Discrete Mathematics–Honors

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1 Chapter 1

1.1 Predicate Logic

1.1.1 Lecture 1–August 23, 2022

There are three basic logical connectives: **and**, **or**, **not** which are denoted by $\wedge$, $\vee$, and $\neg$ respectively. The negation of a proposition p, written $\neg p$, is true if p is false and false if p is true.

1 **Example**
"Less than 80 students are enrolled in CS311H" is a proposition. The negation of this is at least 80 students are in CS311H

Conjunction of two propositions p and q is written $p \wedge q$

2 **Example**
The conjunction of p = "It is Tuesday" and q = "it is morning" is $p \wedge q$ = "It is Tuesday and it is morning"

- Disjunction is written $p \vee q$ and the disjunction between $p \vee q$ for p = "It is Tuesday" and q = "it is morning" is $p \vee q$ = "It is Tuesday or it is morning"
- If your formula has n variables then your truth table has $n + 1$ columns because you have n variables and one column for the truth value of the formula.
- The number of rows is given by the formula 2^n
- Other connectives: exclusive or $\oplus$, implication $\rightarrow$, biconditional $\leftrightarrow$

1.1.2 Lecture 2–August 25, 2022

Let p = "I major in CS", q = "I will find a good job", r = "I can program"

- "I will not find a good job unless I major in CS or I can program": $(\neg p \wedge \neg r) \rightarrow \neg q$
- "I will not find a good job unless I major in CS and I can program": $(\neg p \vee \neg r) \rightarrow \neg q$
- The **inverse** of an implication $p \rightarrow q$ is $\neg p \rightarrow \neg q$. Therefore, "If I'm a CS major then I can program" has an inverse of "If I am not a CS Major then I'm not able to program."
- The **converse** of an implication $p \rightarrow q$ is $q \rightarrow p$.

3 **Definition** (Contrapositive)
The contrapositive of an implication of $p \rightarrow q$ is $\neg q \rightarrow \neg p$
The contrapositive of "if CS major then I can program" is "if I can't program, then I'm not a CS major"

p	q	$p \to q$	$\neg q \to \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

A converse and it's inverse are always the same.

4 **Definition** (Biconditionals)

$p \leftrightarrow q = p \to q \land q \to p = \neg(p \oplus q)$

5 **Example** (Operator precedence)

Given a formula $p \land q \lor r$ do we parse this as $(p \land q) \lor r$ or $p \land (q \lor r)$?

1. Negation $\neg$ has the highest precedence
2. Conjunction ($\land$) has the next highest precedence
3. Disjunction ($\lor$) has the next highest precedence
4. Implication ($\to$) has the next highest precedence
5. Biconditional ($\leftrightarrow$) has the lowest precedence
6. Make sure to explicitly use parentheses for $\oplus$

1.2 Validity and Satisfiability

Validity and satisfiability

- The truth value depends on truth assigments to variables
- Example: $\neg p$ evaluates to true under the assignment $p = F$ and to false under $p = T$
- Some formulas evaluate to true for all assignments–these are called **tautologies** or **valid formulas**
- Some formulas evaluate to false for all assignments–these are called **contradictions** or **unsatisfiable formulas**

6 **Definition** (Interpretation)

An interpretation I for a formula F is a mapping from each propositional value to exactly one truth value.

$$I : \{p \mapsto \text{true}, q \mapsto \text{false}, \dots, \}$$

Each interpretation corresponds to one row in the truth table so there are 2^n interpretations for a formula with n variables.

If the formula is true under interpretation I then we write $I \models F$ and if the formula is false then we write $I \not\models F$.

Theorem: $I \models F$ if and only if $I \not\models \neg F$.

7 **Example**

Consider the formula $F : p \land q \to \neg p \land \neg q$

Let I_1 be the interpretation such that $[p \mapsto \text{true}, q \mapsto \text{true}]$

What does F evaluate to under I_1? Answer: true

8 **Example**

Let F_1 and F_2 be two propositional formulas. Suppose F_1 is true under I. Then, $F_2 \neg F_1$ evaluates to false under I (the "and" shortcuts and forces the whole equation to be false).

Satisfiability, Validy

- F is **satisfiable** iff there exists interpretation $I | I \models F$
- F is **valid** iff for all interpretations $I, I \models F$
- F is **unsatisfiable** iff for all interpretations $I, I \not\models F$
- F is **contingent** if it is satisfiable, but not valid.

9 **Example** (Are the following statements true of false?)

- If a formula is valid, then it is also satisfiable? True. All interpretations are satisfiable.
- If a formula is satisfiable, then its negation is unsatisfiable. False.
- If F_1 and F_2 are satisfiable, then $F_1 \wedge F_2$ is also satisfiable. False.
- If F_1 and F_2 are satisfiable, then $F_1 \vee F_2$ is also satisfiable. True.

10 **Theorem** (Duality Between Validity and Unsatisfiability)

F is valid iff $\neg F$ is unsatisfiable.

Proof. Definition: F is valid iff for all interpretations $I, I \models F$

Theorem: $I \models F \leftrightarrow I \not\models \neg F$

This is very easy to prove: just map all outputs of F to true. □

Question: How can we prove that a propositional formula is a tautology is true?

Answer: We can use the **truth table method** and prove that the formula is true for all possible truth assignments.

11 **Example** (Tautology)

$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$ is a tautology.

$(p \wedge q) \vee \neg p$ is not a tautology.

1.2.1 Lecture–August 30, 2022

Implication: Formula F_1 implies F_2 (written $F_1 \implies F_2$) iff $\forall I, I \models F_1 \rightarrow F_2$

12 **Example** (Implication Removal)

Is $(p \wedge q) \rightarrow p$ true? False. Let $p = F, q = T$

p	q	$p \rightarrow q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

13 **Definition** (Implication Removal)

$p \rightarrow q$ is equivalent to $\neg p \vee q$

Important equivalences

- Law of double negation: $\neg\neg p \equiv p$
- Identity laws: $p \wedge T \equiv p$, $p \vee F \equiv F$
- Domination Laws: $p \vee T \equiv T$, $p \wedge F \equiv p$
- Idempotent Laws: $p \wedge p \equiv p$, $p \vee p \equiv p$
- Negation Laws: $p \wedge \neg p \equiv F$, $p \vee \neg p \equiv T$

14 **Note** (Commutativity and Distributivity Laws)

- Commutative Laws: $p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$
- Distributivity Law 1: $(p \vee (q \wedge r)) \equiv ((p \vee q) \wedge (p \vee r))$
- Distributivity Law 2: $(p \wedge (q \vee r)) \equiv ((p \wedge q) \vee (p \wedge r))$
- Associativity Laws:

$$p \vee (q \vee r) \equiv (p \vee q) \vee r$$

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$

- Absorption 1: $p \wedge (p \vee q) \equiv p$
- Absorption 2: $p \vee (p \wedge q) \equiv p$

15 **Definition** (De Morgan's Laws)

Let a = "John took CS311" and b = "John took CS312". What does $\neg(a \wedge b)$ mean? It means "John did not take both CS311 and CS312". Therefore, John didn't take either CS311 or CS312.

$$\neg(a \wedge b) \equiv \neg a \vee \neg b$$

16 **Example** (Prove $\neg(p \wedge (\neg p \wedge q)) \equiv \neg p \wedge \neg q$)

p	$\neg p$	q	$p \wedge (\neg p \wedge q)$	$\neg(p \wedge (\neg p \wedge q))$
T	F	T	T	F
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T

17 **Example**

If Jill carries an umbrella, it is raining. Jill is not carrying an umbrella. Therefore, it is not raining.

$$((u \to r) \wedge (\neg u)) \to \neg r$$

This can be counter-modeled with r = true, u = false.

1.3 First Order Logic

- The building blocks of propositional logic were propositions
- In first-ordre logic there are three kinds of basic building blocks: constants, variables, predicates.
- Constants: refer to specific objects
- Examples: George, 6, Austin, CS311, …
- If a universe of discourse is cities in Texas, x can represent Houston, Houston, etc.

- **Predicates** describe properties of objects or relationships between objects.
- A predicate $P(c)$ is true or false depending on whether property P holds for c.
- The truth value of `even(2)` = true
- Another example: Suppose $Q(x, y)$ denotes $x = y + 3$ what is the value of $Q(3, 0)$? true