Last Class!

Revisit interactions + Intro to Poisson Regression

Announcements

- Do you all get requests for the formal course evaluations?

- Remaining grading
  - Homework 4, 5, and 6 redos
  - Project Analysis, Report, and Presentation

- Anything I'm missing?

Class 17 Learning Objectives

Not as formal today:

1. Calculate odds ratios for variables (categorical x continuous) involved in interactions

    - If we want, we can look at categorical x categorical and continuous x continuous

2. Run and interpret a simple Poisson Regression

Odds Ratio in the Presence of Interaction (I)

- When interaction exists between a risk factor (F) and another variable (X), the estimate of the odds ratio for F depends on the value of X

- When an interaction term (F*X) exists in the model
  - $\mathrm{OR}_F \neq \exp(\beta_F)$ in general

- Assume we want to compute the odds ratio for ($F = f_1$ and $\mathrm{F} = f_0$), the correct model-based estimate is

$$\widehat{\mathrm{OR}}_F = \exp\big(\hat{g}(F = f_1, x) - \hat{g}(F = f_0, x)\big)$$

Originally from Class 9 Notes

Odds Ratio in the Presence of Interaction (II)

- We may write the two logits for given x as below:

$$\hat{g}(f_1, x) = \hat{\beta}_0 + \hat{\beta}_1 f_1 + \hat{\beta}_2 x + \hat{\beta}_3 f_1 \times x$$

$$\hat{g}(f_0, x) = \hat{\beta}_0 + \hat{\beta}_1 f_0 + \hat{\beta}_2 x + \hat{\beta}_3 f_0 \times x$$

- The difference in two logits is:

$$\hat{g}(f_1, x) - \hat{g}(f_0, x) = \hat{\beta}_1 (f_1 - f_0) + \hat{\beta}_3 x (f_1 - f_0)$$

- Therefore,

$$\widehat{OR}(F = f_1, F = f_0, X = x)$$
$$= exp\left[\hat{\beta}_1 (f_1 - f_0) + \hat{\beta}_3 x (f_1 - f_0)\right]$$

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Odds Ratio in the Presence of Interaction (III)

To find the confidence interval for the $\widehat{OR}$, we need to find confidence interval for $\hat{g}(f_1, x) - \hat{g}(f_0, x)$ first

$$Var(\hat{g}(f_1, x) - \hat{g}(f_0, x))$$
$$= (f_1 - f_0)^2 \times \mathrm{Var}(\hat{\beta}_1) + [x(f_1 - f_0)]^2 \times var(\hat{\beta}_3) + 2x(f_1 - f_0)^2 \times cov(\hat{\beta}_1, \hat{\beta}_3)$$

- We can follow the same procedure as we did previously to construct 95% CI for $\hat{g}(f_1, x) - \hat{g}(f_0, x)$, denoted by $[L_{g01}, U_{g01}]$.

$$\widehat{OR} = \exp(\hat{g}(f_1, x) - \hat{g}(f_0, x))$$

The 95% CI for $\widehat{OR}$ is: $\left[\exp\left(L_{g01}\right), \exp\left(U_{g01}\right)\right]$

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Example: GLOW – Computing OR under interaction

Step 1: The two sets of values of the covariates are (priorfrac = 1, Age = a) compared to (priorfrac = 0, Age = a)

Step 2: Substituting these values into the general expression

Step 3: Taking the difference in the two functions

Step 4: Exponentiating the result

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Example: GLOW – Step 1

Step 1: The two sets of values of the covariates are (priorfrac = 1, Age = a) compared to (priorfrac = 0, Age = a)

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Example: GLOW – Step 2

Step 2: Substituting these values into the general expression

$$\operatorname{logit}\big(\pi(\boldsymbol{x})\big) = \hat{\beta}_0 + \hat{\beta}_1 \mathrm{PF} + \hat{\beta}_2 \mathrm{Age} + \hat{\beta}_3 \mathrm{PF} \times \mathrm{Age}$$

$$\begin{aligned}\operatorname{logit}\big(\pi(\mathrm{PF}=1, \mathrm{Age}=a)\big) &= \hat{\beta}_0 + \hat{\beta}_1 \times 1 + \hat{\beta}_2 a + \hat{\beta}_3 \times 1 \times a \\ &= \hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2 a + \hat{\beta}_3 a\end{aligned}$$

$$\begin{aligned}\operatorname{logit}\big(\pi(\mathrm{PF}=0, \mathrm{Age}=a)\big) &= \hat{\beta}_0 + \hat{\beta}_1 \times 0 + \hat{\beta}_2 a + \hat{\beta}_3 \times 0 \times a \\ &= \hat{\beta}_0 + \hat{\beta}_2 a\end{aligned}$$

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Example: GLOW – Step 3

Step 3: Taking the difference in the two functions

$$\left[\text{logit}\big(\pi(\text{PF} = 1, \text{Age} = a)\big)\right] - \left[\text{logit}\big(\pi(\text{PF} = 0, \text{Age} = a)\big)\right]$$

$$= \left[\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2 a + \hat{\beta}_3 a\right] - \left[\hat{\beta}_0 + \hat{\beta}_2 a\right]$$

$$= \hat{\beta}_1 + \hat{\beta}_3 a$$

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Example: GLOW – Step 4

Step 4: Exponentiating the result

Model		Variable	Coeff.	Std. Err.	z	p	95% CI	
3	$\hat{\beta}_1$	PRIORFRAC	4.961	1.8102	2.74	0.006	1.413,	8.509
	$\hat{\beta}_2$	AGE	0.063	0.0155	4.04	<0.001	0.032,	0.093
	$\hat{\beta}_3$	PRIORFRAC × AGE	−0.057	0.0250	−2.29	0.022	−0.106,	−0.008
	$\hat{\beta}_0$	Constant	−5.689	1.0841	−5.25	<0.001	−7.814,	−3.565

$$\widehat{OR}[(\text{PF} = 1, \text{Age} = a), (\text{PF} = 0, \text{Age} = a)] = \exp\left(\hat{\beta}_1 + \hat{\beta}_3 a\right)$$
$$= \exp(4.961 - 0.057a)$$

If we let $a = 60$, i.e., compute OR for age = 60, then

- $\widehat{OR}_{a=60} = \exp(4.961 - 0.057 * 60) = 4.669$

If we let $a = 70$, i.e., compute OR for age = 70, then

- $\widehat{OR}_{a=70} = \exp(4.961 - 0.057 * 70) = 2.64$

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Example: GLOW – Table of ORs

- Sometimes a Table listing the estimated odds ratio and 95% CI at different value of X will also be helpful and informative

Table 3.13 Estimated Odds Ratios for Prior Fracture as a Function of Age from Model 3 in Table 3.12

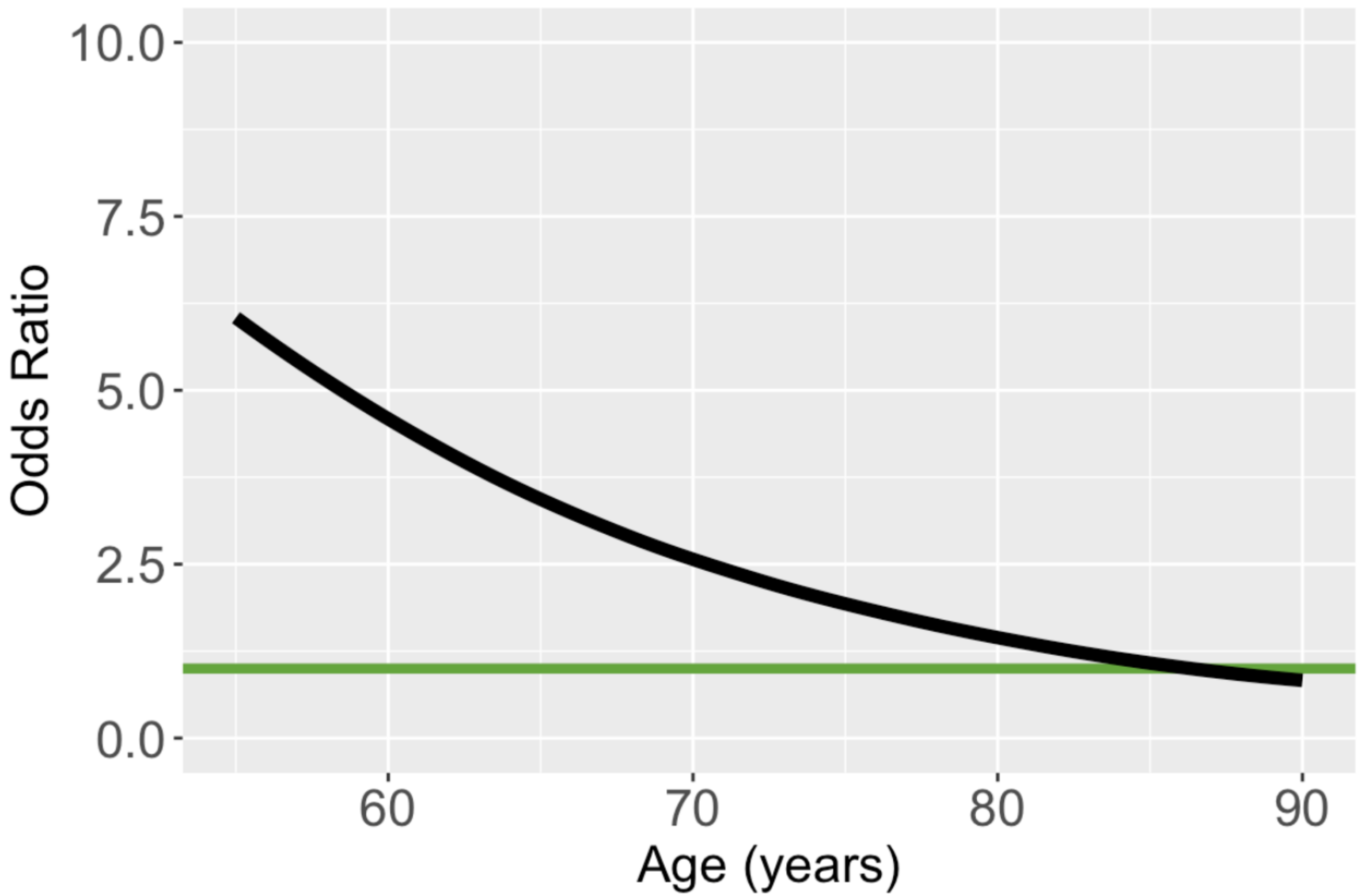
Age	Odds Ratio	95% CI
55	6.1	2.38, 15.53
60	4.6	2.20, 9.49
65	3.4	1.96, 5.99
70	2.6	1.63, 4.06
75	1.9	1.20, 3.11
80	1.4	0.79, 2.65

Originally from Class 9 Notes

Table from Hosmer & Lemeshow, page 77

Plot of Odds Ratio Involving Interaction Term

- Odds ratio comparing prior fracture to no prior fracture



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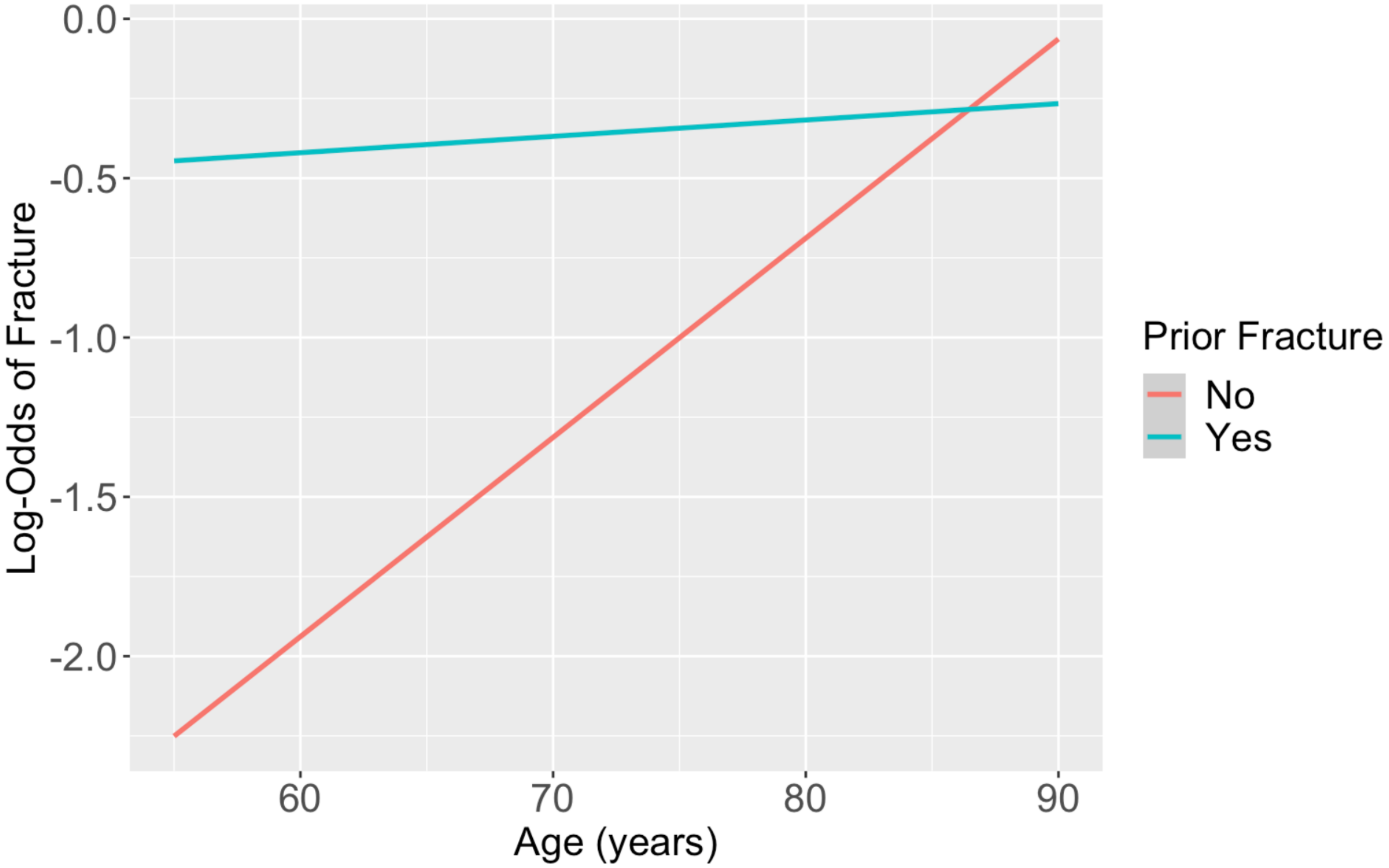
Graphically Showing Interaction **Revisited**

- One easy way to see the nature of the interaction between F and X is to plot the two logit functions
- Using the GLOW fracture example:

$$\text{logit}\big(\pi(\text{PF} = 1, \text{Age} = a)\big)$$
$$= \hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2 a + \hat{\beta}_3 a$$
$$= \big(\hat{\beta}_0 + \hat{\beta}_1\big) + \big(\hat{\beta}_2 + \hat{\beta}_3\big) a$$

$$\text{logit}\big(\pi(\text{PF} = 0, \text{Age} = a)\big)$$
$$= \hat{\beta}_0 + \hat{\beta}_2 a$$



Originally from Class 9 Notes

Simple Poisson Regression

Poisson Distribution (I)

- This distribution is often used to model count data

- Examples:
  - Distribution of number of deaths due to lung cancer
  - Distribution of number of individuals diagnosed with leukemia
  - Distribution of number of hospitalizations

Poisson Distribution (II)

- The probability function of Poisson distribution:

$$P(Y = y|\mu) = \frac{e^{-\mu}\mu^{y}}{y!}$$

  - Where y are non-negative integers $y = 0, 1, 2, \ldots$
  - Where μ is the mean of Y, that is $\mathrm{E(Y)} = \mu$
  - And also, $\mathrm{Var}(Y) = \mu$

- For a Poisson distribution, $Y \sim Poisson(\mu)$
  - Range: $[0, \infty)$

Poisson Distribution (III)

- If we look at the probability of y events <u>in a time period t</u> for a Poisson random variable, we could write:

$$P(Y = y|\mu) = \frac{e^{-\mu}\mu^{y}}{y!}$$

  - Where y are non-negative integers $y = 0, 1, 2, \ldots$
  - And $\mu = \lambda t$, where λ is the expected number of events per unit time
  - Then μ is the expected number of events over time t

Poisson Distribution (IV)

- What does λ represent here?
  - A **rate**, the expected number of events in a given population over a given period time
    - **Example:** Number of patient arrivals into the Emergency Room *per hour*
  - The Poisson distribution is the prototype for assigning probabilities of observing any number of events

Poll Everywhere Question 1

Why Person-Years? (I)

- In the example of number of patient arrivals, an event does not conclude the study
  - If someone arrives within the first minute of the study, then we keep counting
  - We may be able to study the association of arrivals with qualities of the hospital, but we can’t measure qualities of the individuals arriving
- What happens if we want to measure qualities of the individual?
  - We can measure a hospitalization rate

Why Person-Years? (II)

- If we are measuring at the individual level and counting something that is “terminal” then our count will always be 0 or 1
  - Example: Number of individuals diagnosed with leukemia
    - This only happens once, so how do we measure the rate here?
- Since rate involves the counts and time – we can use the time to diagnosis to estimate the rate
  - Often expressed in units such as events per thousand person-years

Calculating Person-Year

- One **person-year** is a unit of time defined as **one person being followed for one year**
- Person-years for a sample of *n* subjects is calculated as the total years followed for the *n* subjects, where each subject could have different follow-up time

- **Example:** suppose we have 5 subjects, two of the subjects were followed for 2 years, and two of them are followed for 3 years and the fifth subject was followed for 3.8 years

$$\begin{aligned} &\text{person} - \text{years} \\ &= 2\text{ people} \times 2\text{ years} + 2\text{ people} \times 3\text{ years} \\ &+ 1\text{ person} \times 3.8\text{ years} = 13.8\text{ person} - \text{years} \end{aligned}$$

Calculating Rate (II)

- Suppose that we observe one event during the follow-up period, then

$$\text{Rate of event} = \frac{\#\ \text{events}}{\text{person} - \text{years}} = \frac{1\ \text{event}}{13.8\ \text{person} - \text{years}}$$

$$= 0.072\ \text{events per person} - \text{year}$$

$$= 72\ \text{events per}\ 1000\ \text{person} - \text{years}$$

Stop here?

If we do:

- Good tutorial in R: https://www.dataquest.io/blog/tutorial-poisson-regression-in-r/

Review: Simple Logistic Regression

- Let Y is the dependent variable of interest and x is a predictor variable,
  - In simple logistic regression, we have

$$\log\left(\frac{\pi(x_i)}{1-\pi(x_i)}\right) = \beta_0 + \beta_1 x_i$$

where $\pi(x_i) = \mathrm{P}(\mathrm{Y_i} = 1|\mathrm{x_i})$

Simple Poisson Regression Model

- What do we model in a Poisson regression?

- Log of conditional mean of Y given x

  - Let **Y be a Poisson count for a given unit of time**, then
    $\mu(x) \ = \lambda(x)$

  - In a simple Poisson regression, we have

$$\ln\big(\mu(x_i)\big) = \ln\big(\lambda(x_i)\big) = \beta_0 + \beta_1 x_i$$

  - So this is also called a **log-linear** model

Parameter Interpretation (I)

- In simple Poisson regression:

$$\ln\big(\mu(x_i)\big) = \ln\big(\lambda(x_i)\big) = \beta_0 + \beta_1 x_i$$

- When $\boldsymbol{x}$ **is a binary variable**: How do we interpret β_0 and β_1?
  - When $x_i = 0$:

$$\ln\big(\mu(x_i = 0)\big) = \ln\big(\lambda(x_i = 0)\big) = \beta_0$$

    - $\mu(x_i = 0) = \exp(\beta_0)$: the mean (rate) of Y_i when $x_i = 0$

  - When $x_i = 1$:

$$\ln\big(\mu(x_i = 1)\big) = \ln\big(\lambda(x_i = 1)\big) = \beta_0 + \beta_1$$

    - $\mu(x_i = 1) = \exp(\beta_0 + \beta_1)$: the mean (rate) of Y_i when $x_i = 1$

Parameter Interpretation (II)

- When $\boldsymbol{x}$ **is a binary variable**: How do we interpret β_0 and β_1?

  - By subtraction, we have

$$\beta_1 = \ln\left(\frac{E(Y_i|x_i = 1)}{E(Y_i|x_i = 0)}\right) = \ln\left(\frac{\lambda(x_i = 1)}{\lambda(x_i = 0)}\right)$$

  - β_1: Log-rate ratio

  - **And $\exp(\boldsymbol{\beta_1})$ is the rate ratio**

Further reading / tutorials on Poisson regression

- Good tutorial in R: https://www.dataquest.io/blog/tutorial-poisson-regression-in-r/

- When people are followed for different amounts of time, we should include an offset
  - Poisson Regression Modeling Using Rate Data: section from above site that discusses offsets

- We can use Wald test and LRT in the same way as logistic regression to test our coefficients and variables

Wrap-up

- 4-minute exit ticket
- Thanks for a great quarter!

Class 17 Exit Ticket

https://forms.office.com/r/5q7YxDi58s