习题五

① 习题五

7.6

Give an example of a linear program in two variables whose feasible region is infinite, but such that there is an optimum solution of bounded cost.

Objective function min $x_1 + x_2$

Constraints $x_1, x_2 \geq 0$

① 习题五

7.7

(a) Find necessary and sufficient conditions on the reals a and b under which the linear program max x + y ax + by ≤ 1 x, y ≥ 0 is infeasible

发现$x = 0, y = 0$时恒满足约束条件，所以无论a, b取何值始终是feasible

① 习题五

7.7

(b) Find necessary and sufficient conditions on the reals a and b under which the linear program max x + y ax + by ≤ 1 x, y ≥ 0 is unbounded.

$a \le 0$时取$x \to +\infty, y = 0$恒满足约束条件

$b \le 0$时取$x = 0, y \to +\infty$恒满足约束条件

当$a > 0$且$b > 0$时，有$x \le \frac{1}{a}, y \le \frac{1}{b}$，$x + y \le \frac{1}{a} + \frac{1}{b}$有界

所以$a \le 0$或$b \le 0$时该线性规划无界

① 习题五

7.7

(c) Find necessary and sufficient conditions on the reals a and b under which the linear program max x + y ax + by ≤ 1 x, y ≥ 0 has a unique optimal solution.

当$a > 0$且$b > 0$时，该线性规划有最优解

当$\mathrm{a} = \mathrm{b}$时，该线性规划的最优解有无数个

所以当$a > 0, b > 0$且$a \neq b$时该线性规划有唯一最优解

7.9

A quadratic programming problem seeks to maximize a quadratric objective function (with terms like $3x_1^2$ or $5x_1x_2$) subject to a set of linear constraints. Give an example of a quadratic program in two variables x_1, x_2 such that the feasible region is nonempty and bounded, and yet none of the vertices of this region optimize the (quadratic) objective.

Objective function min $x_1^2 + x_2^2$

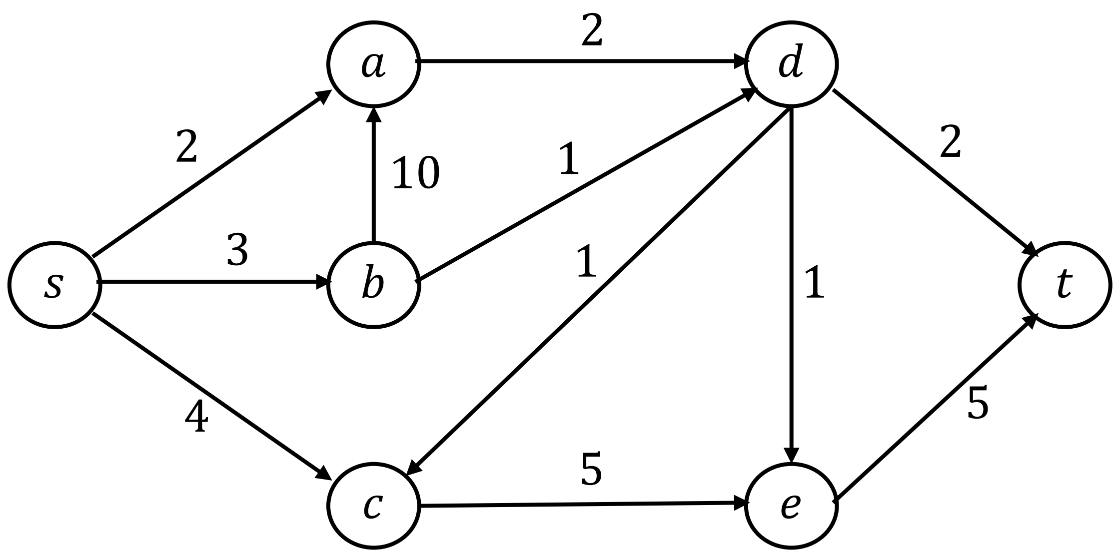
Constraints $10 \geq x_1, x_2 > 0$

1 习题五

7.21

An edge of a flow network is called critical if decreasing the capacity of this edge results in a decrease in the maximum flow. Give an efficient algorithm that finds a critical edge in a network.

首先求最大流，再构造出残量图。显然，临界边的容量肯定被占满，所以在残量图中，临界边的顶点对之间只存在反向边。



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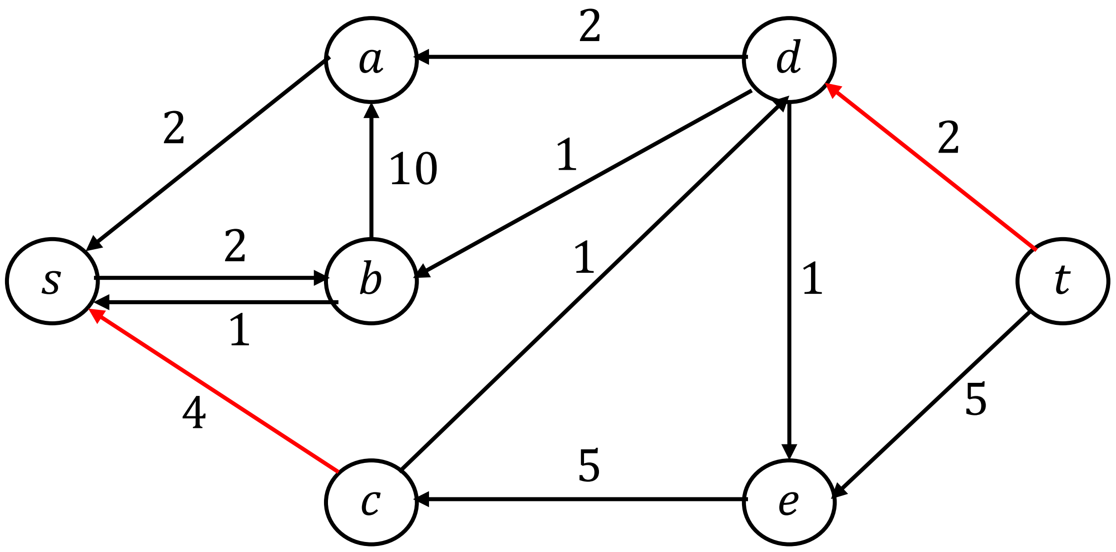
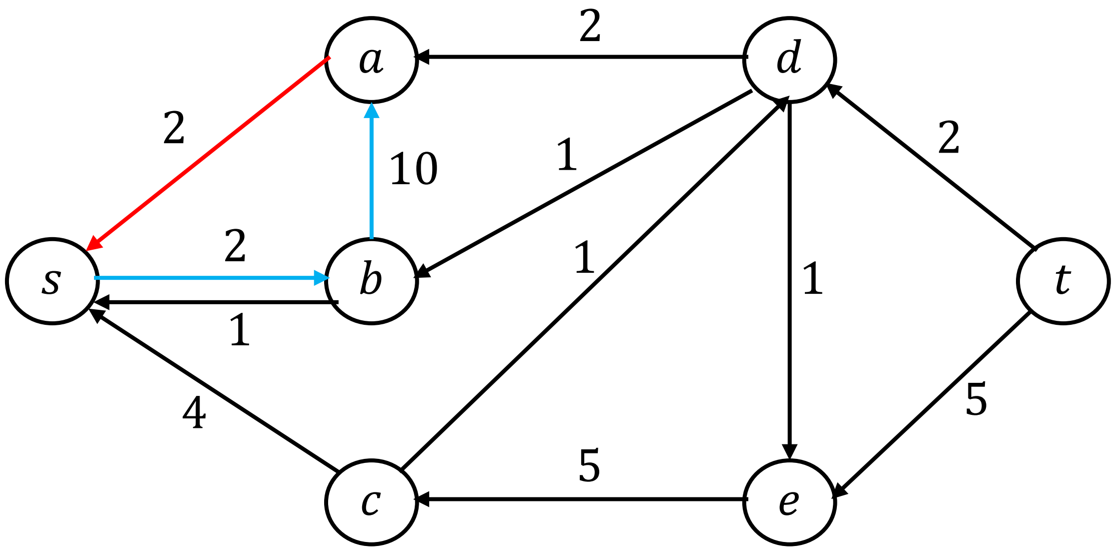


① 习题五

7.21

An edge of a flow network is called critical if decreasing the capacity of this edge results in a decrease in the maximum flow. Give an efficient algorithm that finds a critical edge in a network.

但在残量图中只存在反向边的顶点对不一定都是临界边的顶点，可能某条在最大流中流满的边，其容量下降后，最大流中缺少的流量可以从其他边的流量得到补充

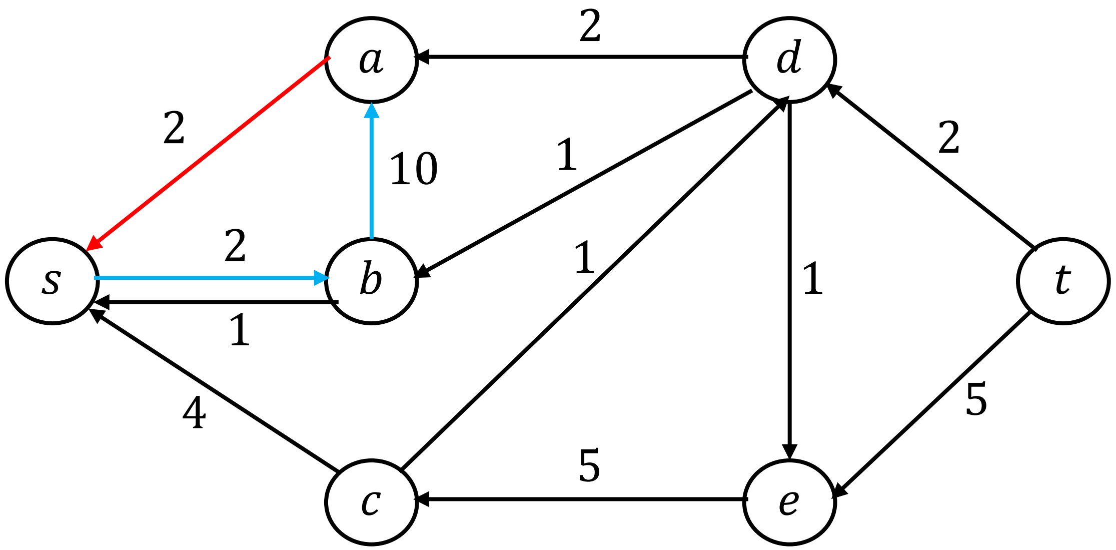


1 习题五

7.21

An edge of a flow network is called critical if decreasing the capacity of this edge results in a decrease in the maximum flow. Give an efficient algorithm that finds a critical edge in a network.

在残量图中以S为源点进行DFS，则所经过的反向边均不是临界边（参考答案）



① 习题五

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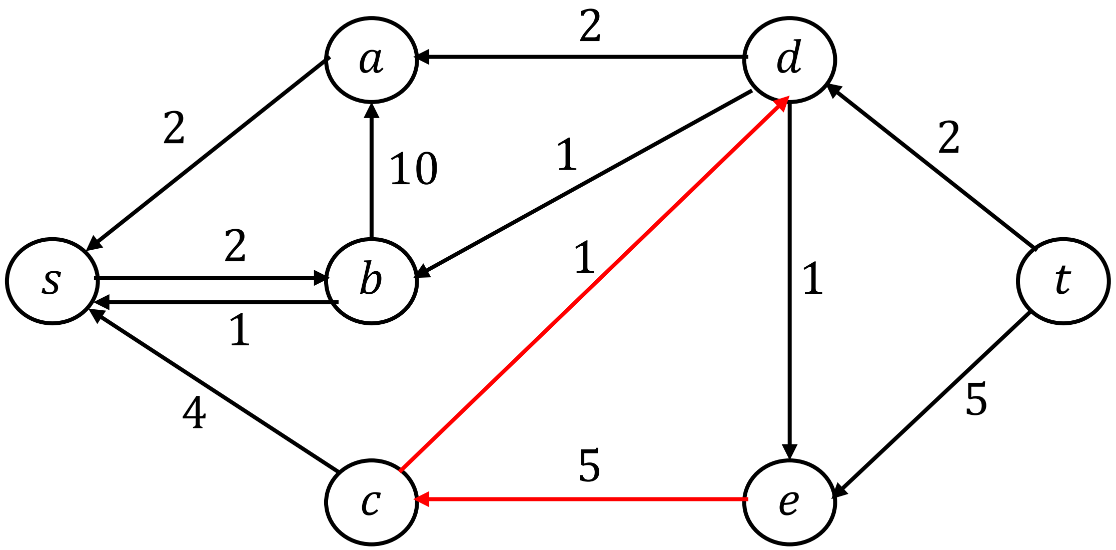
An edge of a flow network is called critical if decreasing the capacity of this edge results in a decrease in the maximum flow. Give an efficient algorithm that finds a critical edge in a network.

在残量图中以S为源点进行 DFS ，则所经过的反向边均不是临界边——不正确，因为没去除掉所有的非临界边的反向边



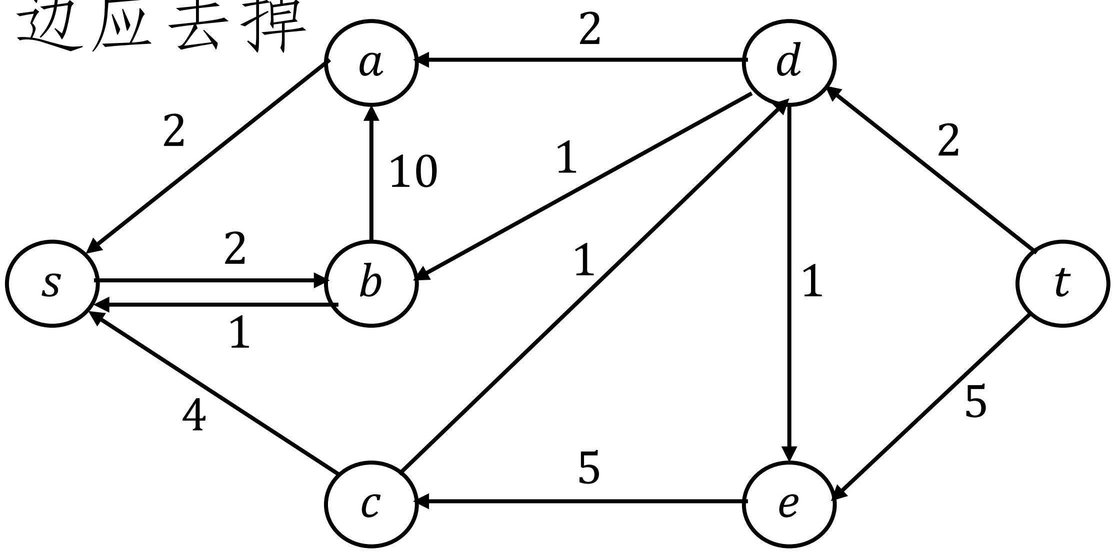
① 习题五

7.21

An edge of a flow network is called critical if decreasing the capacity of this edge results in a decrease in the maximum flow. Give an efficient algorithm that finds a critical edge in a network.

（核心）容量下降后，最大流中缺少的流量可以从其他边的流量得到补充

因此对每条反向边，从汇点DFS，若能DFS到源点，就说明该反向边应去掉



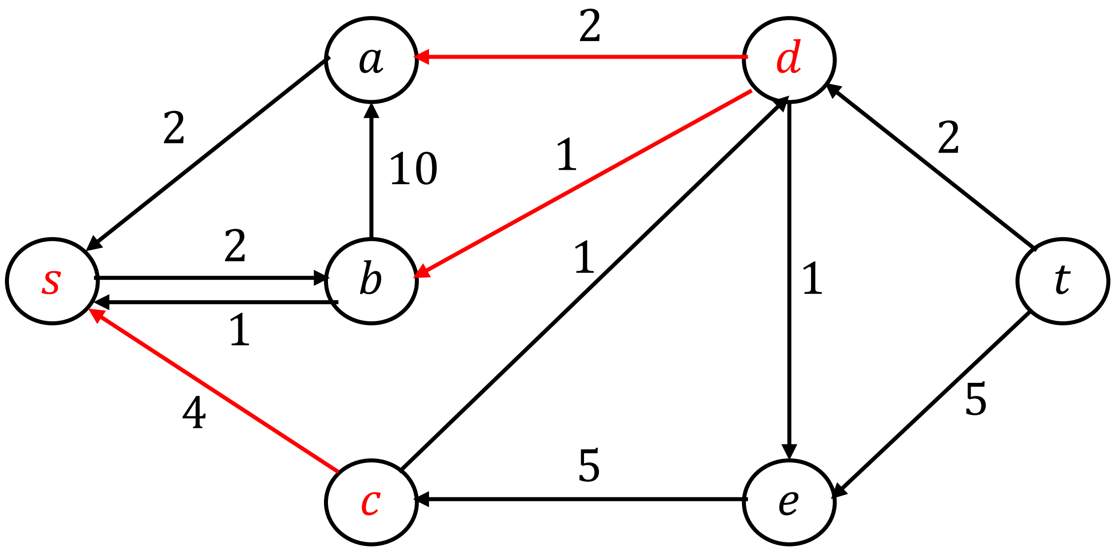
① 习题五

7.21

An edge of a flow network is called critical if decreasing the capacity of this edge results in a decrease in the maximum flow. Give an efficient algorithm that finds a critical edge in a network.

（更好的方法）源点的DFS因最小割而无法继续继续

$e: a \to d, e: b \to d$和$e: s \to c$均为临界边，为继续求出不合格的反向边，需要以为c, d为次级源点继续DFS



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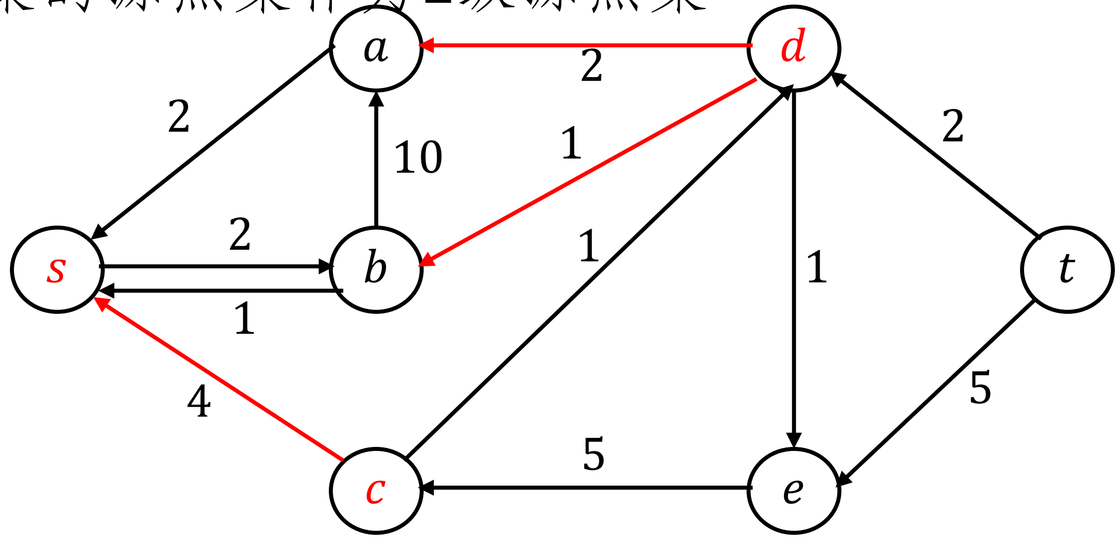
An edge of a flow network is called critical if decreasing the capacity of this edge results in a decrease in the maximum flow. Give an efficient algorithm that finds a critical edge in a network.

（更好的方法）从源点开始DFS，遍历到的点集和未遍历到的点集形成图的一个割，割边集中所有的反向边均对应一条临界边；去掉遍历到的子图和割边集，得到了原图的一个未被遍历的子图，割边集的源点集作为2级源点集



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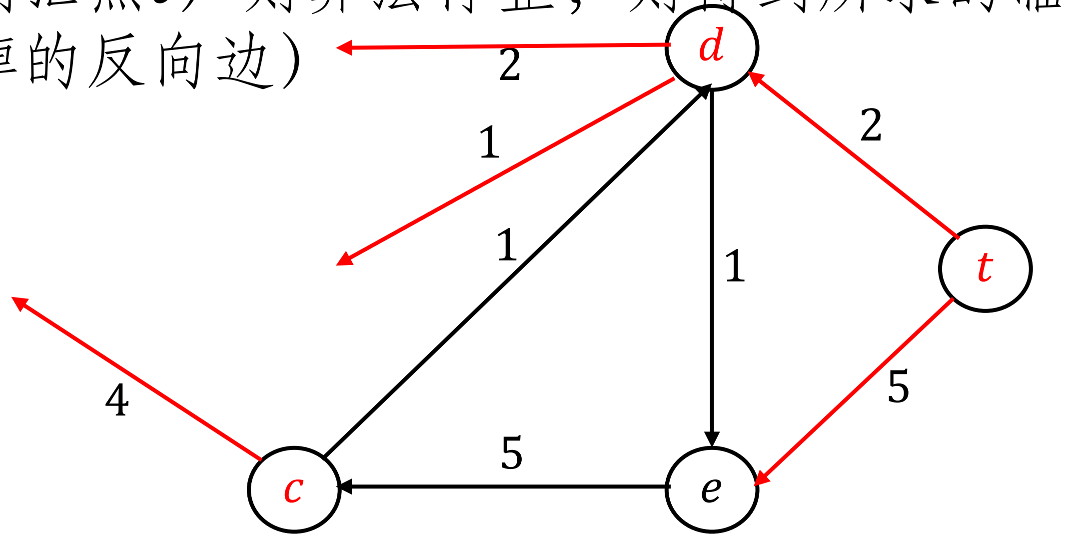
An edge of a flow network is called critical if decreasing the capacity of this edge results in a decrease in the maximum flow. Give an efficient algorithm that finds a critical edge in a network.

（更好的方法）从i级源点继续DFS，若遍历到的点集和未遍历到的点集同样形成图的一个割，则重复上述操作，得到割边集、新的未被遍历的子图，作为 $i+1$ 级源点集；若遍历完整张（子）图（即遍历到汇点t）则算法停止，则得到所求的临界边集（或所有应该去掉的反向边）

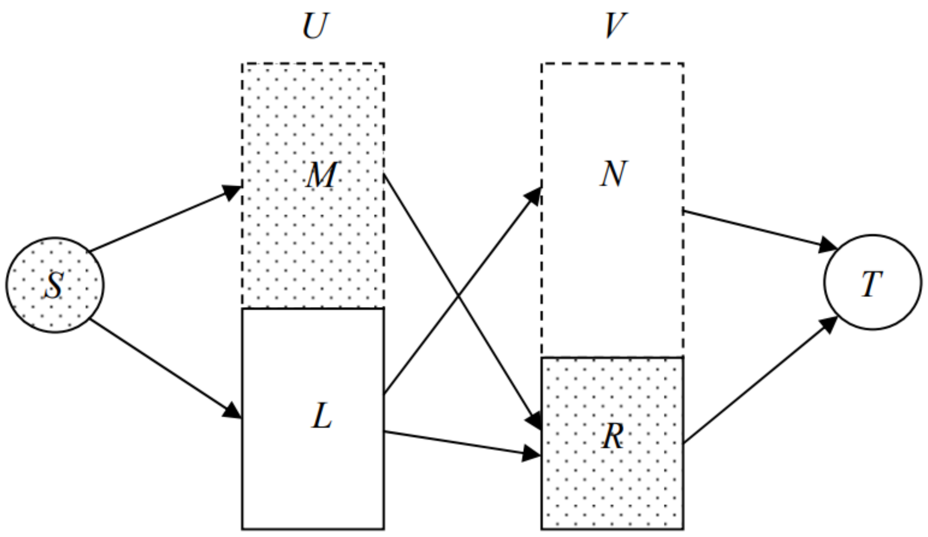


① 习题五

7.23

Show that the problem of finding the minimum vertex cover in a bipartite graph reduces to maximum flow. (Hint: Can you relate this problem to the minimum cut in an appropriate network?)

答案的缺陷：找到的一组割$(S \cup M \cup R, L \cup N \cup T)$ =最小点覆盖，但未证明该割即为最小割，因此还需要证明对任意一种割，其割边数≤割$(S \cup M \cup R, L \cup N \cup T)$的割边数（证明思路：讨论不属于$S \rightarrow L$和$R \rightarrow T$的割边，则添加新的割边会导致$S \rightarrow L$和$R \rightarrow T$中的割边减少，将割边数转移到节点数，利用最小点覆盖说明总割边数不减性）



① 习题五

7.23

Show that the problem of finding the minimum vertex cover in a bipartite graph reduces to maximum flow. (Hint: Can you relate this problem to the minimum cut in an appropriate network?)

思路二：最小点覆盖 ⇒最小割

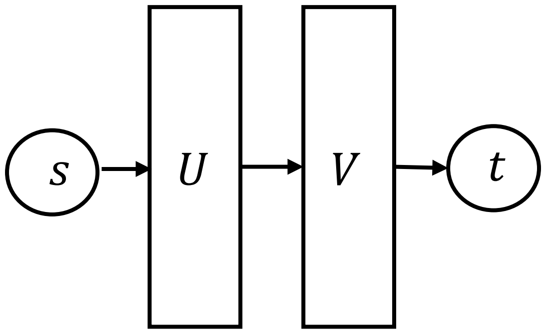
设二分图$G=(U\cup V,E)$，则考察如下的线性规划：

Objective function min $\sum_{u\in U} a_u + \sum_{v\in V} a_v$

Constraints $u\in U, v\in V$

$\forall i\in U\cup V, a_i\in\{0,1\}$

$a_u + a_v \geq 1$，若$e: u\to v\in E$



可知该线性规划等价于G的最小点覆盖

且如右图所示 网络，各边权为1，则该线性规划同样等价于该网络的最小割（证明，若$e: u\to v\in E$，对流$s\to u\to v\to t$，则$s\to u$，$u\to v$和$v\to t$三边至少取走一边，对应点u,v至少取走一个，a_u和a_v至少一个为1，即对应$a_u + a_v\geq 1$）

① 习题五

7.23

Show that the problem of finding the minimum vertex cover in a bipartite graph reduces to maximum flow. (Hint: Can you relate this problem to the minimum cut in an appropriate network?)

思路三：最小点覆盖 ⇒ 最大匹配 ⇒最大流

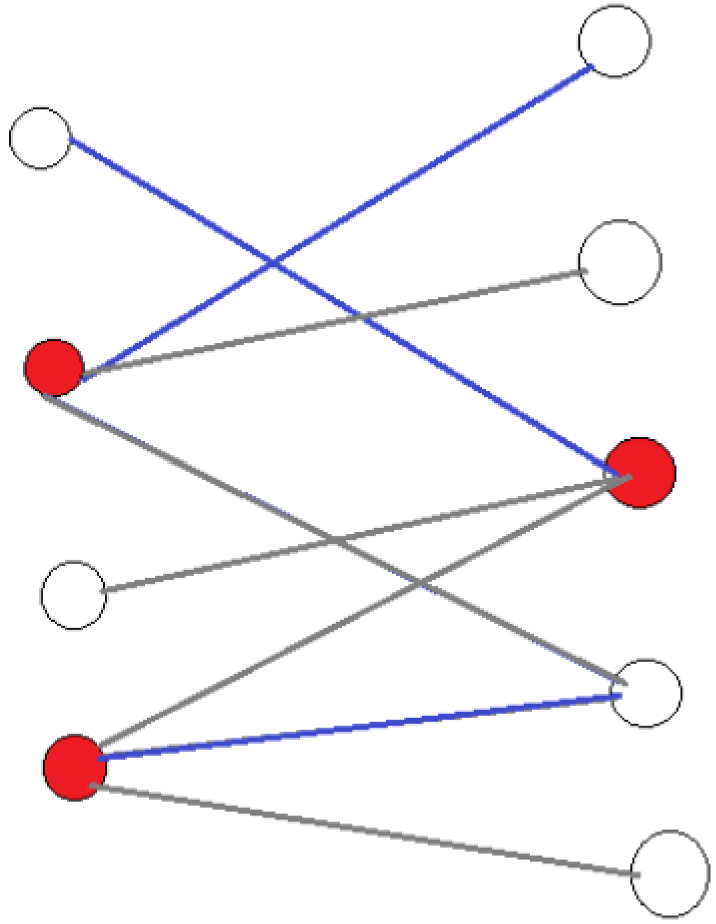
最小点覆盖数≥最大匹配数

从左边非匹配点开始，延非匹配边走向右边的点；右边的点再延匹配边走向左边，标记所有走过的点

左边未标记过的点和右边标记过的点集构成一个最小点覆盖（证明：

1、选出来的点集大小=最大匹配

若左边选出的点存在非匹配点，则该点会作为起点被标记；若右边选出的点存在非匹配点，则存在一对非匹配点相连，得到一条新的匹配边，与最大匹配矛盾）



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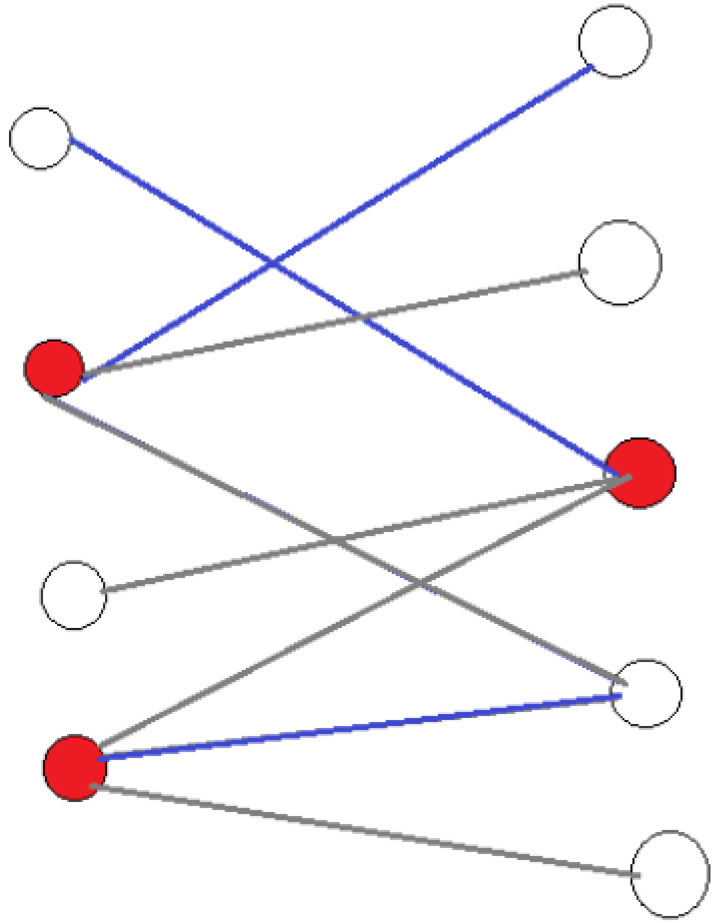

思路三：最小点覆盖 ⇒ 最大匹配 ⇒最大流

左边未标记过的点和右边标记过的点集构成一个最小点覆盖（证明：

2、选出来的点集可以覆盖所有边

因为不可能存在某一条边，其右端点没有标记，而左端点有标记的。

如果这条边不属于匹配边，那么右端点就可以通过这条边得到标记；如果这条边属于匹配边，那么左端点不可能是一条路径的起点，于是其标记只能是从这条边的右端点过来的，右端点就应该有标记。）

① 习题五

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左边未标记过的点和右边标记过的点集构成一个最小点覆盖（证明：

1、选出来的点集大小=最大匹配

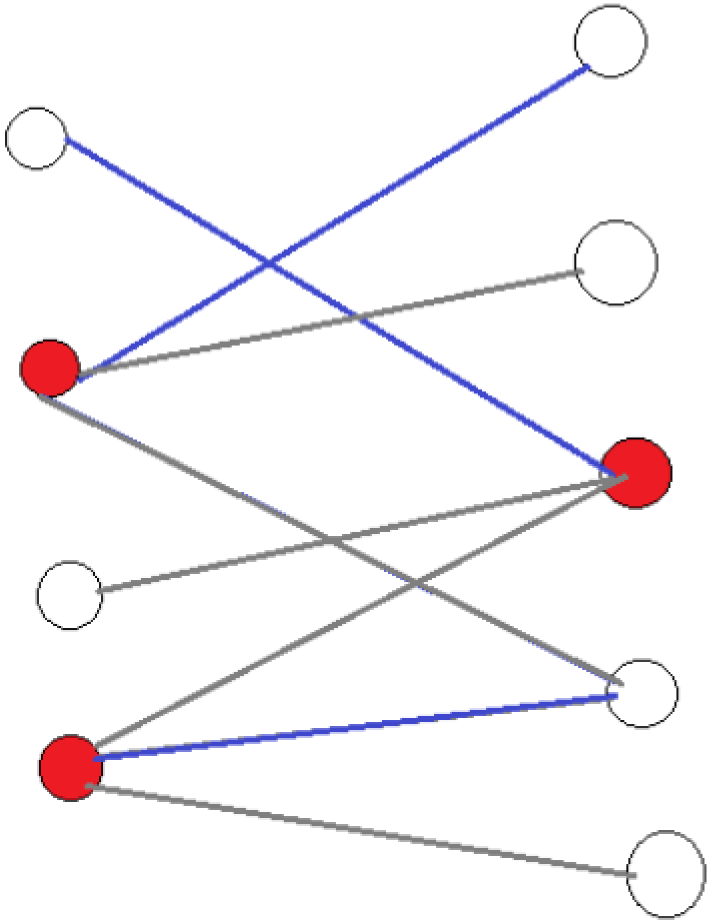
2、选出来的点集可以覆盖所有边

3、最小点覆盖数≥最大匹配数

）

所以选出来的点集即构成最小点覆盖

最小点覆盖⇒最大匹配



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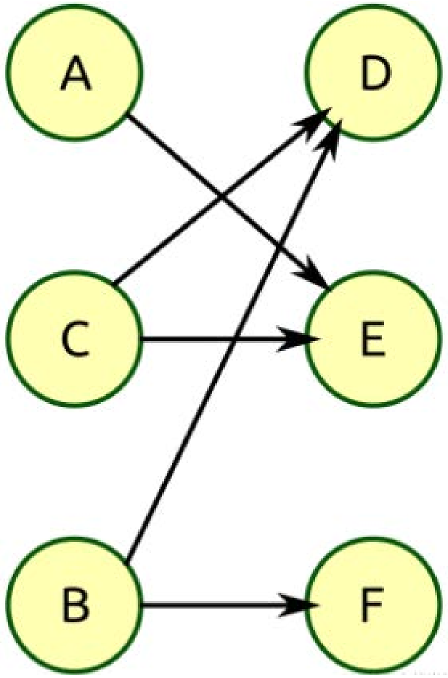
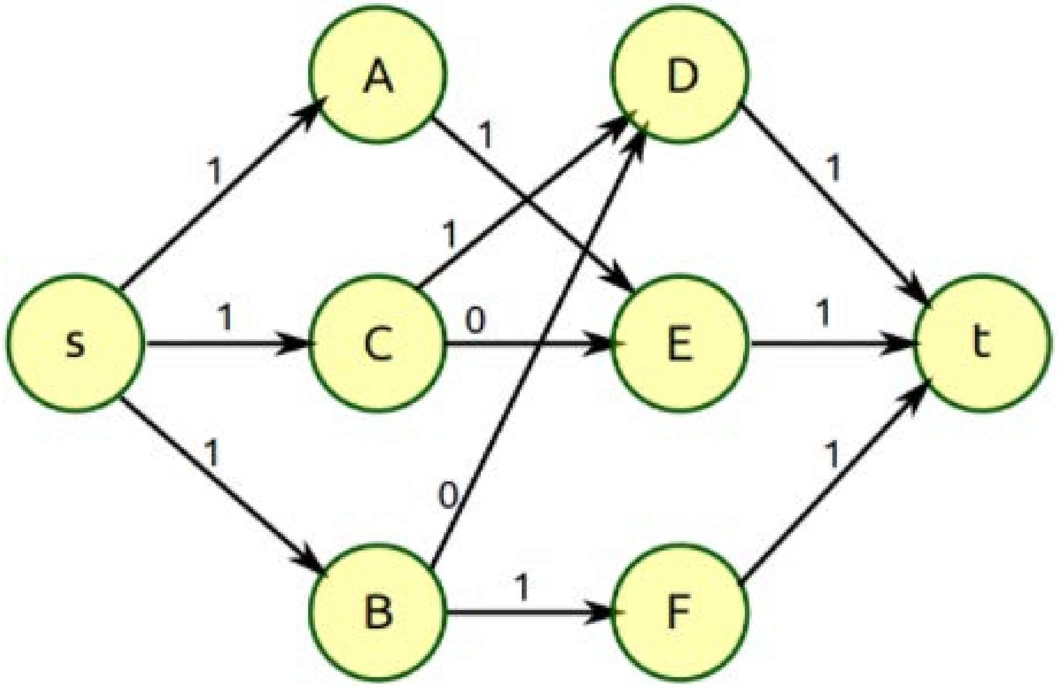
最大匹配⇒最大流





① 习题五

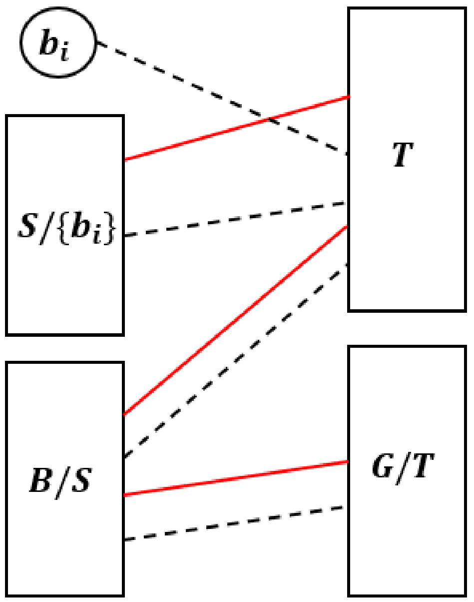
7.30

Hall's theorem. Returning to the matchmaking scenario of Section 7.3, suppose we have a bipartite graph with boys on the left and an equal number of girls on the right. Hall's theorem says that there is a perfect matching if and only if the following condition holds: any subset S of boys is connected to at least |S| girls.
Prove this theorem. (Hint: The max-flow min-cut theorem should be helpful.)

必要性：（显然）

若存在 boys 子集 S，与 S 相连的 girls 集合为 T，满足$|T|<|S|$，必然存在 $b_i\in S$ 在 S 与 T 的最大匹配中没有连边。

考察 boys 全集 B 和 girls 全集 G，B 可拆分为S 和 $B\backslash S$，其中 S 只与 T 相连，$B\backslash S$ 既与 $G\backslash T$ 相连也可能与 T 相连。由于 b_i 与 $G\backslash T$ 不连通，所以 b_i 在 B 与 G 的最大匹配中也没有连边。所以 B 与 G 不存在完美匹配。



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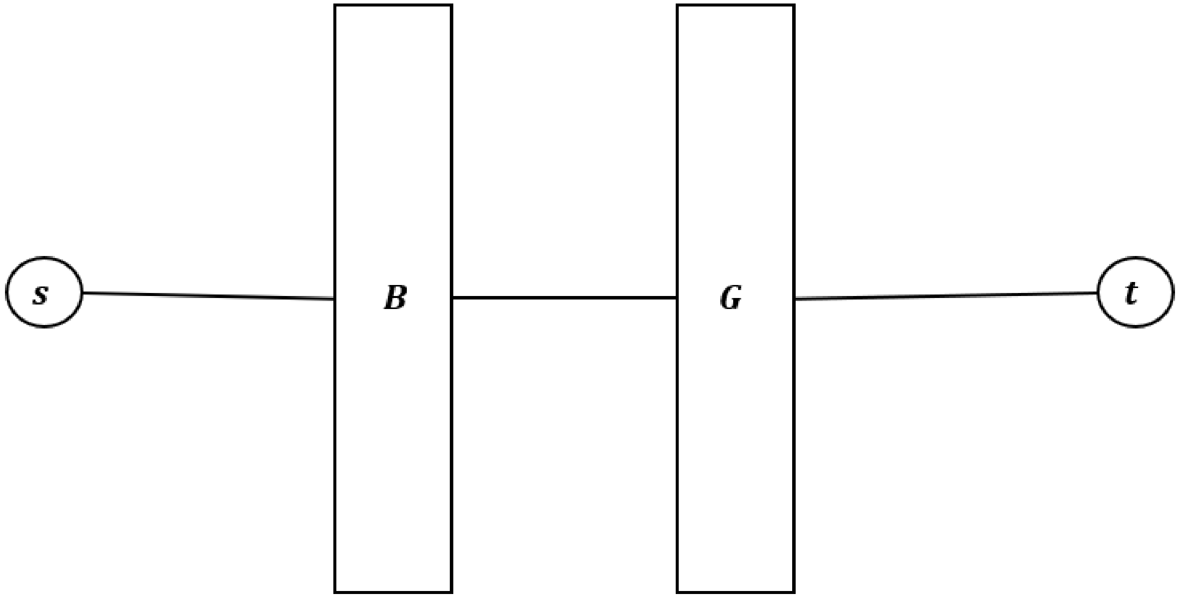
Prove this theorem. (Hint: The max-flow min-cut theorem should be helpful.)

充分性：(最大流最小割定理法)
构造如下网络流图：
源点 s 与每一个 boy 连边
汇点 t 与每一个 girl 连边
B、G 间为二分图
边权均为1
考察 s 到 t 的最大流
显然最大流$\leq |B| = |G| = n$



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充分性：(最大流最小割定理法)

考察该网络流图的任意一个割$(s+S+T, B/S+G/T+t)$

割边集为$e_1 \cup e_2 \cup e_3 \cup e_4$

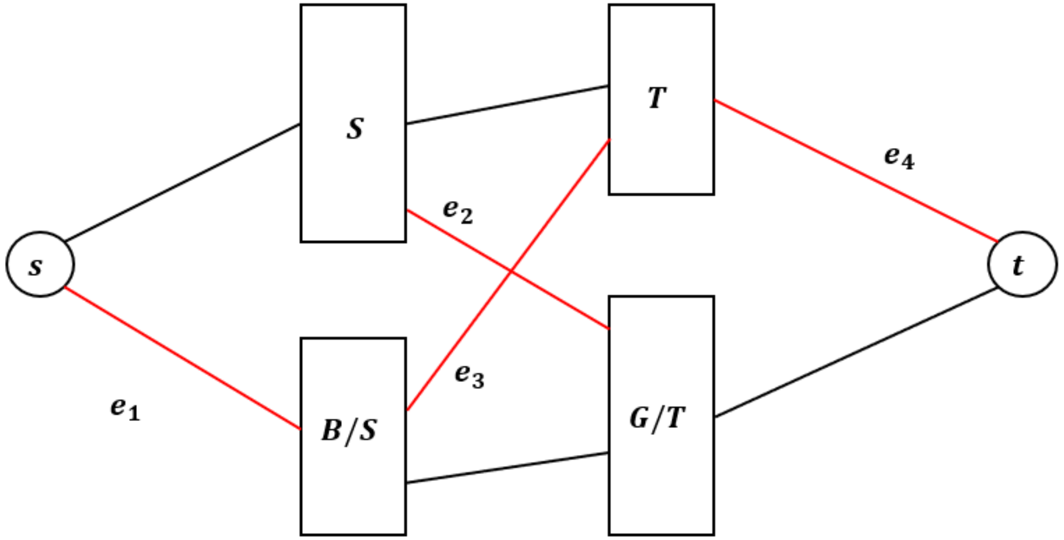
$|e_1| = n - |S|, |e_4| = |T|$

当$|S| \geq |T|$时

$|e_2| \geq |S| - |T|$，$|e_3| \geq 0$

当$|S| < |T|$时

$|e_2| \geq 0$，$|e_3| \geq |T| - |S|$



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割边集为$e_1 \cup e_2 \cup e_3 \cup e_4$

割大小$cut = |e_1| + |e_2| + |e_3| + |e_4|$

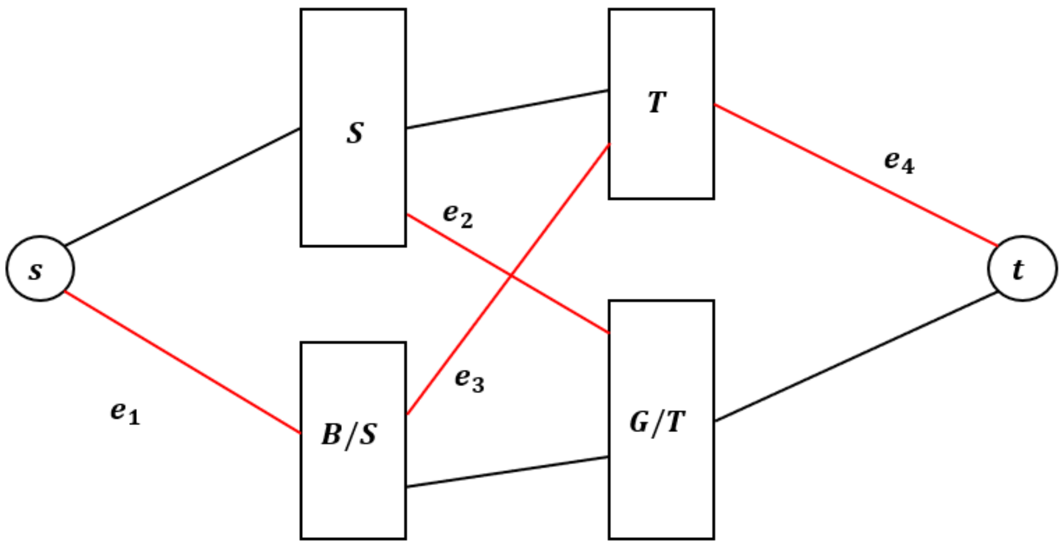
当$|S| \geq |T|$时

$cut \geq n - |S| + (|S| - |T|) + |T| = n$

当$|S| < |T|$时

$cut \geq n - |S| + (|T| - |S|) + |T|$

$= n - 2(|T| - |S|) > n$



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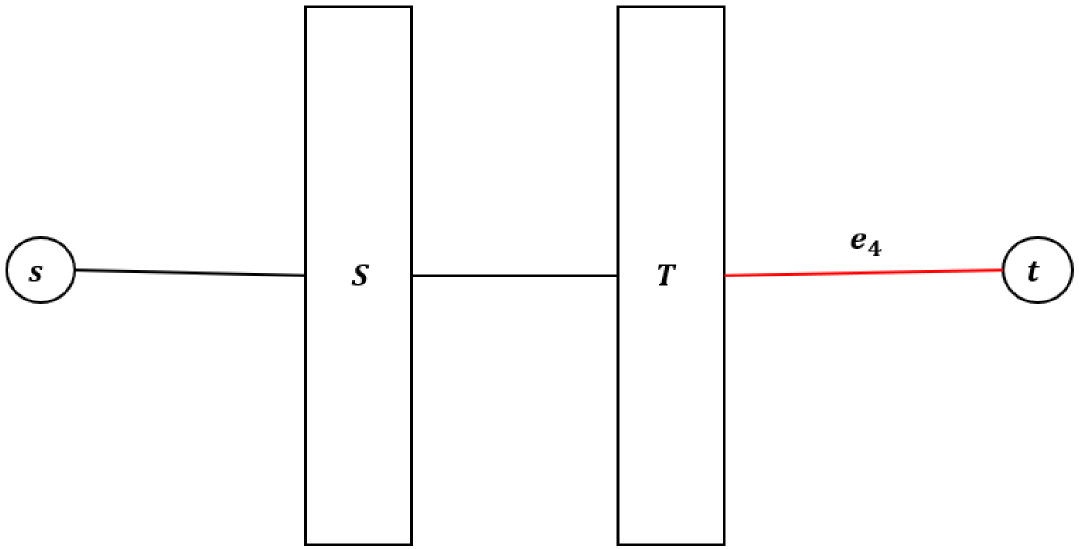
充分性：(最大流最小割定理法)
考察该网络流图的任意一个割$(s+S+T, B/S+G/T+t)$
对该网络流图的任意一个割
均满足割大小$\geq n$
大小为n的割显然存在
当$S=B$、$T=G$，割边为e_4
所以最小割为n
最大流也为n
完美匹配一定存在



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Prove this theorem. (Hint: The max-flow min-cut theorem should be helpful.)

充分性：(归纳法)

当$n=1$时，显然成立

假设当$n=k$时结论成立，即当$|B|=|G|=k$，且对B的任意子集S,与S相连的G的子集T满足$|T|\geq|S|$时，存在B到G的完美匹配

当$n=k+1$时，即当$|B|=|G|=k+1$，且对B的任意子集S,与S相连的G的子集T满足$|T|\geq|S|$时，下证存在B到G的完美匹配

① 习题五

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Hall's theorem. Returning to the matchmaking scenario of Section 7.3, suppose we have a bipartite graph with boys on the left and an equal number of girls on the right. Hall's theorem says that there is a perfect matching if and only if the following condition holds: any subset S of boys is connected to at least |S| girls.
Prove this theorem. (Hint: The max-flow min-cut theorem should be helpful.)

充分性：(归纳法)
当$n=1$时，显然成立
假设当$n=k$时结论成立，即当$|B|=|G|=k$，且对B的任意子集S,与S相连的G的子集T满足$|T|\geq|S|$时，存在B到G的完美匹配
当$n=k+1$时，即当$|B|=|G|=k+1$，且对B的任意子集S,与S相连的G的子集T满足$|T|\geq|S|$时，下证存在B到G的完美匹配
对boy b_i，定义$N(b_i)$为与b_i相连的girl 集合
同理对girl g_i，$N(g_i)$为与g_i相连的boy集合

1 习题五

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Prove this theorem. (Hint: The max-flow min-cut theorem should be helpful.)

充分性：(归纳法)

考察连接boy数最小的girl g_1，即$|N(g_1)|$最小

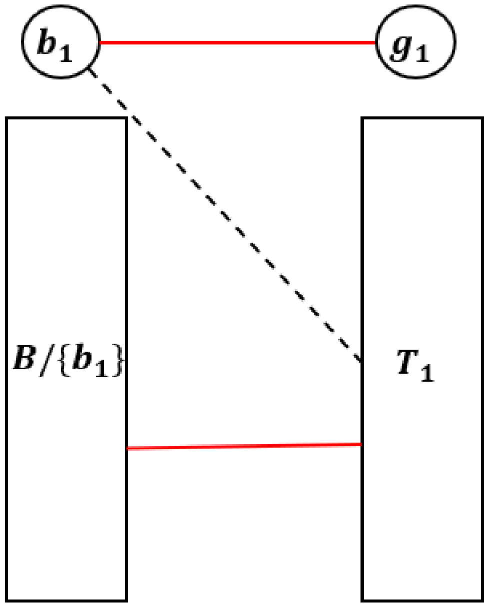
若$|N(g_1)| = 1$

设唯一与g_1相连的boy为b_1

则对B的子集$B/\{b_1\}$，与$B/\{b_1\}$相连的girl集合为T_1，有$|T_1| \geq |B/\{b_1\}| = n$

由于$N(g_1) = \{b_1\}$，所以$T_1 = G/\{g_1\}$

由归纳假设知，存在$B/\{b_1\}$到T_1的完美匹配P_1，则$P_1 \cup \{(b_1, g_1)\}$构成B到G的完美匹配



7.30

Hall's theorem. Returning to the matchmaking scenario of Section 7.3, suppose we have a bipartite graph with boys on the left and an equal number of girls on the right. Hall's theorem says that there is a perfect matching if and only if the following condition holds: any subset S of boys is connected to at least |S| girls.

Prove this theorem. (Hint: The max-flow min-cut theorem should be helpful.)

充分性：(归纳法)

考察连接boy数最小的girl g_1，即$|N(g_1)|$最小

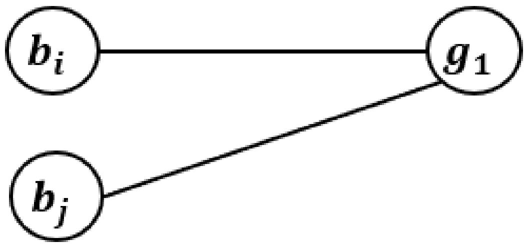
若$|N(g_1)| \geq 2$

不妨设$N(g_1) = \{b_1, b_2 \ldots, b_{|N(g_1)|}\}$

则$|N(b_1)|, |N(b_2)|, \ldots, |N(b_{|N(g_1)|})|$中最多只存在1个为1

否则，$\exists b_i, b_j \in N(g_1)$，$N(b_i) = N(b_j) = \{g_1\}$

则与集合$\{b_i, b_j\}$相连的girl仅有g_1一人，不满足条件



1 习题五

7.30

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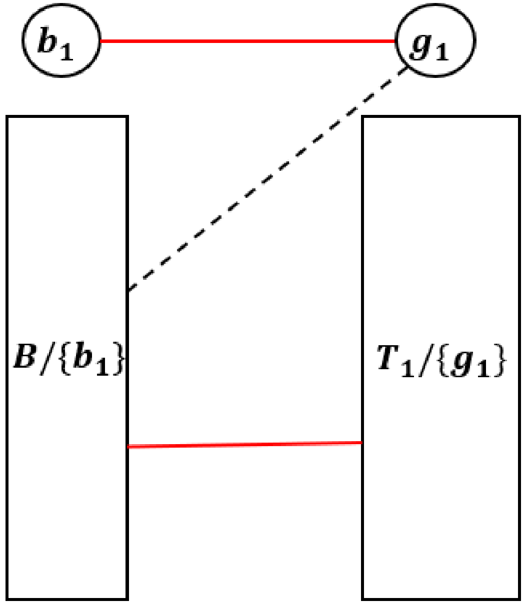
Prove this theorem. (Hint: The max-flow min-cut theorem should be helpful.)

充分性：(归纳法)
考察连接boy数最小的girl g_1，即$|N(g_1)|$最小
若$|N(g_1)| \geq 2$
不妨设$N(g_1) = \{b_1, b_2 \dots, b_{|N(g_1)|}\}$
若$|N(b_1)|, |N(b_2)|, \dots, \left|N\left(b_{|N(g_1)|}\right)\right|$中存在1个为1
不妨设$|N(b_1)| = 1$
则对B的子集$B/\{b_1\}$相连的girl集合为$T_1 = G$
则$|T_1/\{g_1\}| = n$，且$B/\{b_1\}$同样与$T_1/\{g_1\}$相连
由归纳假设知，存在$B/\{b_1\}$到$T_1/\{g_1\}$的完美匹配P_1，则$P_1 \cup \{(b_1, g_1)\}$构成B到G的完美匹配



① 习题五

7.30

Hall's theorem. Returning to the matchmaking scenario of Section 7.3, suppose we have a bipartite graph with boys on the left and an equal number of girls on the right. Hall's theorem says that there is a perfect matching if and only if the following condition holds: any subset S of boys is connected to at least |S| girls.
Prove this theorem. (Hint: The max-flow min-cut theorem should be helpful.)

充分性：(归纳法)

考察连接boy数最小的girl g_1，即$|N(g_1)|$最小

若$|N(g_1)| \geq 2$

不妨设$N(g_1) = \{b_1, b_2 \dots, b_{|N(g_1)|}\}$

若$|N(b_1)|, |N(b_2)|, \dots, |N(b_{|N(g_1)|})| \geq 2$

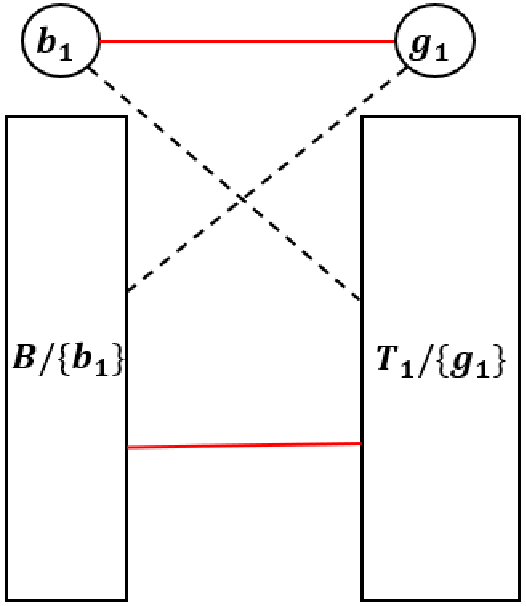
则对B的子集$B/\{b_1\}$相连的girl集合为$T_1 = G$，否则

存在g_i只与b_1相连，$|N(g_i)| = 1$,与$|N\{g_1\}|$最小矛盾

则$|T_1/\{g_1\}| = n$，且$B/\{b_1\}$同样与$T_1/\{g_1\}$相连

由归纳假设知，存在$B/\{b_1\}$到$T_1/\{g_1\}$的完美匹配P_1，

则$P_1 \cup \{(b_1, g_1)\}$构成B到G的完美匹配



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Prove this theorem. (Hint: The max-flow min-cut theorem should be helpful.)

充分性：(归纳法)
考察连接boy数最小的girl g_1，即$|N(g_1)|$最小
对$N(g_1)$的所有可能情况，均证明存在从B到G的完美匹配
证明当$n = k + 1$，即$|B| = |G| = k + 1$，且对B的任意子集S,与S相连的G的子集T满足$|T| \geq |S|$时，存在B到G的完美匹配
由归纳原理可知，对任意大小的集合B和G，当与B的任意子集S相连的G的子集T满足$|T| \geq |S|$时，存在B到G的完美匹配
得证

习题七

9.2 Devise a backtracking algorithm for the RUDRATA PATH problem from a fixed vertex s. To fully specify such an algorithm you must define:

(a) What is a subproblem?

(b) How to `choose` a subproblem.

(c) How to `expand` a subproblem.

Argue briefly why your choices are reasonable.

回溯算法的过程：

```
Start with some problem P_0
Let S = {P_0}, the set of active subproblems
Repeat while S is nonempty:
  choose a subproblem P ∈ S and remove it from S
  expand it into smaller subproblems P_1, P_2, ..., P_k
  For each P_i:
    If test(P_i) succeeds:  halt and announce this solution
    If test(P_i) fails:  discard P_i
    Otherwise:  add P_i to S
Announce that there is no solution
```

9.2

给定$G=(V,E)$和源点s，输出一条从s开始长为$|V|$的$RUDRATA\ PATH$

(a) 定义子问题：$P=\{s,v_1,v_2,\dots,v_t\}$，表示路径$s\to v_1\to v_2\to\cdots\to v_t$且该路径为一条合法的$RUDRATA\ PATH$，$P_0=\{s\}$

(b) 选择子问题：选择$P=argmax_{P\in S}|P|$，减少S中储存的子问题数量，有利于算法提前终止

(c) 扩展子问题：对选定的子问题P，取路径的终点v_t，$\forall v_i\in V$，$(v_t,v_i)\in E$，得到新的子问题$P_i=\{s,\dots,v_t,v_i\}$

给定$G=(V,E)$和源点s，输出一条从s开始长为$|V|$的$RUDRATA\ PATH$

(a) 定义子问题：$P=\{s,v_1,v_2,\dots,v_t\}$，表示路径$s\to v_1\to v_2\to\cdots\to v_t$且该路径为一条合法的$RUDRATA\ PATH$，$P_0=\{s\}$

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9.2 Devise a backtracking algorithm for the RUDRATA PATH problem from a fixed vertex s. To fully specify such an algorithm you must define:

(a) What is a subproblem?

(b) How to `choose` a subproblem.

(c) How to `expand` a subproblem.

Argue briefly why your choices are reasonable.

对每一个新的子问题$P_i = \{s, \dots, v_t, v_i\}$

1、若$|P_i| = |V|$，得到答案

2、若$v_i \in \{s, \dots, v_t\}$，丢弃子问题

3、若$v_i \notin \{s, \dots, v_t\}$，$S = S \cup \{P_i\}$

② 习题七

9.7 In the MULTIWAY CUT problem, the input is an undirected graph $G = (V, E)$ and a set of terminal nodes $s_1, s_2, \ldots, s_k \in V$. The goal is to find the minimum set of edges in E whose removal leaves all terminals in different components.

(a) Show that this problem can be solved exactly in polynomial time when $k = 2$.

(b) Give an approximation algorithm with ratio at most 2 for the case $k = 3$.

(c) Design a local search algorithm for multiway cut.

(a) $k = 2$，求s_1和s_2两点的最小割问题，用最大流算法求解 $Edmond - Karp$算法复杂度$O(|V| \cdot |E|)$

(b) $k = 3$

分别求s_1和s_2的最小割Cut_1、s_2和s_3的最小割Cut_2以及s_2和s_3的最小割Cut_3，显然Cut_1、Cut_2和Cut_3中只需要保留2个割即可

不妨设$Cut_1 \le Cut_2 \le Cut_3$，近似解$\mathcal{A} = Cut_1 + Cut_2$

对最优解Opt，显然有$Opt \ge Cut_3 \ge Cut_2 \ge Cut_1$

所以近似比$\alpha_{\mathcal{A}} = \max \frac{\mathcal{A}}{Opt} \le 2$

② 习题七

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(c) Design a local search algorithm for multiway cut.

(c)

```
let s be any initial solution
while there is some solution s' in the neighborhood of s
   for which cost(s') < cost(s):  replace s by s'
return s
```

9.7 In the MULTIWAY CUT problem, the input is an undirected graph $G=(V,E)$ and a set of terminal nodes $s_1, s_2, \ldots, s_k \in V$. The goal is to find the minimum set of edges in E whose removal leaves all terminals in different components.

(a) Show that this problem can be solved exactly in polynomial time when $k=2$.

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(c) Design a local search algorithm for multiway cut.

(c) $\forall v \in V$，设$Label(v)$为点v的标签

初始时，$s_i \in \{s_1, s_2, \ldots, s_k\}$，$Label(s_i) = l_i$，$v \notin \{s_1, s_2, \ldots, s_k\}$，$Label(v) = 0$

以s_1为起点进行BFS，遍历到节点v，若$Label(v) = 0$，则修改$Label(v) = l_1$，并继续BFS；若$Label(v) \neq 0$，则回溯。

对$s_2, \ldots, s_k$重复上述过程直到所有点$Label(v) \neq 0$

$S = \{e(v_i, v_j) | e(v_i, v_j) \in E, Label(v_i) \neq Label(v_j)\}$为初始解

② 习题七

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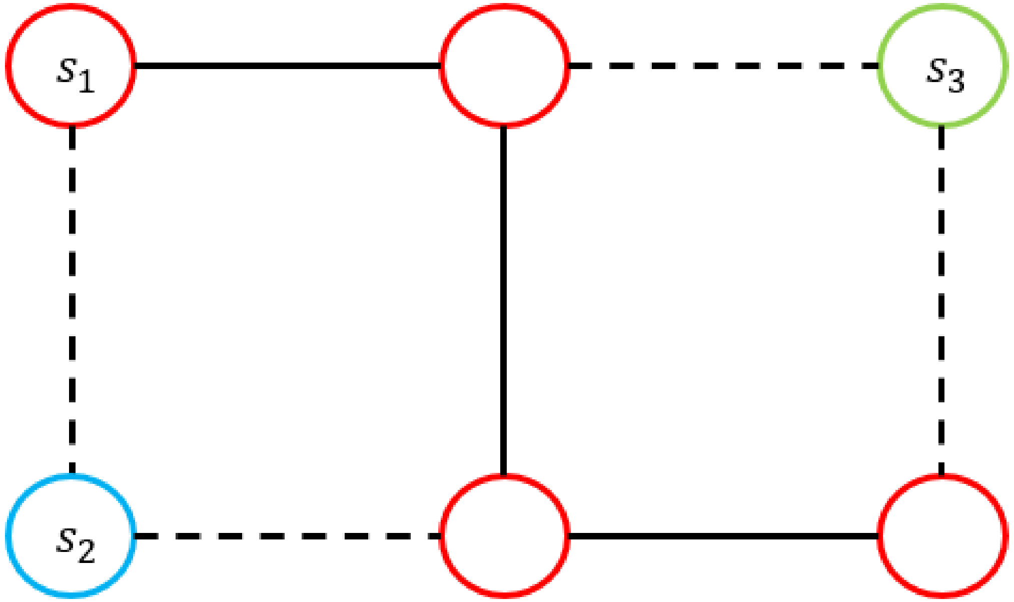

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(c) 取$\forall e(v_i, v_j) \in S$，分别计算修改$Label(v_i) = Label(v_j)$和$Label(v_j) = Label(v_i)$时新的割边边集$S_1$和$S_2$

若$|S| > |S_1|$，$S = S_1$；若$|S| > |S_2|$，$S = S_2$

直到S不再改变为止

② 习题七

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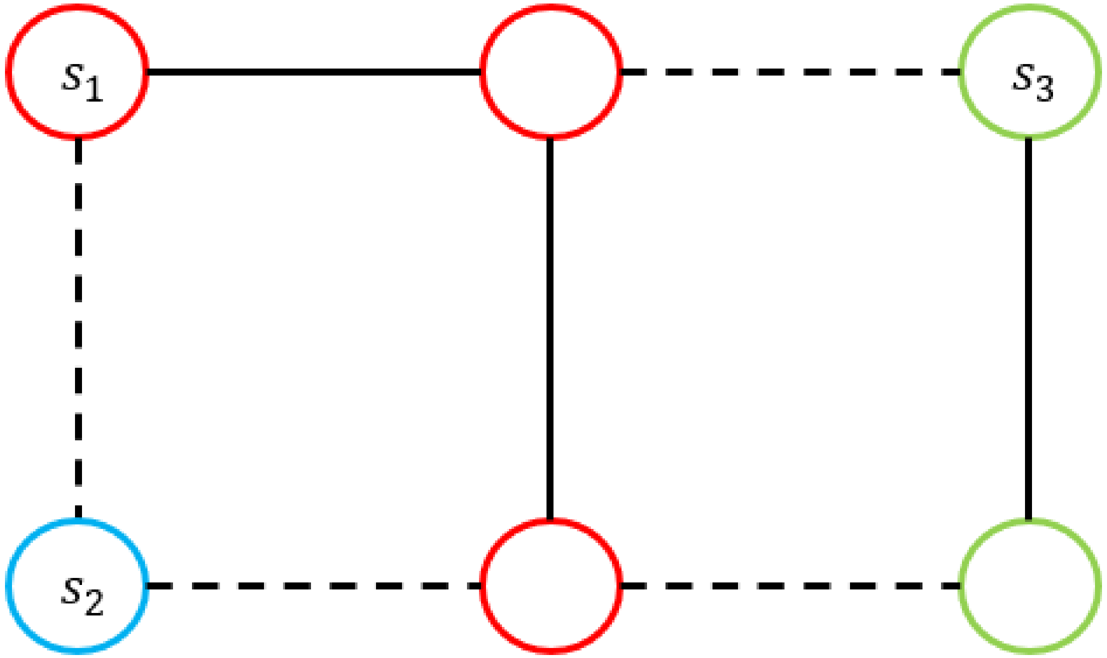

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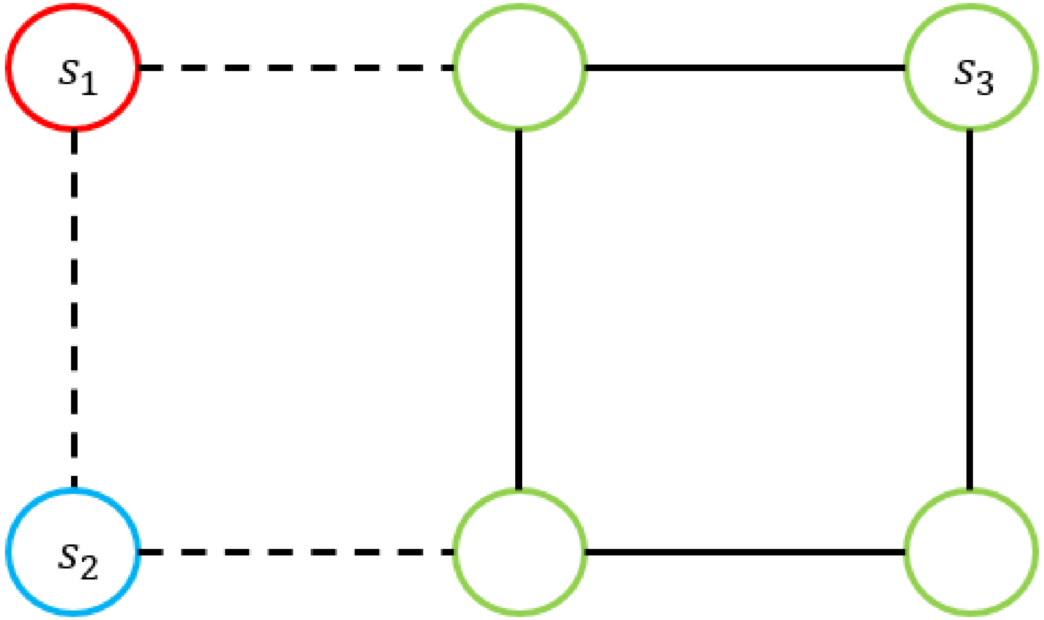

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(c) Design a local search algorithm for multiway cut.

(c) 取$\forall e_1, e_2 \in S$，对$E/S \cup \{e_1, e_2\}$，若$s_1, \ldots, s_k$之间的最小割≤ 1且包含公共边e_3时，$S = S/\{e_1, e_2\} \cup \{e_3\}$

直到S不再改变为止

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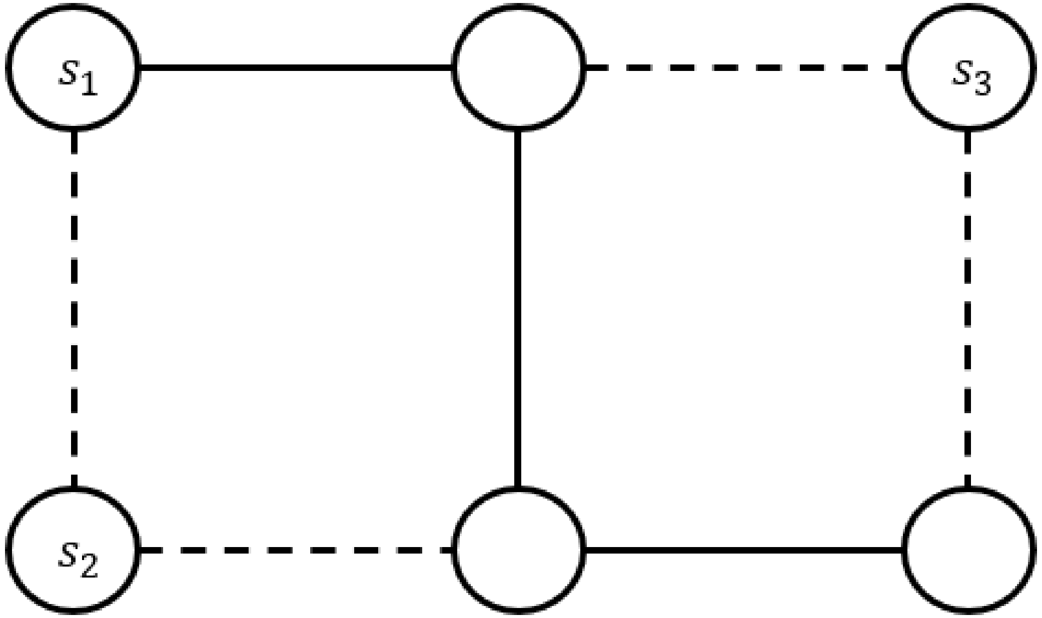

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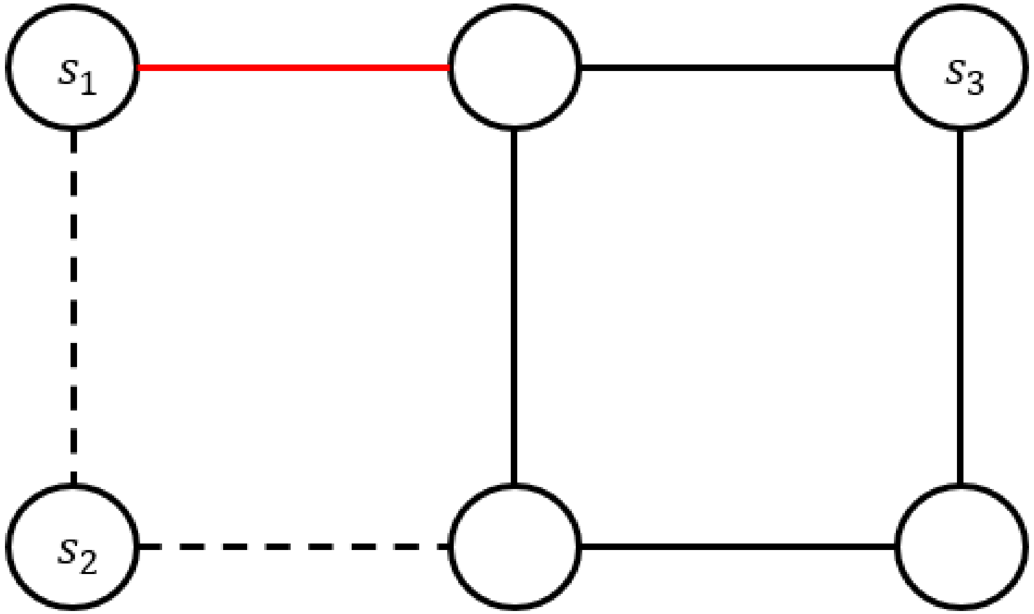

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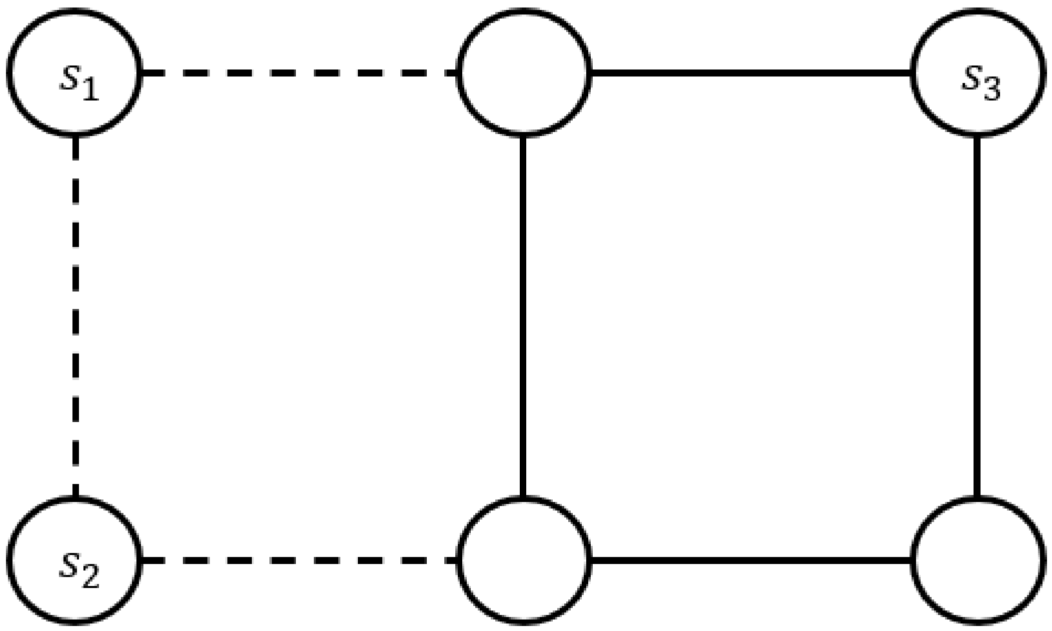

② 习题七

9.9 In the MAXIMUM CUT problem we are given an undirected graph $G = (V, E)$ with a weight $w(e)$ on each edge, and we wish to separate the vertices into two sets S and $V - S$ so that the total weight of the edges between the two sets is as *large* as possible.

For each $S \subseteq V$ define $w(S)$ to be the sum of all $w(e)$ over all edges $\{u, v\}$ such that $|S \cap \{u, v\}| = 1$. Obviously, MAX CUT is about maximizing $w(S)$ over all subsets of V.

Consider the following local search algorithm for MAX CUT:

```
start with any S ⊆ V
while there is a subset S' ⊆ V such that
  ||S'| − |S|| = 1 and w(S') > w(S) do:
    set S = S'
```

(a) Show that this is an approximation algorithm for MAX CUT with ratio 2.

(b) But is it a polynomial-time algorithm?

9.9

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    set $S = S'$

(a) Show that this is an approximation algorithm for MAX CUT with ratio 2.

(b) But is it a polynomial-time algorithm?

(a) 假设该算法得到的解为$(S_1, V/S_1)$，最优解为$(S_o, V/S_o)$

$S_1 \cap S_o = A, S_1 \cap (V/S_o) = B,$

$(V/S_1) \cap S_o = C, (V/S_1) \cap (V/S_o) = D$

$A \cup B = S_1, A \cup C = S\_o$

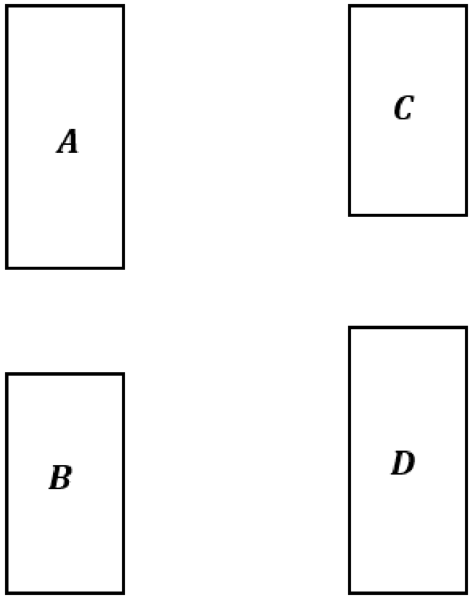
设$w(A, B) = \sum_{e(u,v) \in E, u \in A, v \in B} w(e)$

由于S_1为局部搜索算法所得解，满足

$\forall v \in S_1, w(\{v\}, V/S_1) \geq w(\{v\}, S_1/\{v\})$

$w(A, B) = \sum_{v \in A} w(\{v\}, B) \leq \sum_{v \in A} w(\{v\}, S_1/\{v\})$

$\leq \sum_{v \in A} w(\{v\}, C \cup D) = w(A, C \cup D)$



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Consider the following local search algorithm for MAX CUT:

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      set S = S'
```

(a) Show that this is an approximation algorithm for MAX CUT with ratio 2.

(b) But is it a polynomial-time algorithm?

(a) $w(A,B) \le w(A, C \cup D)$

$w(C,D) \le w(A \cup B, C)$

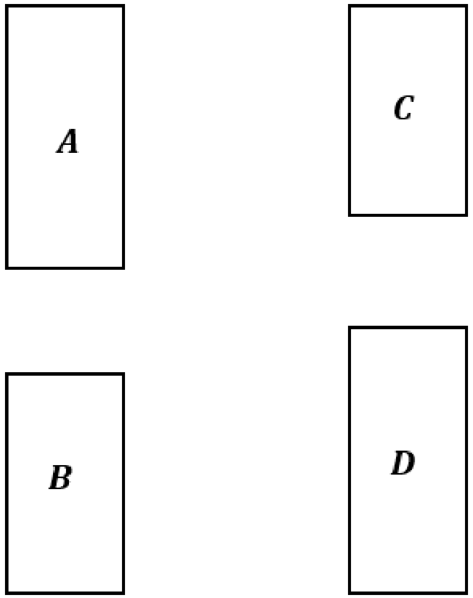
最优解$w(S_o) = w(A,B) + w(A,D) + w(B,C) + w(C,D)$

$\le w(A, C \cup D) + w(A \cup B, D) + w(B, C \cup D) + w(A \cup B, C)$

$= 2 \cdot w(A \cup B, C \cup D) = 2w(S)$

即$w(S_o) \le 2w(S)$

所以近似比$\alpha_S = \min \frac{S}{S_o} = \frac{1}{2}$



② 习题七

9.9

For each $S \subseteq V$ define $w(S)$ to be the sum of all $w(e)$ over all edges $\{u, v\}$ such that $|S \cap \{u, v\}| = 1$. Obviously, MAX CUT is about maximizing $w(S)$ over all subsets of V.

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  ||S'| − |S|| = 1 and w(S') > w(S) do:
    set S = S'
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(a) Show that this is an approximation algorithm for MAX CUT with ratio 2.

(b) But is it a polynomial-time algorithm?

(b) 每一轮最多考虑$|V|$个点是否改变位置

每一轮如果S更新，则割边数量至少+1，当S不再更新时算法结束，因此最多更新$|E|$轮

从一个确定的S开始时，算法复杂度为$O(|V| \cdot |E|)$