Vector Calculus – Honors

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1 Chapter 1

1.1 Lecture 1–August 22, 2022

Riemann integrals deal with functions that are basically continuous. You should use the notation (x, y, z) or $x\hat{i} + y\hat{j} + z\hat{k}$

There are several coordinates: Cartesian, cylindrical, and spherical.

Spherical coordinates are given by (r, θ, ϕ)

1 **Example**

Let f by a continuous functions. Suppose $f(x, y, z) = g(\sqrt{x^2 + y^2 + z^2})$.

Let 1. $f(x) = g(\sqrt{x^2 + y^2 + z^2})$ and 2. $f(x, y, z) = h_1(|x|)h_2(|y|) + h_3(|z|)$

how many such functions satisfy this?

2 **Definition**

Some useful integrals

- Continuous: $dxdydz$
- Cylindrical: $rdrd\theta$
- Spherical: $r^2 \sin\theta drd\theta d\phi$

3 **Definition** (Vectors)

Cross product: $\vec{x} \wedge \vec{y} = -\hat{y} \wedge \hat{x}$ is a vector operation

Dot product: $\vec{x} \cdot \vec{y} = \sum x_i y_i$ is a scalar operation

1.2 Lecture 2–August 23, 2022

4 **Definition** (Coordinate Systems)

$\vec{v} = (x, y, z) \to \vec{x} = (x_1, x_2, x_3)$

Sometimes will not include $\vec{}$ symbol–we want to think more abstractly in order to build to higher concepts. Spherical coordinates: (r, θ, ϕ), and θ is always the polar angle (from z-axis).

Orientation is the order of (x, y, z) which comes into play with $\boxed{\text{change of variables.}}$

5 **Remark** (Property of the determinant)

The determinant is always + or −

There are several volume form differentials:

$$dxdydz = r^2 dr \sin\theta d\theta d\phi$$

and note that $\sin\theta$ is always positive

6 **Remark** (Normal model)

Normal distribution pdf:

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

With $\mu = 0$ and $\sigma = 0$ The error function is the simplest example of a function that is "not integrable in elementary terms"

$$2\int_0^\infty e^{-x^2} dx = \sqrt{\pi}$$

There are connections with physics i.e. the uncertainty principle and the quantum mechanics harmonic oscillator.

$$\begin{aligned} A &= \int_{-\infty}^{\infty} e^{-x^2} dx \\ A^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dy dx \\ &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r dr d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\int_{\infty}^{0} e^{-u} du d\theta \\ &= \frac{1}{2}\int_0^{2\pi} 1 d\theta \\ &= \pi \\ &\boxed{A = \sqrt{\pi}} \end{aligned}$$

Differentials tell you how to compute an integral.

1. The integral is linear (also the derivative)

$$\int (af + bg) dm = a\int f dm + b\int g dm$$

2. For a non-negative function $f(x) \geq 0$ any way that you can calculate a finite value for the integral gives you the "correct answer."
3. Dilation:

$$\int_0^\infty f(ax) dx = \frac{1}{a}\int_0^\infty f(x) dx$$

7 **Remark** (For proving estimates for the dot or scalar product)

Estimate: $|x \cdot y| = |\sum x_i y_i| \leq ||\vec{x}||||\vec{y}||, x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$

3 simple arguments:

1. Euclidian geometry
2. Arithmetic
3. Adding a variable $\longleftarrow$ the best way and expands view to another parameter

Properties of vectors: vector products (may be a scalar $\vec{x} \cdot \vec{y}$ or a vector $\vec{x} \wedge \vec{y}$) and representation of data in terms of partial derivatives.

Partial derivatives–suppose we have a function $F(x, y, z)$ then we can have

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$$

For in the future–2nd partial derivatives–leads to a square matrix called a Hessian matrix.

8 **Definition** (Gradient)

Gradient: $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$