Linear Algebra

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April 26, 2021

Contents

1 Single Value Decomposition

Key takeaways:

- v's are eigenvectors of $A^T A$

- σ^2's are eigenvalues of $A^T A$.

- The orthonormal columns of U and V respectively are eigenvectors of AA^T and $A^T A$

- U's are eigenvectors of AA^T

A reading on Medium can be found here[1]. Now we will get into the SVD itself. Since the columns of V are orthonormal...

- $V^T = eigenvector(A^T A)^T$

- $U = \begin{bmatrix} \frac{1}{\sigma 1 Av}1 & \frac{1}{\sigma_2} & \frac{N(A^T)}{|N(A^T)|} \end{bmatrix}$

$$U^T U = V V^T = I$$

$$A = U \sum V^T$$

Where do we get the columns of V and U from?

$$A^T = (U \sum U^T)^T = V \sum^2 V^T$$

diagnolization of $A^T A$, so the $\sigma = \sqrt{\text{eigenvalues of} A^T A}$

After we find the v's there is a shortcut for the u's.

$$Av_i = \sigma_i \vec{u}_i$$

Example 1: Find the SVD for $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$

- We can find $\sigma = \sqrt{5}, 3\sqrt{5}$ and eigenvalues of $5, 45$

- The full worked example can be found in your notes.

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[1] <https://blog.statsbot.co/singular-value-decomposition-tutorial-52c695315254>

Example 2: Using the left SVD to find SVD of:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

- Since this matrix is diagonal, the eigenvalues are $0, 1, 4, 9$. Then $\sigma = 0, 1, 2, 3$ respectively.

Example 3 (from MIT OCW):

Find the singular value decomposition of

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

Step 1: We calculate

$$A^T A = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

Step 2: Find the eigenvectors of this matrix to get the vectors v_i, and the eigenvalues for σ_i.

For our example, the two orthogonal eigenvectors are $(1, 1)$ and $(1, -1)$. Our orthonormal basis, then is

$$v_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), v_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$$

Ok, the above notes are like not the best - since I have no idea if they are correct or not (as of writing, I still have no clue what SVD is). Below is the official example from an MIT video (which happens to be extremely helpful).

If we are trying to find the SVD of the matrix

$$C = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}$$

We are trying to find $C = U \sum V_T$

We will need a few equations. These equations are:

$$C^T C = V \sum^T \sum V^T$$

$$CV = U \sum$$

First, we compute

$$C^C = \begin{bmatrix} 26 & 18 \\ 18 & 74 \end{bmatrix}$$

Notice that this is just a diagnolization of $C^T C$.

1.1 4/26/2021

Ok, these notes are probably more important, since they are actually "correct". So cool thing, note that the orthonormal columns of U and V respectively are the eigenvectors of AA^T and $A^T A$. Don't forget to normalize them. Furthermore, note that S is simply all the σs. This can include the value 0. Finally, remember the shortcut of $Av_i = \sigma_i \vec{u}_i$

Now, time to get into norms and condition numbers.

Some norm properties:

- $||A|| \geq 0$ for any square matrix A.

- $||A|| = 0$ if and only if the matrix $A = 0$

- $||kA|| = |k|||A||$, for any scalar k

- $||A + B|| \leq ||A|| + ||B||$

- In this class, we define the norm to be

$$||A|| = \max_{x \neq 0} \frac{||Ax||}{||x||}$$

There are several important shortcuts we can take.

- The norm of a diagonal matrix is its largest entry by absolute value.

- The norm of a positive definite matrix is its largest eigenvalue.

- The norm of a symmetrix matrix is its largest eigenvalue.

Recall that

$$||A^T A|| = ||A||^2 = \max_{x \neq 0} \frac{||Ax||^2}{||x||^2}$$

The square root of the largest λ of $A^T A$ is norm of A.

The condition number is given by $c = ||A||||A^{-1}||$