



## Algorithm Design XII

Linear Programming I

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# **An Introduction to Linear Programming**



A linear programming problem gives a set of variables, and assigns real values to them so as to

- 1 satisfy a set of linear equations and/or linear inequalities involving these variables, and
- 2 maximize or minimize a given linear objective function.

## Example: Profit Maximization



A boutique chocolatier has two products:

- triangular chocolates, called **Pyramide**,
- and the more decadent and deluxe **Pyramide Nuit**.

**Q:** How much of each should it produce to **maximize** profits?

- Every box of **Pyramide** has a profit of **\$1**.
- Every box of **Nuit** has a profit of **\$6**.
- The daily demand is limited to at most **200** boxes of **Pyramide** and **300** boxes of **Nuit**.
- The current workforce can produce a total of at most **400** boxes of chocolate per day.

## LP Formulation



$$\begin{array}{ll}\text{Objective function} & \max x_1 + 6x_2 \\ \text{Constraints} & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0\end{array}$$

A linear equation in  $x_1$  and  $x_2$  defines a line in the two-dimensional (2D) plane, and a linear inequality designates a half-space, the region on one side of the line.

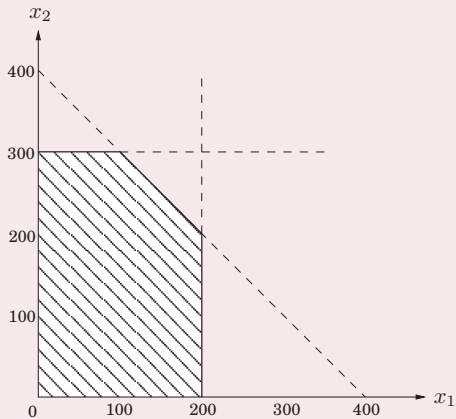
The set of all feasible solutions of this linear program is the intersection of five half-spaces.

It is a convex polygon.

# The Convex Polygon



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## The Optimal Solution



We want to find the point in this polygon at which the objective function is **maximized**.

The points with a profit of  $c$  dollars lie on the line  $x_1 + 6x_2 = c$ , which has a **slope** of  $-1/6$ .

As  $c$  increases, this “**profit line**” moves parallel to itself, up and to the right.

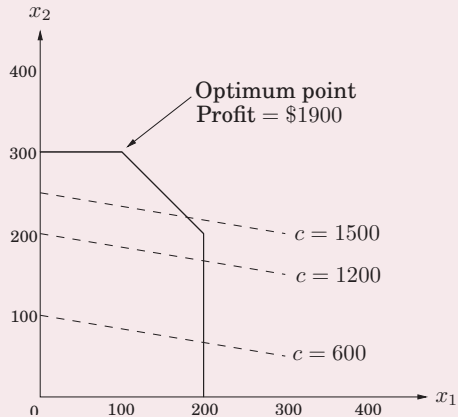
Since the goal is to **maximize**  $c$ , we must move the line as far up as possible, while still touching the **feasible region**.

The optimum solution will be the very last feasible point that the profit line sees and must therefore be a **vertex of the polygon**.

# The Convex Polygon



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It is a general rule of linear programs that the optimum is achieved at a vertex of the feasible region.

The only exceptions are cases in which there is **no optimum**; this can happen in two ways:

- 1 The linear program is **infeasible**; that is, the constraints are so tight that it is impossible to satisfy all of them.
  - For instance,  $x \leq 1, x \geq 2$ .
- 2 The constraints are so loose that the feasible region is **unbounded**, and it is possible to achieve arbitrarily high objective values.
  - For instance,  $\max x_1 + x_2$
  - $x_1, x_2 \geq 0$

# Solving Linear Programs



Linear programs (LPs) can be solved by the **simplex method**, devised by **George Dantzig** in 1947.

This algorithm starts at a **vertex**, and repeatedly looks for an **adjacent vertex** of better objective value.

It does **hill-climbing** on the vertices of the polygon, walking from neighbor to neighbor so as to **steadily increase profit** along the way.

Upon reaching a vertex that has no better neighbor, simplex declares it to be optimal and halts.



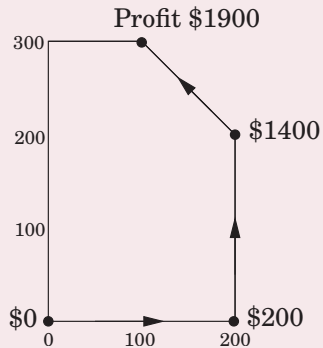
Q: Why does this local test imply **global optimality**?

By simple **geometry**. Since all the vertex's neighbors lie below the line, the rest of the **feasible polygon** must also lie below this line.

## The Example



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The chocolatier introduces a third and even more exclusive chocolates, called **Pyramide Luxe**. One box of these will bring in a profit of \$13.

Let  $x_1, x_2, x_3$  denote the number of boxes of each chocolate produced daily, with  $x_3$  referring to **Luxe**.

The old constraints on  $x_1$  and  $x_2$  persist. The labor restriction now extends to  $x_3$  as well: the sum of all three variables is at most 400.

**Nuit** and **Luxe** require the same packaging machinery. Luxe uses it **three times** as much, which imposes another constraint  $x_2 + 3x_3 \leq 600$ .



$$\max x_1 + 6x_2 + 13x_3$$

$$x_1 \leq 200$$

$$x_2 \leq 300$$

$$x_1 + x_2 + x_3 \leq 400$$

$$x_2 + 3x_3 \leq 600$$

$$x_1, x_2, x_3 \geq 0$$

The space of solutions is now **three-dimensional**.

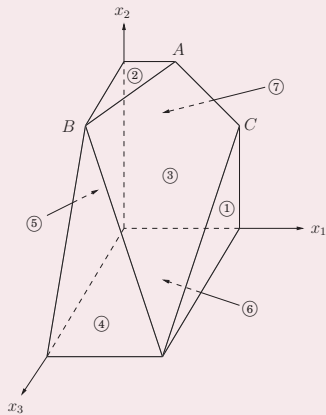
Each **linear equation** defines a **3D plane**, and **each inequality** a **half-space** on one side of the plane.

The feasible region is an intersection of seven half-spaces, a **polyhedron**.

A profit of  $c$  corresponds to the plane  $x_1 + 6x_2 + 13x_3 = c$ .

As  $c$  increases, this profit-plane moves parallel to itself, further into the positive **orthant** until it no longer touches the feasible region.

## The Example





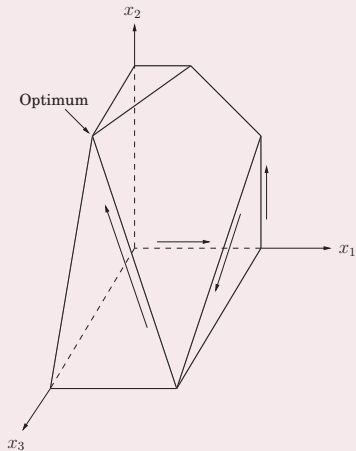
The point of final contact is the **optimal vertex**:  $(0, 300, 100)$ , with total **profit** \$3100.

**Q:** How would the **simplex** algorithm behave on this modified problem?

A possible **trajectory**

$$\frac{(0, 0, 0)}{\$0} \rightarrow \frac{(200, 0, 0)}{\$200} \rightarrow \frac{(200, 200, 0)}{\$1400} \rightarrow \frac{(200, 0, 200)}{\$2800} \rightarrow \frac{(0, 300, 100)}{\$3100}$$

## The Example



## Integer Linear Programming and Rounding

## Example: Production Planning



The company makes **handwoven carpets**, a product for which the demand is extremely seasonal.

Our analyst has just obtained demand estimates for all months of the next calendar year:  
 $d_1, d_2, \dots, d_{12}$ , ranging from 440 to 920.

Currently with 30 employees, each of whom makes 20 carpets per month and gets a monthly salary of \$2000.

With no initial surplus of carpets.

## Example: Production Planning



Q: How can we handle the **fluctuations in demand**? There are three ways:

- 1 **Overtime**. Overtime pay is **80%** more than regular pay. Workers can put in at most **30%** overtime.
- 2 **Hiring and firing**, costing **\$320** and **\$400**, respectively, per worker.
- 3 **Storing surplus production**, costing **\$8** per carpet per month. Currently without stored carpets on hand, and without any carpets stored at the end of year.



- $w_i$  = number of workers during  $i$ -th month;  $w_0 = 30$ .
- $x_i$  = number of carpets made during  $i$ -th month.
- $o_i$  = number of carpets made by overtime in month  $i$ .
- $h_i, f_i$  = number of workers hired and fired, respectively, at beginning of month  $i$ .
- $s_i$  = number of carpets stored at end of month  $i$ ;  $s_0 = 0$ .



All variables must be **nonnegative**:

$$w_i, x_i, o_i, h_i, f_i, s_i \geq 0, i = 1, \dots, 12$$

The total number of carpets made per month consists of regular production plus overtime:

$$x_i = 20w_i + o_i$$

$$i = 1, \dots, 12.$$

The **number of workers** can potentially change at the start of each month:

$$w_i = w_{i-1} + h_i - f_i$$



The number of carpets stored at the end of each month is what we started with, plus the number we made, minus the demand for the month:

$$s_i = s_{i-1} + x_i - d_i$$

And overtime is limited:

$$o_i \leq 6w_i$$





The **objective function** is to minimize the total cost:

$$\min 2000 \sum_i w_i + 320 \sum_i h_i + 400 \sum_i f_i + 8 \sum_i s_i + 180 \sum_i o_i$$



The optimum solution might turn out to be **fractional**; for instance, it might involve hiring **10.6** workers in the month of March.

This number would have to be **rounded** to either **10** or **11** in order to make sense, and the overall cost would then increase correspondingly.

In the example, most of the variables take on fairly large values, and thus **rounding** is unlikely to affect things too much.



There are other LPs, in which rounding decisions have to be made very carefully to end up with an integer solution of reasonable quality.

There is a tension in linear programming between the ease of obtaining fractional solutions and the desirability of integer ones.

In NP problems, finding the optimum integer solution of an LP is an important but very hard problem, called integer linear programming.

## Duality

## Profit Maximization Revisit



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Recall:

$$\begin{aligned}\max \quad & x_1 + 6x_2 \\ & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0\end{aligned}$$

Simplex declares the optimum solution to be  $(x_1, x_2) = (100, 300)$ , with objective value 1900.

Can this answer be checked somehow?

We take the first inequality and add it to six times the second inequality:

$$x_1 + 6x_2 \leq 2000$$

## Profit Maximization Revisit



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Multiplying the three inequalities by 0, 5, and 1, respectively, and adding them up yields

$$x_1 + 6x_2 \leq 1900$$

# Multipliers



Let's investigate the issue by describing what we expect of these three multipliers, call them  $y_1$ ,  $y_2$ ,  $y_3$ .

Multiplier	Inequality
$y_1$	$x_1 \leq 200$
$y_2$	$x_2 \leq 300$
$y_3$	$x_1 + x_2 \leq 400$

These  $y_i$ 's must be **nonnegative**, otherwise they are unqualified to multiply inequalities.

After the multiplication and addition steps, we get the bound:

$$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$

We want the left-hand side to look like the **objective function**  $x_1 + 6x_2$  so that the right-hand side is an upper bound on the **optimum solution**.



$$x_1 + 6x_2 \leq 200y_1 + 300y_2 + 400y_3$$

if

$$y_1, y_2, y_3 \geq 0$$

$$y_1 + y_3 \geq 1$$

$$y_2 + y_3 \geq 6$$



# The Dual Program



We can easily find  $y$ 's that satisfy the inequalities on the right by simply making them large enough, for example  $(y_1, y_2, y_3) = (5, 3, 6)$ .

These particular multipliers tell us that the optimum solution of the LP is at most

$$200 \cdot 5 + 300 \cdot 3 + 400 \cdot 6 = 4300$$

What we want is a bound as tight as possible, so we minimize

$$200y_1 + 300y_2 + 400y_3$$

subject to the preceding inequalities. This is a new linear program!

# The Dual Program



$$\begin{aligned}\min & 200y_1 + 300y_2 + 400y_3 \\ & y_1 + y_3 \geq 1 \\ & y_2 + y_3 \geq 6 \\ & y_1, y_2, y_3 \geq 0\end{aligned}$$

Any feasible value of this **dual LP** is an **upper bound** on the **original primal LP**.

If we find a pair of primal and dual feasible values that are equal, then they must both be **optimal**.

Here is just such a pair:

- **Primal:**  $(x_1, x_2) = (100, 300)$ ;
- **Dual:**  $(y_1, y_2, y_3) = (0, 5, 1)$ .

They both have value **1900** and certify each other's optimality.

## Matrix-Vector Form and Its Dual



Primal LP

$$\begin{aligned} \max \quad & c^T \mathbf{x} \\ & A\mathbf{x} \leq b \\ & \mathbf{x} \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & \mathbf{y}^T b \\ & \mathbf{y}^T A \geq c^T \\ & \mathbf{y} \geq 0 \end{aligned}$$

Primal LP:

$$\begin{aligned} \max \quad & c_1 x_1 + \cdots + c_n x_n \\ & a_{i1} x_1 + \cdots + a_{in} x_n \leq b_i \quad \text{for } i \in I \\ & a_{i1} x_1 + \cdots + a_{in} x_n = b_i \quad \text{for } i \in E \\ & x_j \geq 0 \quad \text{for } j \in N \end{aligned}$$

Dual LP:

$$\begin{aligned} \min \quad & b_1 y_1 + \cdots + b_m y_m \\ & a_{1j} y_1 + \cdots + a_{mj} y_m \geq c_j \quad \text{for } j \in N \\ & a_{1j} y_1 + \cdots + a_{mj} y_m = c_j \quad \text{for } j \notin N \\ & y_i \geq 0 \quad \text{for } i \in I \end{aligned}$$

## Matrix-Vector Form and Its Dual



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$$\begin{aligned}\max & x_1 + 6x_2 \\ & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0\end{aligned}$$

$$\begin{aligned}\min & 200y_1 + 300y_2 + 400y_3 \\ & y_1 + y_3 \geq 1 \\ & y_2 + y_3 \geq 6 \\ & y_1, y_2, y_3 \geq 0\end{aligned}$$



## Theorem (Duality)

*If a linear program has a **bounded optimum**, then so does its **dual**, and the two optimum values **coincide**.*

## Complementary Slackness



The number of variables in the dual is equal to that of constraints in the primal and the number of constraints in the dual is equal to that of variables in the primal.

An inequality constraint has slack if the slack variable is positive.

The **complementary slackness** refers to a relationship between the slackness in a primal constraint and the associated dual variable.



$$\max x_1 + 6x_2$$

$$x_1 \leq 200$$

$$x_2 \leq 300$$

$$x_1 + x_2 \leq 400$$

$$x_1, x_2 \geq 0$$

$$x_1 = 100, x_2 = 300$$

$$\min 200y_1 + 300y_2 + 400y_3$$

$$y_1 + y_3 \geq 1$$

$$y_2 + y_3 \geq 6$$

$$y_1, y_2, y_3 \geq 0$$

$$y_1 = 0, y_2 = 5, y_3 = 1$$



## Theorem

Assume LP problem  $(P)$  has a solution  $x^*$  and its dual problem  $(D)$  has a solution  $y^*$ .

- 1 If  $x_j^* > 0$ , then the  $j$ -th constraint in  $(D)$  is binding.
- 2 If the  $j$ -th constraint in  $(D)$  is not binding, then  $x_j^* = 0$ .
- 3 If  $y_i^* > 0$ , then the  $i$ -th constraint in  $(P)$  is binding.
- 4 If the  $i$ -th constraint in  $(P)$  is not binding, then  $y_i^* = 0$ .



## A Concrete Example for Duality

# Brewery Problem



Small brewery produces ale and beer.

- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.

Beverage	Corn(pounds)	Hops(ounces)	Malt(pounds)	Profit(\$)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

## LP and its Dual



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$$\begin{aligned} \max \quad & 13x_1 + 23x_2 \\ \text{s.t.} \quad & 5x_1 + 15x_2 \leq 480 \\ & 4x_1 + 4x_2 \leq 160 \\ & 35x_1 + 20x_2 \leq 1190 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$x_1^* = 12, x_2^* = 28$$

**Brewer:** find optimal mix of beer  
and ale to maximize profits.

$$\begin{aligned} \min \quad & 480y_1 + 160y_2 + 1190y_3 \\ \text{s.t.} \quad & 5y_1 + 4y_2 + 35y_3 \geq 13 \\ & 15y_1 + 4y_2 + 20y_3 \geq 23 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

$$y_1^* = 1, y_2^* = 2, y_3^* = 0$$

**Entrepreneur:** buy individual  
resources from brewer at min cost.

## LP Duality: Sensitivity Analysis



Q. How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?

A. corn \$1, hops \$2, malt \$0.

Q. Suppose a new product “light beer” is proposed. It requires 2 corn, 5 hops, 24 malt. How much profit must be obtained from light beer to justify diverting resources from production of beer and ale?

A. At least  $2 (\$1) + 5 (\$2) + 24 (\$0) = \$12$  / barrel.