Discrete Mathematics–Honors

Neo Wang
Lecturer: Isil Dilig

Last Updated: August 25, 2022

1 Chapter 1

1.1 Predicate Logic

1.1.1 Lecture 1–August 23, 2022

There are three basic logical connectives: **and**, **or**, **not** which are denoted by $\wedge$, $\vee$, and $\neg$ respectively. The negation of a proposition p, written $\neg p$, is true if p is false and false if p is true.

1 **Example**

"Less than 80 students are enrolled in CS311H" is a proposition. The negation of this is at least 80 students are in CS311H

Conjunction of two propositions p and q is written $p \wedge q$

2 **Example**

The conjunction of p = "It is Tuesday" and q = "it is morning" is $p \wedge q$ = "It is Tuesday and it is morning"

- Disjunction is written $p \vee q$ and the disjunction between $p \vee q$ for p = "It is Tuesday" and q = "it is morning" is $p \vee q$ = "It is Tuesday or it is morning"
- If your formula has n variables then your truth table has $n + 1$ columns because you have n variables and one column for the truth value of the formula.
- The number of rows is given by the formula 2^n
- Other connectives: exclusive or $\oplus$, implication $\rightarrow$, biconditional $\leftrightarrow$

1.1.2 Lecture 2–August 25, 2022

Let p = "I major in CS", q = "I will find a good job", r = "I can program"

- "I will not find a good job unless I major in CS or I can program": $(\neg p \wedge \neg r) \rightarrow \neg q$
- "I will not find a good job unless I major in CS and I can program": $(\neg p \vee \neg r) \rightarrow \neg q$
- The **inverse** of an implication $p \rightarrow q$ is $\neg p \rightarrow \neg q$. Therefore, "If I'm a CS major then I can program" has an inverse of "If I am not a CS Major then I'm not able to program."
- The **converse** of an implication $p \rightarrow q$ is $q \rightarrow p$.

3 **Definition** (Contrapositive)

The contrapositive of an implication of $p \rightarrow q$ is $\neg q \rightarrow \neg p$

The contrapositive of "if CS major then I can program" is "if I can't program, then I'm not a CS major"

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

A converse and it's inverse are always the same.

4 **Definition** (Biconditionals)

$p \leftrightarrow q = p \to q \land q \to p = \neg(p \oplus q)$

5 **Example** (Operator precedence)

Given a formula $p \land q \lor r$ do we parse this as $(p \land q) \lor r$ or $p \land (q \lor r)$?

1. Negation $\neg$ has the highest precedence
2. Conjunction ($\land$) has the next highest precedence
3. Disjunction ($\lor$) has the next highest precedence
4. Implication ($\to$) has the next highest precedence
5. Biconditional ($\leftrightarrow$) has the lowest precedence
6. Make sure to explicitly use parentheses for $\oplus$

1.2 Validity and Satisfiability

Validity and satisfiability

- The truth value depends on truth assigments to variables
- Example: $\neg p$ evaluates to true under the assignment $p = F$ and to false under $p = T$
- Some formulas evaluate to true for all assignments–these are called **tautologies** or **valid formulas**
- Some formulas evaluate to false for all assignments–these are called **contradictions** or **unsatisfiable formulas**

6 **Definition** (Interpretation)

An interpretation I for a formula F is a mapping from each propositional value to exactly one truth value.

$$I : \{p \mapsto \text{true}, q \mapsto \text{false}, \ldots, \}$$

Each interpretation corresponds to one row in the truth table so there are 2^n interpretations for a formula with n variables.

If the formula is true under interpretation I then we write $I \models F$ and if the formula is false then we write $I \not\models F$.

Theorem: $I \models F$ if and only if $I \not\models \neg F$.

7 **Example**

Consider the formula $F : p \land q \to \neg p \land \neg q$

Let I_1 be the interpretation such that $[p \mapsto \text{true}, q \mapsto \text{true}]$

What does F evaluate to under I_1? Answer: true

8 **Example**

Let F_1 and F_2 be two propositional formulas. Suppose F_1 is true under I. Then, $F_2 \neg F_1$ evaluates to false under I (the "and" shortcuts and forces the whole equation to be false).

Satisfiability, Validy

- F is **satisfiable** iff there exists interpretation $I | I \models F$
- F is **valid** iff for all interpretations $I, I \models F$

- F is **unsatisfiable** iff for all interpretations $I, I \not\models F$
- F is **contingent** if it is satisfiable, but not valid.

9 **Example** (Are the following statements true of false?)

- If a formula is valid, then it is also satisfiable? True. All interpretations are satisfiable.
- If a formula is satisfiable, then its negation is unsatisfiable. False.
- If F_1 and F_2 are satisfiable, then $F_1 \land F_2$ is also satisfiable. False.
- If F_1 and F_2 are satisfiable, then $F_1 \lor F_2$ is also satisfiable. True.

10 **Theorem** (Duality Between Validity and Unsatisfiability)

F is valid iff $\neg F$ is unsatisfiable.

Proof. Definition: F is valid iff for all interpretations $I, I \models F$

Theorem: $I \models F \leftrightarrow I \not\models \neg F$

This is very easy to prove: just map all outputs of F to true. □

Question: How can we prove that a propositional formula is a tautology is true?

Answer: We can use the **truth table method** and prove that the formula is true for all possible truth assignments.

11 **Example** (Tautology)

$(p \to q) \leftrightarrow (\neg q \to \neg p)$ is a tautology.

$(p \land q) \lor \neg p$ is not a tautology.