Discrete Mathematics–Honors

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Contents

1 Logic and Sets

1.1 Lecture –August 23, 2022

1.1.1 Predicate Logic

There are three basic logical connectives: **and**, **or**, **not** which are denoted by $\wedge$, $\vee$, and $\neg$ respectively. The negation of a proposition p, written $\neg p$, is true if p is false and false if p is true.

1 **Example**
"Less than 80 students are enrolled in CS311H" is a proposition. The negation of this is at least 80 students are in CS311H

Conjunction of two propositions p and q is written $p \wedge q$

2 **Example**
The conjunction of p = "It is Tuesday" and q = "it is morning" is $p \wedge q$ = "It is Tuesday and it is morning"

- Disjunction is written $p \vee q$ and the disjunction between $p \vee q$ for p = "It is Tuesday" and q = "it is morning" is $p \vee q$ = "It is Tuesday or it is morning"
- If your formula has n variables then your truth table has $n + 1$ columns because you have n variables and one column for the truth value of the formula.
- The number of rows is given by the formula 2^n
- Other connectives: exclusive or $\oplus$, implication $\longrightarrow$, biconditional $\leftrightarrow$

1.2 Lecture–August 25, 2022

Let p = "I major in CS", q = "I will find a good job", r = "I can program"

- "I will not find a good job unless I major in CS or I can program": $(\neg p \wedge \neg r) \longrightarrow \neg q$
- "I will not find a good job unless I major in CS and I can program": $(\neg p \vee \neg r) \longrightarrow \neg q$
- The **inverse** of an implication $p \longrightarrow q$ is $\neg p \longrightarrow \neg q$. Therefore, "If I'm a CS major then I can program" has an inverse of "If I am not a CS Major then I'm not able to program."
- The **converse** of an implication $p \longrightarrow q$ is $q \longrightarrow p$.

3 **Definition** (Contrapositive)
The contrapositive of an implication of $p \longrightarrow q$ is $\neg q \longrightarrow \neg p$
The contrapositive of "if CS major then I can program" is "if I can't program, then I'm not a CS major"

p	q	$p \to q$	$\neg q \to \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

A converse and it's inverse are always the same.

4 **Definition** (Biconditionals)

$p \leftrightarrow q = p \to q \wedge q \to p = \neg(p \oplus q)$

5 **Example** (Operator precedence)

Given a formula $p \wedge q \vee r$ do we parse this as $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$?

1. Negation $\neg$ has the highest precedence
2. Conjunction ($\wedge$) has the next highest precedence
3. Disjunction ($\vee$) has the next highest precedence
4. Implication ($\to$) has the next highest precedence
5. Biconditional ($\leftrightarrow$) has the lowest precedence
6. Make sure to explicitly use parentheses for $\oplus$

1.2.1 Validity and Satisfiability

Validity and satisfiability

- The truth value depends on truth assigments to variables
- Example: $\neg p$ evaluates to true under the assignment $p = F$ and to false under $p = T$
- Some formulas evaluate to true for all assignments–these are called **tautologies** or **valid formulas**
- Some formulas evaluate to false for all assignments–these are called **contradictions** or **unsatisfiable formulas**

6 **Definition** (Interpretation)

An interpretation I for a formula F is a mapping from each propositional value to exactly one truth value.

$$I : \{p \mapsto \text{true}, q \mapsto \text{false}, \ldots, \}$$

Each interpretation corresponds to one row in the truth table so there are 2^n interpretations for a formula with n variables.

If the formula is true under interpretation I then we write $I \models F$ and if the formula is false then we write $I \not\models F$.

Theorem: $I \models F$ if and only if $I \not\models \neg F$.

7 **Example**

Consider the formula $F : p \wedge q \to \neg p \wedge \neg q$

Let I_1 be the interpretation such that $[p \mapsto \text{true}, q \mapsto \text{true}]$

What does F evaluate to under I_1? Answer: true

8 **Example**

Let F_1 and F_2 be two propositional formulas. Suppose F_1 is true under I. Then, $F_2 \neg F_1$ evaluates to false under I (the "and" shortcuts and forces the whole equation to be false).

Satisfiability, Validy

- F is **satisfiable** iff there exists interpretation $I | I \models F$
- F is **valid** iff for all interpretations $I, I \models F$
- F is **unsatisfiable** iff for all interpretations $I, I \not\models F$
- F is **contingent** if it is satisfiable, but not valid.

9 **Example** (Are the following statements true of false?)

- If a formula is valid, then it is also satisfiable? True. All interpretations are satisfiable.
- If a formula is satisfiable, then its negation is unsatisfiable. False.
- If F_1 and F_2 are satisfiable, then $F_1 \wedge F_2$ is also satisfiable. False.
- If F_1 and F_2 are satisfiable, then $F_1 \vee F_2$ is also satisfiable. True.

10 **Theorem** (Duality Between Validity and Unsatisfiability)

F is valid iff $\neg F$ is unsatisfiable.

Proof. Definition: F is valid iff for all interpretations $I, I \models F$

Theorem: $I \models F \leftrightarrow I \not\models \neg F$

This is very easy to prove: just map all outputs of F to true. □

Question: How can we prove that a propositional formula is a tautology is true?

Answer: We can use the **truth table method** and prove that the formula is true for all possible truth assignments.

11 **Example** (Tautology)

$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$ is a tautology.

$(p \wedge q) \vee \neg p$ is not a tautology.

1.3 Lecture–August 30, 2022

Implication: Formula F_1 implies F_2 (written $F_1 \implies F_2$) iff $\forall I, I \models F_1 \rightarrow F_2$

12 **Example** (Implication Removal)

Is $(p \wedge q) \rightarrow p$ true? False. Let $p = F, q = T$

p	q	$p \rightarrow q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

13 **Definition** (Implication Removal)

$p \rightarrow q$ is equivalent to $\neg p \vee q$

1.3.1 Important equivalences

- Law of double negation: $\neg\neg p \equiv p$

- Identity laws: $p \wedge T \equiv p$, $p \vee F \equiv F$
- Domination Laws: $p \vee T \equiv T$, $p \wedge F \equiv p$
- Idempotent Laws: $p \wedge p \equiv p$, $p \vee p \equiv p$
- Negation Laws: $p \wedge \neg p \equiv F$, $p \vee \neg p \equiv T$

14 **Note** (Commutativity and Distributivity Laws)

- Commutative Laws: $p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$
- Distributivity Law 1: $(p \vee (q \wedge r)) \equiv ((p \vee q) \wedge (p \vee r))$
- Distributivity Law 2: $(p \wedge (q \vee r)) \equiv ((p \wedge q) \vee (p \wedge r))$
- Associativity Laws:

$$p \vee (q \vee r) \equiv (p \vee q) \vee r$$

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$

- Absorption 1: $p \wedge (p \vee q) \equiv p$
- Absorption 2: $p \vee (p \wedge q) \equiv p$

15 **Definition** (De Morgan's Laws)

Let a = "John took CS311" and b = "John took CS312". What does $\neg(a \wedge b)$ mean? It means "John did not take both CS311 and CS312". Therefore, John didn't take either CS311 or CS312.

$$\neg(a \wedge b) \equiv \neg a \vee \neg b$$

16 **Example** (Prove $\neg(p \wedge (\neg p \wedge q)) \equiv \neg p \wedge \neg q$)

p	$\neg p$	q	$p \wedge (\neg p \wedge q)$	$\neg(p \wedge (\neg p \wedge q))$
T	F	T	T	F
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T

17 **Example**

If Jill carries an umbrella, it is raining. Jill is not carrying an umbrella. Therefore, it is not raining.

$$((u \rightarrow r) \wedge (\neg u)) \rightarrow \neg r$$

This can be counter-modeled with r = true, u = false.

1.3.2 First Order Logic

- The building blocks of propositional logic were propositions
- In first-ordre logic there are three kinds of basic building blocks: constants, variables, predicates.
- Constants: refer to specific objects
- Examples: George, 6, Austin, CS311, …
- If a universe of discourse is cities in Texas, x can represent Houston, Houston, etc.
- **Predicates** describe properties of objects or relationships between objects.
- A predicate $P(c)$ is true or false depending on whether property P holds for c.
- The truth value of `even(2)` = true

- Another example: Suppose $Q(x, y)$ denotes $x = y + 3$ what is the value of $Q(3, 0)$? true

1.4 Lecture–September 1, 2022

- In propositional logic, the truth value depends on a truth assignment
- In FOL, truth depends on interpretation over some domain D
- Universe of discourse (domain) + what elements in the domain the variables map to

18 **Example** (Semantics of First-Order Logic)

Consider a FOL formula $\neg P(x)$

$$D = \{A, B\}, P(A) = \text{true}, P(B) = \text{false}, x = A$$

This is a falsifying interpretation

19 **Example**

Consider I over domain $D = \{1, 2\}$

- $P(1, 1) = P(1, 2) = \text{true}, P(2, 1) = P(2, 2) = \text{false}$
- $Q(1) = \text{false}, Q(2) = \text{true}$
- $x = 1, y = 2$
- What is $P(x, y) \land Q(y)$ under I? True.
- What is truth value of $P(y, x) \to Q(y)$ under I? True.
- Waht is truth value of $P(x, y) \to Q(x)$ under I? False.

1.4.1 Quantifiers

- Real power of first-order logic over propositional logic: quantifiers.
- There are two quantifiers in first-order logic:
  1. Universal quantifier (for **all** objects): $\forall x P(x)$
  2. Existential quantifier (for **some** object): $\exists x P(x)$

20 **Example**

Let $D = \{a, b\}, P(a) = \text{true}, P(b) = \text{false}$ then $\forall x.P(x)$ is false.

21 **Example**

Consider $D = \mathbb{R}$ and $P(x) = x^2 \geq x$ then $\forall x.P(x)$ is false.

- In first-order logic, domain is required to be **non-empty**.

22 **Example**

Consider the domain of reals and predicate $P(x)$ with interpretation $x < 0$. Then, $\exists x.P(x)$ is true.

- $\forall x.P(x)$ is true iff $P(o_1) \land P(o_2) \land \ldots \land P(o_n)$ is true
- $\exists x.P(x)$ is true iff $P(o_1) \lor P(o_2) \lor \ldots \lor P(o_n)$ is true

$\exists x.(\text{even}(x) \wedge \text{gt}(x, 100))$ is a valid formula in FOL.

23 **Example** (What is the truth value of the following formulas?)

- $\forall x.(even(x) \rightarrow div4(x))$ False. $x = 2$ is a counter-model.
- $\exists x.(\neg div4(x) \wedge even(x))$ True.
- $\exists x.(\neg div4(x) \rightarrow even(x))$ True.

24 **Example** (Translating English into formulas)

Assuming $\texttt{freshman}(x)$ means "x is a freshman" and $\texttt{inCS311}(x)$ to be x is taking CS311, then "someone in CS311 is a freshman" is $\exists x.(\texttt{freshman}(x) \wedge \texttt{inCS311}(x))$.

No one in CS311 is a freshman: $\forall x.(freshman(x) \rightarrow \neg inCS311(x))$

Everyone taking CS311 are freshmen: $\forall x.(inCS311(x) \rightarrow freshman(x))$

All freshmen take CS311: $\forall x.(freshman(x) \rightarrow inCS311(x))$

1.4.2 DeMorgan's Laws for Propositional Logic

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
$$\neg \forall x.P(x) \equiv \exists x.\neg P(x)$$
$$\neg \exists x.P(x) \equiv \forall x.\neg P(x)$$

25 **Example**

We can change $\neg \exists x.(inCS311(x) \wedge freshman(x))$ to $\forall x.(\neg inCS311(x) \vee \neg freshman(x))$ which is equivalent to $\forall x.(inCS311(x) \rightarrow \neg freshman(x))$.

1.4.3 Nested Quantifiers

- Sometimes may be necessary to use multiple quantifiers
- For example, can't express "EEverybody loves someone" using a single quantifier.
- Suppose predicate $L(x, y)$ means "x loves y".
- What does $\forall x.\exists y.L(x, y)$ mean? "Everybody loves someone"
- What does $\exists y.\forall x.L(x, y)$ mean? "There is someone who is loved by everybody"

26 **Example** (More Nested Quantifier Examples)

- "Someone loves everyone" $\exists x.\forall y.L(x, y)$
- "There is someone who doesn't love anyone' $\exists x.\forall y.\neg L(x, y)$
- "There is someone who is not loved by anyone" $\exists x.\forall y.\neg L(y, x)$
- "Everyone loves everyone" $\forall x.\forall y.L(x, y)$
- "Someone doesn't love themselves": $\exists x.\neg L(x, x)$

1.5 Lecture–September 6, 2022

27 **Example**

- Every UT student has a friend: $\forall x.(atUT(x) \land student(x) \rightarrow \exists y.friends(x, y))$
- $\exists x.(atUT(x) \land student(x)) \land \forall y.\neg friends(x, y)$
- $\forall x \forall y(atUT(x) \land student(x) \land atUT(y) \land student(y)) \rightarrow friends(x, y))$

1.5.1 Satisfiability and validity in FOL

- The concepts of satisfiability validty also important in FOL
- FOL F is satisfiable if there exists some domain and some interpretation such that F is true.
- Example: Prove that $\forall x.(P(x) \rightarrow Q(x))$ is satisfiable. Solution: Let $P(x)$ be false. Let the domain $D = \{x\}$
- Example: Prove that $\forall x.(P(x) \rightarrow Q(x))$ is satisfiable. Solution: Let $P(x)$ be true, let $Q(x)$ be false. Let the domain $D = \{x\}$

1.5.2 Equivalence

- Two formulas F_1 and F_2 are equivalent iff $F_1 \leftrightarrow F_2$ is valid.
- We could prove equivalence using truth tables but not possible in FOL.
- However, we can still use known equivalences to rewrite one as the other.

28 **Example**

Prove that

$$\neg(\forall x.(P(x) \rightarrow Q(x))) \equiv \exists x.(P(x) \land \neg Q(x))$$

1.5.3 Rules of Inference

- We can prove validity in FOL by using **proof rules**
- Proof rules are written as **rules of inference**
- An example inference rule:

$$\begin{array}{c} F_1 \\ F_2 \\ \hline \therefore F_1 \land F_2 \end{array}$$

Modus Ponens

The most basic inference rule is modus ponens:

$$\begin{array}{c} F_1 \\ F_1 \rightarrow F_2 \\ \hline \therefore F_2 \end{array}$$

- Modus ponens applicable to both propositional logic and first-order logic.

Modus Tollens

- Second important inference rule is **modus tollens**:

$$\begin{array}{c} F_1 \rightarrow F_2 \\ \neg F_2 \\ \hline \therefore \neg F_1 \end{array}$$

Hypothetical Syllogism

Implication is transitive.

$$\frac{\begin{array}{c} F_1 \to F_2 \\ F_2 \to F_3 \end{array}}{\therefore F_1 \to F_3}$$

Or Introduction

$$\frac{F_1}{\therefore F_1 \lor F_2}$$

Or Elimination

$$\frac{\begin{array}{c} F_1 \lor F_2 \\ \neg F_2 \end{array}}{\therefore F_1}$$

And Introduction

$$\frac{\begin{array}{c} F_1 \\ F_2 \end{array}}{\therefore F_1 \land F_2}$$

Resolution

$$\frac{\begin{array}{c} F_1 \lor F_2 \\ \neg F_1 \lor \neg F_3 \end{array}}{\therefore F_2 \lor F_3}$$

Proof: ϕ_1 must be either true or false. If ϕ_1 is true, then ϕ_3 must be true. If ϕ_1 is false then ϕ_2 must be true. Therefore either ϕ_2 or ϕ_3 must be true.

29 **Example**

Assume the following:

S, C, L, H

$$\neg S \land C$$
$$L \to S$$
$$\neg L \to H$$
$$H \to back$$

We know that $\neg S$ is true, so S is false. Therefore, for $L \to S$ to be true, L must be false. In order for $\neg L \to H$ to be true, H must be true. Since H is true we know we must be back by sunset because that's the only way to make the last expression true.

1.6 Lecture–September 8, 2022

- Generalization and the other one is called instantiation

1.6.1 Universal Instantiation

- If we know that something is true for all members of a group we can conclude is also true for a specific member of this group.
- This idea is called **universal instantiation**

$$\frac{\forall x.(F(x))}{F(a)}$$

30 **Example**

Consider predicates $man(X)$ and $mortal(x)$ and the hypotheses:

- All men are mortal: $\forall x.(man(x \to mortal(x)))$
- Socrates is a man: $man(socrates)$
- Prove mortal(Socrates)
- $$man(socrates) \to mortal(socrates)$$
  $$mortal(socrates)(2, 3, modus\ ponens)$$

1.6.2 Universal Generalization

- Prove a claim for an **arbitrary** element in the domain.
- Since we've made no assupmtions proof should apply to all elements in the domain.
- The correct reasoning is captured by **universal generalization**
- "arbitrary" means an objects introduced through universal instantiation.

$$\frac{P(c)\text{for arbitrary c}}{\forall x.P(x)}$$

31 **Example**

Prove $\forall x.Q(x)$ from the hypothesis:

1. $\forall x.(P(x) \to Q(x))$
2. $\forall x.P(x)$
3. $P(a)$ (2, U-inst)
4. $P(a) \to Q(a)$ (1, U-inst)
5. $Q(a)$ (3, 4, MP)
6. $\forall x.Q(x)$ (5, U-gen)

Caveats about universal generalization

- When using universal generalization need to ensure that c is truly arbitrary
- If you prove something about a specific person Mary, you cannot make generalizations about all people.

1.6.3 Existential Instantiation

- Consider formula $\exists x.P(x)$
- We know there is an element c in the domain for which $P(c)$ is true.
- This is called **existential instantiation**

$$\frac{\exists x.P(x)}{P(c)}$$

- Here c is a **fresh** name (i.e. not used in the original formula)

32 **Example**

Prove $\exists x.P(x) \land \forall x.\neg P(x)$ is unsatisfiable.

1. $\exists x.P(x)$ (and elimination)
2. $\forall x.\neg P(x)$ (and elimination)
3. $P(a)$
4. $\neg P(a)$
5. False

1.6.4 Existential Generalization

- Suppose we know $P(c)$ is true for someone constant c
- Then there exists an element for which P is true.
- Thus we canc conlude $\exists x.P(x)$
- This inference rule is called **existential generalization**

$$\frac{P(c)}{\exists x.P(x)}$$

1.7 Lecture–September 13, 2022

Some terminology

- Important mathematical statements that can be shown to be true are **theorems**
- Many famous mathematical theormes, e.g., Pythagoraean theorem, Fermat's Last Theorem
- Pythagorean theorem: $a^2 + b^2 = c^2$
- Fermat's Last Theorem: $a^n + b^n = c^n$ has no solutions for $n > 2$

Theorems, Lemmas, and propositions

- Lemma: minor auxilary result aids in the proof of a theorem.
- Corollary: a result whose proof follows immediately from a theorem or proposition

Conjectures vs. Theorems

- Conjecture is a statement that is suspected to be true by experts but not proven.
- Goldman's Conjecture: Every even integer greater than 2 can be expressed as the sum of two prime numbers
- One of the most famous unsolved problems in mathematics

General Strategies for Proving Theorems:

- Direct proof: $p \to q$ proved by directly showing that if p then q.
- Proof by contraposition: $p \to q$ proved by showing that if $\neg q$ then $\neg p$.

33 **Example**

If n is an odd integer then n^2 is also odd.

Assume n is odd.

Proof. $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2k' + 1$.

$\therefore n^2$ is odd □

34 **Example**

In proof by contraposition, you prove $p \rightarrow q$ by assuming $\neg q$ and $\neg p$ follows. For example: n is an odd integer, then n^2 is also odd. Or, you can prove if n is not odd, then n^2 is not odd.

Proof. $n = 2k$

$n^2 = 4k^2$

$2(2k^2)$ is even □

Proof by contradiction: A formula ϕ is valid iff $\neg\phi$ is unsatisfiable.

Assume $\neg(p \rightarrow q)$ is unsatisfiable. If you can prove that it is unsatisfiable then you have proved that $p \rightarrow q$ is valid.

35 **Example**

Prove by contradiction that if $3n + 2$ is odd, then n is odd.

Proof. Assume $3n + 2$ is odd and n is even. Since n is even, $3n + 2$ can be written as $6k + 2 | k \in \mathbb{Z}$ which contradicts our assumption. □