SLR: More inference + Evaluation

Nicky Wakim

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Learning Objectives

1. Identify different sources of variation in an Analysis of Variance (ANOVA) table
2. Using the F-test, determine if there is enough evidence that population slope β_1 is not 0
3. Calculate and interpret the coefficient of determination
4. Describe the model assumptions made in linear regression using ordinary least squares

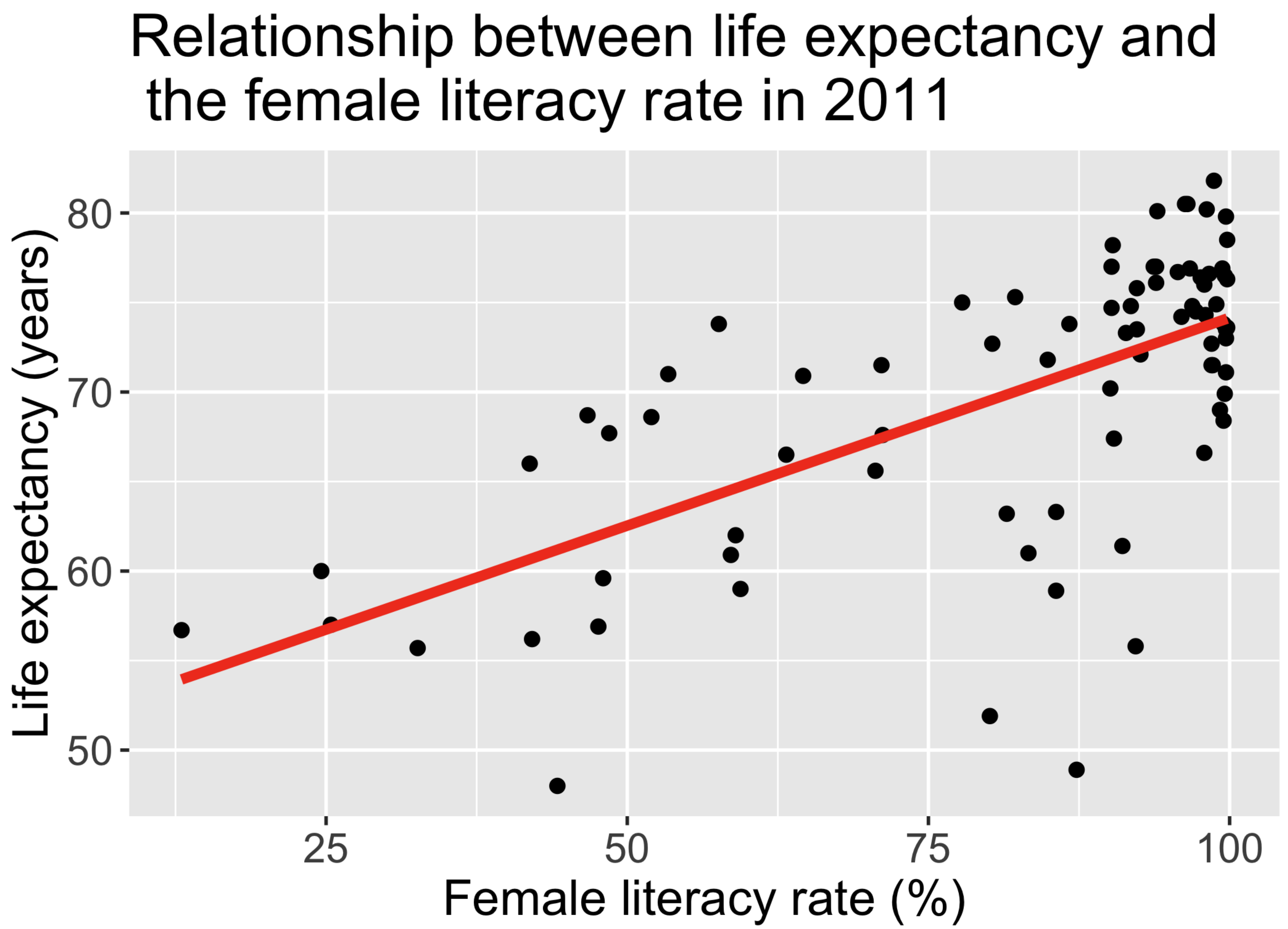
So far in our regression example...

Lesson 1 of SLR:

- Fit regression line
- Calculate slope & intercept
- Interpret slope & intercept

Lesson 2 of SLR:

- Estimate variance of the residuals
- Inference for slope & intercept: CI, p-value
- Confidence bands of regression line for mean value of Y|X



$$\widehat{Y} = \widehat{\beta}_0 + \widehat{\beta}_1 \cdot X$$

$$\widehat{\text{life expectancy}} = 50.9 + 0.232 \cdot \text{female literacy rate}$$

Let's revisit the regression analysis process

Model Selection

- Building a model
- Selecting variables
- Prediction vs interpretation
- Comparing potential models

Model Fitting

- Find best fit line
- Using OLS in this class
- Parameter estimation
- Categorical covariates
- Interactions

Model Evaluation

- Evaluation of model fit
- Testing model assumptions
- Residuals
- Transformations
- Influential points
- Multicollinearity

Model Use (Inference)

- Inference for coefficients
- Hypothesis testing for coefficients
- Inference for expected Y given X
- Prediction of new Y given X

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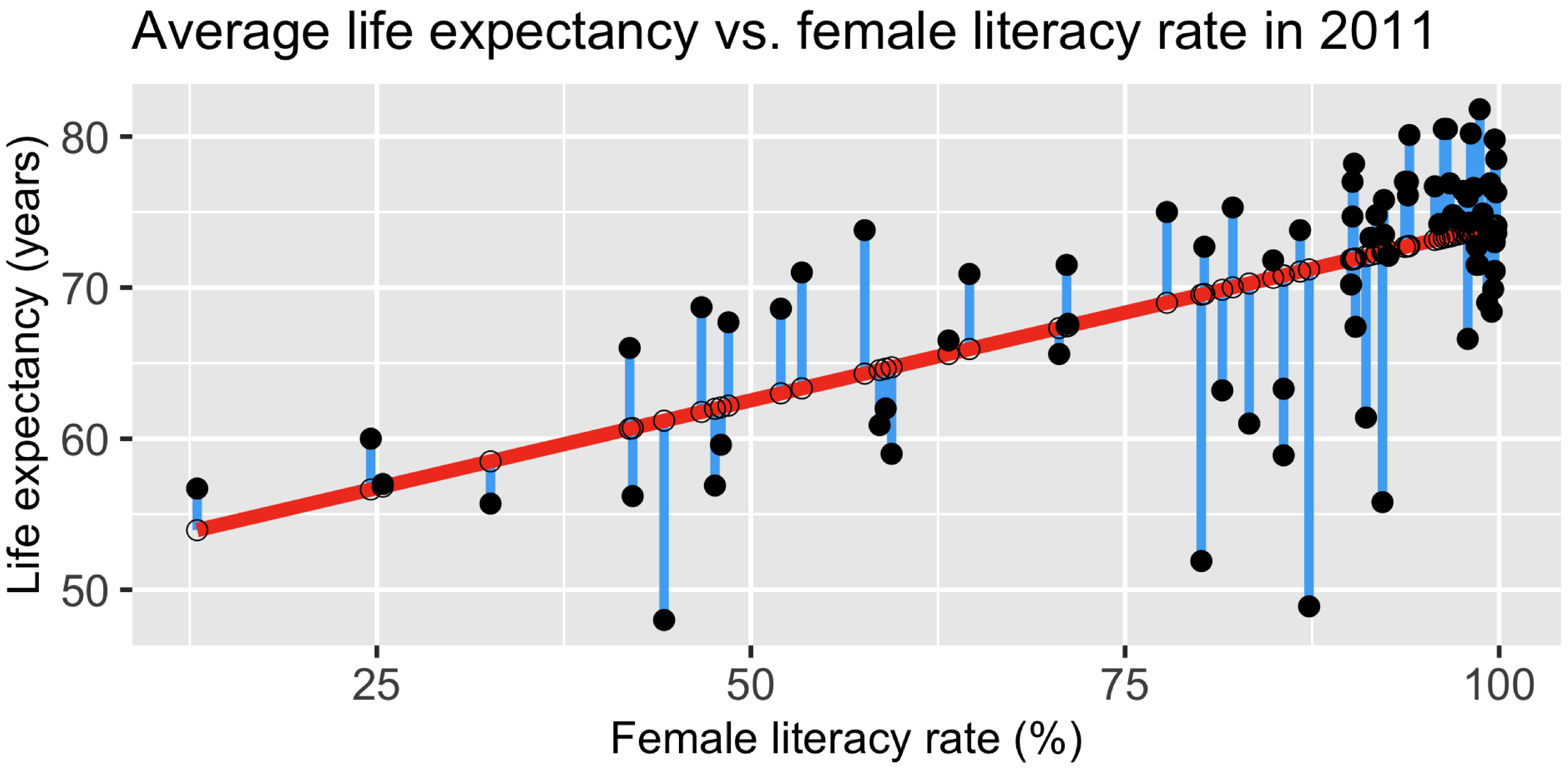
Getting to the F-test

The F statistic in linear regression is essentially a proportion of the variance explained by the model vs. the variance not explained by the model

1. Start with visual of explained vs. unexplained variation
2. Figure out the mathematical representations of this variation
3. Look at the ANOVA table to establish key values measuring our variance from our model
4. Build the F-test

Explained vs. Unexplained Variation



$$Y_i - \overline{Y} = (Y_i - \hat{Y}_i) + (\hat{Y}_i - \overline{Y})$$

$$\text{Total unexplained variation} = \text{Variation due to regression} + \text{Residual variation after regression}$$

More on the equation

$$Y_i - \overline{Y} = (Y_i - \hat{Y}_i) + (\hat{Y}_i - \overline{Y})$$

- $Y_i - \overline{Y}$ = the deviation of Y_i around the mean $\overline{Y}$
  - (the **total** amount deviation unexplained at X_i).
- $Y_i - \hat{Y}_i$ = the deviation of the observation Y around the fitted regression line
  - (the amount deviation **unexplained** by the regression at X_i).
- $\hat{Y}_i - \overline{Y}$ = the deviation of the fitted value $\hat{Y}_i$ around the mean $\overline{Y}$
  - (the amount deviation **explained** by the regression at X_i)

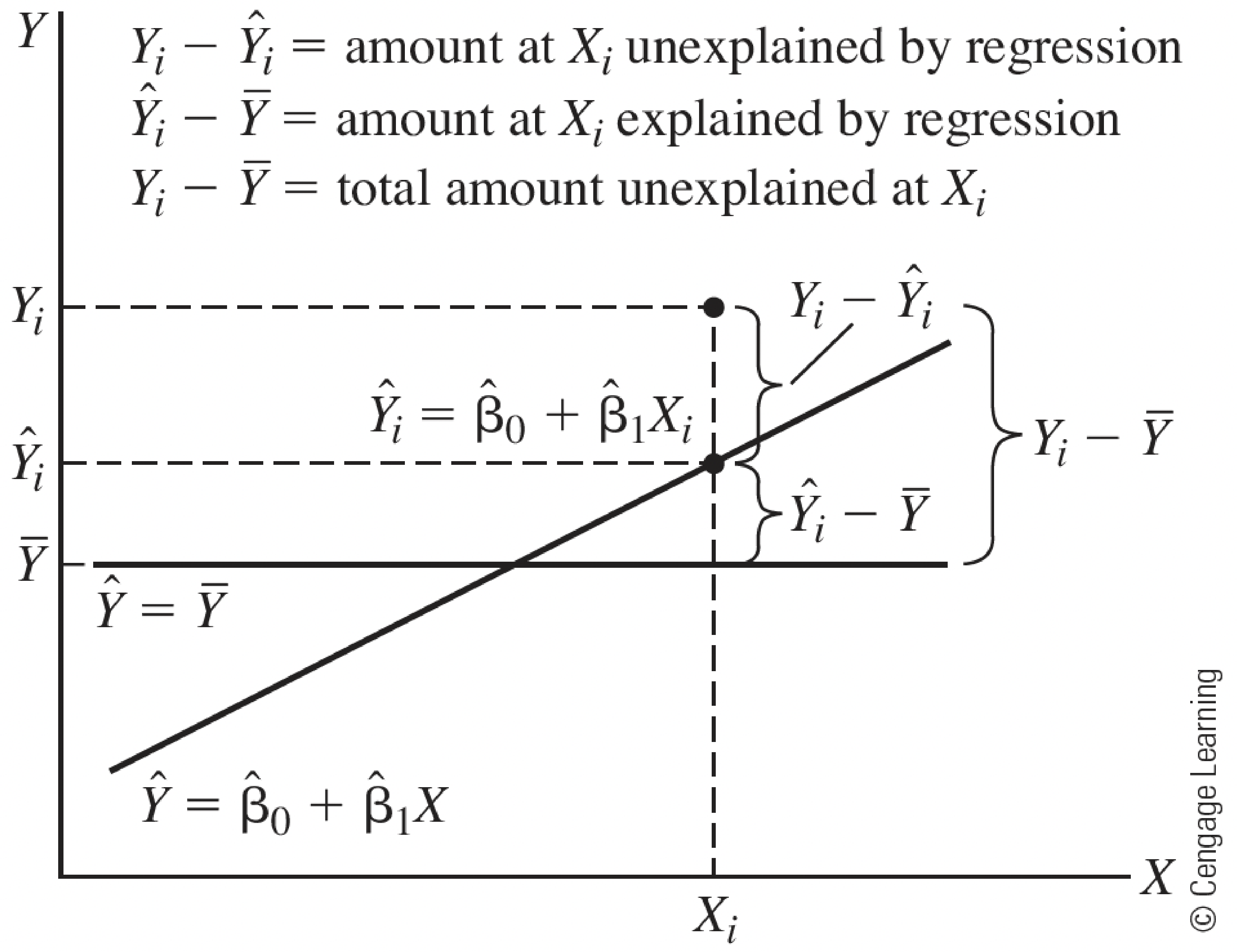


FIGURE 7.1 Variation explained and unexplained by straight-line regression

Poll Everywhere Question 1

How is this actually calculated for our fitted model?

$$Y_i - \overline{Y} = (Y_i - \hat{Y}_i) + (\hat{Y}_i - \overline{Y})$$

$$\text{Total unexplained variation} = \text{Variation due to regression} + \text{Residual variation after regression}$$

$$\sum_{i=1}^{n}(Y_i - \overline{Y})^2 = \sum_{i=1}^{n}(\hat{Y}_i - \overline{Y})^2 + \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2$$

$$SSY = SSR + SSE$$

$$\text{Total Sum of Squares} = \text{Sum of Squares due to Regression} + \text{Sum of Squares due to Error (residuals)}$$

ANOVA table:

Variation Source	df	SS	MS	test statistic	p-value
Regression	1	SSR	$MSR = \frac{SSR}{1}$	$F = \frac{MSR}{MSE}$	
Error	$n-2$	SSE	$MSE = \frac{SSE}{n-2}$		
Total	$n-1$	SSY			

Analysis of Variance (ANOVA) table in R

```r
# Fit regression model:
model1 <- lm(life_expectancy_years_2011 ~ female_literacy_rate_2011,
                 data = gapm)

anova(model1)
```

```
Analysis of Variance Table

Response: life_expectancy_years_2011
                          Df Sum Sq Mean Sq F value    Pr(>F)
female_literacy_rate_2011  1 2052.8 2052.81  54.414 1.501e-10 ***
Residuals                 78 2942.6   37.73
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```r
anova(model1) %>% tidy() %>% gt() %>%
  tab_options(table.font.size = 40) %>%
  fmt_number(decimals = 3)
```

term	df	sumsq	meansq	statistic	p.value
female_literacy_rate_2011	1.000	2,052.812	2,052.812	54.414	0.000
Residuals	78.000	2,942.635	37.726	NA	NA

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What is the F statistic testing?

$$F = \frac{MSR}{MSE}$$

- It can be shown that

$$E(MSE) = \sigma^2 \text{ and } E(MSR) = \sigma^2 + \beta_1^2 \sum_{i=1}^{n} (X_i - \overline{X})^2$$

- Recall that σ^2 is the variance of the residuals
- Thus if
  - $\beta_1 = 0$, then $F \approx \frac{\hat{\sigma}^2}{\hat{\sigma}^2} = 1$
  - $\beta_1 \neq 0$, then $F \approx \frac{\hat{\sigma}^2 + \hat{\beta}_1^2 \sum_{i=1}^{n} (X_i - \overline{X})^2}{\hat{\sigma}^2} > 1$
- So the F statistic can also be used to test β_1

F-test vs. t-test for the population slope

The square of a t-distribution with $df = \nu$ is an F-distribution with $df = 1, \nu$

$$T_\nu^2 \sim F_{1,\nu}$$

- We can use either F-test or t-test to run the following hypothesis test:

$$\begin{aligned} H_0 &: \beta_1 = 0 \\ \text{vs. } H_A &: \beta_1 \neq 0 \end{aligned}$$

- Note that the F-test does not support one-sided alternative tests, but the t-test does!

Planting a seed about the F-test

We can think about the hypothesis test for the slope...

Null H_0	Alternative H_1
$\beta_1 = 0$	$\beta_1 \neq 0$

in a slightly different way...

Null model ($\beta_1 = 0$)	Alternative model ($\beta_1 \neq 0$)
• $Y = \beta_0 + \epsilon$ • Smaller (reduced) model	• $Y = \beta_0 + \beta_1 X + \epsilon$ • Larger (full) model

- In multiple linear regression, we can start using this framework to test multiple coefficient parameters at once
  - Decide whether or not to reject the smaller reduced model in favor of the larger full model
  - Cannot do this with the t-test!

Poll Everywhere Question 2

F-test: general steps for hypothesis test for population slope β_1

1. For today's class, we are assuming that we have met the underlying assumptions

2. State the null hypothesis.

Often, we are curious if the coefficient is 0 or not:

$$\begin{aligned} H_0 &: \beta_1 = 0 \\ \text{vs. } H_A &: \beta_1 \neq 0 \end{aligned}$$

3. Specify the significance level.

Often we use $\alpha = 0.05$

4. Specify the test statistic and its distribution under the null

The test statistic is F, and follows an F-distribution with numerator $df = 1$ and denominator $df = n - 2$.

5. Compute the value of the test statistic

The calculated **test statistic** for $\widehat{\beta}_1$ is

$$F = \frac{MSR}{MSE}$$

6. Calculate the p-value

We are generally calculating: $P(F_{1,n-2} > F)$

7. Write conclusion for hypothesis test

- Reject: $P(F_{1,n-2} > F) < \alpha$

We (reject/fail to reject) the null hypothesis that the slope is 0 at the $100\alpha\%$ significiance level. There is (sufficient/insufficient) evidence that there is significant association between (Y) and (X) (p-value = $P(F_{1,n-2} > F)$).

Life expectancy example: hypothesis test for population slope β_1

- Steps 1-4 are setting up our hypothesis test: not much change from the general steps

1. For today's class, we are assuming that we have met the underlying assumptions 2. State the null hypothesis.

We are testing if the slope is 0 or not:

$$\begin{aligned} H_0 &: \beta_1 = 0 \\ \text{vs. } H_A &: \beta_1 \neq 0 \end{aligned}$$

3. Specify the significance level.

Often we use $\alpha = 0.05$

4. Specify the test statistic and its distribution under the null

The test statistic is F, and follows an F-distribution with numerator $df = 1$ and denominator $df = n - 2 = 80 - 2$.

```
nobs(model1)
```

```
[1] 80
```

Life expectancy example: hypothesis test for population slope β_1 (2/4)

5. Compute the value of the test statistic

```
anova(model1) %>% tidy() %>% gt() %>%
  tab_options(table.font.size = 40)
```

term	df	sumsq	meansq	statistic	p.value
female_literacy_rate_2011	1	2052.812	2052.81234	54.4136	1.501286e-10
Residuals	78	2942.635	37.72609	NA	NA

- **Option 1:** Calculate the test statistic using the values in the ANOVA table

$$F = \frac{MSR}{MSE} = \frac{2052.81}{37.73} = 54.414$$

- **Option 2:** Get the test statistic value (F) from the ANOVA table

I tend to skip this step because I can do it all with step 6

Life expectancy example: hypothesis test for population slope β_1 (3/4)

6. Calculate the p-value

- As per Step 4, test statistic F can be modeled by a F-distribution with $df1 = 1$ and $df2 = n - 2$.
  - We had 80 countries' data, so $n = 80$
- **Option 1:** Use `pf()` and our calculated test statistic

```r
# p-value is ALWAYS the right tail for F-test
pf(54.414, df1 = 1, df2 = 78, lower.tail = FALSE)
```

```
[1] 1.501104e-10
```

- **Option 2:** Use the ANOVA table

```r
anova(model1) %>% tidy() %>% gt() %>%
  tab_options(table.font.size = 40)
```

term	df	sumsq	meansq	statistic	p.value
female_literacy_rate_2011	1	2052.812	2052.81234	54.4136	1.501286e-10
Residuals	78	2942.635	37.72609	NA	NA

Life expectancy example: hypothesis test for population slope β_1 (4/4)

7. Write conclusion for the hypothesis test

We reject the null hypothesis that the slope is 0 at the 5% significance level. There is sufficient evidence that there is significant association between female life expectancy and female literacy rates (p-value < 0.0001).

Did you notice anything about the p-value?

The p-value of the t-test and F-test are the same!!

- For the t-test:

```
tidy(model1) %>% gt() %>%
  tab_options(table.font.size = 40)
```

term	estimate	std.error	statistic	p.value
(Intercept)	50.9278981	2.66040695	19.142898	3.325312e-31
female_literacy_rate_2011	0.2321951	0.03147744	7.376557	1.501286e-10

- For the F-test:

```
anova(model1) %>% tidy() %>% gt() %>%
  tab_options(table.font.size = 40)
```

term	df	sumsq	meansq	statistic	p.value
female_literacy_rate_2011	1	2052.812	2052.81234	54.4136	1.501286e-10
Residuals	78	2942.635	37.72609	NA	NA

This is true when we use the F-test for a single coefficient!

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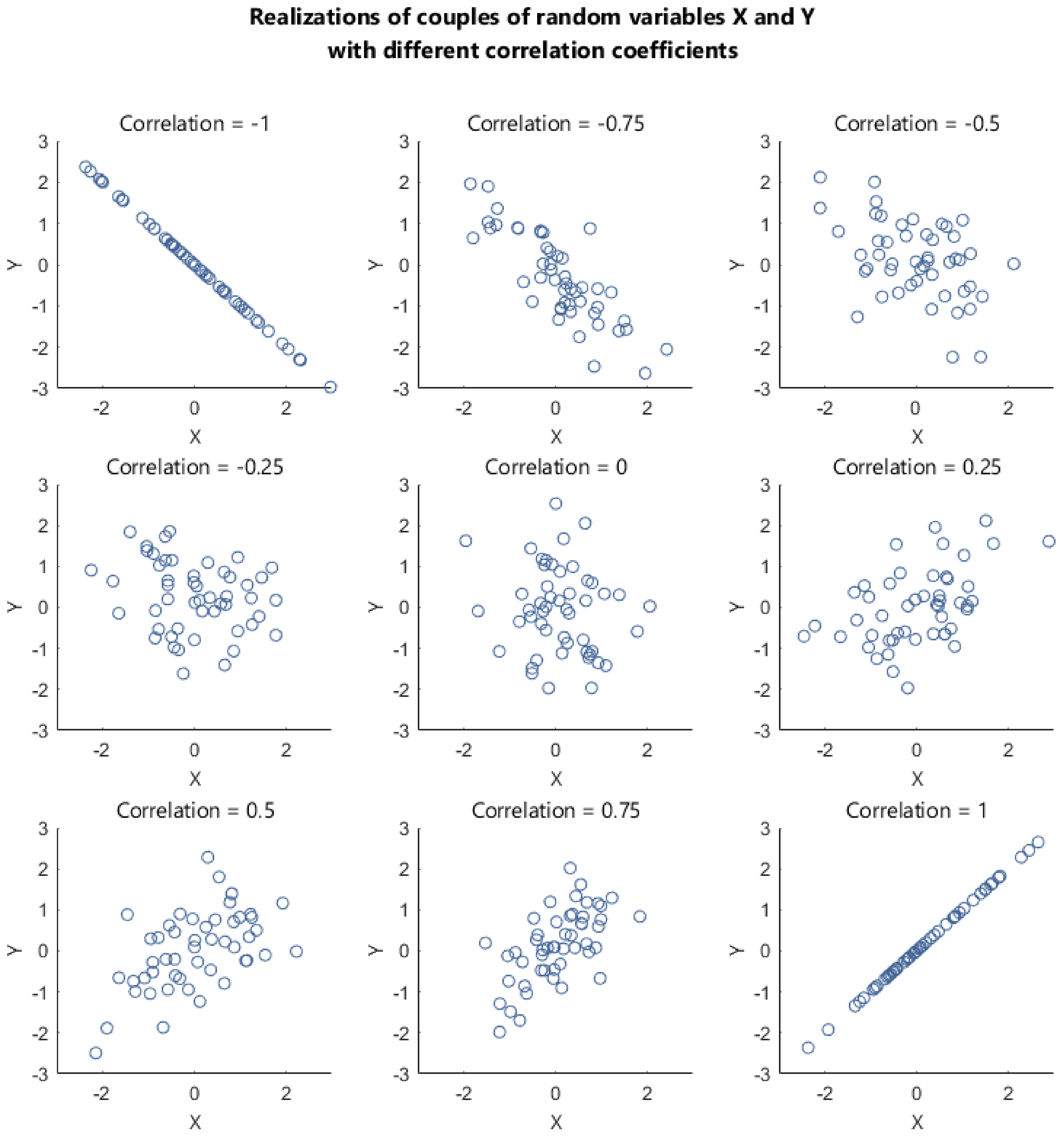
Correlation coefficient from 511

Correlation coefficient r can tell us about the strength of a relationship

- If $r = -1$, then there is a perfect negative linear relationship between X and Y
- If $r = 1$, then there is a perfect positive linear relationship between X and Y
- If $r = 0$, then there is no linear relationship between X and Y

Note: All other values of r tell us that the relationship between X and Y is not perfect. The closer r is to 0, the weaker the linear relationship.



Correlation coefficient

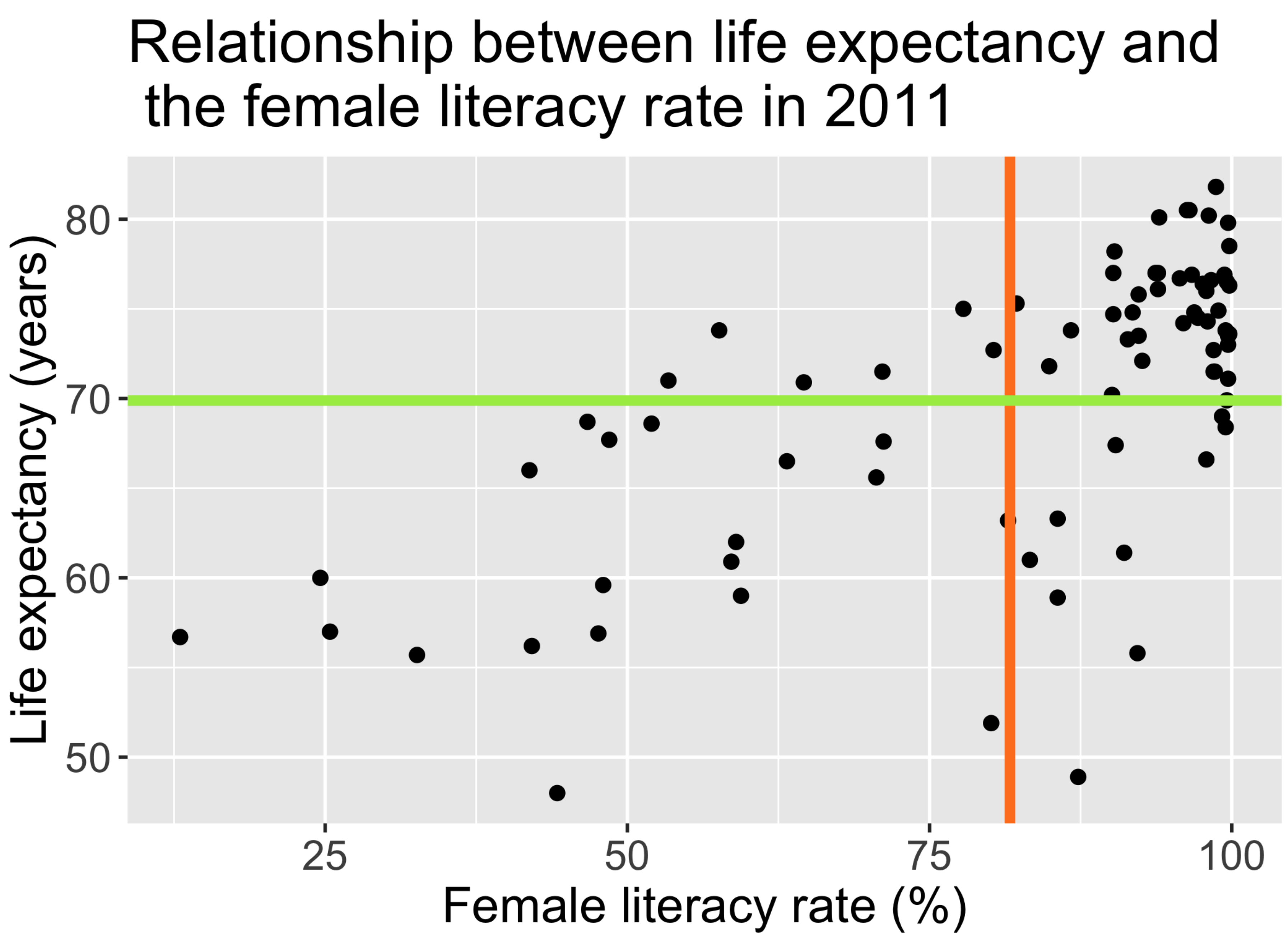
The (Pearson) correlation coefficient r of variables X and Y can be computed using the formula:

$$r = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\left(\sum_{i=1}^{n}(X_i - \overline{X})^2 \sum_{i=1}^{n}(Y_i - \overline{Y})^2\right)^{1/2}}$$

$$= \frac{SSXY}{\sqrt{SSX \cdot SSY}}$$

we have the relationship

$$\widehat{\beta}_1 = r\frac{SSY}{SSX}, \quad \text{or,} \quad r = \widehat{\beta}_1\frac{SSX}{SSY}$$



Coefficient of determination: R^2

It can be shown that the square of the correlation coefficient r is equal to

$$R^2 = \frac{SSR}{SSY} = \frac{SSY - SSE}{SSY}$$

- R^2 is called the **coefficient of determination**.
- **Interpretation**: The proportion of variation in the Y values explained by the regression model
- R^2 measures the strength of the linear relationship between X and Y:
  - $R^2 = \pm 1$: Perfect relationship
    - Happens when $SSE = 0$, i.e. no error, all points on the line
  - $R^2 = 0$: No relationship
    - Happens when $SSY = SSE$, i.e. using the line doesn't not improve model fit over using $\overline{Y}$ to model the Y values.

Poll Everywhere Question

Life expectancy example: correlation coeffiicent r and coefficient of determination R^2

```r
(r = cor(x = gapm$life_expectancy_years_2011,
     y = gapm$female_literacy_rate_2011,
     use =  "complete.obs"))
```

```
[1] 0.6410434
```

```r
r^2
```

```
[1] 0.4109366
```

```r
(sum_m1 = summary(model1)) # for R^2 value
```

```

Call:
lm(formula = life_expectancy_years_2011 ~ female_literacy_rate_2011,
    data = gapm)

Residuals:
    Min      1Q  Median      3Q     Max 
-22.299  -2.670   1.145   4.114   9.498 

Coefficients:
```

```r
sum_m1$r.squared
```

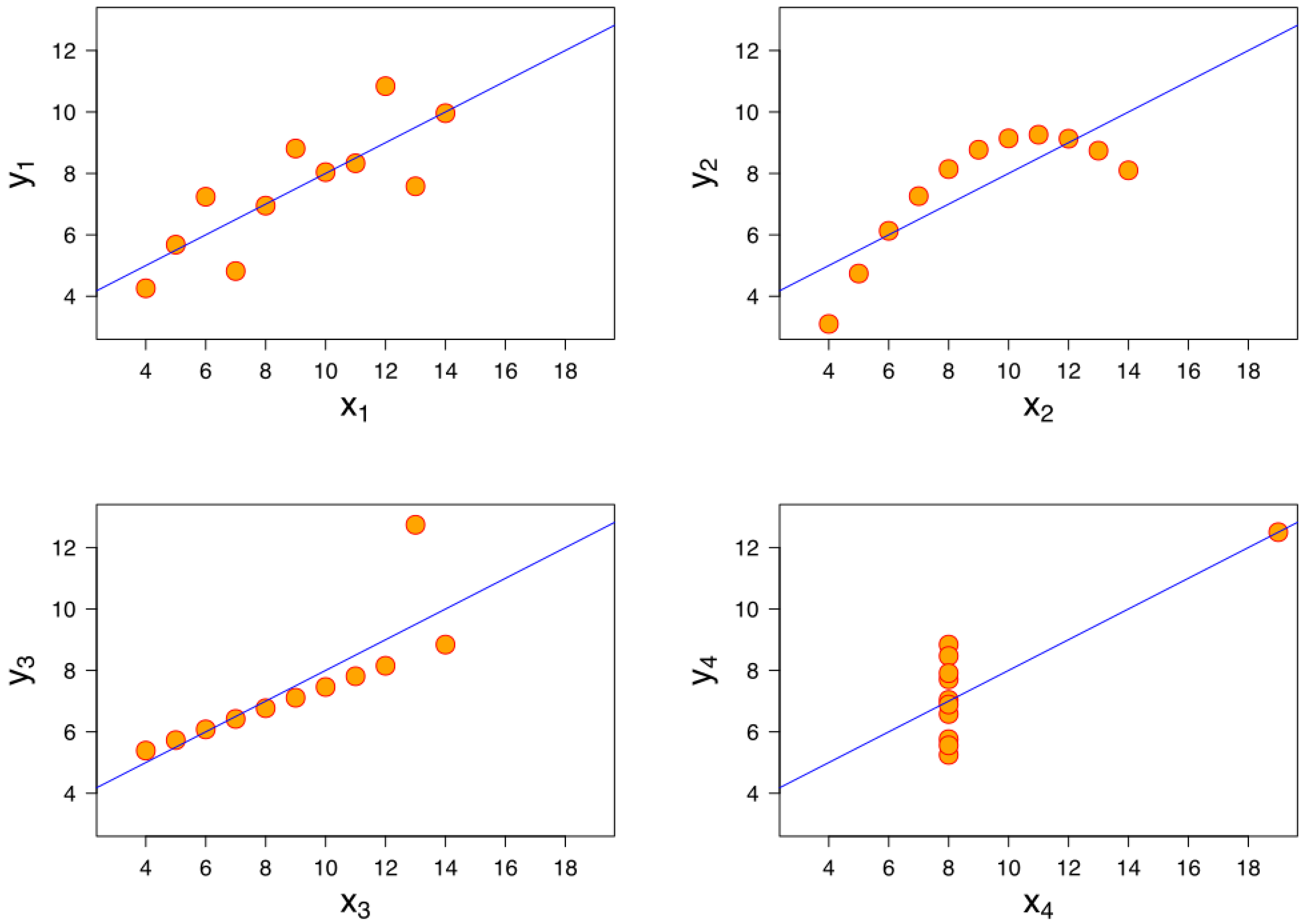
```
[1] 0.4109366
```

Interpretation

41.1% of the variation in countries' average life expectancy is explained by the linear model with female literacy rate as the independent variable.

What does R^2 not measure?

- R^2 is not a measure of the magnitude of the slope of the regression line
  - Example: can have $R^2 = 1$ for many different slopes!!
- R^2 is not a measure of the appropriateness of the straight-line model
  - Example: figure



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Least-squares model assumptions: eLINE

These are the model assumptions made in ordinary least squares:

- **e** xistence: For any X, there exists a distribution for Y
- **L** inearity of relationship between variables
- **I** ndependence of the Y values
- **N** ormality of the Y's given X (residuals)
- **E** quality of variance of the residuals (homoscedasticity)

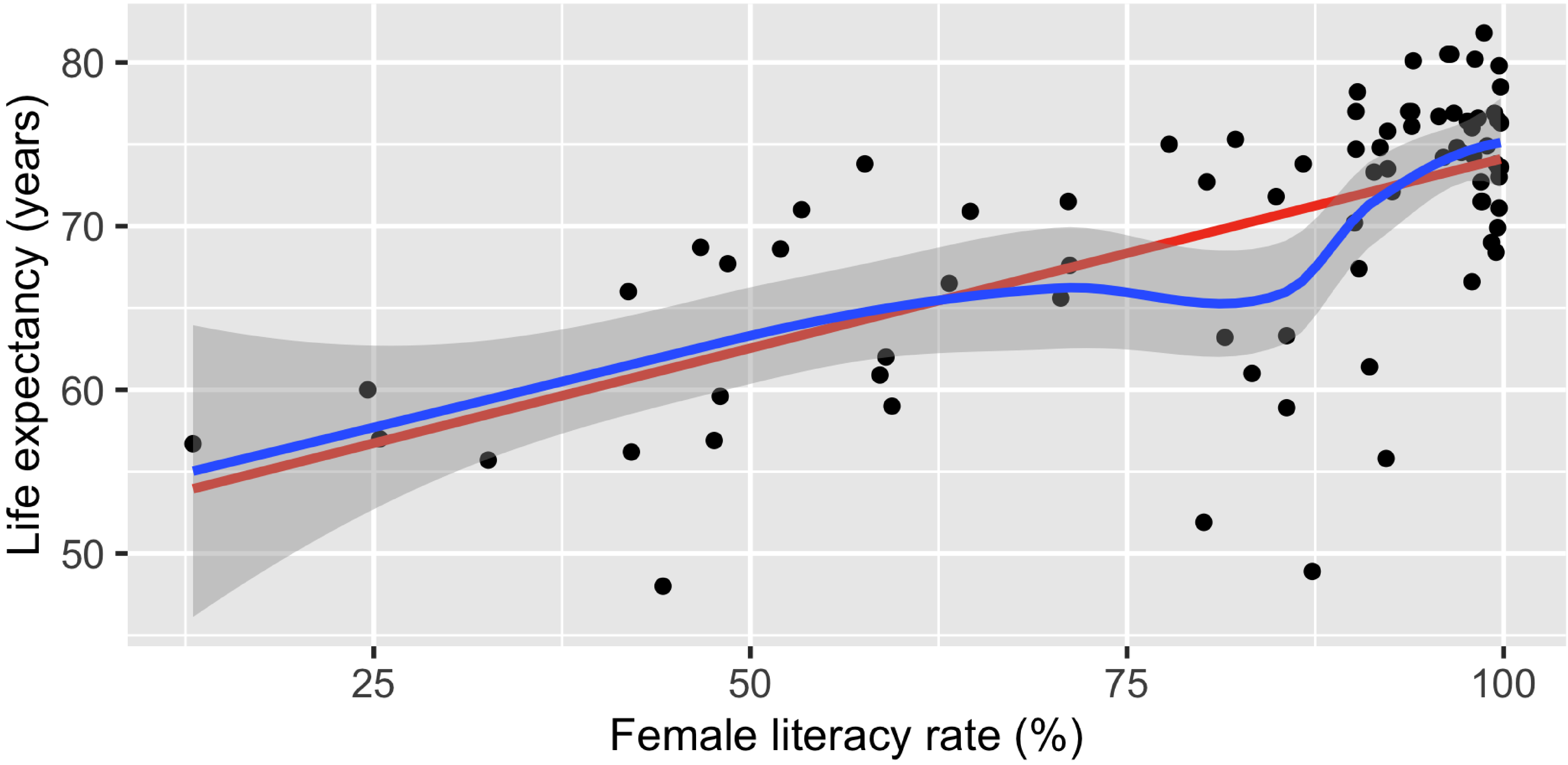
e: Existence of Y's distribution

- For any fixed value of the variable X, Y is a
  - random variable with a certain probability distribution
  - having finite
    - mean and
    - variance
- This leads to the normality assumption
- Note: This is not about Y alone, but $Y|X$

L: Linearity

- The relationship between the variables is linear (a straight line):
  - The mean value of Y given X, $\mu_{y|x}$ or $E[Y|X]$, is a straight-line function of X

$$\mu_{y|x} = \beta_0 + \beta_1 \cdot X$$



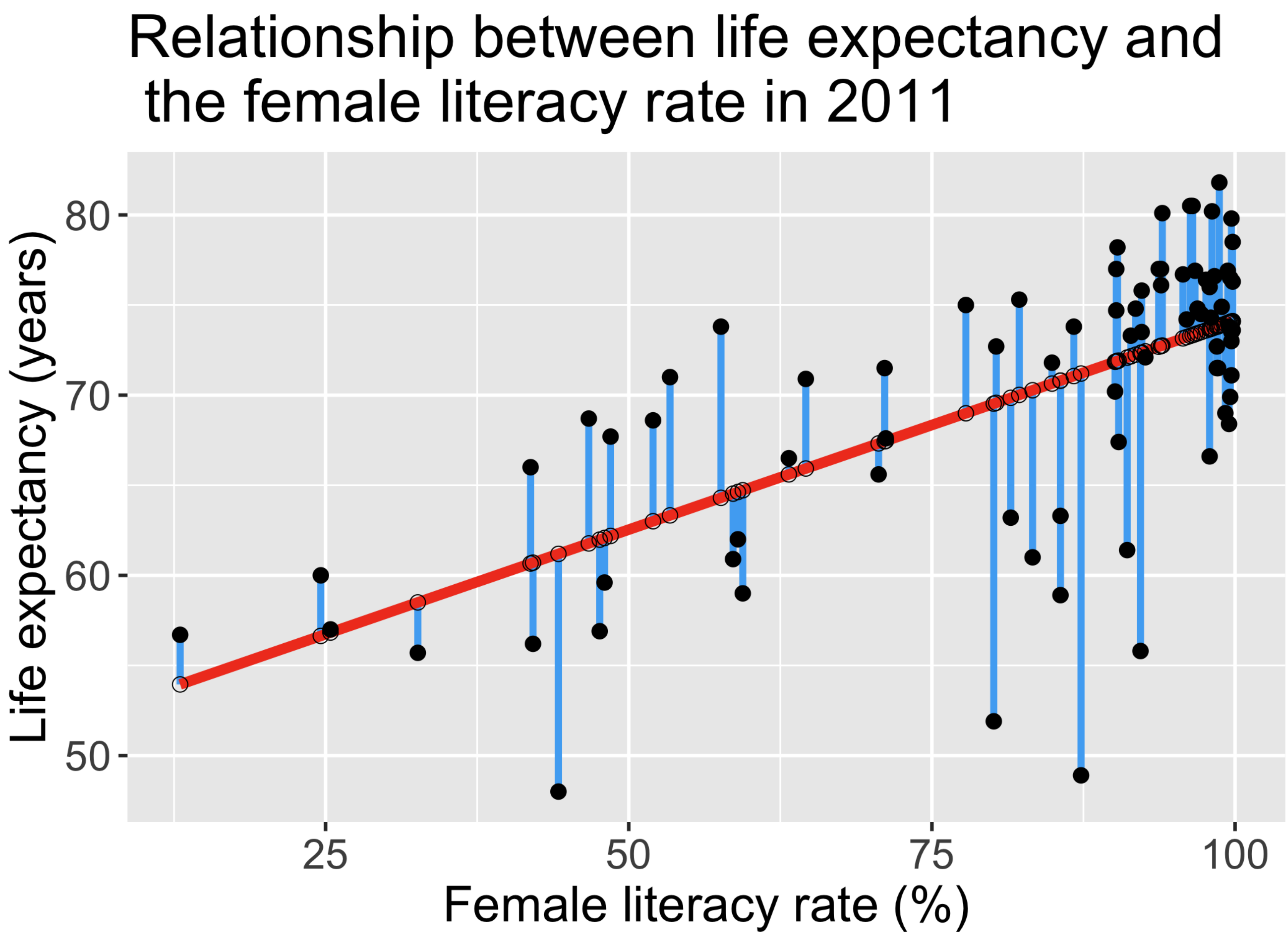
I: Independence of observations

- The Y-values are statistically independent of one another
- Examples of when they are *not* independent, include
  - repeated measures (such as baseline, 3 months, 6 months)
  - data from clusters, such as different hospitals or families
- This condition is checked by reviewing the study *design* and not by inspecting the data
- How to analyze data using regression models when the Y-values are not independent is covered in BSTA 519 (Longitudinal data)

Poll Everywhere Question

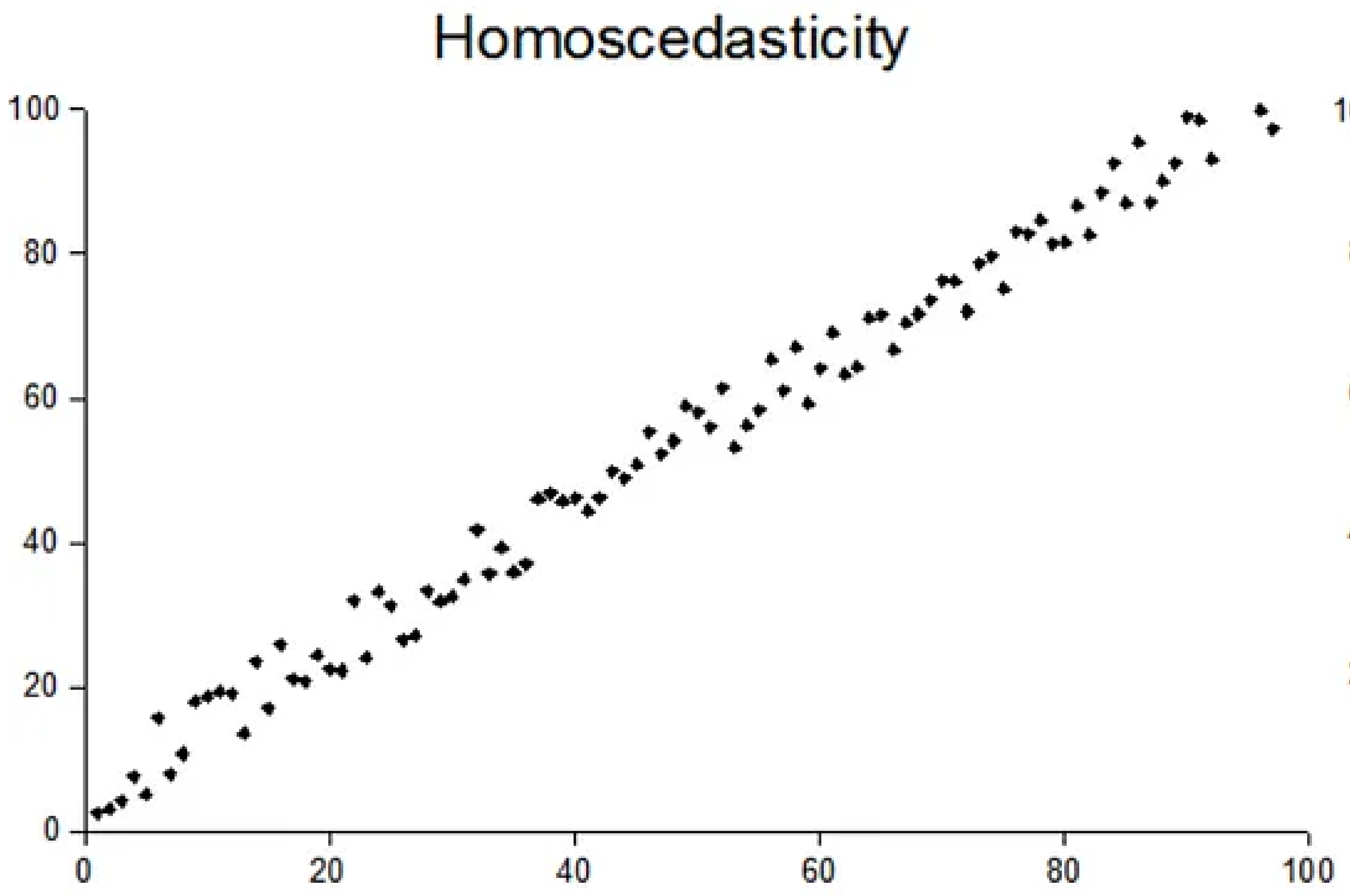
N: Normality

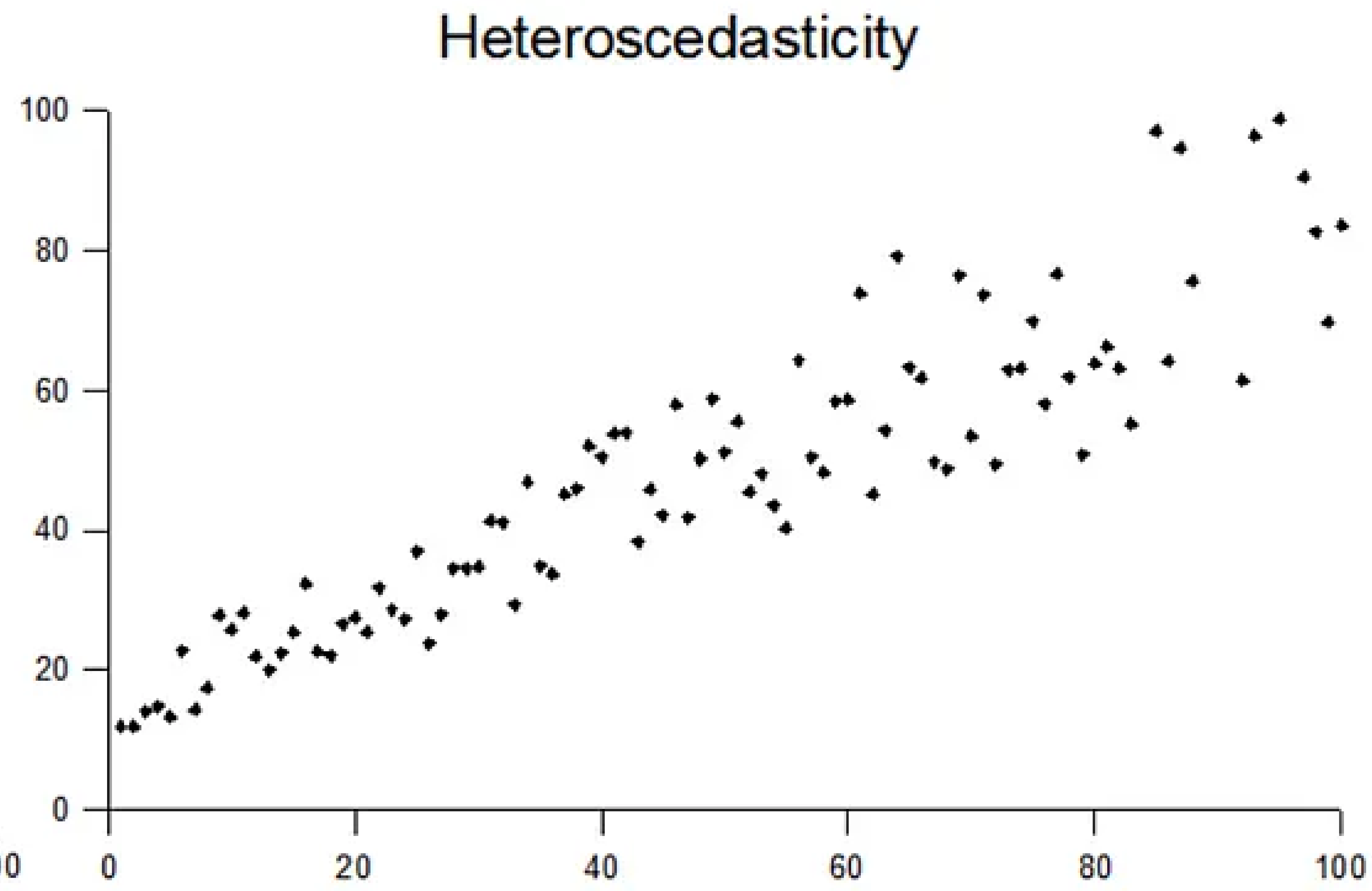
- For any fixed value of X, Y has normal distribution.
  - Note: This is not about Y alone, but $Y|X$
- Equivalently, the measurement (random) errors ϵ_i 's normally distributed
  - This is more often what we check
- We will discuss how to assess this in practice in Chapter 14 (Regression Diagnostics)



E: Equality of variance of the residuals

- The variance of Y given X ($\sigma^2_{Y|X}$), is the same for any X
  - We use just σ^2 to denote the common variance
- This is also called **homoscedasticity**
- We will discuss how to assess this in practice in Chapter 14 (Regression Diagnostics)





Summary of eLINE model assumptions

- Y values are independent (check study design!)

- The distribution of Y given X is
  - normal
  - with mean $\mu_{y|x} = \beta_0 + \beta_1 \cdot X$
  - and common variance σ^2

- This means that the residuals are
  - normal
  - with mean = 0
  - and common variance σ^2

Anscombe's Quartet