Linear Algebra - Neater

Neo Wang

April 28, 2021

Contents

1 Computer Graphics

Points are stored like

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_0 & y_0 & z_0 & 1 \end{bmatrix}$$

Rotations in 2D then, are given by the matrix

$$T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2 Fractions in Modular Arithmetic

- The modular inverse of a number a is denoted by the number a^{-1} such that $a \cdot a^{-1} \equiv 1 \mod n$.
- Such inverses do not exist if a and n are not coprime.
- If we have a fraction like $\frac{2}{3} \equiv x \mod 5$, then we must decompose as: $2 \cdot 3^{-1} \equiv x \mod 5$. Since $3^{-1} \equiv 2 \mod 5$, $2 \cdot 2 \equiv 4 \mod 5$.

3 Incidence Matrices

3.1 Construction

Given the graph A with 5 nodes, $1 \to 2$, $1 \to 3$, $2 \to 5$, $3 \to 5$, $5 \to 4$, $1 \to 5$. We can construct the graph. Notice that there is one 1 and -1 in each row; this represents an edge.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Recall that the b in $Ax = b$ is the change in voltage.

3.2 Properties

Elimination can reduce every graph to a tree. Closed loops produce dependent rows.

4 Encoding and Decoding

- Put the numbers in left to right in an n by m matrix. For example, the message *himyname* in a $2 \times \ldots$ matrix would equate to

$$\begin{bmatrix} h & i & m & y \\ n & a & m & e \end{bmatrix}$$

5 Complex Numbers

- Please don't be dumb and remember that $a + bi$ in polar is $(\sqrt{a^2 + b^2}, \tanh(a, b))$

5.1 Hermitian Matrices

- $z^h = \bar{z}^T$
- The inner product of real or complex vectors u and v is $u^h v$.
- This applies to complex matrices as well.
- An inner product of zero tells us that two vectors are perpendicular.
- When factorizing such matrices into $A = Q\Lambda Q^{-1}$

$$Q = \begin{bmatrix} \frac{x_1}{||x_1||} & 0 \\ 0 & \frac{x_2}{||x_2||} \end{bmatrix}$$

- Recall the property that $Q^T = Q^{-1}$

6 Unitary Matrix Properties

- A matrix is unitary if and only if it is invertible and its inverse is equal to its conjugate transpose.
- All unitary matrices have orthonormal columns.

7 Singular Value Decomposition

- v's are eigenvectors of $A^T A$
- σ^2's are eigenvalues of $A^T A$.
- The orthonormal columns of U and V respectively are eigenvectors of AA^T and $A^T A$

- U's are eigenvectors of AA^T.
- The eigenvalues of $A^TA = AA^T$ are the same (for nonzero eigenvalues).
- The matrix of A^TA is positive definite.
- Recall the shortcut $Av_i = \sigma_i u_i$. This can also be rewritten as $u_i = \frac{1}{\sigma_i} Av_i$

8 Norms and Condition Numbers

8.1 Norm Properties

- $||A|| \geq 0$ for any square matrix A.
- $||A|| = 0$ if and only if the matrix $A = 0$
- $||kA|| = |k|||A||$, for any scalar k
- $||A + B|| \leq ||A|| + ||B||$
- In this class, we define the norm to be

$$||A|| = \max_{x \neq 0} \frac{||Ax||}{||x||}$$

8.2 Norm Shortcuts

- The norm of a diagonal matrix is its largest entry by absolute value.
- The norm of a positive definite matrix is its largest eigenvalue.
- The norm of a symmetrix matrix is its largest eigenvalue.

8.3 Condition numbers

Recall that

$$||A^TA|| = ||A||^2 = \max_{x \neq 0} \frac{||Ax||^2}{||x||^2}$$

The square root of the largest λ of A^TA is norm of A.

The condition number is given by $c = ||A||||A^{-1}||$