The Creation and Application of RLC Circuits

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***Abstract*—The goal of this case study was to discover the uses of RLC circuits. This involved the usage of Kirchoff's Voltage Law, Ohm's Law, linear dynamical systems, and MATLAB programming to solve circuits and create solutions for audio problems. We determined precise numbers for frequency, inductance, and capacitance in a circuit to model the resonance of a tuning fork and filter audio. The insights from this case study can be used to remove white noise from audio recordings.**

I. INTRODUCTION

Understanding the functions of a circuit's individual components and the ways in which they interact is crucial to mastering electrical and systems engineering. This project is a great introduction to how resistors, inductors, and capacitors (abbreviated "R," "L," and "C," respectively) affect voltages across a circuit. In parts one and two, we demonstrate how a resistor functions in conjunction solely with a capacitor (part one) and solely with an inductor (part two), and how the voltage drops across the capacitor and inductor change over time. In part three, we combine all these components together, demonstrating how the capacitor and inductor interact by pushing against each other to create an oscillating voltage. In part four, we demonstrate some practical, everyday applications of RLC circuits: how they can be used to produce an oscillating output voltage at a certain frequency to produce a fixed note (like a tuning fork) and filter out background noises so that target audio can be heard more clearly.

II. METHODS

A. *Modeling an RC Circuit*

Part one of this case study involves the creation of an RC circuit, which contains a voltage source, a resistor, and a capacitor. For now, we assumed the voltage source was providing a constant voltage of 1 volt. According to Kirchoff's voltage law, the input voltage of the circuit subtracted by the voltage of the capacitor, subtracted by the voltage of the resistor equals 0 as shown in (1).

$$V_{in} - V_C - V_R = 0 \quad (1)$$

This equation when combined with the equation for capacitance allows for the derivation of the following linear dynamical system (2).

$$V_{C,k+1} = V_{C,k} + \frac{h}{RC} V_{in,k} \quad (2)$$

In this equation, R (resistance) is set to .001 ohms, C (capacitance) is set to .000001 Farads, and h is the change in time between each timestep. This h variable, when chosen correctly results in a plot showing the charging of the capacitor over time (Fig. 1). We found 10^{-6} to be a good h value and 10^{-3} to be a bad h-value.

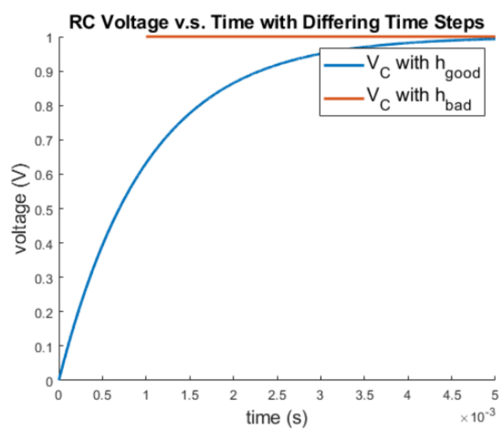

Fig. 1. Plot of voltage across a capacitor using a linear dynamical system with differing time steps.

Because h refers to the time difference between iterations of the linear dynamical system, the larger the value of h, the more data is not being accounted for by the system. This has the effect of fitting the entirety of the capacitor charging within a single iteration, making it seem as if the capacitor were fully charged instantly. Thus, the data for bad choices of h is not useful for examining how a capacitor charges over time, but it is not necessarily inaccurate for the choice of timestep.

Next, we compared the linear dynamical system with a viable choice of h to the theoretical equation for voltage cross a capacitor yielding the graph below (Fig. 2). From observation, the theoretical and simulated plots are identical.

$$V_C(t) = 1 - e^{-\frac{t}{RC}} \quad (3)$$

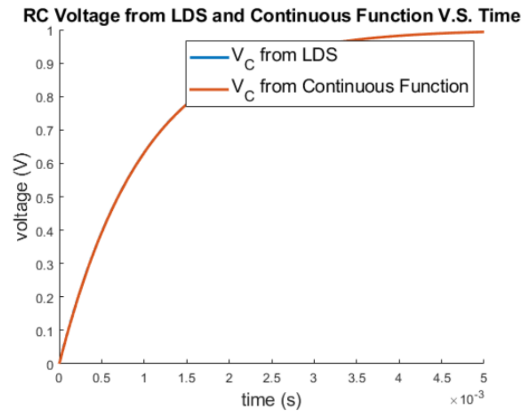

Fig. 2. Plot of voltage across a capacitor comparing the use of a linear dynamical system simulation and a differential equation.

When looking at the theoretical equation for voltage across a capacitor as it charges, we can rewrite it as shown below in (4) to better understand how quickly the capacitor charges. The quantity τ quantifies how long it takes to charge the capacitor (i.e., if τ=RC, the capacitor is 63.21% charged; if τ =2RC, the capacitor is 86.47% charged, etc.).

$$\tau = RC \Rightarrow V_C(t) = 1 - e^{-\frac{t}{\tau}} \quad (4)$$

Modeling an RL Circuit

Part two of this case study has us create an RL circuit, which is a circuit containing a voltage source, a resistor, and an inductor. The addition of an inductor has us apply Faraday's law to find the voltage across it as given below (5).

$$V_{L,k} = \frac{L\left(I_{k+1} - I_k\right)}{h} \quad (5)$$

The inductance of the circuit (L) was set as .1 Farads, the input voltage was a constant 1 volt, the resistance was set as 100 ohms, h was set as 10^{-6} seconds, and the current (I) will change depending on time. Equation (5) can be rewritten by Ohm's law to get the equation below which was used to simulate the linear dynamical system (6).

$$I_{k+1} = \left(1 - \frac{hR}{L}\right) I_k + \frac{h}{l} v_{in,k} \quad (6)$$

Once this is calculated, the voltage across the inductor can be calculated using Kirchoff's voltage law as shown below (7). The plot of this linear dynamical system is shown in Fig. 3.

$$V_L = V_{in,k} - I_{k+1} R \quad (7)$$

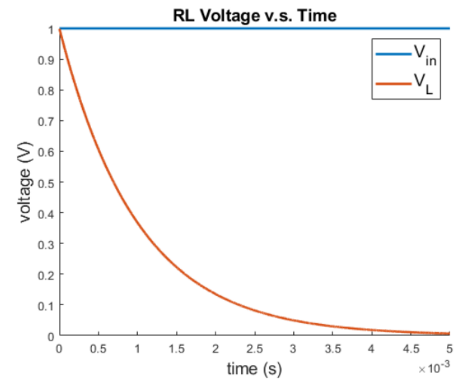

Fig. 3. Plot of voltage across an inductor in an RL circuit.

Comparing the data from Fig. 3 to Fig. 1 it appears that capacitors and inductors act like opposites. The steady state voltage across a capacitor after a long time is equal to the input voltage whereas the steady stage voltage across an inductor after a long time is equal to 0. Aditionally, the steady state current across a capacitor after a long time is equal to 0, whereas the steady state current across an inductor is equal to as if the inductor were a wire (steady state currents can be found by applying Ohm's law and Kirchoff's voltage law). Simply, in the short term, capacitors act like wires and inductors act like open circuits, and in the long term, capacitors act like open circuits and inductors act like wires.

Modeling an RLC Circuit

Part 3 of this case study asks us to create a circuit with a voltage source, resistor, capacitor, and inductor. First, we derived a linear dynamical system. For our approach, we started by wanting to create a linear dynamical system of the form below in (8).

$$\begin{pmatrix} V_{C,k+1} \\ I_{k+1} \end{pmatrix} = A \begin{pmatrix} V_{C,k} \\ I_k \end{pmatrix} + BV_{in} \quad (8)$$

In (8), the A matrix is a 2x2 matrix while B is a 2x1 vector. By combining equation 9 from the case study document and Ohm's law, we can create the following equation.

$$V_{C,k+1} = V_{C,k} + \frac{h}{C}\left(I_k\right) + V_{in} \quad (9)$$

Now for the formula for current, we can combine Kirchoff's voltage law and equation 16 from the case study document to get the equation below (9).

$$I_{k+1} = I_k + \frac{h}{L}\left(V_{in} - I_k R - V_{C,k}\right) \quad (10)$$

The above two equations form the groundwork for the A matrix. The input voltage is not added to the voltage of the capacitor, so the first element of the B vector is 0. Because the previous I is added to the new I, the second element of the B matrix is equal to h/L. The rest of the equation below comes from rewriting the equations above (without the input voltage parts) in the form we wanted by using matrix-vector multiplication.

$$\begin{pmatrix} V_{C,k+1} \\ I_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{h}{C} \\ -\frac{h}{L} & 1-\frac{hR}{L} \end{pmatrix} \begin{pmatrix} V_{C+k} \\ I_k \end{pmatrix} + V_{in} \begin{pmatrix} 0 \\ \frac{h}{L} \end{pmatrix} \quad (11)$$

Through experimentation, general trends for voltage as an effect of changing resistance, capacitance, and inductance can be observed. When lower values of resistance are introduced to the circuit, the waves begin at lower amplitudes and end at higher amplitudes. Higher values of resistance cause the waves to start at higher amplitudes and end at lower amplitudes as shown below in Fig. 4 when resistances are changed, but everything else is left constant. Listening to the sounds created by the three waves in Fig. 4, the growing and decaying waves sound like short bursts of a note, and the wave that neither grows nor decays holds a note for a long time.

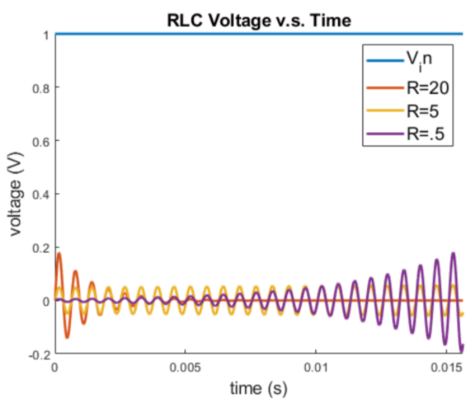

Fig. 4. Plot of voltage across an RLC circuit with changing resistances.

As inductance increases, the frequency of the waves decreases and the amplitudes approach zero more slowly as demonstrated by Fig. 5 below where we changed L values while keeping the rest of the circuit the same. Lower values of inductance increase frequency and start at high amplitudes, but those amplitudes quickly fall.

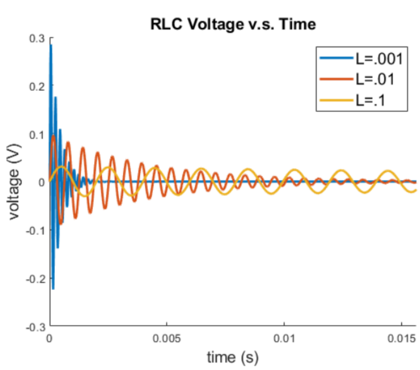

Fig. 5. Plot of voltage across an RLC circuit with changing inductance values.

Fig 6. below shows that increasing capacitance while leaving all other circuit elements constant increases frequency and makes the amplitudes approach zero more quickly whereas decreasing capacitance causes frequency to decrease and amplitudes to approach zero more slowly. This behavior is the exact opposite of the inductor, which is consistent with our prior findings.

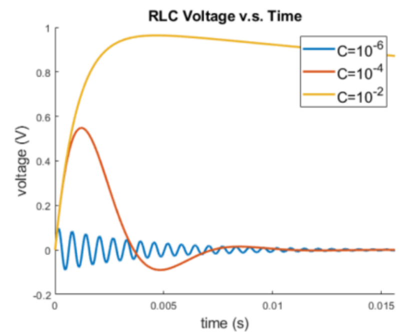

Fig. 6. Plot of voltage across an RLC circuit with changing capacitance values.

By experimenting with different frequencies, we can develop ways of filtering audio using RLC circuits. This process involves modifying the input voltage using a sinusoidal wave using the following formula. We chose the following values for our circuit components: $R = 100$, $L=.1$, $C=10^{-7}$. Additionally, we chose our h value to be 1/192000.

$$V_{in} = \sin(2\pi ft) \quad (12)$$

First, we tinkered with values of input frequency, creating the three plots below in Fig. 7. From listening to the waves' sounds, we observed that higher frequencies create higher notes and lower frequencies create lower notes. Additionally, increasing the amplitude of the waves increases the volume of the sound and decreasing the amplitude of the waves decreases the volume of the sound.

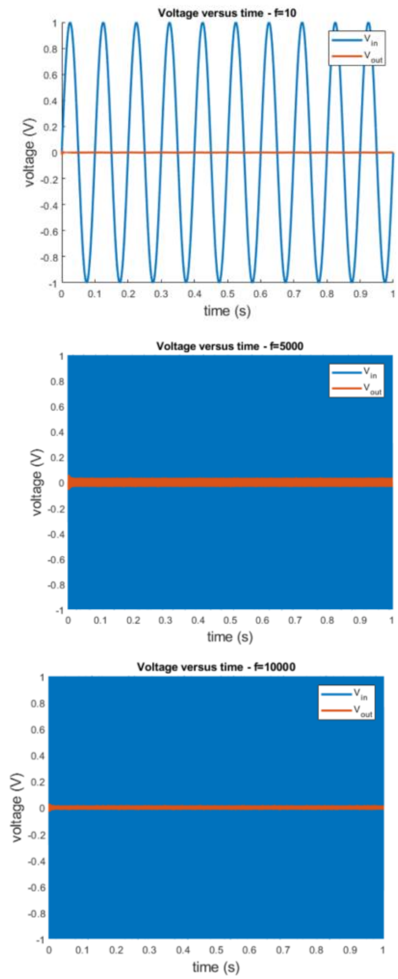

Fig. 7. Plot of voltage across an RLC circuit with input voltages of different frequencies.

Observing the data in Fig. 7, we find that the RLC circuit with the chosen component values is a bandpass filter since it accepts almost none of the wave when the frequency is 10 Hz or 10000 Hz, but accepts much more in between at 5000 Hz.

Applications of RLC Circuits

Part 4.1 of this case study involves tuning an RLC circuit to produce a voltage output that oscillates at a certain frequency (we chose 440 Hz) by tuning the RLC values in the circuit. To obtain our result, we used the equations given to us to derive an equation for frequency (f) in terms of our inductor and capacitor values (L and C, respectively). If the resistor and voltage source do not alter the resonant frequency since only the capacitor and inductor cause oscillations in voltage, we can turn the RLC circuit into an LC circuit. Applying Kirchoff's loop rule to this circuit and rewriting everything in terms of charge gives (11).

$$0 = V_C + V_L = \frac{Q(t)}{C} - L\frac{dI}{dt} = \frac{Q(t)}{C} - L\frac{d^2Q}{dt^2} \quad (11)$$

From there, we can solve the second order differential equation by assuming that the solution Q(t) is of the form $e^{\lambda t}$, which creates the characteristic equation and its solution as shown in (12) where i is the square root of -1.

$$0 = \frac{1}{C} - L\lambda^2 \Rightarrow \lambda = \frac{\pm i}{\sqrt{LC}} \quad (12)$$

Since we assumed the solution Q(t) was of the form $e^{\lambda t}$ we can plug in our lambda value and use Euler's identity to get the solution (13) where c_1 and c_2 are constant.

$$Q(t) = c_1 \cos\left(\frac{t}{\sqrt{LC}}\right) + c_2 \sin\left(\frac{t}{\sqrt{LC}}\right) \quad (13)$$

Observe that the angular frequency within the sin and cos in (z) are both $\frac{1}{\sqrt{LC}}$, which can be rewritten as done in (14) to find the resonant frequency of the RLC circuit.

$$\omega = 2\pi f = \frac{1}{\sqrt{LC}} \Rightarrow f = \frac{1}{2\pi\sqrt{LC}} \quad (14)$$

Since we only knew the frequency value and wanted to find values for L and C, we decided to maintain the ratio between L and C given to us in part 3, allowing us to set $L = C*10^{-6}$, thus allowing us to solve for L (which we got to be 0.361716 Henries) and C (which we got to be 0.361716 $*10^{-6}$ Farads). After we had solved for L and C, we then needed to find a resistance such that the amplitude of the oscillating output voltage (Vout) remained constant after the circuit was supplied an impulse voltage. Using a resistance value that was too high and would quickly cause the Vout oscillation to converge to 0; using too low of an R would cause the amplitude of the Vout oscillation to diverge. To do this, we implemented the same matrix multiplication from part three, with the minor change of adding a condition such that if the input voltage ended, the input voltage would be automatically set to 0 so that the "tuning fork" could continue to ring. With this, we then began to test different R values, plotting Vout as a function of time and adjusting R accordingly. After tinkering with the resistance, we eventually found that if we set R equal to 14.4 ohms, the Vout would continue to oscillate at 440 Hz for over a minute when given an impulse voltage (an input voltage of $1\text{x}10^{-6}$ V over a single time step followed by an indefinite 0 input voltage allowed the Vout to oscillate with a very small but consistent Vout for over a minute), and would produce a constant ringing as long as the amplitude was scaled so that it could be heard (which MATLAB's "soundsc" function automatically does).

Next, we designed an audio sensor to detect when Ingenuity is flying. We were told that Ingenuity's rotors create a sound centered around 84 Hz when it is flying, so we decided to create a bandpass filter that would only allow frequencies between 80 and 90 Hz. We want the center frequency of 84 Hz to be equal to the resonant frequency of the circuit (so the sensor most easily picks up the center frequency), giving us (15).

$$84 = \frac{1}{2\pi\sqrt{LC}} \tag{15}$$

We chose C to be 1 microfarad, giving us an L of 3.590 Henrys. Next, we needed to choose a value for R. To do this, we had to find the ratio between the center frequency and the difference in the highest and lowest frequencies that pass the filter. This ratio is the "quality factor" which measures the energy loss of the circuit and is defined by (16) [1]. This makes sense as a method to determine resistance since theoretically, all the energy loss occurring in an RLC circuit should come from the resistor converting voltage (electrical potential) into heat.

$$q = \frac{84}{90-80} = \sqrt{\frac{L}{CR^2}} \tag{16}$$

Plugging in all our results thus far we get R equal to 225.560 ohms. At this point, we have the values of all the RLC components needed to make the bandpass circuit, so we can model it as a linear dynamical system as we did the previous RLC circuits.

Finally, we designed an RLC circuit to filter out extra noise from the Hallelujah Chorus. First, we looked at the sheet music for the Hallelujah Chorus and found that the part of the song we were assigned to filter was the first 5 measures. Within those measures, the highest note is a D_5 and the lowest note is a D_3. A D_5 has a frequency of 590 Hz and D_3 has a frequency of 145 Hz, so we decided to make a bandpass filter that filters out frequencies that are not in between those 2 bounds [2]. From there, we found the center frequency of the range by taking the geometric mean of the bounds [1], giving us a center frequency of 292.489 Hz. Finally, the process for finding the values of capacitance, inductance, and resistance was the same as it was for designing the sensor, giving us a capacitance of 1 microfarad, an inductance of .296 Henries, and a resistance of 827.866 ohms.

III. Results and Discussion

For section 4.1, after much tinkering with the RLC values, we eventually found values that seemed to work perfectly to make the Vout oscillate at a constant amplitude for a long time. To test our function and know how to tinker our values to get closer to the ultimate values we used, we created a separate testing MATLAB script which ran the function. While we were able to derive an equation that can be used to tune an RLC circuit to any frequency as well as a good ratio between our L and C values (allowing for very little work to have to be done to be able to tune the L and C values our RLC circuit to tune it to any other frequency), we couldn't find an equation to derive the desired R value for this other circuit, forcing us to have to use guess and check to obtain the R value for our frequency, and to also have to use guess and check to find a new R value if we wanted to tune our circuit to a different frequency (since the R value would likely change). Another limitation of our project is that with the computer we used to test the RLC values, each time we tested new values, running the function took a very long time. Because of this, not only could we not test our function for very long time periods (one minute was our longest which took over night).

For section 4.2, we derived equations that gave results for the components of our RLC circuit such that when listening to the audio of Ingenuity, most of the wind sounds are cut out. Additionally, when observing the graph of the original versus the sensed audio (Fig. 8), the sensed audio preserves the general shape of the original audio while still cutting out undesirable frequencies.

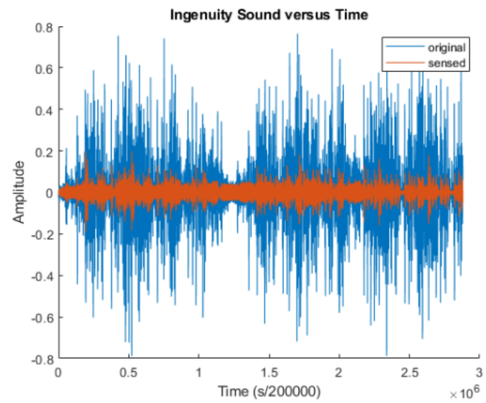

Fig. 8. Amplitude versus time of Ingenuity, comparing the original audio to the audio the sensor picked up.

For section 4.3, the filtered audio cuts out much of the white noise. Unfortunately, the filtered music sounds strangely dull and tinny. Additionally, we found that when removing the filtered parts of the music from the original audio file, a significant amount of music (in addition to white noise) was contained in the original file that was not contained in the filtered file. This is demonstrated via the original audio vector minus the filtered audio vector still containing a large amount of music. This error comes from the constructive interference of sound waves creating higher frequencies than those that are sung by the performers. This phenomenon known as overtone frequently occurs when all members performing are in tune, allowing for perfect constructive interference. Thus, when we cut out all frequencies above 590 Hz, we also cut out the overtones, losing some of the music and causing what remains to sound empty.

IV. Conclusion

Although the results from this case study were by no means perfect, they demonstrated the extraordinary versatility of RLC circuits to create and filter out sounds. The tuning fork can hold an A_4 pitch for over one minute, the sensor can detect when Ingenuity is flying by accepting frequencies between 80 and 90 Hz, and the audio filter can remove frequencies that are not within the note range of a song. Shortcomings not previously mentioned include ignoring the existence of displacement current in capacitors, assuming the circuit's wires have a negligible resistance, and the use of a linear dynamical system to model a circuit rather than a differential equation, which can create results that violate conservation of energy as shown in Fig. 4. Further related projects may include audio filtering that includes overtones, creating RLC filters with parallel circuits rather than series circuits, and more thoroughly researching the connection between voltage waves and sound waves.

REFERENCES

[1] James W. Nilsson and Susan Riedel, *Electric Circuits (Eighth Edition)*, 2008.

[2] https://pages.mtu.edu/~suits/notefreqs.html, "Frequencies for equal-tempered scale, $A_4 = 440$ Hz".