data points = **perfect fit**
      - Basically every individual has their own covariate
    - Perfect fit = maximum possible likelihood

- **All fitted models will have likelihood less than saturated model**

LRT: what is Deviance?

- The **deviance** is mathematically defined as:

$$D = -2[L_{\text{fitted}} - L_{\text{saturated}}]$$

- An alternative way to write it is:

$$D = -2ln\left[\frac{\text{likelihood of the fitted model}}{\text{likelihood of the saturated model}}\right]$$

- Using '-2' is to make the deviance follow a chi-square distribution

Deviance to Likelihood Ratio Test

- In the LRT, we are NOT comparing the likelihood of saturated model to the fitted model

- We ARE comparing the Deviance of the model with x and the model without x
  - We just use the saturated model to calculate Deviance
  - Both are considered fitted models with their own respective Deviance

- So our LRT is:

$$G = D\left(\text{model without } x\right) - D\left(\text{model with } x\right)$$

For reference

$$G = D\left(\text{model without } x\right) - D\left(\text{model with } x\right)$$

$$G = -2ln\left[\frac{\text{likelihood of model without } x}{\text{likelihood of saturated model}}\right] - \left(-2ln\left[\frac{\text{likelihood of model with } x}{\text{likelihood of saturated model}}\right]\right)$$

$$G = -2ln\left[\frac{\text{likelihood of model without } x}{\text{likelihood of saturated model}} \times \frac{\text{likelihood of saturated model}}{\text{likelihood of model with } x}\right]$$

$$G = -2ln\left[\frac{\text{likelihood of model without } x}{\text{likelihood of model with}}\right]$$

$$G = 2ln\left[\frac{l\left(\hat{\beta}_0, \hat{\beta}_1\right)}{l(\hat{\beta}_0)}\right]$$

Poll Everywhere Question 3

Likelihood ratio test (2/3)

- To assess the significance of a continuous/binary covariate's coefficient in the simple logistic regression, we compare the deviance (D) with and without the covariate

$$G = D\,(\text{model without } x) - D\,(\text{model with } x)$$

- For a continuous or binary variable, this is equivalent to test: $H_0 : \beta_j = 0$ vs. $H_1 : \beta_j \neq 0$
- Test statistic for LRT:

$$G = -2ln\left[\frac{\text{likelihood without } x}{\text{likelihood with } x}\right] = 2ln\left[\frac{l\left(\hat{\beta}_0, \hat{\beta}_1\right)}{l(\hat{\beta}_0)}\right]$$

Likelihood ratio test (3/3)

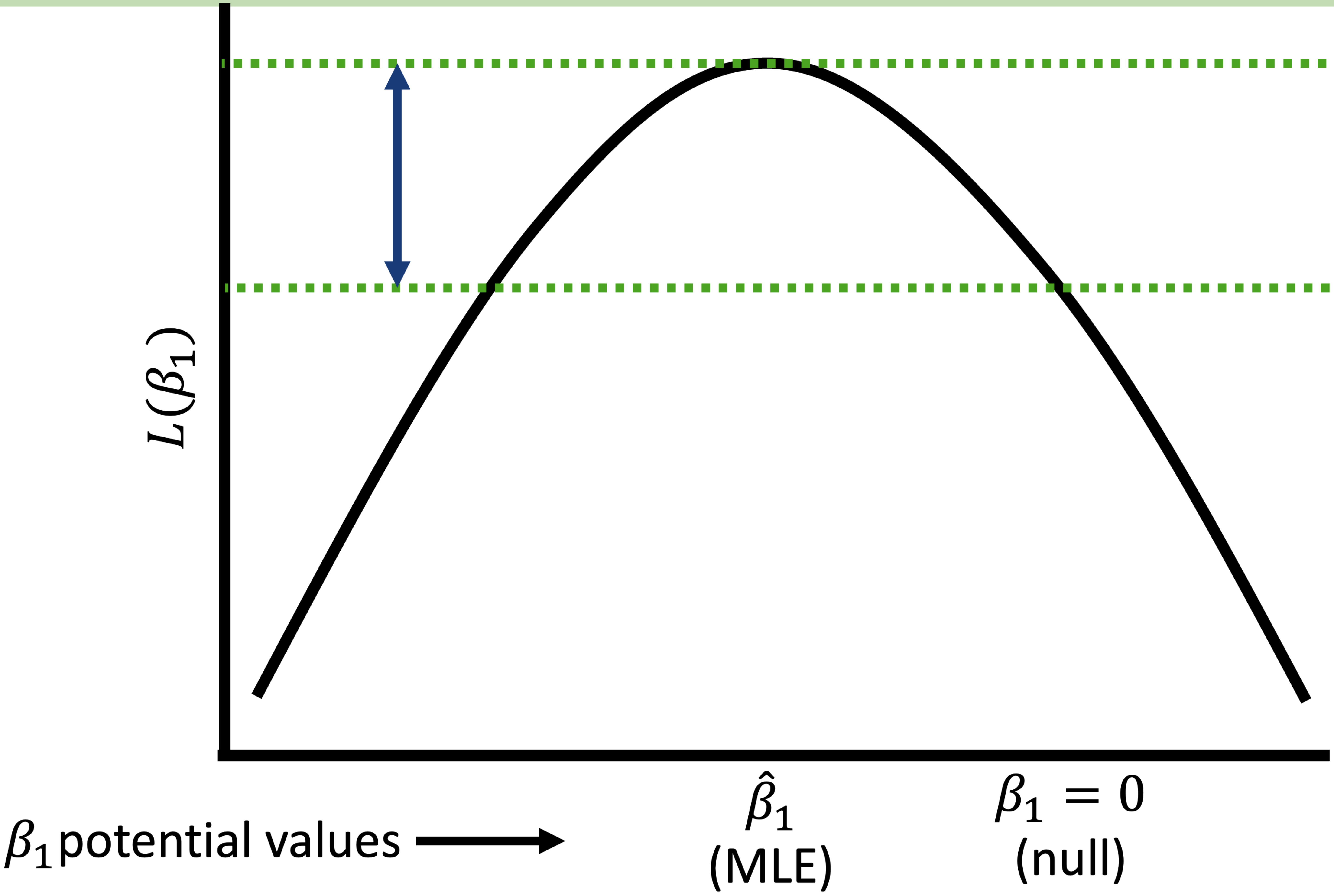
- Under the null hypothesis, with adequate sample size, LRT statistic follows a chi-square distribution:

$$G \sim \chi^2(df)$$

  - $df = (\#\text{coefficients in larger model}) - (\#\text{coefficients in smaller model})$

- If we are testing a single coefficient, like age, then $df = 1$

Likelihood ratio test



LRT procedure

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2. Specify the **null** (H_0) and **alternative** (H_A) **hypotheses**
   - In symbols
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4. Write a **conclusion** to the hypothesis test
   - Do we reject or fail to reject H_0?
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Example: BC diagnosis and age

LRT

Determine if the model including age is more likely than model without age. Aka: Is age associated with late stage breast cancer diagnosis?

Needed steps:

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Example: BC diagnosis and age

LRT

Determine if the model including age is more likely than model without age. Aka: Is age associated with late stage breast cancer diagnosis?

3. Calculate the **test statistic** and **p-value**

```
library(lmtest)
bc_age = glm(Late_stage_diag ~ Age_c, data = bc, family = binomial)
bc_int = glm(Late_stage_diag ~ 1, data = bc, family = binomial)
lmtest::lrtest(bc_age, bc_int)
```

```
Likelihood ratio test

Model 1: Late_stage_diag ~ Age_c
Model 2: Late_stage_diag ~ 1
  #Df  LogLik Df  Chisq Pr(>Chisq)
1   2 -5754.8
2   1 -5930.5 -1 351.27  < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Example: BC diagnosis and age

LRT

Determine if the model including age is more likely than model without age. Aka: Is age associated with late stage breast cancer diagnosis?

4. Write a **conclusion** to the hypothesis test

We reject the null hypothesis that the coefficient corresponding to age is 0 ($p-value << 0.05$). There is sufficient evidence that there is an association between age and late stage breast cancer diagnosis.

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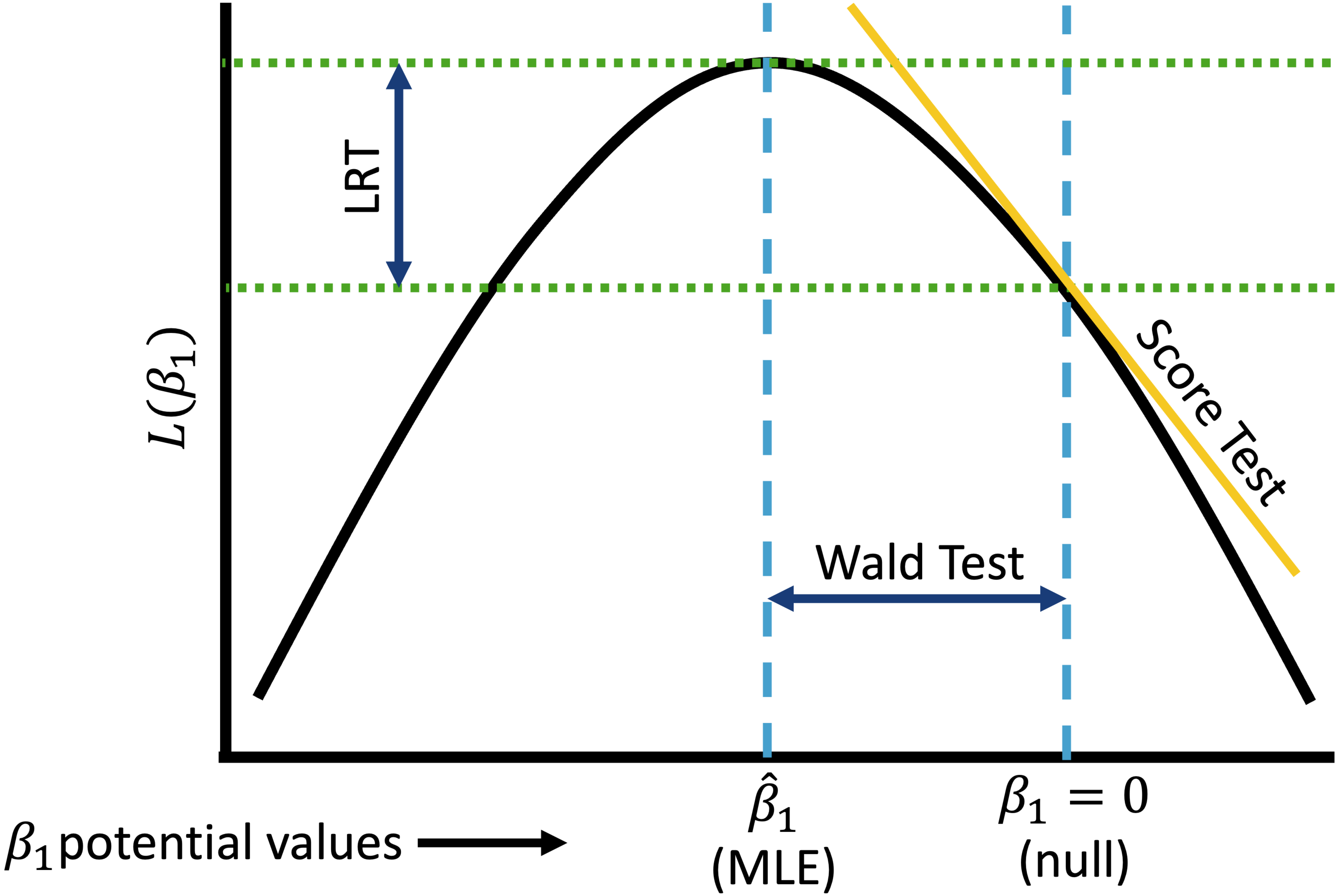
All three tests together



UCLA FAQ on Tests

Which test to use?

- All three tests are asymptotically equivalent
  - As sample approaches infinity
- For testing significance of single covariate coefficient:
  - LRT
    - Wald and score are only approximations of LRT
    - For smaller samples, LRT better
  - Wald test is very convenient
    - Automatically performed in R
    - Does not need to estimate two models (LRT does)
    - Good for constructing confidence intervals of coefficients and odds ratios
  - Score test
    - Does not need to estimate two models (LRT does)
    - I don’t really see people use this…

Poll Everywhere Question 4

Poll Everywhere Question 5

Poll Everywhere Question 6

Poll Everywhere Question 7

Poll Everywhere Question 8

Tests and what they're used for

	Wald test	Score test	LRT
Used to test significance of single coefficient			
Can be used to report confidence interval for a single coefficient			
Confidence interval reported by R for a single coefficient (and most commonly used)			
Use to test significance/contribution to outcome prediction of multi-level categorical covariate			
Used for comparing two models with different (but nested) covariates			

So how would the Wald test and LRT show up in research?

- Wald test
  - Often used when reporting estimates
  - Generally presented using a forest plot or table of ORs or RRs
    - Then we highlight the specific variable of interest in text
    - Will include the OR/RR estimate (not the coefficient like we saw today) with the 95% CI and proper interpretation of result

- LRT
  - Often when performing model selection and comparing two models
    - Reporting model selection Cross Validated post
  - Often does not show up explicitly in our reports, but is essential to get to our final model!!