ESE 105 Case Study 3: Circuits as Resonators, Sensors, and Filters

Fall 2021

DUE: Tues. Nov. 23 at 5 PM to Canvas

In this case study you will use the skills and methods you have learned in linear algebra and MATLAB to simulate the current flows in an electrical circuit and their effects on audio signals.

Similar to Case Study 2, we will use a linear dynamical system model to (approximately) describe a physical system; here, the quantities of interest will be the current flows and node voltages within an electrical circuit. In such a circuit, the flow of electrons performs work, such as heating a resistor, moving a speaker cone, or producing light within a light-emitting diode (LED). Your goals are fourfold:

1. Implement a linear dynamical system model for a simple RC circuit. Explore how the sampling interval in time affects the accuracy of the model.
2. Compare the simple RC circuit to the behavior of a simple RL circuit.
3. Derive the update equations for a series RLC circuit. Explore how the circuit responds to a step input and sinusoidal input. Explore our perception of the circuit responses as audio signals.
4. Design an RLC circuit to operate as a resonator, sensor, and audio filter in a class-wide competition.

1 Part 1: Model an RC circuit

1.1 Background

A capacitor is an electrical component that stores potential energy in an electric field. Its capacitance C quantifies how much charge Q (electrons) can be stored at a given voltage v across the capacitor:

$$C = \frac{Q}{v}. \tag{1}$$

If the voltage changes with time, then

$$i = C\frac{dv}{dt}, \tag{2}$$

where i is the current through the capacitor and $\frac{dv}{dt}$ is the first derivative of the voltage across the capacitor with respect to time.

To approximate the behavior of the capacitor as a discrete-time linear dynamical system, we rewrite Equation (2) as a difference equation:

$$i_k = C\frac{v_{C,k+1} - v_{C,k}}{t_{k+1} - t_k}, \tag{3}$$

where i_k is the current through the capacitor (to avoid confusion with the identity matrix) and $v_{C,k}$ is the voltage across the capacitor at time index k. We now define an integer index $k = t_k/h$ to represent *discrete* time steps with an interval (spacing) of h (sec). That is, $k = 1$ represents h sec., $k = 2$ represents $2h$ sec., etc.

$$\implies i_k = C\frac{v_{C,k+1} - v_{C,k}}{h} \tag{4}$$

$$\implies v_{C,k+1} = v_{C,k} + \frac{h}{C}i_k. \tag{5}$$

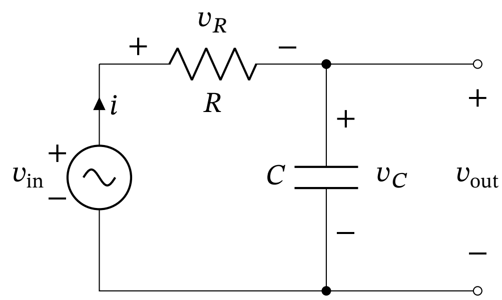


Figure 1: Resistor-capacitor (RC) circuit A

Consider RC circuit A in Figure 1. Kirchoff's voltage law (KVL, conservation of energy) tells us that the sum of all voltages around any loop must be zero. For this circuit,

$$v_{\text{in}} - v_R - v_C = 0, \tag{6}$$

where v_R and v_C are the voltages across the resistor and capacitor, respectively.

Further, using Ohm's law on the resistor implies

$$v_R = iR, \tag{7}$$

where i is the current through the resistor and capacitor. (We assume very little current flows through the load across the circuit's output terminals, which may or may not be valid depending on the situation.)

Putting these equations together, we have

$$v_{R,k} = v_{\text{in},k} - v_{C,k} \tag{8}$$

$$v_{C,k+1} = v_{C,k} + \frac{h}{C}\frac{v_{R,k}}{R} \tag{9}$$

$$\implies v_{C,k+1} = \left(1 - \frac{h}{RC}\right)v_{C,k} + \frac{h}{RC}v_{\text{in},k}. \tag{10}$$

For any input voltage vector v_{in} and initial condition $v_{C,0}$, the voltages across the resistor v_R and capacitor v_C can be calculated using Equations (8) and (10), respectively.

1.2 Tasks

1. Implement Equations (8) and (10) in MATLAB to simulate circuit A (Figure 1) with $R = 1$ kΩ and $C = 1$ µF. Simulate charging the capacitor using a step input: set $v_C = 0$ V at $t = 0$ and $v_{\text{in}} = 1$ V for $t > 0$. Choose a suitable h to model the charging process accurately.

   Plot v_{in} and v_C vs. time to show the charging of the capacitor. You should observe a charging curve similar to Figure 2.

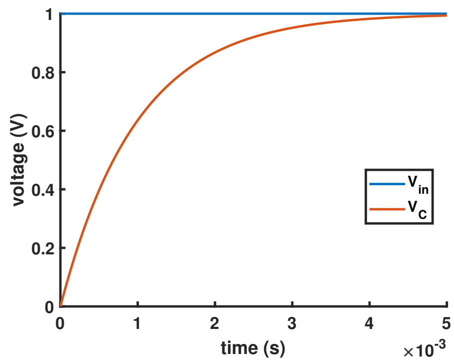


Figure 2: Voltage measured across a capacitor in response to a constant 1 V input

2. Now, run several versions of your simulation for various temporal sampling intervals h. As h gets larger or smaller, how does your simulation's prediction change? Why is this happening? Does the charging behavior of a "real" capacitor change as a function of your choice of h?

   Plot the predicted v_{out} using 1) an "accurate" choice of h and 2) an "inaccurate" choice of h and 3) the theoretical charging curve

   $$v_C(t) = 1 - \exp(-t/RC) \text{ [Volts]}. \tag{11}$$

   Discuss what happens for the "inaccurate" choice of h.

   Note: Be careful to compare the three curves using correct time axes.

3. Relative to the charging curve of the capacitor $v_C(t)$, how do you interpret the meaning of $\tau = RC$, called the RC time constant of the circuit?

2 Part 2: Model an RL circuit

2.1 Background

An inductor is an electrical component that stores potential energy in a magnetic field. Its inductance L quantifies how much magnetic flux Φ_B (the surface integral of the magnetic field through a cross-section of the inductor) is generated by a current i flowing through the inductor:

$$L = \frac{\Phi_B}{i}. \tag{12}$$

Using Faraday's law, we can relate the change in magnetic flux to the voltage across the inductor using

$$v = -L\frac{di}{dt}, \tag{13}$$

where $\frac{di}{dt}$ is the first derivative of the current through the inductor with respect to time. Here, the negative sign signifies that the inductor *opposes* the change in current.

However, we must be careful: If i increases because, for example, v_{in} increases, then the *voltage drop* v across the inductor must *increase* to *oppose* the change in current. Thus, we use a positive sign for the prefactor L and approximate the inductor as a discrete-time linear dynamical system by writing Equation (13) as a difference equation:

$$\begin{aligned} v_{L,k} &= L\frac{i_{k+1} - i_k}{t_{k+1} - t_k} && (14)\\ &= L\frac{i_{k+1} - i_k}{h} && (15)\\ \implies i_{k+1} &= i_k + \frac{h}{L}v_{L,k}, && (16) \end{aligned}$$

where $v_{L,k}$ is the voltage across the inductor and i_k is the current through the inductor at time index k. The integer index $k = t_k/h$ has the same meaning as in Section 1.

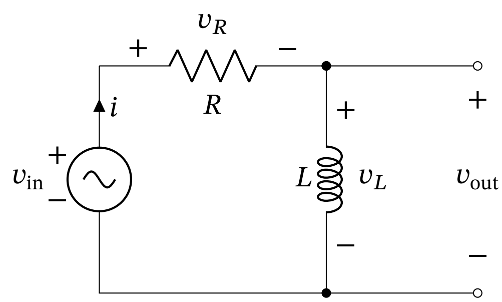


Figure 3: Resistor-inductor (RL) circuit B

Consider RL circuit B in Figure 3. For this circuit,

$$v_{\text{in}} - v_R - v_L = 0, \tag{17}$$

where v_R and v_L are the voltages across the resistor and inductor, respectively. Combining Equation (17) with Ohm's law (Equation (7)) results in

$$\begin{aligned} v_{L,k} &= v_{\text{in},k} - v_{R,k} && (18)\\ \implies i_{k+1} &= i_k + \frac{h}{L}(v_{\text{in},k} - Ri_k) && (19)\\ \implies i_{k+1} &= \left(1 - \frac{hR}{L}\right)i_k + \frac{h}{L}v_{\text{in},k}. && (20) \end{aligned}$$

For any input voltage vector v_{in} and initial condition i_0, the current within the inductor can be calculated using Equation (20). To calculate the voltage across the resistor and inductor, one can further apply Equations (7) and (18), respectively.

2.2 Tasks

1. Implement Equation (20) in MATLAB to simulate circuit B (Figure 3) with $R = 100\ \Omega$ and $L = 100$ mH. Simulate charging the inductor using a step input: set $i = 0$ A at $t = 0$ and $v_{\text{in}} = 1$ V for $t > 0$. Choose a suitable h to model the charging process accurately.

   Plot the voltage across the inductor v_L vs. time to show the charging of the inductor. You should observe a charging curve similar to Figure 4.

2. Capacitors and inductors exhibit complementary characteristics. Use your MATLAB simulations to demonstrate this phenomena. What is the steady-state voltage across the capacitor after it is fully-charged? How about that of the inductor?

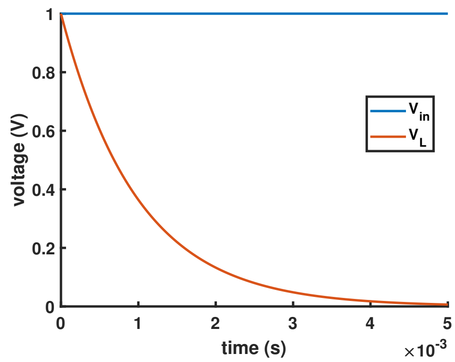


Figure 4: Voltage measured across an inductor in response to a constant 1 V input

Similarly, what is the steady-state current through the capacitor after it is fully-charged? How about that of the inductor?

Note: In your report, be sure to support your observations using data from your MATLAB simulations, not what you found in a textbook or read on the internet!

3 Part 3: Putting it all together in an RLC circuit

3.1 Background

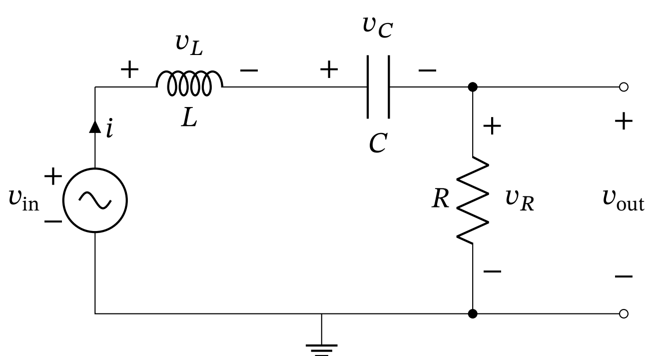


Figure 5: Resistor-inductor-capacitor (RLC) circuit C

Consider circuit C composed of a voltage source v_{in}, an inductor L, a capacitor C, and a resistor R in series as shown in Figure 5. We may model this circuit as a discrete-time linear dynamical system with a state vector $x_k = (v_{C,k}, i_k)^T \in \mathbb{R}^2$ and an input $u_k = v_{\text{in},k} \in \mathbb{R}^1$ such that

$$x_{k+1} = Ax_k + Bu_k. \tag{21}$$

Expressions for matrix A and vector B may be found in terms of the circuit parameters by utilizing Equations (5), (7) and (16) and KVL.

3.2 Tasks

Note: For this entire section, set the sampling interval $h = 1/(192\text{ kHz})$.

1. Derive mathematical expressions for the matrix A and vector B in Equation (21) in terms of the circuit components R, L, C, and v_{in} and the sampling interval h. Include this derivation in your report.

2. *Fun with oscillations.* Similar to Sections 1 and 2, simulate the response of circuit C to a step input: set $v_C = 0$ V and $i = 0$ A at $t = 0$ and $v_{\text{in}} = 1$ V for $t > t_0$.

   Systematically explore the values of R, L, and C. What do you observe? How does each component value affect the circuit response? Use plots of the output voltage of the circuit v_R vs. time to justify your answers.

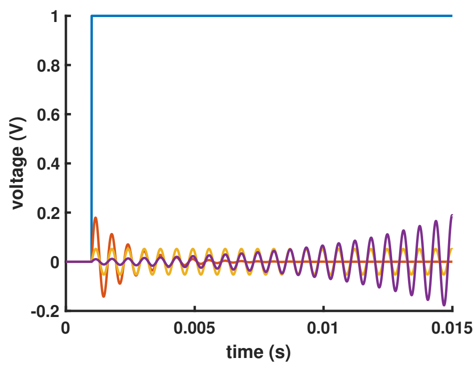


Figure 6: Various responses of an RLC circuit to a 1 V step input (blue). (Red) Ringing with a quick decay. (Yellow) Tuned oscillation with a very slow decay. (Purple) Unstable oscillation growing in amplitude.

   It is possible to tune this "oscillator" circuit to respond in significantly different ways (Figure 6). In one case, the circuit simply rings but decays quickly to zero. It is also possible to "tune" the circuit to ring for many cycles (thousands). In yet another case, the ringing amplitude rises out of control, as if in a positive feedback loop. *This final example is actually a numerical artifact caused by the discrete-time approximation of our simulation.*

   Tune your circuit to reproduce these responses, and state the component values needed to produce each condition in your report.

   Listen to the 3 different responses using the function `soundsc()`. Interpret what you hear in terms of what you see in the voltage vs. time graphs (e.g., in Figure 6). What does the fast oscillation in the graph sound like? What does the growing or decaying oscillation sound like?

3. *Sinusoidal response.* We now wish to understand how such circuits affect audio signals. We could of course build the circuit and attach them to speakers in the laboratory, but our linear algebraic model allows us to simulate the circuits and "listen" to their outputs in MATLAB.

   Explore sending sinusoidal voltages (Equation (22)) at various frequencies f from 10 Hz to 10 kHz to RLC circuit C (Figure 5), where $R = 100\ \Omega$, $L = 100$ mH, and $C = 0.1\ \mu$F.

$$v_{\text{in}}(t) = \sin(2\pi f t) \text{ [Volts]} \tag{22}$$

   For various frequencies f, do the output signals "look the same" as the input signals? (Note that the circuit responses in Figure 6 look nothing like the step input you sent to the circuit in the previous section.) How are the shapes and/or amplitudes different?

For what frequencies f is the output approximately the same amplitude as the input? And, for what frequencies is it much bigger? Or much smaller? Would you call this circuit a "lowpass," "bandpass," or "highpass" filter? Use your simulation data to justify your responses.

Listen to the output of the circuit using the function `playSound(y, Fs)` included in the case study materials. Interpret what you hear in terms of what you see in the voltage vs. time graphs. What do small or large amplitudes sound like? What do small, medium, and large frequencies sound like?

Hint: Systematically measuring the amplitude of the input and output sine waves as a function of f could be very helpful in understanding the circuit response.

4 Part 4: Competition

RLC circuit C (Figure 5) is extremely versatile, and your job for the competition is to tune it to accomplish a variety of functions.

4.1 Tasks

1. *A tuning fork.* Choose your favorite musical tone from the list of piano key frequencies. For example, symphony orchestras tune to A_4 (440 Hz). Tune your RLC oscillator to "ring" at your favorite frequency after receiving a short voltage/current pulse. Report the circuit component values you used in your design.

   Implement your circuit resonator as the function `myResonatorCircuit(Vsound,h)` stored in `myResonatorCircuit.m`.

   Describe the procedure you followed to tune your circuit.

   Demonstrate the ringing of your circuit. Comment on the ability of your circuit to function as a tuning fork. How strong of an input pulse (i.e., what voltage, what duration) is needed to make it ring for ~5 seconds?

2. *An audio sensor.* NASA recently landed Perseverance on Mars, the first rover to contain a microphone on board. The first audio captured from Mars is publicly available. Perseverance carried Ingenuity, the first drone and helicopter to fly completely autonomously on Mars.

   When Ingenuity is flying, its rotors create a sound centered around 84 Hz. Unfortunately, Martian winds can obscure this sound, especially when the helicopter is far away.

   Your job is to design an RLC circuit to "detect" when Ingenuity is flying. If the sound of the rotors is contained within v_{in}, then the circuit output v_{out} should contain a "clean" version of the rotor sound, removing wind or other noise from covering up the rotor sound.

   If there is no rotor sound within v_{in}, or if the rotors stop spinning, then the circuit should output "quiet" audio, i.e., $v_{\text{out}} \approx 0$.

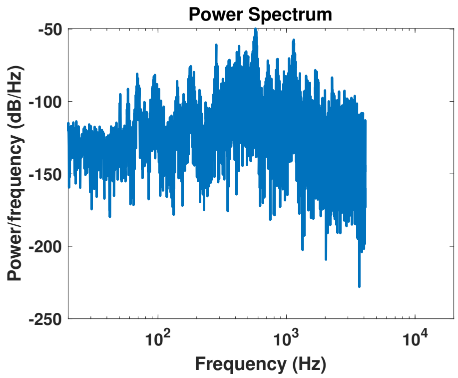
   Implement your circuit sensor as the function `mySensorCircuit(Vsound,h)` stored in `mySensorCircuit.m`.

   Describe the procedure you followed to design your circuit.

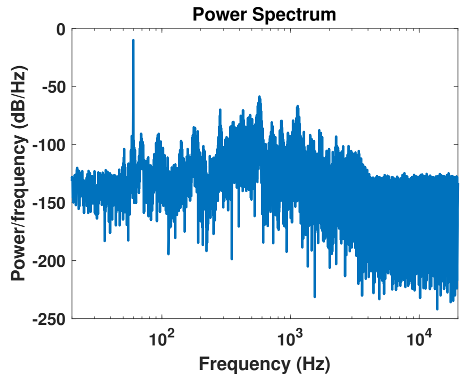
   Demonstrate your circuit "detecting" the rotor sound within the provided audio. Comment on its ability to filter out the wind noise so that the rotor sound can be heard.

   Note: The 84 Hz hum is extremely hard to hear if you are in a noisy room or if you are without good speakers/headphones. Watch the neat video here if you want a sample of the helicopter audio!

3. *A music filter.* Cheap or low-quality audio amplifiers can add noise to an audio recording that is distracting and undesired. Two examples are a white-noise "hiss" across all frequencies, which arises from electrons that randomly move in response to thermal fluctuations, and a 60 Hz "hum", which comes from the electrical grid itself (power lines and electrical wires within buildings) coupling inductively into the circuit.



(a) `handel.mat`



(b) `noisyhandel.mat`

Figure 7: Power spectra of two audio signals. Note the additional 60 Hz and high-frequency noise in `noisyhandel.mat`.

Our objective is to design an RLC filter to remove these noise sources while allowing the audio itself (e.g., tones from musical instruments and speech) to get through. The time-series vector v_{in} will represent the noisy music, which could be carried on an audio cable from a phone, laptop, or stereo system, and v_{out} represents the filtered audio signal from your circuit.

Design your circuit to remove noise while allowing the audio signal itself (e.g., tones from musical instruments and speech) to get through.

Implement your circuit filter as the function `myFilterCircuit(Vsound,h)` stored in `myFilterCircuit.m`.

Justify your design choices in your report. Comment on the ability of your circuit to reduce noise in the provided music in `noisyhandel.mat`.

4.2 Scoring

1. Scores will be tabulated for each of the 3 tasks. The 3 scores will be combined equally to represent a final competition score. No points will be awarded for code that contains errors.

2. *Tuning fork:* Points will be awarded for resonators that ring for the longest duration with the smallest energy input. Circuits that are unstable (i.e., violate conservation of energy) will be disqualified.

3. *Sensor:* Points will be awarded for circuits that provide a clean rotor sound when the helicopter is running, while also providing quiet audio when the helicopter is not running.

4. *Music filter:* Several noisy and noiseless audio signals will be tested. Points will be awarded for maximizing the signal-to-noise ratio, i.e., maximizing the transmission of the desired audio while minimizing the transmission of undesired noise.

5 Tips

- Use matrix and vector operations within your code where possible. You may choose to use ss() and lsim(), or you can directly use matrix multiplication and a `for` loop to implement the circuit.

- No code is provided for Parts 1-3.

- Skeleton code is only provided for the competition, consisting of

  1. `competitionTest.m` for testing your proposed circuit
  2. `myResonatorCircuit.m`, `mySensorCircuit.m`, and `myFilterCircuit.m` for implementing your circuits for each task of the competition
  3. `plotPowerSpectrum.m` for calculating the power spectrum of an audio signal
  4. `playSound.m` for playing an unscaled audio signal. Use `soundsc()` to have MATLAB automatically normalize the audio signal so that it is loud enough to hear.
  5. `convertAudio.m` for converting a digital music file (MP3 or similar) to a `.mat` file for use in this case study

  Sample audio signals are provided in `handel.mat`, `noisyhandel.mat`, and `MarsHelicopter_noisy.mat`.

  In order for these functions and files to work with each other, they must be saved in a common directory, and your MATLAB "current folder" should be set to this same directory.

- You may test your circuit using the example code provided in `competitionTest.m`. The code loads a voltage sequence `Vsound` from one of several `.mat` files, and uses `myResonatorCircuit.m`, `mySensorCircuit.m`, or `myFilterCircuit.m` to compute the output voltage sequence from your circuit. The `competitionTest.m` script will plot the power spectrum of the original sound (see examples in Figure 7) and filtered sound to show the effect of your filter. The script will also play the original and filtered audio signals one after the other through your computer speakers.

- Be sure to form a group with your partner on Canvas under the Case Study 3 group tab on the People page. This will ensure that you'll both get credit for the assignment!

6 What to turn in

1. Any MATLAB `.m` files you write or modify, including `myResonatorCircuit.m`, `mySensorCircuit.m`, and `myFilterCircuit.m` for the competition

2. PDFs or Word DOCs for each of the MATLAB files above. Use the `publish()` command similar to what you've produced for previous MATLAB homeworks.

3. A 4-5 page report that explains your analysis of your circuits and your design choices in the competition. *Use the provided Word/LaTeX template.* Make sure that your report clearly states/presents:

   (a) Answers to each question within the case study
   (b) Justification to your answers using calculations or plots from your MATLAB code when possible
   (c) Design choices that you've made

4. A signed version of the provided honor code

7 Rubric

- Correctness of MATLAB code - 40%
  - Simulation of RC, RL, and RLC circuits as discrete-time linear dynamical systems
  - Calculation of circuit outputs in response to various inputs: step input, sinusoidal input, and simulated audio
- Presentation - 20%
  - Plots are easy to read and interpret, with appropriate font sizes, line widths, axis labels, etc.
  - Report should be well-organized, concise, and clearly written.
- Programming style - 20%
- Study design - 20%
  - Rationale for circuit designs
  - Interpretation of circuit behaviors based upon data from model