

# **Vector Calculus – Honors**

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# 1 Chapter 1

## 1.1 Lecture 1–August 22, 2022

Riemann integrals deal with functions that are basically continuous. You should use the notation  $(x, y, z)$  or  $x\hat{i} + y\hat{j} + z\hat{k}$

There are several coordinates: Cartesian, cylindrical, and spherical.

Spherical coordinates are given by  $(r, \theta, \phi)$

### 1 Example

Let  $f$  be a continuous function. Suppose  $f(x, y, z) = g(\sqrt{x^2 + y^2 + z^2})$ .

Let 1.  $f(x) = g(\sqrt{x^2 + y^2 + z^2})$  and 2.  $f(x, y, z) = h_1(|x|)h_2(|y|) + h_3(|z|)$

how many such functions satisfy this?

### 2 Definition

Some useful integrals

- Continuous:  $dx dy dz$
- Cylindrical:  $r dr d\theta$
- Spherical:  $r^2 \sin \theta dr d\theta d\phi$

### 3 Definition (Vectors)

Cross product:  $\vec{x} \wedge \vec{y} = -\vec{y} \wedge \vec{x}$  is a vector operation

Dot product:  $\vec{x} \cdot \vec{y} = \sum x_i y_i$  is a scalar operation

## 1.2 Lecture 2–August 23, 2022

### 4 Definition (Coordinate Systems)

$\vec{v} = (x, y, z) \rightarrow \vec{x} = (x_1, x_2, x_3)$

Sometimes will not include  $\vec{\phantom{x}}$  symbol—we want to think more abstractly in order to build to higher concepts. Spherical coordinates:  $(r, \theta, \phi)$ , and  $\theta$  is always the polar angle (from  $z$ -axis).

Orientation is the order of  $(x, y, z)$  which comes into play with change of variables.

### 5 Remark (Property of the determinant)

The determinant is always + or –

There are several volume form differentials:

$$dx dy dz = r^2 dr \sin \theta d\theta d\phi$$

and note that  $\sin \theta$  is always positive

## 6 Remark (Normal model)

Normal distribution pdf:

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

With  $\mu = 0$  and  $\sigma = 1$  The error function is the simplest example of a function that is “not integrable in elementary terms”

$$2 \int_0^\infty e^{-x^2} dx = \sqrt{\pi}$$

There are connections with physics i.e. the uncertainty principle and the quantum mechanics harmonic oscillator.

$$\begin{aligned} A &= \int_{-\infty}^{\infty} e^{-x^2} dx \\ A^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dy dx \\ &= \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \int_\infty^0 e^{-u} du d\theta \\ &= \frac{1}{2} \int_0^{2\pi} 1 d\theta \\ &= \pi \\ \boxed{A &= \sqrt{\pi}} \end{aligned}$$

Differentials tell you how to compute an integral.

1. The integral is linear (also the derivative)

$$\int (af + bg) dm = a \int f dm + b \int g dm$$

2. For a non-negative function  $f(x) \geq 0$  any way that you can calculate a finite value for the integral gives you the “correct answer.”
3. Dilation:

$$\int_0^\infty f(ax) dx = \frac{1}{a} \int_0^\infty f(x) dx$$

## 7 Remark (For proving estimates for the dot or scalar product)

Estimate:  $|x \cdot y| = |\sum x_i y_i| \leq ||\vec{x}|| ||\vec{y}||$ ,  $x = (x_1, x_2, x_3)$ ,  $y = (y_1, y_2, y_3)$

3 simple arguments:

1. Euclidian geometry
2. Arithmetic
3. Adding a variable  $\leftarrow$  the best way and expands view to another parameter

Properties of vectors: vector products (may be a scalar  $\vec{x} \cdot \vec{y}$  or a vector  $\vec{x} \wedge \vec{y}$ ) and representation of data in terms of partial derivatives.