Algorithm Design V

Divide and Conquer II

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Counting Inversions

Counting Inversions

Music site tries to match your song preferences with others.

- You rank n songs.
- Music site consults database to find people with similar tastes.

Similarity metric: number of inversions between two rankings.

- My rank: $1, 2, \ldots, n$.
- Your rank: $a_1, a_2, \ldots, a_n$.
- Songs i and j are inverted if $i < j$, but $a_i > a_j$.

	A	B	C	D	E
me	1	2	3	4	5
you	1	3	4	2	5

2 **inversions**: 3-2, 4-2

Divide-and-Conquer

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Divide: separate list into two halves A and B.

Conquer: recursively count inversions in each list.

Combine: count inversions (a, b) with $a \in A$ and $b \in B$.

Return sum of three counts.

input

1	5	4	8	10	2	6	9	3	7

count inversions in left half A

1	5	4	8	10
		5-4		

count inversions in right half B

2	6	9	3	7
	6-3	9-3	9-7	

count inversions (a, b) with $a \in A$ and $b \in B$

1	5	4	8	10	2	6	9	3	7

4-2 4-3 5-2 5-3 8-2 8-3 8-6 8-7 10-2 10- 3 10-6 10-7 10-9

output 1+3+13=17

How to Combine Two Subproblems?

Q. How to count inversions (a, b) with $a \in A$ and $b \in B$?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B.
- For each element $b \in B$,
  - binary search in A to find how many elements in A are greater than b.

list A

7	10	18	3	14

list B

20	23	2	11	16

sort A

3	7	10	14	18

sort B

2	11	16	20	23

binary search to count inversions (a, b) with $a \in A$ and $b \in B$

3	7	10	14	18

2	11	16	20	23
5	2	1	0	0

How to Combine Two Subproblems?

Count inversions (a, b) with $a \in A$ and $b \in B$, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i < b_j$, then a_i is not inverted with any element left in B.
- If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

count inversions (a, b) with $a \in A$ and $b \in B$

3	7	10	a_i	18

2	11	b_j	20	23
5	2			

merge to form sorted list C

2	3	7	10	11					

Algorithm Implementation

SORT-AND-COUNT(L) ;
input : List L
output: Number of inversions in L and L in sorted order

if *List L has one element* **then**
  RETURN $(0, L)$;
end
Divide the list into two halves A and B;
$(r_A, A) \leftarrow$ SORT-AND-COUNT (A);
$(r_B, B) \leftarrow$ SORT-AND-COUNT (B);
$(r_{AB}, L) \leftarrow$ MERGE-AND-COUNT (A,B);
RETURN $(r_A + r_B + r_{AB}, L)$;

Proposition

The sort-and-count algorithm counts the number of inversions in a permutation of size n in $O(n \log n)$ time.

Proof.

$$T(n) = 2 \cdot T(\lceil n/2 \rceil) + \Theta(n)$$

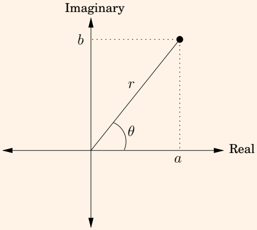
Complex Number

Complex Number

$z = a + bi$ is plotted at position (a, b).

In its polar coordinates, denoted (r, θ), rewrite as

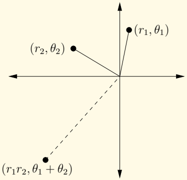
$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$



- length: $r = \sqrt{a^2 + b^2}$.
- angle: $\theta \in [0, 2\pi)$.
- θ can always be reduced modulo 2π.

Basic arithmetic:

- $-z = (r, \theta + \pi)$.
- $(r_1, \theta_1) \times (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2)$.
- If z is on the unit circle (i.e., $r = 1$), then $z^n = (1, n\theta)$.



The n-th Complex Roots of Unity

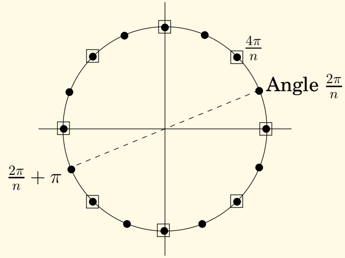
Solutions to the equation $z^n = 1$

- by the multiplication rules: solutions are $z = (1, \theta)$, for θ a multiple of $2\pi/n$.
- It can be represented as

$$1, \omega, \omega^2, \ldots, \omega^{n-1}$$

where

$$\omega = e^{2\pi i/n}$$



For n is even:

- These numbers are plus-minus paired.
- Their squares are the $(n/2)$-nd roots of unity.

Complex Conjugate

The complex conjugate of a complex number $z = re^{i\theta}$ is $z^* = re^{-i\theta}$.

The complex conjugate of a vector (or a matrix) is obtained by taking the complex conjugates of all its entries.

The angle between two vectors $u = (u_0, \ldots, u_{n-1})$ and $v(v_0, \ldots, v_{n-1})$ in $\mathbb{C}^n$ is just a scaling factor times their inner product

$$u \cdot v^* = u_0 v_0^* + u_1 v_1^* + \ldots + u_{n-1} v_{n-1}^*$$

The above quantity is maximized when the vectors lie in the same direction and is zero when the vectors are orthogonal to each other.

The Fast Fourier Transform

Polynomial multiplication

If $A(x) = a_0 + a_1 x + \ldots + a_d x^d$ and $B(x) = b_0 + b_1 x + \ldots + b_d x^d$, their product

$$C(x) = c_0 + c_1 x + \ldots + c_{2d} x^{2d}$$

has coefficients

$$c_k = a_0 b_k + a_1 b_{k-1} + \ldots + a_k b_0 = \sum_{i=0}^{k} a_i b_{k-i}$$

where for $i > d$, take a_i and b_i to be zero.

Computing c_k from this formula take $O(k)$ step, and finding all $2d + 1$ coefficients would therefore seem to require $\Theta(d^2)$ time.

Q: Can we do better?

An alternative representation

Fact: A degree-d polynomial is uniquely characterized by its values at any $d+1$ distinct points.

We can specify a degree-d polynomial $A(x) = a_0 + a_1 x + \ldots + a_d x^d$ by either of the following:

- Its coefficients $a_0, a_1, \ldots, a_d$. (coefficient representation).
- The values $A(x_0), A(x_1), \ldots A(x_d)$ (value representation).

$$\text{coefficient representation} \underset{\text{interpolation}}{\overset{\text{evaluation}}{\rightleftarrows}} \text{value representation}$$

An alternative representation

$$\text{coefficient representation} \underset{\text{interpolation}}{\overset{\text{evaluation}}{\rightleftarrows}} \text{value representation}$$

The product $C(x)$ has degree $2d$, it is determined by its value at any $2d+1$ points.

Its value at any given point z is just $A(z)$ times $B(z)$.

Therefore, polynomial multiplication takes linear time in the value representation.

The algorithm

Input: Coefficients of two polynomials, $A(x)$ and $B(x)$, of degree d
Output: Their product $C = A \cdot B$

Selection
Pick some points $x_0, x_1, \ldots, x_{n-1}$, where $n \geq 2d + 1$.

Evaluation
Compute $A(x_0), A(x_1), \ldots, A(x_{n-1})$ and $B(x_0), B(x_1), \ldots, B(x_{n-1})$.

Multiplication
Compute $C(x_k) = A(x_k)B(x_k)$ for all $k = 0, \ldots, n-1$.

Interpolation
Recover $C(x) = c_0 + c_1 x + \ldots + c_{2d} x^{2d}$

Fast Fourier Transform

The selection step and the multiplications are just linear time:

- In a typical setting for polynomial multiplication, the coefficients of the polynomials are real number.
- Moreover, are small enough that basic arithmetic operations take unit time.

Evaluating a polynomial of degree $d \leq n$ at a single point takes $O(n)$, and so the baseline for n points is $\Theta(n^2)$.

The Fast Fourier Transform (FFT) does it in just $O(n \log n)$ time, for a particularly clever choice of $x_0, \ldots, x_{n-1}$.

Evaluation by divide-and-conquer

Q: How to make it efficient?

First idea, we pick the n points,

$$\pm x_0, \pm x_1, \ldots, \pm x_{n/2-1}$$

then the computations required for each $A(x_i)$ and $A(-x_i)$ overlap a lot, because the even power of x_i coincide with those of $-x_i$.

We need to split $A(x)$ into its odd and even powers, for instance

$$3 + 4x + 6x^2 + 2x^3 + x^4 + 10x^5 = (3 + 6x^2 + x^4) + x(4 + 2x^2 + 10x^4)$$

More generally

$$A(x) = A_e(x^2) + xA_o(x^2)$$

where $A_e(\cdot)$, with the even-numbered coefficients, and $A_o(\cdot)$, with the odd-numbered coefficients, are polynomials of degree $\leq n/2 - 1$.

Evaluation by divide-and-conquer

Given paired points $\pm x_i$, the calculations needed for $A(x_i)$ can be recycled toward computing $A(-x_i)$:

$$\begin{aligned} A(x_i) &= A_e(x_i^2) + x_i A_o(x_i^2) \\ A(-x_i) &= A_e(x_i^2) - x_i A_o(x_i^2) \end{aligned}$$

Evaluating $A(x)$ at n paired points $\pm x_0, \ldots, \pm x_{n/2-1}$ reduces to evaluating $A_e(x)$ and $A_o(x)$ at just $n/2$ points, $x_0^2, \ldots, x_{n/2-1}^2$.

If we could recurse, we would get a divide-and-conquer procedure with running time

$$T(n) = 2T(n/2) + O(n) = O(n \log n)$$

How to choose n points?

Aim: To recurse at the next level, we need the $n/2$ evaluation points $x_0^2, x_1^2, \ldots, x_{n/2-1}^2$ to be themselves plus-minus pairs.

Q: How can a square be negative?

- We use complex numbers.

At the very bottom of the recursion, we have a single point, 1, in which case the level above it must consist of its square roots, $\pm\sqrt{1} = \pm 1$.

The next level up then has $\pm\sqrt{+1} = \pm 1$, as well as the complex numbers $\pm\sqrt{-1} = \pm i$.

By continuing in this manner, we eventually reach the initial set of n points: the complex $n\,th$ roots of unity, that is the n complex solutions of the equation

$$z^n = 1$$

The n-th complex roots of unity

Solutions to the equation $z^n = 1$

- by the multiplication rules: solutions are $z = (1, \theta)$, for θ a multiple of $2\pi/n$.
- It can be represented as
$$1, \omega, \omega^2, \ldots, \omega^{n-1}$$
where
$$\omega = e^{2\pi i/n}$$

For n is even:

- These numbers are plus-minus paired.
- Their squares are the $(n/2)$-nd roots of unity.

The FFT algorithm


```
FFT(A, ω)
```

input : coefficient reprentation of a polynomial $A(x)$ of degree $\leq n-1$, where n is a power of 2;
ω, an n-th root of unity

output: value representation $A(\omega^0), \ldots, A(\omega^{n-1})$

if $\omega = 1$ **then** `return` $A(1)$;
express $A(x)$ in the form $A_e(x^2) + xA_o(x^2)$;
call `FFT(`A_e,ω^2`)` to evaluate A_e at even powers of ω;
call `FFT(`A_o,ω^2`)` to evaluate A_o at even powers of ω;
for $j = 0$ *to* $n-1$ **do**
  compute $A(\omega^j) = A_e(\omega^{2j}) + \omega^j A_o(\omega^{2j})$;
end
`return(`$A(\omega^0), \ldots, A(\omega^{n-1})$`);`

Interpolation

$\texttt{FFT}$ moves from coefficients to values in time just $O(n \log n)$, when the points $\{x_i\}$ are complex n-th roots of unity $(1, \omega, \omega^2, \ldots, \omega^{n-1})$.

That is,

$$\langle value \rangle = \texttt{FFT}(\langle coefficients \rangle, \omega)$$

We will see that the interpolation can be computed by

$$\langle coefficients \rangle = \frac{1}{n}\texttt{FFT}(\langle values \rangle, \omega^{-1})$$

A matrix reformation

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Let's explicitly set down the relationship between our two representations for a polynomial $A(x)$ of degree $\leq n-1$.

$$\begin{bmatrix} A(x_0) \\ A(x_1) \\ \vdots \\ A(x_{n-1}) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ & & \vdots & & \\ 1 & x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Let M be the matrix in the middle, which is a Vandermonde matrix.

- If $x_0, x_1, \dots, x_{n-1}$ are distinct numbers, then M is invertible.
- evaluation is multiplication by M, while interpolation is multiplication by M^{-1}.

A matrix reformation

This reformulation of our polynomial operations reveals their essential nature more clearly.

It justifies an assumption that $A(x)$ is uniquely characterized by its values at any n points.

Vandermonde matrices also have the distinction of being quicker to invert than more general matrices, in $O(n^2)$ time instead of $O(n^3)$.

However, using this for interpolation would still not be fast enough for us..

Interpolation resolved

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In linear algebra terms, the `FFT` multiplies an arbitrary n-dimensional vector, which we have been calling the coefficient representation, by the $n \times n$ matrix.

$$M_n(\omega) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ & & \vdots & & \\ 1 & \omega^j & \omega^{2j} & \dots & \omega^{(n-1)j} \\ & & \vdots & & \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & x^{(n-1)(n-1)} \end{bmatrix}$$

Its (j, k)-th entry (starting row- and column-count at zero) is ω^{jk}

Interpolation resolved

The columns of M are orthogonal to each other, which is often called the Fourier basis.

The FFT is thus a change of basis, a rigid rotation. The inverse of M is the opposite rotation, from the Fourier basis back into the standard basis.

Inversion formula

$$M_n(\omega)^{-1} = \frac{1}{n} M_n(\omega^{-1})$$

Interpolation resolved

Take ω to be $e^{2\pi i/n}$, and think of M as vectors in $\mathbb{C}^n$.

Recall that the angle between two vectors $u = (u_0, \ldots, u_{n-1})$ and $v(v_0, \ldots, v_{n-1})$ in $\mathbb{C}^n$ is just a scaling factor times their inner product

$$u \cdot v^* = u_0 v_0^* + u_1 v_1^* + \ldots + u_{n-1} v_{n-1}^*$$

where z^* denotes the complex conjugate of z.

The above quantity is maximized when the vectors lie in the same direction and is zero when the vectors are orthogonal to each other.

Interpolation resolved

Lemma

The columns of matrix M are orthogonal to each other.

Proof.

- Take the inner product of of any columns j and k of matrix M,

  $$1 + \omega^{j-k} + \omega^{2(j-k)} + \ldots + \omega^{(n-1)(j-k)}$$

  This is a geometric series with first term 1, last term $\omega^{(n-1)(j-k)}$, and ratio ω^{j-k}.
- Therefore, if $j \neq k$, it evaluates to

  $$\frac{1-\omega^{n(j-k)}}{1-\omega^{(j-k)}} = 0$$

- If $j = k$, then it evaluates to n.

Interpolation resolved

Corollary

$MM^* = nI$, *i.e.*,

$$M_n^{-1} = \frac{1}{n} M_n^*$$

The definitive FFT algorithm

The FFT takes as input a vector $a = (a_0, \ldots, a_{n-1})$ and a complex number ω whose powers $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the complex n-th roots of unity.

It multiplies vector a by the $n \times n$ matrix $M_n(\omega)$, which has (j, k)-th entry ω^{jk}.

The potential for using divide-and-conquer in this matrix-vector multiplication becomes apparent when M's columns are segregated into evens and odds.

The product of $M_n(\omega)$ with vector $a = (a_0, \ldots, a_{n-1})$, a size-$n$ problem, can be expressed in terms of two size-$n/2$ problems: the product of $M_{n/2}(\omega^2)$ with $(a_0, a_2, \ldots, a_{n-2})$ and with $(a_1, a_3, \ldots, a_{n-1})$.

This divide-and-conquer strategy leads to the definitive FFT algorithm, whose running time is $T(n) = 2T(n/2) + O(n) = O(nlogn)$.

The general FFT algorithm

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`FFT` (a, ω)

input : An array $a = (a_0, a_1, \ldots, a_{n-1})$ for n is a power of 2; ω, an n-th root of unity

output: $M_n(\omega)a$

if $\omega = 1$ **then** `return` a;

$(s_0, s_1, \ldots, s_{n/2-1})$=`FFT` $((a_0, a_2, \ldots, a_{n-2}), \omega^2)$;

$(s'_0, s'_1, \ldots, s'_{n/2-1})$=`FFT` $((a_1, a_3, \ldots, a_{n-1}), \omega^2)$;

for $j = 0$ *to* $n/2 - 1$ **do**

  r_j=$s_j + \omega^j s'_j$;

  $r_{j+n/2}$=$s_j - \omega^j s'_j$;

end

`return` $(r_0, r_1, \ldots, r_{n-1})$;

Top 10 algorithms of the 20th century

1946: The Metropolis Algorithm

1947: Simplex Method

1950: Krylov Subspace Method

1951: The Decompositional Approach to Matrix Computations

1957: The Fortran Optimizing Compiler

1959: QR Algorithm

1962: Quicksort

1965: Fast Fourier Transform

1977: Integer Relation Detection

1987: Fast Multipole Method