Vector Calculus – Honors

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1 Vectors

1.1 Vector Basics

1.1.1 Lecture–August 22, 2022

Riemann integrals deal with functions that are basically continuous. You should use the notation (x, y, z) or $x\hat{i} + y\hat{j} + z\hat{k}$

There are several coordinates: Cartesian, cylindrical, and spherical.

Spherical coordinates are given by (r, θ, ϕ)

1 **Example**

Let f by a continuous functions. Suppose $f(x, y, z) = g(\sqrt{x^2 + y^2 + z^2})$.

Let 1. $f(x) = g(\sqrt{x^2 + y^2 + z^2})$ and 2. $f(x, y, z) = h_1(|x|)h_2(|y|) + h_3(|z|)$

how many such functions satisfy this?

2 **Definition**

Some useful integrals

- Continuous: $dxdydz$
- Cylindrical: $rdrd\theta$
- Spherical: $r^2 \sin\theta drd\theta d\phi$

3 **Definition** (Vectors)

Cross product: $\vec{x} \wedge \vec{y} = -\hat{y} \wedge \hat{x}$ is a vector operation

Dot product: $\vec{x} \cdot \vec{y} = \sum x_i y_i$ is a scalar operation

1.1.2 Lecture–August 23, 2022

4 **Definition** (Coordinate Systems)

$\vec{v} = (x, y, z) \longrightarrow \vec{x} = (x_1, x_2, x_3)$

Sometimes will not include $\vec{}$ symbol–we want to think more abstractly in order to build to higher concepts. Spherical coordinates: (r, θ, ϕ), and θ is always the polar angle (from z-axis).

Orientation is the order of (x, y, z) which comes into play with $\boxed{\text{change of variables.}}$

5 **Remark** (Property of the determinant)

The determinant is always + or −

There are several volume form differentials:

$$dxdydz = r^2 dr \sin\theta d\theta d\phi$$

and note that $\sin\theta$ is always positive

6 **Remark** (Normal model)

Normal distribution pdf:

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

With $\mu = 0$ and $\sigma = 0$ The error function is the simplest example of a function that is "not integrable in elementary terms"

$$2\int_0^\infty e^{-x^2} dx = \sqrt{\pi}$$

There are connections with physics i.e. the uncertainty principle and the quantum mechanics harmonic oscillator.

$$\begin{aligned} A &= \int_{-\infty}^{\infty} e^{-x^2} dx \\ A^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dydx \\ &= \int_0^{2\pi}\int_0^\infty e^{-r^2} rdrd\theta \\ &= \frac{1}{2}\int_0^{2\pi}\int_\infty^0 e^{-u} dud\theta \\ &= \frac{1}{2}\int_0^{2\pi} 1 d\theta \\ &= \pi \\ &\boxed{A = \sqrt{\pi}} \end{aligned}$$

Differentials tell you how to compute an integral.

1. The integral is linear (also the derivative)

$$\int (af + bg)dm = a\int f dm + b\int g dm$$

2. For a non-negative function $f(x) \geq 0$ any way that you can calculate a finite value for the integral gives you the "correct answer."
3. Dilation:

$$\int_0^\infty f(ax)dx = \frac{1}{a}\int_0^\infty f(x)dx$$

7 **Remark** (For proving estimates for the dot or scalar product)

Estimate: $|x \cdot y| = |\sum x_i y_i| \leq ||\vec{x}||||\vec{y}||, x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$

3 simple arguments:

1. Euclidian geometry
2. Arithmetic
3. Adding a variable ⟵ the best way and expands view to another parameter

Properties of vectors: vector products (may be a scalar $\vec{x} \cdot \vec{y}$ or a vector $\vec{x} \wedge \vec{y}$) and representation of data in terms of partial derivatives.

1.1.3 Discussion–August 24, 2022

Vectors are a directed line segment. A vector in n-dimensional space is an ordered tuple of n real numbers. A vector is denoted $\vec{v} = (a_1, \ldots, a_n), a_i \in \mathbb{R}$

The basic operations:

- Addition of vectors: $\vec{v} + \vec{w} = (a_1 + b_1, \ldots, a_n + b_n)$
- Multiplication by scalar: $c\vec{v} = (ca_1, \ldots, ca_n)$

Properties:

- $\forall \vec{u} \in \mathbb{R}^n \langle \vec{u}, \vec{u} \rangle \geq 0$
- $\langle u, v \rangle = \langle v, u \rangle$
- Dot product is linear:

$$c(u \cdot v) = (cu) \cdot v = u \cdot (cv)$$

1.1.4 Lecture–August 25, 2022

How to think about data: strings, words, “columns”, rectangular arrays.

We can think of strings as slots: $(x_1, x_2, x_3, \ldots, x_n)$

Structures of organization may give insightful information on how to extra information

8 **Definition** (Triangle Inequality)

The length of any side is less than the sum of the lengths of the other two sides.

More abstract setting–use “norm.” Vectors: objects that we can add or subtract scalar mltiples–scalars.

Length–norm: $|\vec{v}| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

9 **Definition** (Homogenous)

A real-valued function $h(x)$ is homogenous of degree λ if $h(\lambda x) = \delta^\lambda h(x)$ where $\lambda \in \mathbb{R}$

The differential $r^2 dr sin\theta d\theta d\phi$ is homogenous of degree 3.

This defines a vector space.

10 **Remark** (Most important property from linear algebra)

Suppose you have a finite collection of vectors which you want to be $\boxed{\text{linearly independent.}}$ Then you can find a basis for the space spanned by the vectors. Then:

$$c_1 u_1 + c_2 u_2 + \ldots + c_n u_n = 0 \Leftrightarrow c_1 = c_2 = \ldots = c_n = 0$$

in $\mathbb{R}^3$, the unit vectors $\vec{i}, \vec{j}, \vec{k}$ are linearly independent. Any vector expression written in terms of these vectors is unique. The only way for us to get the zero vector is if all the coefficients are zero.

A vector space with a norm is called a “normed vector space.”

11 **Example** (Normed vector space)

Consider the set of continuous functions defined on the unit square–with scalars as real numbers, they are vector space. Restrict to all such functions that are square integrable on the unit square:

$$\int_0^1 \int_0^1 |f(x,y)|^2 dxdy < \infty$$

if this is true then we can create a norm on this space by setting

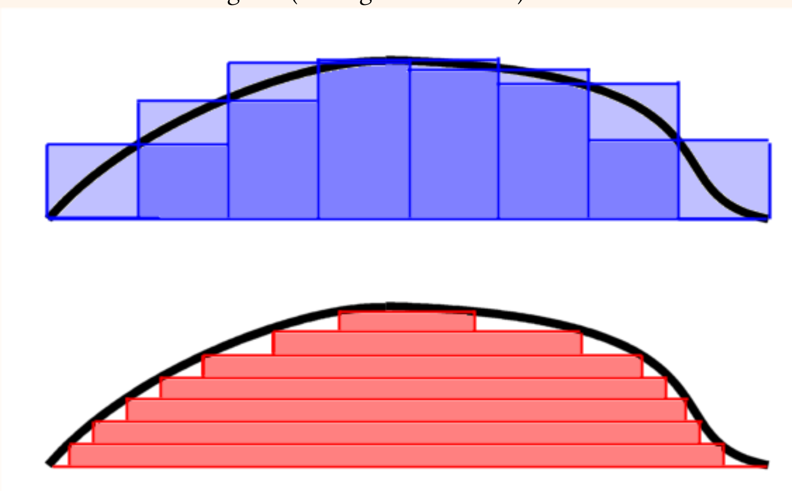
$$||f|| = \sqrt{\int_0^1 \int_0^1 |f(x,y)|^2 dxdy}$$

$$||f||_2 = \int_0^1 \int_0^1 |f(x,y)| dxdy$$

Then we have informally said that f is Lebesgue integrable.

12 **Remark**

The function is defined for a domain and the function takes various values. What do you want to partition? You want to partition the range and you can estimate the function in order to get the Lebesgue integral. We can look at this fundamental difference via the diagram (Lebesgue on bottom):



13 **Fact** (Dot Product)

$\vec{x} \cdot \vec{x} = |\vec{x}|^2$

1.2 Cauchy Schwarz Inequality

14 **Theorem** (Cauchy Schwartz Inequality)

With respect to length–to show that for two vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ then

$$|\vec{x} \cdot \vec{y}| \leq |\vec{x}||\vec{y}|.$$

same argument will work for abstract vector spaces with norm and scalar product.
Multiple proofs: Euclidean geometry, arithmetic, to see the role of length. But proof should capture the spirit of calculating length.

Proof. We will add a parameter λ

$$|\vec{x} - \lambda\vec{y}|$$

Remember that lengths and norms have corners. These are not smooth. For example $f(x = |x|)$ is not smooth at $x = 0$. You would like to smooth it out to something like $g(x) = x^2$. Therefore, square the expression to remove the "corners."
Assume that $\vec{x}$ and $\vec{y}$ are non-zero. Otherwise nothing to show.

$$\begin{aligned} 0 \leq |\vec{x} - \lambda\vec{y}|^2 &= \vec{x} \cdot \vec{x} + \lambda^2\vec{y} \cdot \vec{y} - 2\lambda\vec{x} \cdot \vec{y} \\ &= |\vec{x}|^2 + \lambda^2|\vec{y}|^2 - 2\lambda\vec{x} \cdot \vec{y} \\ &= |y|^2 + [\lambda^2 - 2B\lambda + C] \\ &= \lambda^2 - 2B\lambda + C \end{aligned}$$

If we complete the square we get $\lambda^2 - 2B\lambda + B^2 + C - B^2 \geq 0$

$$(\lambda - B)^2 + C - B^2 \geq 0 \implies \boxed{C - B^2 \geq 0}$$

$$C = \frac{|x|^2}{|y|^2}, B = \frac{x \cdot y}{|y|^2}$$

Expanding this, we get the Cauchy Schwartz inequality. □

We want to extend length of vectors to norms, but also the dot product to scalar products.

- $\vec{x} \cdot \vec{y} \longrightarrow \langle x, y \rangle$ (scalar product)
- $\langle x, y \rangle = \langle y, x \rangle$ symmetry
- $\langle x, \alpha y + \beta z \rangle = \alpha\langle x, y \rangle + \beta\langle x, z \rangle$
- $\langle x, x \rangle \geq 0$ (positive definite)
- $\langle x, x \rangle = \|x\|^2$

1.2.1 Discussion–August 29, 2022

15 **Definition** (Cross Product)

$$\vec{a} = (x_1, y_1, z_1), \vec{b} = (x_2, y_2, z_2)$$

$$\vec{a} \times \vec{b} = \det \begin{bmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \vec{i} \begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \end{bmatrix} - \vec{j} \begin{bmatrix} x_1 & z_1 \\ x_2 & z_2 \end{bmatrix} + \vec{k} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

$\forall \vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. Cross product is linear:
   - $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
   - $\vec{a} \times \alpha\vec{b} = \alpha\vec{a} \times \vec{b}$
   - $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
3. $\vec{a} \times \vec{a} = 0$
4. $i \times j = k, j \times k = i, k \times i = j$ in order $i \longrightarrow j \longrightarrow k \longrightarrow i, \ldots$

16 **Example**

Let $P = (-2, 1, 2), Q = (0, 0, 5), R = (5, 7, -1)$

First, $\vec{u} = Q - P = (2, -1, 3)$ and $\vec{w} = R - P = (7, 6, -3)$

1. What is the area of PQR? $\frac{1}{2}\|u \times w\|$
2. What is the equation of a plane defined by P, Q, R?
   $u \times w = \langle -15, 27, 19\rangle$ is the normal vector to the plane
   $\therefore$ $(v - p) \cdot (u \times w) = -15(x + 2) + 27(y - 1) + 19(z - 2) = 0$ is the equation of the plane

1.2.2 Lecture–August 30, 2022

Error Function,

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\frac{1}{2}t^2} dt$$

$\text{erf}(0) = 0, \text{erf}(\infty) = 1$

Another function with similar character comes from the simplest 2nd order linear differential equation with non-constant coefficients.

$$y'' + xy' + y = 0$$

Since it's a second order equation there are 2 linearly independent solutions and

$$y_1(x) = c_1 e^{-\frac{1}{2}x^2}, y_2(x) = c_2 e^{-\frac{1}{2}x^2} \int_0^x e^{\frac{1}{2}t^2} dt$$

Notice that y_1 is an even function, and that y_2 is an odd function. You should expect this solution to have one even, one odd.

$$\left[\int f(x)g(x)dm\right]^2 + \int \int_{x<y} |f(x)g(y) - f(y)g(x)|^2 dudm = \int |f(x)|^2 dm \int |g(y)|^2 dm$$

1.3 Differentiation and Linear Transformations

Most important concept from calculus is the notion of a limit. The 1st application of this is the definition of a derivative.

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \longrightarrow \text{does this limit exist?}$$

The expression has a removal singularity at $h = 0$.

17 **Example** (Classic Example)

$$\frac{\sin x}{x} = \frac{1}{x} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k}$$

because of the way the factorial function grows, this converges everywhere. Therefore,

$$\int_0^\infty \frac{sinx}{x} dx = \frac{\pi}{2}$$

Write as

$$\lim_{h \to 0} \frac{|f(x + h) - f(x) - f'(x)h|}{h} = 0$$

This is better because it's easier to work with functions that go to 0.

$$f(x + h) = f(x) + f'(x)h + error$$

the error goes to zero faster than the factor h. The 1st non-trivial term in the Taylor approximation for the function $f(x)$. $f(x) + f'(x)h$ = linear approximation for the function. Linear structure is one of our basic tools for solving problems.

$f(x, y, z) \longrightarrow$ what does the derivative mean?

$$\vec{v} = (x, y, z), \vec{v}_0 = (x_0, y_0, z_0)$$

$$f(x_0, y_0, z_0) = (\text{linear transformation}) \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$$

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \longrightarrow \text{able to compute}$$

action takes vectors to numbers. Matrices are arrays of numbers, they act on vectors and act on vectors: a n x m matrix.

18 **Definition** (Gradient)

$$\nabla = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

1.3.1 Lecture–September 1, 2021

19 **Definition** (Laplacian)

The Laplacian is defined by:

$$\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

Fundamental operator for partial differential equations. Keep in mind that Shrodinger equation and Maxwell's equations are partial differential equations.

In physics $\Delta = \nabla \cdot \nabla$

20 **Definition** (Deriative)

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$–define the derivative as the existence of a linear transformation such that the formula holds:

$$\lim_{\|\vec{x}-\vec{a}\| \to 0} \frac{\|f(\vec{x}) - f(\vec{a}) - (Df)\vec{a}(\vec{x} - \vec{a})\|_{\mathbb{R}^m}}{\|\vec{x} - \vec{a}\|_{\mathbb{R}^n}} \longrightarrow 0$$

the linear transformation Df involves data of evaluated at points:

$$Df(\vec{a}) = f(\vec{a})$$

Sometimes this can be written as $(Df)_{\vec{a}}$ or $Df|_{\vec{x}=\vec{a}}$. The transformation Df takes vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$. $\vec{x} - \vec{a}$ is a column vector. If $m = 1$, $(Df)(a)(\vec{x} - \vec{a})$ corresponds to $\nabla f \cdot (\vec{x} - \vec{a})$ = numerical value.

The error term is $\text{Error}(\vec{x}, \vec{a}) = \|f(x) - f(a) - (Df)(a)(x - a)\|$

21 **Definition** (Jacobian Matrix)

$(Df)(\vec{a})$ becomes the Jacobian matrix.

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

For $n = m$ (square matrix) sometimes the **determinant of J is called the Jacobian**. Jacobian acts on vectors in $\mathbb{R}^n$.

In terms of rows:

$$J = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_m \end{bmatrix}$$

$$J(\vec{x} - \vec{a}) = \begin{bmatrix} \nabla f_1 \cdot (\vec{x} - \vec{a}) \\ \nabla f_2 \cdot (\vec{x} - \vec{a}) \\ \vdots \\ \nabla f_m \cdot (\vec{x} - \vec{a}) \end{bmatrix}$$

22 **Remark** (Trace and Divergence)

Note that the trace of J is $\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$ which is also the divergence of the vector field $\nabla \cdot F$.

Formula called the change of variables theorem.

$dudv = |\det J| dxdy \longrightarrow$ connect with change of variables theorem.

Fact not emphasized within the text is that the determinant depends on orientation–this is why we put absolute value.

23 **Definition** (Vector Spaces)

In mathematics, physics, and engineering, a vector space (also called a linear space) is a set whose elements, often called vectors, may be added together and multiplied ("scaled") by numbers called scalars. Scalars are often real numbers, but can be complex numbers or, more generally, elements of any field. The operations of vector addition and scalar multiplication must satisfy certain requirements, called vector axioms. The terms real vector space and complex vector space are often used to specify the nature of the scalars: real coordinate space or complex coordinate space.

Matrices are arrays:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$