

Discrete Mathematics–Honors

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1 Chapter 1

1.1 Predicate Logic

1.1.1 Lecture 1–August 23, 2022

There are three basic logical connectives: **and**, **or**, **not** which are denoted by \wedge , \vee , and \neg respectively. The negation of a proposition p , written $\neg p$, is true if p is false and false if p is true.

1 Example

“Less than 80 students are enrolled in CS311H” is a proposition. The negation of this is at least 80 students are in CS311H

Conjunction of two propositions p and q is written $p \wedge q$

2 Example

The conjunction of p = “It is Tuesday” and q = “it is morning” is $p \wedge q$ = “It is Tuesday and it is morning”

- Disjunction is written $p \vee q$ and the disjunction between $p \vee q$ for p = “It is Tuesday” and q = “it is morning” is $p \vee q$ = “It is Tuesday or it is morning”
- If your formula has n variables then your truth table has $n + 1$ columns because you have n variables and one column for the truth value of the formula.
- The number of rows is given by the formula 2^n
- Other connectives: exclusive or \oplus , implication \rightarrow , biconditional \leftrightarrow

1.1.2 Lecture 2–August 25, 2022

Let p = “I major in CS”, q = “I will find a good job”, r = “I can program”

- “I will not find a good job unless I major in CS or I can program”: $(\neg p \wedge \neg r) \rightarrow \neg q$
- “I will not find a good job unless I major in CS and I can program”: $(\neg p \vee \neg r) \rightarrow \neg q$
- The **inverse** of an implication $p \rightarrow q$ is $\neg p \rightarrow \neg q$. Therefore, “If I’m a CS major then I can program” has an inverse of “If I am not a CS Major then I’m not able to program.”
- The **converse** of an implication $p \rightarrow q$ is $q \rightarrow p$.

3 Definition (Contrapositive)

The contrapositive of an implication of $p \rightarrow q$ is $\neg q \rightarrow \neg p$

The contrapositive of “if CS major then I can program” is “if I can’t program, then I’m not a CS major”

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

A converse and it's inverse are always the same.

4 **Definition** (Biconditionals)

$$p \leftrightarrow q = p \rightarrow q \wedge q \rightarrow p = \neg(p \oplus q)$$

5 **Example** (Operator precedence)

Given a formula $p \wedge q \vee r$ do we parse this as $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$?

1. Negation \neg has the highest precedence
2. Conjunction (\wedge) has the next highest precedence
3. Disjunction (\vee) has the next highest precedence
4. Implication (\rightarrow) has the next highest precedence
5. Biconditional (\leftrightarrow) has the lowest precedence
6. Make sure to explicitly use parentheses for \oplus

1.2 Validity and Satisfiability

Validity and satisfiability

- The truth value depends on truth assignments to variables
- Example: $\neg p$ evaluates to true under the assignment $p = F$ and to false under $p = T$
- Some formulas evaluate to true for all assignments—these are called **tautologies** or **valid formulas**
- Some formulas evaluate to false for all assignments—these are called **contradictions** or **unsatisfiable formulas**

6 **Definition** (Interpretation)

An interpretation I for a formula F is a mapping from each propositional value to exactly one truth value.

$$I : \{p \mapsto \text{true}, q \mapsto \text{false}, \dots\}$$

Each interpretation corresponds to one row in the truth table so there are 2^n interpretations for a formula with n variables.

If the formula is true under interpretation I then we write $I \models F$ and if the formula is false then we write $I \not\models F$.

Theorem: $I \models F$ if and only if $I \not\models \neg F$.

7 **Example**

Consider the formula $F : p \wedge q \rightarrow \neg p \wedge \neg q$

Let I_1 be the interpretation such that $[p \mapsto \text{true}, q \mapsto \text{true}]$

What does F evaluate to under I_1 ? Answer: true

8 **Example**

Let F_1 and F_2 be two propositional formulas. Suppose F_1 is true under I . Then, $F_2 \neg F_1$ evaluates to false under I (the “and” shortcuts and forces the whole equation to be false).

Satisfiability, Validity

- F is **satisfiable** iff there exists interpretation $I | I \models F$
- F is **valid** iff for all interpretations $I, I \models F$

- F is **unsatisfiable** iff for all interpretations $I, I \not\models F$
- F is **contingent** if it is satisfiable, but not valid.

9 **Example** (Are the following statements true or false?)

- If a formula is valid, then it is also satisfiable? True. All interpretations are satisfiable.
- If a formula is satisfiable, then its negation is unsatisfiable. False.
- If F_1 and F_2 are satisfiable, then $F_1 \wedge F_2$ is also satisfiable. False.
- If F_1 and F_2 are satisfiable, then $F_1 \vee F_2$ is also satisfiable. True.

10 **Theorem** (Duality Between Validity and Unsatisfiability)

F is valid iff $\neg F$ is unsatisfiable.

Proof. Definition: F is valid iff for all interpretations $I, I \models F$

Theorem: $I \models F \iff I \not\models \neg F$

TODO: it's trivial so maybe later. □

Question: How can we prove that a propositional formula is a tautology is true?

Answer: We can use the **truth table method** and prove that the formula is true for all possible truth assignments.

11 **Example** (Tautology)

$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$ is a tautology.

$(p \wedge q) \vee \neg p$ is not a tautology.