Algorithm Design VIII

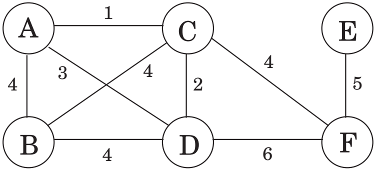
Greedy Algorithms

Guoqiang Li
School of Software

SHANGHAI JIAO TONG UNIVERSITY

Minimum Spanning Trees

Build a Network

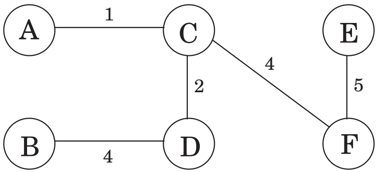


Suppose you are asked to network a collection of computers by linking selected pairs of them.

This translates into a graph problem in which

- nodes are computers,
- undirected edges are potential links, each with a maintenance cost.

Build a Network



The goal is to

- pick enough of these edges that the nodes are connected,
- the total maintenance cost is minimum.

One immediate observation is that the optimal set of edges cannot contain a cycle.

Properties of the Optimal Solutions

Lemma (1)

Removing a cycle edge cannot disconnect a graph.

So the solution must be connected and acyclic: undirected graphs of this kind are called trees.

A tree with minimum total weight, is a minimum spanning tree, MST.

Input: An undirected graph $G = (V, E)$; edge weights w_e

Output: A tree $T = (V, E')$ with $E' \subseteq E$ that minimizes

$$\texttt{weight}(T) = \sum_{e \in E'} w_e$$

Lemma (2)

A tree on n nodes has $n-1$ edges.

To build the tree one edge at a time, starting from an empty graph.

Each of the n nodes is disconnected from the others, in a connected component by itself.

As edges are added, these components merge. Since each edge unites two different components, exactly $n-1$ edges are added by the time the tree is fully formed.

When a particular edge (u, v) comes up, we can be sure that u and v lie in separate connected components, for otherwise there would already be a path between them and this edge would create a cycle.

Lemma (3)

Any connected, undirected graph $G = (V, E)$ with $|E| = |V| - 1$ is a tree.

It is the converse of Lemma (2). We just need to show that G is acyclic.

While the graph contains a cycle, remove one edge from this cycle.

The process terminates with some graph $G' = (V, E'), E' \subseteq E$, which is acyclic and, by Lemma (1), is also connected.

Therefore G' is a tree, whereupon $|E'| = |V| - 1$ by Lemma (2). So $E' = E$, no edges were removed, and G was acyclic to start with.

Lemma (4)

An undirected graph is a tree if and only if there is a unique path between any pair of nodes.

In a tree, any two nodes can only have one path between them; for if there were two paths, the union of these paths would contain a cycle.

On the other hand, if a graph has a path between any two nodes, then it is connected. If these paths are unique, then the graph is also acyclic.

A Greedy Approach

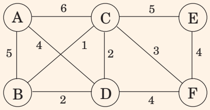
Kruskal's minimum spanning tree algorithm starts with the empty graph and then selects edges from E according to the following rule.

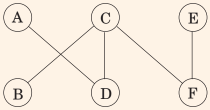
Repeatedly add the next lightest edge that doesn't produce a cycle.

Example

Starting with an empty graph and then attempt to add edges in increasing order of weight

$$B-C; C-D; B-D; C-F; D-F; E-F; A-D; A-B; C-E; A-C$$





The Cut Property

Lemma

Suppose edges X are part of a MST of $G = (V, E)$. Pick any subset of nodes S for which X does not cross between S and $V \setminus S$, and let e be the lightest edge across this partition. Then

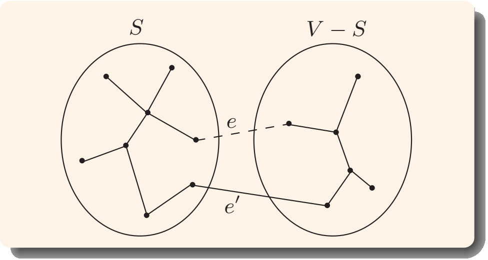
$$X \cup \{e\}$$

is part of some MST.

The Cut Property

A cut is any partition of the vertices into two groups, S and $V \backslash S$.

It is safe to add the lightest edge across any cut, provided X has no edges across the cut.



Proof of the Cut Property

Shanghai Jiao Tong University

Proof:

Edges X are part of some MST T; if the new edge e also happens to be part of T, then there is nothing to prove.

So assume e is not in T. We will construct a different MST T' containing $X \cup \{e\}$ by altering T slightly, changing just one of its edges.

Add edge e to T. Since T is connected, it already has a path between the endpoints of e, so adding e creates a cycle.

This cycle must also have some other edge e' across the cut $(S, V \backslash S)$. If we now remove e'

$$T' = T \cup \{e\} \backslash \{e'\}$$

which we will show to be a tree.

T' is connected by Lemma (1), since e' is a cycle edge. And it has the same number of edges as T; so by Lemma (2) and Lemma (3), it is also a tree.

Proof of the Cut Property

Proof:

T' is a minimum spanning tree, since

$$weight(T') = weight(T) + w(e) - w(e')$$

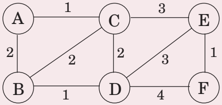
Both e and e' cross between S and $V \backslash S$, and e is the lightest edge of this type. Therefore $w(e) \leq w(e')$, and

$$weight(T') \leq weight(T)$$

Since T is an MST, it must be the case that $weight(T') = weight(T)$ and that T' is also an MST.

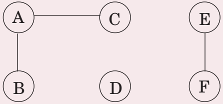
An Example of Cut Property

(a)

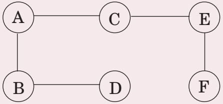


(b)

Edges X:

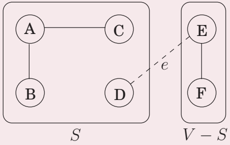


MST T:

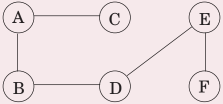


(c)

The cut:



MST T':



Kruskal's Algorithm

SHANGHAI JIAO TONG UNIVERSITY

```
KRUSKAL(G, w)
input  : A connected undirected graph G = (V, E), with edge weight w_e
output: A minimum spanning tree defined by the edges X

for all u ∈ V do
    makeset (u);
end
X = { };
Sort the edges E by weight;
for all (u, v) ∈ E in increasing order of weight do
    if find (u) ≠ find (v) then
        add (u, v) to X;
        union (u,v)
    end
end
```

Data Structure Retailer: Disjoint Sets

`makeset`(x)	create a singleton set containing x	$\lvert V \rvert$
`find`(x)	find the set that x belong to	$2 \cdot \lvert E \rvert$
`union`(x, y)	merge the sets containing x and y	$\lvert V \rvert - 1$

Prim’s Algorithm

A General Kruskal's Algorithm

$X = \{\ \}$;
repeat until $|X| = |V| - 1$;
    pick a set $S \subset V$ for which X has no edges between S and $V - S$;
    let $e \in E$ be the minimum-weight edge between S and $V - S$;
    $X = X \cup \{e\}$;

Prim's Algorithm

A popular alternative to Kruskal's algorithm is Prim's, in which the intermediate set of edges X always forms a subtree, and S is chosen to be the set of this tree's vertices.

On each iteration, the subtree defined by X grows by one edge.

The lightest edge between a vertex in S and a vertex outside S. We can equivalently think of S as growing to include the vertex $v \notin S$ of smallest `cost`:

$$\texttt{cost}(v) = \min_{u \in S} w(u, v)$$

The Algorithm


```
PRIM(G, w)
input  : A connected undirected graph G = (V, E), with edge weights w_e
output: A minimum spanning tree defined by the array prev

for all u ∈ V do
    cost(u) = ∞;
    prev(u) = nil;
end
pick any initial node u_0;
cost(u_0) = 0;
H =makequeue(V)\\ using cost-values as keys;
while H is not empty do
    v=deletemin(H);
    for each (v, z) ∈ E do
        if cost(z) > w(v, z) then
            cost(v) = w(v, z); prev(z) = v;
            decreasekey (H,z);
        end
    end
end
```

Dijkstra's Algorithm


```
DIJKSTRA(G, l, s)
input  : Graph G = (V, E), directed or undirected; positive edge length {l_e | e ∈ E};
         Vertex s ∈ V
output: For all vertices u reachable from s, dist(u) is the set to the distance from s to
         u

for all u ∈ V do
    dist(u) = ∞;
    prev(u) = nil;
end
dist(s) = 0;
H =makequeue(V) \\ using dist-values as keys;
while H is not empty do
    u=deletemin(H);
    for all edge (u, v) ∈ E do
        if dist(v) > dist(u) + l(u, v) then
            dist(v) = dist(u) + l(u, v);  prev(v) = u;
            decreasekey(H,v);
        end
    end
end
```

Think About

Let C be a cycle with no red edges, and select an uncolored edge of C of max cost and color it red.

Let D be a edge set crossing a cut with no blue edges, and select an uncolored edge in D of min cost and color it blue.

Apply the red and blue rules nondeterministically until all edges are colored.

The blue edges form an MST.

Set Cover

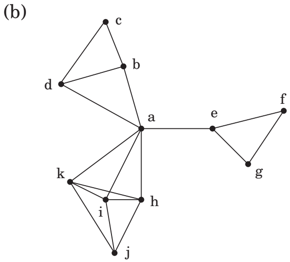
The Problem

A county is in its early stages of planning and is deciding where to put schools.

There are only two constraints:

- each school should be in a town,
- and no one should have to travel more than 30 miles to reach one of them.

Q: What is the minimum number of schools needed?



The Problem

This is a typical (cardinality) set cover problem.

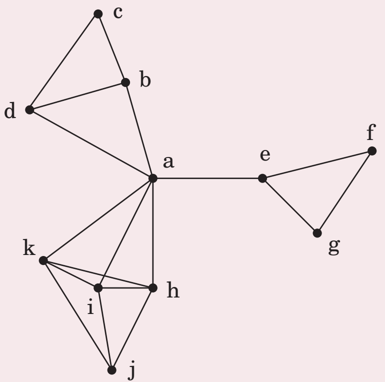
- For each town x, let S_x be the set of towns within 30 miles of it.
- A school at x will essentially "cover" these other towns.
- The question is then, how many sets S_x must be picked in order to cover all the towns in the county?

Set Cover Problem

Set Cover

- Input: A set of elements B, sets $S_1, \ldots, S_m \subseteq B$
- Output: A selection of the S_i whose union is B.
- Cost: Number of sets picked.

The Example



Performance Ratio

SHANGHAI JIAO TONG UNIVERSITY

Lemma

Suppose B contains n elements and that the optimal cover consists of OPT sets. Then the greedy algorithm will use at most $\ln n \cdot OPT$ sets.

Proof.

Let n_t be the number of elements still not covered after t iterations of the greedy algorithm (so $n_0 = n$).

Since these remaining elements are covered by the optimal OPT sets, there must be some set with at least n_t/OPT of them.

Therefore, the greedy strategy will ensure that

$$n_{t+1} \leq n_t - \frac{n_t}{OPT} = n_t(1 - \frac{1}{OPT})$$

which by repeated application implies

$$n_t \leq n_0(1 - \frac{1}{OPT})^t$$

Performance Ratio

A more convenient bound can be obtained from the useful inequality

$$1 - x \le e^{-x} \text{ for all } x$$

with equality if and only if $x = 0$,

Thus

$$n_t \le n_0 \left(1 - \frac{1}{OPT}\right)^t < n_0 \left(e^{-\frac{1}{OPT}}\right)^t = n e^{-\frac{t}{OPT}}$$

At $t = \ln n \cdot OPT$, therefore, n_t is strictly less than $ne^{-\ln n} = 1$, which means no elements remain to be covered.

Why Greedy Does Not Work: Coin Changing

Coin changing

Goal. Given U. S. currency denominations $\{1, 5, 10, 25, 100\}$, devise a method to pay amount to customer using fewest coins.

Example $\$34$.

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Example $\$2.89$.

Cashier's algorithm


```
CASHIERS-ALGORITHM(x, c_1, c_2, ..., c_n)

SORT n coin denominations so that 0 < c_1 < c_2 < ... < c_n ;
S ← ∅;
while x > 0 do
    k ← largest coin denomination c_k such that c_k ≤ x;
    if no such k then  RETURN no solution;
    else
        x ← x − c_k;
        S ← S ∪ {k};
    end
end
RETURN S;
```

Quiz

Is the cashier's algorithm optimal?

A Yes, greedy algorithms are always optimal.

B Yes, for any set of coin denominations $c_1 < c_2 < \ldots < c_n$ provided $c_1 = 1$.

C Yes, because of special properties of U.S. coin denominations.

D No.

Cashier's algorithm (for arbitrary coin denominations)

Q. Is cashier's algorithm optimal for any set of denominations?

A. No. Consider U.S. postage: $1, 10, 21, 34, 70, 100, 350, 1225, 1500$.

- Cashier's algorithm: $\$140 = 100 + 34 + 1 + 1 + 1 + 1 + 1 + 1$.
- Optimal: $\$140 = 70 + 70$.

A. No. It may not even lead to a feasible solution if $c_1 > 1$: $7, 8, 9$.

- Cashier's algorithm: $\$15 = 9 + ?$.
- Optimal: $\$15 = 7 + 8$.

Properties of any optimal solution (for U.S. coin denominations)

SHANGHAI JIAO TONG UNIVERSITY

Property. Number of pennies ≤ 4.

Proof. Replace 5 pennies with 1 nickel.

Property. Number of nickels ≤ 1.

Property. Number of quarters ≤ 3.

Property. Number of nickels + number of dimes ≤ 2.

Proof.

- Recall: ≤ 1 nickel.
- Replace 3 dimes and 0 nickels with 1 quarter and 1 nickel;
- Replace 2 dimes and 1 nickel with 1 quarter.

Optimality of cashier's algorithm (for U.S. coin denominations)

SHANGHAI JIAO TONG UNIVERSITY

Theorem

Cashier's algorithm is optimal for U.S. coins $\{1, 5, 10, 25, 100\}$.

A rather formal proof

Proof. by induction on amount to be paid x

Consider optimal way to change $c_k \leq x \leq c_{k+1}$: greedy takes coin k.

Claim that any optimal solution must take coin k.

- if not, it needs enough coins of type $c_1, \ldots, c_{k-1}$ to add up to x.
- table below indicates no optimal solution can do this

Problem reduces to coin-changing $x - c_k$ cents, which, by induction, is optimally solved by cashier's algorithm.

k	c_k	all optimal solutions must satisfy	max value of $c_1, c_2, \ldots c_{k-1}$ in any optimal solution
1	1	$P \leq 4$	none
2	5	$N \leq 1$	4
3	10	$N + D \leq 2$	$4 + 5 = 9$
4	25	$Q \leq 3$	$20 + 4 = 24$
5	100	no limit	$75 + 24 = 99$