Cheatsheet Analysis 2

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1 Differential equations

Ordinary Differential Equation (ODE)

In general an **ordinary** differential equation (ODE) relates a function $f(x)$ at x to the values of its derivatives at x. I.e. it's an equation of the Form

$$F(x, f(x), f'(x), f''(x), ..., f^{(n)}(x)) = 0$$

The order of the diff. equation is the highest order of derivative that appears in the equation.
A partial diff. equation is a diff. equation for a function of several variables. (It involves "partial derivatives").

$f'(x+2) = f(x)$ is not an **ordinary** differential equation.

Linear ODE

A linear ODE of order k on I, is an equation of the form

$$y^{(k)} + a_{k-1}(x)y^{(k-1)} + ... + a_1(x)y' + a_0(x)y = b(x)$$

where $b, a_1, ..., a_{k-1}$ are continous functions of x defined on I with values in $\mathbb{C}$.
If $b(x) = 0, \forall x \in I$, we call the ODE **homogeneous** and otherwise **inhomogeneous**.

Recognising a linear ODE

- no coefficients before the highest order derivative (**what about constants?**)
- alle coefficients are continous functions
- no products of y and it's derivatives
- y and all of it's derivatives occur with the power one
- Neither y nor it's derivatives are *inside* another function.

Solutions of Linear ODE's

Let $I \subset \mathbb{R}$ open interval, $k \geq 1, k \in \mathbb{N}$.

$$y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_0 y = b$$

is a linear ODE over I with continous coefficients.
Then

1. The set of solutions S_0 for the associated **homogeneous** ODE (when $b = 0$), is a vector space of dimension k.
2. For any initial conditions (i.e. any choice of $x_0 \in I$ and $(y_0, ..., y_{k-1}) \in \mathbb{C}^k$) there exists an **unique** solution $f \in S_0$ s.t. $f(x_0) = y_0, f'(x_0) = y_1, ..., f^{(k-1)}(x_0) = y_{k-1}$.
3. For any arbitrary $b(x)$, the set of solutions of the ODE is
$$S_b = \{f + f_p \mid f \in S_0\}$$
where f_p is a **paritcular** solution of the ODE.
4. For any initial condition there is a unique condition there is a unique solution $f \in S_b$.

S_b is **not** a Vector Space! (It's an affine Space.)

1.1 Linear ODE's of order 1

$I \subset \mathbb{R}$ be an open interval.
We consider the diff. equation of the form

$$y' + a(x)y = b(x)$$

1. (Homogeneous solution)
$$\begin{aligned} y' + a(x)y &= 0 \\ \frac{y'}{y} &= -a(x) \quad (\text{assuming } y \neq 0, \forall x \in I) \\ \ln(|y|) &= -A(x) + C \\ y &= e^C \cdot e^{-A(x)} = Ke^{-A(x)}, K \in \mathbb{C} \end{aligned}$$
If an initial condition is given, we can determine K.
2. (Particular solution)
Use either "Variation of parameters" or "Educated guess".

1.2 Variation of parameters

We assume that the particular solution is of the form $f_p = K(x)e^{-A(x)}$ for a function $K : I \to \mathbb{C}$. Then we can insert our guess into the ODE and see what it forces K to satisfy. We get

$$\begin{aligned} b(x) &= (K(x)e^{-A(x)})' + a(x)(K(x)e^{-A(x)}) \\ b(x) &= K'(x)e^{-A(x)} - a(x)K(x)e^{-A(x)} + a(x)K(x)e^{-A(x)} \\ b(x) &= K'(x)e^{-A(x)} \\ K'(x) &= b(x)e^{A(x)} \end{aligned}$$

and thus

$$K(x) = \int_{x_0}^{x} b(t)e^{A(t)}\, dt$$

Therefore we get

$$f_p = \left(\int_{x_0}^{x} b(t)e^{A(t)}\, dt\right) \cdot e^{-A(t)}$$

The method with the "Integration factor" gives the same particular solution!

1.3 Educated Guess for general case

(1) If $b(x) = x^d e^{\beta x}$ and β **is not a root of the companion Polynomial** P, then we try $f_p(x) = Q(x)e^{\beta x}$, where Q is a Polynomial of degree d.

(2) If $b(x) = x^d \cos(\beta x)$ or $b(x) = x^d \sin(\beta x)$ and β **is not a root of the companion Polynomial** P, we try
$$f_p(x) = Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x)$$
where Q_1, Q_2 are polynomials of degree d.

(3) If b is of the form of the previous two examples but β **is a root with multiplicity** j **of the companion polynomial**, then one tries the same functions, except that the Polynomials Q (or Q_1, Q_2 resp.) have degree $d + j$.

(4) The special case $\beta = 0$ in cases $(1), (2), (3)$ corresponds to the situation where b is a polynomial of degree d. Therefore we try f_p as a polynomial of degree d, unless 0 is a root with multiplicity j, then we try f_p as a polynomial of degree $d + j$.

1.4 Educated Guess for constant coefficients

If $b(x)$ is of a specific form, we try following f_p, where we insert the f_p into the ODE, which gives us a system of equations for the constants:

$b(x)$	Ansatz
$a \cdot e^{\alpha x}$	$b \cdot e^{\alpha x}$
$a\sin(\beta x)$	$c\sin(\beta x) + d\cos(\beta x)$
$b\cos(\beta x)$	$c\sin(\beta x) + d\cos(\beta x)$
$ae^{\alpha x}\sin(\beta x)$	$e^{\alpha x}\Big(c\sin(\beta x) + d\cos(\beta x)\Big)$
$be^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\Big(c\sin(\beta x) + d\cos(\beta x)\Big)$
$P_n(x) \cdot e^{\alpha x}$	$R_n(x) \cdot e^{\alpha x}$
$P_n(x) \cdot e^{\alpha x}\sin(\beta x)$	$e^{\alpha x}\left(R_n(x)\sin(\beta x) + S_n(x)\cos(\beta x)\right)$
$P_n(x) \cdot e^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\left(R_n(x)\sin(\beta x) + S_n(x)\cos(\beta x)\right)$

P_n, R_n and S_n are Polynomials of degree n.

1. If $b(x)$ is a linear combination of any of the base functions, try that linear combination of 'Ansatz' functions.
2. If $b(x) = c \cdot e^{\alpha x}, \alpha \in \mathbb{C}$ for which α is a zero of the characteristic polynomial (i.e. α is a root and $b(x)$ a solution for the homogeneous equation), then we try $f_p = d \cdot x^m \cdot e^{\alpha x}$, where m is the multiplicity of the root α.

1.5 Linear ODE's with constant coefficients

We consider an ODE of the form

$$y^{(k)} + a_{k-1}y^{(k-1)} + \ldots + a_1 y' + a_0 y = b(x)$$

We search for a homogeneous solution of the form $e^{\lambda x}$. Now we can solve the characteristic polynomial:

$$\begin{aligned} P(\lambda) = e^{\lambda x}\left(\lambda^k + a_{k-1}\lambda^{k-1} + \ldots + a_0\right) = 0 \\ \implies 0 = \lambda^k + a_{k-1}\lambda^{k-1} + \ldots + a_0 \end{aligned}$$

- The roots of $P(\lambda)$ are the Eigenvalues λ_i, with corresponding multiplicity m_r. Thus the functions $f_{i,r} : x \to x^r e^{\lambda_i x}, 0 \leq r < m_r$ span the Vector Space S_0.
- If $\lambda = \beta + \gamma i$ is a complex of $P(\lambda)$, then the complex conjugation, i.e. $\bar{\lambda} = \beta - \gamma i$ is also a root. Thus $f_1 = e^{\lambda x}$ and $f_2 = e^{\bar{\lambda} x}$ are solutions to the homogeneous equation.
- We realize that $f_1 = e^{\lambda x} = e^{\beta x}(\cos(\gamma x) + i\sin(\gamma x))$ and $f_2 = e^{\bar{\lambda} x} = e^{\beta x}(\cos(\gamma x) - i\sin(\gamma x))$.
- We can thus replace f_1 and f_2 by $\tilde{f_1} = e^{\beta x}\cos(\gamma x)$ and $\tilde{f_2} = e^{\beta x}\sin(\gamma x)$. (Note that $f_1 = \tilde{f_1} + i\tilde{f_2}$ and $f_2 = \tilde{f_1} - i\tilde{f_2}$)
- Note that we are often only interested in finding real-valued solutions if the coefficients are all real valued.
- If $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = 0$ only has real coefficients, every pair of complex conjugated roots $\beta_j \pm \gamma_j i$ with multiplicity m_j leads to a solution

  $$x^l e^{\beta_j x}\Big(\cos(\gamma_j x) + i\sin(\gamma_j x)\Big) \quad \text{for } 0 \leq l < m_j$$

  of which then the real part can be extracted.

To find a particular solution f_p we can as in the general case use **Varation of parameters** or **Educated guess**. We will now show an simple example with 2 basis functions:
Consider the Linear ODE $y'' + ya_1y' + a_0y = b$

(1) Assume the space of homogeneous solutions S_0 is spanned by f_1, f_2, i.e. $f_0 = f_1 + f_2$ is also a solution

(2) Now we try $f_p = z_1(x)f_1 + z_2(x)f_2$

(3) We first insert f_p into the ODE and we require the additional constraint that $z_1'(x)f_1 + z_2'(x)f_2 = 0$ to find a concrete solution.

Therefore we get the following system of equations:

$$\begin{aligned} z_1'(x)f_1 + z_2'(x)f_2 = 0 \\ z_1'(x)f_1' + z_2'(x)f_2' = b(x) \end{aligned}$$

We can solve this as follows:

$$\begin{aligned} W &= f_1 f_2' - f_2 f_1' \neq 0 \\ \Rightarrow z_1' &= \frac{-f_2 b}{W}, z_2' = \frac{f_1 b}{W} \\ \Rightarrow f_p &= \left(\int \frac{-f_2 b}{W} dt\right) f_1 + \left(\int \frac{f_1 b}{W} dt\right) f_2 \end{aligned}$$

1.6 Seperation of Variables

Consider a differential equation of the form

$$y'(x) = b(x)g(y)$$

Assume $g(y(x)) \neq 0$. If $\exists y_0$ s.t. $g(y_0(x)) = 0$ then $y = y_0$ is a solution.

$$\begin{aligned} \frac{y'(x)}{g(y(x))} &= b(x) \\ \int \frac{y'(x)}{g(y(x))}\, dx &= \int b(x)\, dx \end{aligned}$$

Applying substitution with $u = y(x)$ we obtain

$$\int \frac{1}{g(u)}\, du = \int b(x)\, dx$$

We can then determine both integrals and solve for $u = y$.

2 Derivations in $\mathbb{R}^n$

Monomial in $\mathbb{R}^n$

A Monomial of degree e is a function $f : \mathbb{R}^n \mapsto \mathbb{R}$:

$$\begin{aligned} (x_1, \ldots, x_n) \mapsto \alpha x_1^{d_1} \cdot \ldots \cdot x_n^{d_n} \\ e = d_1 + \ldots + d_n \end{aligned}$$

$\to$ i.e. a Polynomial that only has one term.

Polynomial in $\mathbb{R}^n$

A Polynomial with n variables of degree d is a finite sum of Monomials of degree $e \leq d$.

2.1 Convergence

1. Dot product: $\langle x, y\rangle = \sum_{i=0} x_i \cdot y_i$
2. Euclidean norm: $||x|| := \sqrt{x_2^1 + \cdots + x_n^2}$ with the following properties:
   a) $||x|| \geq 0, ||x|| = 0 \iff x = 0$
   b) $||\lambda x|| = |\lambda| \cdot ||x||, \forall \lambda \in \mathbb{R}$
   c) $||x + y|| \leq ||x|| + ||y||$
   d) $|\langle x, y\rangle| \leq ||x|| \cdot ||y||$

Definition Convergence

Let $(x_k)_{k\in\mathbb{N}}, x_k \in \mathbb{R}^n$. The following definitions for $\lim_{k\to\infty} x_k = y$ equivalent:

1. $\forall \epsilon > 0 \exists N \geq 1$ such that $\forall k \geq N$ $||x_k - y|| < \epsilon$.
2. For every $i, 1 \leq i \leq n$ the sequence $(x_{k,i})_k$ of real numbers converges to y_i.
3. The sequence $||x_k - y||$ of real numbers converges to 0.

2.2 Continuity

Definition Continuity

Let $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}^m$ and $x_0 \in \mathcal{X}$.
f is continous in x_0, if one of the following conditions is fulfilled:

1. $\forall \epsilon > 0$ $\exists \delta > 0$ such that for all $x \in \mathcal{X}$

   $$||x - x_0|| < \delta \implies ||f(x) - f(x_0)|| < \epsilon$$

2. $\forall$ sequences (x_k) in X with $\lim_{k\to\infty} x_k = x_0$ we have

   $$\lim_{k\to\infty} f(x_k) = f\left(\lim_{k\to\infty} x_k\right) = f(x_0)$$

f is continous on $\mathcal{X} \iff f$ is continous at every point $x_0 \in \mathcal{X}$.
In Addition we have the following:

1. Cartesian product of continous functions is continous.
2. $f : \mathbb{R}^n \mapsto \mathbb{R}^m$
   $(x_1, \ldots, x_n) \mapsto (f_1(x), \ldots, f_m(x))$
   is continous $\iff f_i : \mathbb{R}^n \to \mathbb{R}$ continous $\forall i = 1, \ldots, m$.
3. Linear Maps $x \mapsto Ax$ are continous.
4. Finite sums and products of continous functions are continous.
5. Functions with seperated Variables are continous if each factor is continous. (i.e. $f(x_1, ..., x_n) = f_1(x_1)f_2(x_2) \cdot ... \cdot f_n(x_n)$ is continous if $f_1, f_2, ..., f_n$ are continous.)

6. In particular Polynomials are continous.
7. The composition of continous functions is continous.
8. If $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is continous. For an arbitrary fixed $y_0 \in \mathbb{R}$ we can define $g_{y_0}(x) := f(x, y_0)$. Since g_{y_0} is a composition of continous functions it is also continous.
9. Warning! The converse is not true. g_{y_0} continous for all $y_0 \in \mathbb{R}$ does **not** imply that f is continous!

Sandwich-Lemma

If $f, g, h : \mathbb{R}^n \to \mathbb{R}$ are functions with $f(x) < g(x) < h(x)$ $\forall x \in \mathbb{R}^n$. Let $a \in \mathbb{R}^n$:

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \implies \lim_{x\to a} g(x) = L$$

2.3 Properties of sets

A set $\mathcal{X} \subset \mathbb{R}^n$ is

- **bounded**, if the set $\{||x|| \mid x \in \mathcal{X}\}$ is bounded in $\mathbb{R}$(i.e. $\exists K \geq 0, \forall x \in \mathcal{X} : ||x|| \leq K$).
- **closed**, if every sequence $(x_k)_{k\in\mathbb{N}} \subset \mathcal{X}$, that converges to some Vector $y \in \mathbb{R}^n$, we have $y \in \mathcal{X}$ (i.e. limits of sequences in X are also in X).
- **compact**, if its closed and bounded.
- **open** if, for any $x = (x_1, x_2, ..., x_n) \in \mathcal{X}$, there exists $\delta > 0$ such that the set

  $$\{y = (y_1, ..., y_n) \in \mathbb{R}^n \mid |x_i - y_i| < \delta, \forall 1 \leq i \leq n\}$$

  is contained in $\mathcal{X}$.
- **convex**, if $\forall x, y \in \mathcal{X} : \lambda x + (1-\lambda)y \in \mathcal{X}, \forall 0 \leq \lambda \leq 1$ (the line segment between x, y is contained in $\mathcal{X}$).
- **open**, if and only if the complement $Y = \mathbb{R}^n \setminus \mathcal{X}$ is **closed**. (Equivalent definition)

Important examples:

- $(a, b) \subset \mathbb{R}$ is open.
- $[a, b) \subset \mathbb{R}$ is neither open nor closed.
- $\mathbb{R}^n$ and $\varnothing$ are both open and closed. There exists no other set in $\mathbb{R}^n$ which is both open and closed.
- If $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ are both bounded (rsp. closed/compact) then $X \times Y \subseteq \mathbb{R}^{n+m}$ is bounded (rsp. closed/compact)
- In particular the cartesian product of compact intervals $I_i \in \mathbb{R}$: $I_1 \times I_2 \times ... \times I_n = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_i \in I_i\}$ is compact (i.e. closed and bounded).
- Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be continous. Then for every closed(/open) set $Y \subseteq \mathbb{R}^m$, the set $f^{-1}(Y)$ is closed(/open).

Bolzano-Weierstrass

Every bounded sequence in $\mathbb{R}^n$ has a converging partial sequence.

Min-Max-Theorem

Let $\mathcal{X} \subset \mathbb{R}^n, \mathcal{X} \neq \varnothing$ be compact and $f : \mathcal{X} \to \mathbb{R}$ a continous function. Then f is bounded and achieves a max and a min. I.e. $\exists x^+, x^- \in \mathcal{X}$, such that

$$f(x^+) = \sup_{x\in\mathcal{X}} f(x) \quad f(x^-) = \inf_{x\in\mathcal{X}} f(x)$$

2.4 Partial Derivatives

Partial Derivative

To find the partial derivative of $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ (whereby $\mathcal{X}$ open) with respect to $x_j, 1 \leq j \leq n$ at a point $x_0 \in \mathcal{X}$ we define:

$$\partial_j f(x_0) = \frac{\partial f}{\partial x_j}(x_0) = \lim_{h\to 0} \frac{f(x_0 + h \cdot e_j) - f(x_0)}{h}$$

where e_j is the j-th canonical basis vector of $\mathbb{R}^n$.

For $f : \mathbb{R}^n \to \mathbb{R}^m, x_0 \in \mathbb{R}^n$ we have

$$\frac{\partial f(x_0)}{\partial x_j} := \begin{pmatrix} \frac{\partial}{\partial x_j} f_1(x_0) \\ \vdots \\ \frac{\partial}{\partial x_j} f_m(x_0) \end{pmatrix}$$

Partial derivatives have following properties:

1. $\partial_j(f + g) = \partial_j(f) + \partial_j g$
2. $\partial_j(f \cdot g) = \partial_j(f) \cdot g + \partial_j(g) \cdot f$
3. $\partial_j(f/g) = \frac{\partial_j(f)\cdot g - \partial_j(g)\cdot f}{g^2}$ for $g \neq 0$

Jacobi-Matrix

Let $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathcal{X}$ an open set. The Jacobi-Matrix is the $m \times n$ Matrix:

$$J_f = \left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1\leq j\leq n \\ 1\leq i\leq m}}$$

Gradient

In the special case of a function $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$, the Jacobi-Matrix is a row vector which transposed gives us ∇f. The geometric interpretation is a vectorfield, defined by ∇f, which indicates the direction and magnitude of the biggest growth of f.

2.5 Differentiability

Differentiability

Let $\mathcal{X} \subset \mathbb{R}^n$ be open, $x_0 \in \mathcal{X}$. We have $f : \mathcal{X} \to \mathbb{R}^m$.
We say that f is **differentiable** at x_0, with the differential u, if there exists a **linear map** $u : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\substack{x\to x_0 \\ x\neq x_0}} \frac{f(x) - f(x_0) - u \cdot (x - x_0)}{||x - x_0||} = 0$$

We denote $u = df(x_0) = d_{x_0} f$.

- If f is differentiable at all points $x_0 \in \mathcal{X}$, then f is differentiable on $\mathcal{X}$.
- Having all partial derivatives defined is not sufficient to conclude Differentiability.
- If all partial derivatives are defined and continous, then f is differentiable.

Conclusions from Differentiability

If f, g are differentiable in $x_0 \in \mathcal{X}$ we have:

1. f is continous in x_0
2. f has all partial derivatives at x_0 and the matrix of the linear map $df(x_0) : x \mapsto Ax$ is given by the **Jacobi-Matrix** of f at x_0, i.e. $A = J_f(x_0)$
3. $d(f + g)(x_0) = df(x_0) + dg(x_0)$
4. If $m = 1$, then $f \cdot g$ is differentiable. If additionally $g \neq 0$, then f/g is also differentiable. (Product rule and Quotient rule apply)
5. (Chain rule): Let $\mathcal{X} \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ be open.

   If $f : \mathcal{X} \to Y, g : Y \to \mathbb{R}^p$ are both differentiable, we have $d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$. Furthermore

   $$J_{g\circ f}(x_0) = J_g(f(x_0)) \cdot J_f(x_0)$$

   Therefore

   $$d(g \circ f)(x_0) : \mathcal{X} \to \mathbb{R}^p \qquad x \mapsto J_g(f(x_0)) \cdot J_f(x_0) \cdot x$$

Tangent Space

The **tangent space** at x_0 of f is given by the graph of the affine linear map $g(x) = f(x_0) + df(x_0)(x - x_0)$.
I.e.

$$\{(x, g(x)) \in \mathbb{R}^n \times \mathbb{R}^m \mid g(x) = f(x_0) + df(x_0)(x - x_0)\}$$

Directional Derivative

Let $f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $x_0 \in \mathcal{X}$. For any $v \in \mathbb{R}^n, v \neq 0$ the **directional Derivative** of f at x_0 exists and is defined as

$$\partial_v f(x_0) = \lim_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h} = J_f(x_0) \cdot v$$

Change of variables (Bijection)

Let $\mathcal{X} \subset \mathbb{R}^n$ be open and $f : \mathcal{X} \to \mathbb{R}^n$ differentiable. f is a **change of variable** around $x_0 \in \mathcal{X}$ if there exists a radius $r > 0$ such that the Ball

$$B_r(x_0) := \{x \in \mathbb{R}^n \mid ||x - x_0|| < r\}$$

has the property that the Image $Y = f(B_r(x_0))$ is open and there exists a differentiable map $g : Y \to B_r(x_0)$ such that $f \circ g = g \circ f = \text{id}$.
We find that if $\det(J_f(x_0)) \neq 0$ (i.e. $J_f(x_0)$ is invertible), then f is a change of variables around x_0. Moreover the Jacobian of the inverse map g is determined by

$$J_g(f(x_0)) = J_f(x_0)^{-1}$$

(Analog to the fact that a function $h : I \subseteq \mathbb{R} \to \mathbb{R}$ is bijective from I to its image if $h' > 0$ or $h' < 0$)

2.6 Higher derivatives

Notation for higher partial derivatives

For a function $f : \mathbb{R}^n \to \mathbb{R}^t$ we denote higher order partial derivatives with the following:
First let $m = (m_1, m_2, ..., m_n)$ and $|m| = m_1 + m_2 + ... + m_n$.
We write

$$\frac{\partial^{|m|} f_j}{\partial x_1^{m_1} \partial x_2^{m_2} \cdot \ldots \cdot \partial x_n^{m_n}} = \partial_{x^m}^{|m|} f_j, \; 1 \leq j \leq t$$

Differential Classes

Let $\mathcal{X} \subset \mathbb{R}^n$ be open, $f : \mathcal{X} \to \mathbb{R}^m$.

- We say that f is differentiable of class C^1 if f is differentiable on $\mathcal{X}$ and all its partial derivatives are continous. The set of all C^1 functions from $\mathcal{X}$ to $\mathbb{R}^m$ are denoted by $C^1(\mathcal{X} : \mathbb{R}^m)$.
- Let $k \geq 2$. We say $f \in C^k(\mathcal{X} : \mathbb{R}^m)$ (i.e. f is of class C^k) if its differentiable and each $\partial_{x_i} f : \mathcal{X} \to \mathbb{R}^m$ $(1 \leq i \leq n)$ is of class C^{k-1}.
- We say that f is smooth or of class C^∞ if $f \in C^k$, $\forall k \in \mathbb{N}$.
- All polynomials, trigonometric and exponential functions are of class C^∞.
- If $f \in C^k, k \geq 2$ then all partial derivatives of order $\leq k$ are commutative.

$$\frac{\partial^2 f}{\partial x_i \, \partial x_j} = \frac{\partial^2 f}{\partial x_j \, \partial x_i}$$

Hessian

Let $\mathcal{X} \subset \mathbb{R}^n$ be open and $f : \mathcal{X} \to \mathbb{R}$ a C^2 function.
For an $x_0 \in \mathcal{X}$, the **Hessian matrix** of f at x_0 is the symmetric $n \times n$ matrix that denotes the second derivative:

$$\text{Hess}_f(x_0) := \left(\frac{\partial^2 f(x_0)}{\partial x_i \, \partial x_j} \right)_{1 \leq i,j \leq n}$$

Sometimes we also denote it by $\nabla^2 f(x_0)$ or $H_f(x_0)$.

2.7 Taylor polynomials

Let $k \leq 1$ and $f : \mathcal{X} \mapsto \mathbb{R}$ be a funciton of class C^k on $\mathcal{X}$, and fix $x_0 \in \mathcal{X}$. The k-th Taylor polynomial of f at the point x_0 is the polynomial in n variables of degree $\leq k$ given by

$$T_k f(y; x_0) = f(x_0) + \sum_{i=1}^{n} \partial_i f(x_0) \cdot y_i + \ldots$$
$$+ \sum_{m_1 + \cdots + m_n = k} \frac{1}{m_1! \cdots m_n!} \frac{\partial^k f(x_0)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} \cdot y_1^{m_1} \cdots y_n^{m_n}$$

Using our new notation for higher-order derivatives we can denote the Taylor polynomial by

$$T_k(y; x_0) = \sum_{|m| \leq k} \frac{1}{m!} \partial_{x^m}^{|m|} f(x_0) \cdot y^m$$

with $m! = m_1! m_2! \cdot ... \cdot m_n!$

Examples

$$T_1 f(\vec{x}; x_0) := f(x_0) + \nabla f(x_0) \cdot \vec{x}$$
$$T_2 f(\vec{x}; x_0) := T_1 f + \frac{1}{2} \cdot \vec{x}^\top \cdot \text{Hess}_f(x_0) \cdot \vec{x}$$

2.8 Extrema

Local Extrema

Let $f : \mathcal{X} \subset \mathbb{R}^n \mapsto \mathbb{R}$ be differentiable and $\mathcal{X}$ open.
Then $x_0 \in \mathcal{X}$ is a **local Maximum (Minimum)** if there exists an $r > 0, r \in \mathbb{R}$ and $B_{x_0}(r) = \{x \in \mathbb{R}^n \mid ||x - x_0|| < r\} \subset \mathcal{X}$ such that:

$$\forall x \in B_{x_0}(r) : f(x) \leq (\geq) f(x_0)$$

If $x_0 \in \mathcal{X}$ is a local extrema, we additionally have $\nabla f(x_0) = 0$.

Critical point

A point $x_0 \in \mathcal{X}$ with $\nabla f(x_0) = 0$ is a **critical point**.
Critical points are candidates for local extrema.
If additionally $\det(\text{Hess}_f(x_0)) \neq 0$, then x_0 is a **non-degenerate** critical point.

Saddle point

If a critical point is neither a maximum nor a minimum, we call it a **saddle point**.

Global Extrema

Let $f : K \mapsto \mathbb{R}$ and K compact, then a global extrema of f exists and is either at a point x_0 in the interior of K or on the boundary of K. To determine such an global extrema we split K into it's interior $\mathcal{X}$ and the boundary B.
First we determine the critical points of $\mathcal{X}$. To determine the Maximas/Minimas B, we will need Knowledge from Analysis I (redefine the boundary as a union of sets dependend on 1 variable, i.e. Line-segments).

Testing critical points

Let $f : \mathcal{X} \subseteq \mathbb{R}^n \mapsto \mathbb{R}, \mathcal{X}$ open and $f \in C^2$. Let x_0 be a **non-degenerate critical point** of f. Then we:

1. $\text{Hess}_f(x_0)$ pos. def. $\implies$ x_0 is a local Minimum.
2. $\text{Hess}_f(x_0)$ neg. def. $\implies$ x_0 is a local Maximum.
3. $\text{Hess}_f(x_0)$ indefinite $\implies$ x_0 is a saddle point.

If x_0 is a **degenerate critical point**, we can't conclude anything in general. In such a case we would have to verify the signs in the neighborhood of x_0. (Not much information found on how to do that in multi-variable calculus)

Critical points with constraints

If we want to determine the Minimas/Maximas of a function $f : \mathcal{X} \mapsto \mathbb{R}$ with the constraint $g(x) = 0, g : \mathcal{X} \mapsto \mathbb{R}$, we can use the Lagrange multipliers.

Lagrange multipliers

Let $\mathcal{X} \subset \mathbb{R}^n$ be open and $f, g : \mathcal{X} \mapsto \mathbb{R}$ functions of C^1. If x_0 is a local extremum of f restricted to the set

$$Y = \{x \in \mathcal{X} \mid g(x) = 0\}$$

then either $\nabla g(x_0) = 0$, or there exists $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0) = \lambda \cdot \nabla g(x_0)$$

and $g(x_0) = 0$.

2.9 Definite

A symmetric (non-singular) matrix A, $\det A \neq 0$ is

- **positive definite** $\iff$ all Eigenvalues are positive $\iff$ all principal minors of A are positive
- **negativ definite** $\iff$ all E.V. are negative $\iff$ $-A$ is positive definite.
- **indefinite** if it has positive and negative Eigenvalues.

Eigenvalues can be found with the characterstic polynomial:

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$
$$\Rightarrow ad - (a+d)\lambda + \lambda^2 - bc = 0$$

For non-symmetric Matrices we have to test for all Vectors v, if $v^\top A v > 0$ (rsp. < 0).

3D Determinant

$$a \cdot \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \cdot \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Principal Minor

The k-th leading principal minor of A is given by

$$M_k = \det\left((A)_{1:k,1:k}\right)$$

3 Integrals in $\mathbb{R}^n$

3.1 Simple Integrals

For $f : \mathbb{R} \mapsto \mathbb{R}^n$ we define the integral of f as

$$\int_a^b f(t)dt = \begin{pmatrix} \int_a^b f_1(t)dt \\ \vdots \\ \int_a^b f_n(t)dt \end{pmatrix}$$

3.2 Line Integrals (Path Integrals)

A parameterized curve

A parameterized curve in $\mathbb{R}^n$ is a continous map $\gamma : [a, b] \mapsto \mathbb{R}^n$ that is piecewise in C^1, i.e. $\exists k > 1, k \in \mathbb{N}$ and a partition $a = t_0 < t_1 < ... < t_k = b$, such that

$$\gamma|_{[t_{i-1}, t_i]} \in C^1, \forall 1 \leq i \leq k.$$

A parameterized curve does not have to be injective.

Useful trick:
In general if $\gamma : [a, b] \to \mathbb{R}^n (t \mapsto \gamma(t))$ is a curve, then $\alpha : [a, b] \to \mathbb{R}^n$ with $\alpha(t) := \gamma(b + a - t)$ traces the same curve in the opposite direction.

Line(Path) Integrals

Let $\gamma : [a, b] \mapsto \mathbb{R}^n$ be a parameterized curve and $\mathcal{X} \subset \mathbb{R}^n$ a set containing the image of γ, and let $f : \mathcal{X} \to \mathbb{R}^n$ be a continous function. The **Line Integral (Path Integral)** is defined as

$$\int_\gamma f(s)\, ds = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

For Notation s represents $\gamma(t)$ and ds represents $\gamma'(t)\, dt$.

Line Integrals have following properties:

1. It is independent of orientation preserving reparameterization of the curve:
$$\begin{aligned} &\gamma : [a,\ b] \mapsto \mathbb{R}^n \\ &\sigma : [c,\ d] \mapsto [a,\ b],\ \sigma'(t) > 0\ \forall t \in (c,\ d) \\ &\tilde{\gamma} : [c,\ d] \mapsto \mathbb{R}^n \\ &\tilde{\gamma} = \gamma \circ \sigma = \gamma(\sigma) \\ \Rightarrow &\int_\gamma f(s)\ ds = \int_{\tilde{\gamma}} f(s)\ ds \end{aligned}$$
2. We have $\gamma_1 : [a,\ b] \to \mathcal{X} \subset \mathbb{R}^n, \gamma_2 : [c,\ d] \to \mathcal{X}$ with $\gamma_1(b) = \gamma_2(c)$. We can now concatenate these 2 curves to $\gamma_1 + \gamma_2$
$$\gamma_1 + \gamma_2 := \begin{cases} \gamma_1(t) & t \in [a,\ b] \\ \gamma_2(t - b + c) & t \in [b,\ b + (d - c)] \end{cases}$$
$$\int_{\gamma_1+\gamma_2} f(s)\ ds = \int_{\gamma_1} f(s)\ ds + \int_{\gamma_2} f(s)\ ds$$
3. Let $\gamma : [a, b] \mapsto \mathbb{R}^n$ be a path and $-\gamma : [a, b] \to \mathbb{R}^n$ the same path in the opposite direction (i.e. $(-\gamma)(t) = \gamma(a + b - t)$). Then we have
$$\int_{-\gamma} f(s)ds = -\int_\gamma f(s)ds$$

3.3 Potential

A differentiable function $g : \mathcal{X} \subset \mathbb{R}^n \mapsto \mathbb{R}$ with $\nabla g = f, f : \mathcal{X} \mapsto \mathbb{R}^n$ is called a **potential** for f. This can be used as follows:

$$\begin{aligned} \int_\gamma f\ ds &= \int_a^b f(\gamma(t)) \cdot \gamma'(t)\ dt \\ &= \int_a^b \nabla g(\gamma(t)) \cdot \gamma'(t)\ dt \\ &= \int_a^b \frac{d}{dt}(g \circ \gamma)\ dt \\ &= (g \circ \gamma)(b) - (g \circ \gamma)(a) \end{aligned}$$

For a function $f : \mathbb{R}^n \to \mathbb{R}^n$ there is not always a potential g! And continuity of f is not sufficient for the existence of a potential for f. (Counterexample: $f(x, y) = (2xy^2, 2x)$)

3.4 Conservative Vector fields

Conservative Vector fields

$f : \mathcal{X} \to \mathbb{R}^n$ continous Vector field. If for any $x_1, x_2 \in \mathcal{X}$ the line integrals $\int_\gamma f\, ds$ for any curve between x_1, x_2 are equal, f is called **conservative**.

Let $\mathcal{X}$ be open and a path-connected subset of $\mathbb{R}^n$. Let $f : \mathcal{X} \subset \mathbb{R}^n \mapsto \mathbb{R}^n$ be a continous vector field. The following are equivalent:

1. f is the gradient of a function $g : \mathcal{X} \to \mathbb{R}$, i.e. $f = \nabla g$.
2. The line integral of f is independent of the path between any 2 points.
3. The line integral of f along any closed path is always 0. (A closed path $\gamma : [a, b] \to \mathcal{X}$ fulfills $\gamma(a) = \gamma(b)$)

We additionally have this necessary but not sufficient condition:

$$f \text{ is conservative} \implies \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad \forall i,\ j$$

Path-connected set

Let $\mathcal{X} \subset \mathbb{R}^n$ be open. $\mathcal{X}$ is path-connected, if for every pair of points $x, y \in \mathcal{X}$ there exists a parameterized curve $\gamma : [0,1] \mapsto \mathcal{X}$, such that $\gamma(0) = x, \gamma(1) = y$.

Starshaped set

A subset $\mathcal{X} \subset \mathbb{R}^n$ is starshaped if there $\exists x_0 \in \mathcal{X}$ such that, $\forall x \in \mathcal{X}$ the line segment joining x_0 to x is contained in $\mathcal{X}$.

$$\mathcal{X} \text{ convex} \implies \mathcal{X} \text{ starshaped}$$

If $\mathcal{X}$ is a starshaped, open subset of $\mathbb{R}^n$ and $f \in C^1$ a vector field, we have:

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad \forall i, j \quad \Rightarrow \quad f \text{ is conservative}$$

For $f : \mathcal{X} \subset \mathbb{R}^3 \to \mathbb{R}^3, f \in C^1$ we also have:

$$\operatorname{curl}(f) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad f \text{ is conservative}$$

(As above only for $f : \mathcal{X} \subset \mathbb{R}^3 \to \mathbb{R}^3$)
$\operatorname{curl}(f)$ is defined as

$$\operatorname{curl}(f) := \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

3.5 Riemann Integral in $\mathbb{R}^2$

Partition in zwei Dimensionen

Eine Partition P eines abgeschlossenen Rechtecks $R = [a,b] \times [c,d]$ ist eine Menge von Rechtecken. Für jede Partition P_x : $a = x_0 < \ldots < x_n = b$ von $[a,b]$ und P_y (analog) erhalten wir eine Partition $P_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ von R mit der Fläche $\mu(P_{i,j} = (x_i - x_{i-1})(y_j - y_{j-1}))$.

Mit den Hilfsdefinitionen

$$f_{i,j} = \inf_{P_{i,j}} f(x,y), \quad F_{i,j} = \sup_{P_{i,j}} f(x,y)$$

können wir die Unter- und Obersumme bestimmen:

$$s(P_x \times P_y) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_{i,j} \cdot \mu(P_{i,j})$$

$$S(P_x \times P_y) = \sum_{i=1}^{n} \sum_{j=1}^{m} F_{i,j} \cdot \mu(P_{i,j})$$

Sei $f : R \mapsto \mathbb{R}$ beschränkt. f ist auf R integrierbar, falls $\sup_{(P_x, P_y)} s(P_x, P_y)$ $\inf_{(P_x,P_y)} S(P_x, P_y)$ gilt. Dieser Wert ist dann definiert als:

$$\int_R f(x,y)\, d(x,y) \text{ oder } \int\int_R f(x,y)\, d(x,y)$$

Nicht-Quadratische Flächen

Sei $A \subset R$ eine Fläche. $f : A \subset R \mapsto \mathbb{R}$ ist auf A integrierbar, falls $f \cdot \mathcal{X}_A$ auf R integrierbar ist.

$$\int_R f(x,y) \cdot \mathcal{X}_A(x,y)\, d(x,y) \text{ oder } \int_A f(x,y)\, d(x,y)$$

$\mathcal{X}_A$ ist die charakteristische Funktion von A.

Eigenschaften des Integrals

Sei $f, g : A \subset R \mapsto \mathbb{R}$ auf A integrierbar, dann gilt folgendes:

1. $\alpha, \beta \in \mathbb{R} : \alpha f + \beta g$ ist integrierbar:
$$\int_A \alpha f + \beta g\, d(x,y) = \alpha \int_A f\, d(x,y) + \beta \int_A g\, d(x,y)$$
2. Falls $\forall (x,y) \in A : f(x,y) \leq g(x,y)$, dann gilt:
$$\int_A f(x,y)\, d(x,y) \leq \int_A g(x,y)\, d(x,y)$$
3. Falls $f(x,y) \geq 0$ und $B \subset A$, dann gilt:
$$\int_B f(x,y)\, d(x,y) \leq \int_A f(x,y)\, d(x,y)$$
4. Dreiecksungleichung:
$$\left| \int_A f(x,y)\, d(x,y) \right| \leq \int_A |f(x,y)|\, d(x,y)$$
5. Falls $f = 1$, dann gilt:
$$\int_A f(x,y)\, d(x,y) = \int_A 1\, d(x,y) = \gamma(A)$$

Satz von Fubini

Für eine Region $D \subset \mathbb{R}^2 := \{(x,y) \mid a \leq x \leq b, g(x) < y < h(x)\}$ gilt:

$$\int_D f(x,y)\, dx\, xy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\, dy \right) dx$$

Für eine Region $D \subset \mathbb{R}^2 := \{(x,y) \mid c \leq y \leq d, G(y) < x < H(y)\}$ gilt:

$$\int_D f(x,y)\, dx\, dy = \int_c^d \left(\int_{G(y)}^{H(y)} f(x,y)\, dx \right) dy$$

Satz von Stolz

Sei $f : R \mapsto \mathbb{R}$ integrierbar auf $R = [a,b] \times [c,d]$. Sei $y \mapsto f(x,y)$ integrierbar auf $[c,d]$ für jedes $x \in [a,b]$. Dann folgt:

$$\int_R f(x,y)\, (\, d(x,y)) = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

Nullmenge

Eine Nullmenge $X \subset R \subset \mathbb{R}^2$ ist eine Menge, so dass für alle $\epsilon > 0$ eine endliche Menge an Rechtecken $R_k, 1 \leq k \leq n$ existiert, so dass:

$$X \subset \bigcup_{k=1}^{n} R_k, \; \sum_{k=1}^{n} \mu(R_k) < \epsilon$$

(Informell: Wir können die ganze Menge mit einer endlichen Menge an Rechtecken beliebiger Grösse überdecken.)

Weitere Integrationskriterien

1. Sei R ein kompaktes Rechteck und $f : R \mapsto \mathbb{R}$ ist stetig. Dann ist f integrierbar auf R.
2. Sei $f : R \mapsto \mathbb{R}$ beschränkt und X die Menge aller nicht stetigen Punkte von f. Wenn X eine Nullmenge ist, dann ist f auf R integrierbar.
3. Sei $\varphi_1, \varphi_2 : [a,b] \mapsto \mathbb{R}$ stetig mit $\forall x \in [a,b] : \varphi_1(x) \leq \varphi_2(x)$ und $A = \{(x,y) \mid a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}$. Falls $f : A \mapsto \mathbb{R}$ stetig ist, so ist f auf A integrierbar und es folgt dass
$$\int_A f(x,y)\, d(x,y) = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy \right) dx$$

Lipschitz-Kurve

Eine Kurve $\varphi : [0,1] \mapsto \mathbb{R}^2$ ist Lipschitz, falls

$$||\varphi(s) - \varphi(t)|| \leq M \cdot |s - t| \; \forall s, t \in [0,1]$$

Es folgt ausserdem, dass $\varphi([0,1]) \subset \mathbb{R}^2$ eine Nullmenge ist.

3.6 Variablenwechsel

Sei ∂A der Rand einer Menge A:

$$\partial A = \Big\{ (x,y) \in \mathbb{R}^2 \mid \forall \delta > 0,$$
$$R = (x - \delta, x + \delta) \times (y - \delta, y + \delta),$$

$$A \cap R \neq \emptyset \text{ und } \mathbb{R}^2 \backslash A \cap R \neq \emptyset \Big\}$$

Sei $\varphi : B \mapsto A$ eine stetige Abbildung, wobei $A = A_0 \cup \partial A$, $B = B_0 \cup \partial B$ kompakte Mengen mit A_0, B_0 offen und $\partial A, \partial B$ Nullmengen sind. Wenn $\varphi : B \setminus \mathbb{N} \mapsto A$ injektiv ist und $N \subset B$ die Nullmenge ist, dann folgt

$$\int_A f(x,y)\, d(x,y) = \int_B f(\varphi(u,v)) \cdot |\det J_\varphi(u,v)|\, d(u,v)$$

1. Polarkoordinaten: $dx\, dy = r\, dr\, d\theta$
2. Zylindrische Koordinaten: $dx\, dy\, dz = r\, dr\, d\theta\, dz$
3. Kugelkoordinaten: $dx\, dy\, dz = r^2 \sin(\varphi)\, dr\, d\theta\, d\varphi$

Achtung: Multiplikation mit der Determinante von Jacobi-Matrix nicht vergessen!

Beispiel mit Kettenregel

Sei $f : \mathbb{R}^3 \mapsto \mathbb{R}$ eine glatte Funktion mit $\nabla f\left(-\frac{\sqrt{3}}{2}, \frac{3}{2}, 7\right) = (6, 2, 0)$. Wenn wir nun $\frac{\partial f}{\partial r}\left(\sqrt{3}, \frac{2}{3}\pi, 7\right)$ mit zylindrischen Koordinaten berechnen wollen, dann ist

$$\frac{\partial f(g(r,\theta,z))}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial g_1}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial g_2}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial g_3}{\partial r}$$

Nun können wir die obige Information brauchen, um für $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ einzusetzen.

3.7 Satz von Green

Der Satz von Green stellt eine Beziehung zwischen Linienintegralen und Doppelintegralen über einen von einer parametrisierten Kurve umschlossenen Bereich her.
Eine parametrisierte Jordan-Kurve ist eine geschlossene parametrisierte Kurve $\gamma : [a,b] \mapsto \mathbb{R}$, wobei $\gamma :]a,b] \mapsto \mathbb{R}$ injektiv ist. Eine Jordan-Kurve in $\mathbb{R}^2$ ist das Bild einer parametrisierten Jordan-Kurve.
Seien b_1, b_2 die Basisvektoren von $\mathbb{R}^2$. Dann ist die Orientierung genau dann positiv, wenn die Matrix $[b_1, b_2]$ eine positive Determinante hat.
Ein reguläres Gebiet ist eine offene, beschränkte Teilmenge $A \subset \mathbb{R}^2$, deren Rand ∂A endliche Vereinungen von disjunkten Jordan-Kurven ist.
Eine parametrisierte Jordan-Kurve γ, die eine Randkomponente von A bildet, hat einen positiven Umlaufsinn, falls $(n(t), \gamma'(t))$ eine positiv orientierte Basis von $\mathbb{R}^2$ bildet. Dabei ist $n(t)$ der Einheitsvektor, welcher orthogonal zu $\gamma'(t)$ steht und von A weg zeigt. (Intuitiv: wenn die umschlossene Menge immer "links" liegt.)

Satz von Green

Sei $A \subset \mathbb{R}^2$ ein reguläres Gebiet und $F : U \mapsto \mathbb{R}^2$ ein Vektorfeld der Klasse C^1, wobei $(A \cup \partial A) \subset U \subset \mathbb{R}^2$. Dann gilt

$$\int_{\partial a} F(x)\, ds = \int\int_A (\partial_x f_2 - \partial_y f_1)\, dx\, dy$$

Um Flächen mit dem Satz von Green zu berechnen, benutzen wir ein Vektorfeld mit $\mathrm{curl}(f) = 1$, beispielsweise

$$f = (0, x) \text{ oder } f(-y, 0)$$

4 Themen aus Analysis I

Partielle Integration

$$\int f'(x)g(x)\ \mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\ \mathrm{d}x$$

- Grundsätzlich gilt: Polynome ableiten ($g(x)$), wo das Integral periodisch ist ($\sin, \cos, e^x$,...) integrieren ($f'(x)$)
- Teils ist es nötig, mit 1 zu multiplizieren, um partielle Integration anwenden zu können (z.B. im Fall von $\int \log(x)\ \mathrm{d}x$)
- Muss eventuell mehrmals angewendet werden

Substitution

Um $\int_a^b f(g(x))\ \mathrm{d}x$ zu berechnen: Ersetze $g(x)$ durch u und integriere $\int_{g(a)}^{g(b)} f(u) \frac{\mathrm{d}u}{g'(x)}$.

- $g'(x)$ muss sich irgendwie herauskürzen, sonst nutzlos.
- Grenzen substituieren nicht vergessen.
- Alternativ kann auch das unbestimmte Integral berechnet werden und dann u wieder durch x substituiert werden.

Partialbruchzerlegung

Seien $p(x), q(x)$ zwei Polynome. $\int \frac{p(x)}{q(x)}$ wird wie folgend berechnet:

1. Falls $\deg(p) \geq \deg(q)$, führe eine Polynomdivision durch. Dies führt zum Integral $\int a(x) + \frac{r(x)}{q(x)}$.
2. Berechne die Nullstellen von $q(x)$.
3. Pro Nullstelle: Einen Partialbruch erstellen.
   - Einfach, reell: $x_1 \to \frac{A}{x - x_1}$
   - n-fach, reell: $x_1 \to \frac{A_1}{x-x_1} + \ldots + \frac{A_r}{(x-x_1)^r}$
   - Einfach, komplex: $x^2 + px + q \to \frac{Ax+B}{x^2+px+q}$
   - n-fach, komplex: $x^2 + px + q \to \frac{A_1x+b_1}{x^2+px+q} + \ldots$
4. Parameter $A_1, \ldots, A_n$ (bzw. $B_1, \ldots, B_n$) bestimmen. (x jeweils gleich Nullstelle setzen, umformen und lösen).

5 Trigonometrie

5.1 Regeln

5.1.1 Doppelwinkel

- $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$
- $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 1 - 2\sin^2(\alpha)$
- $\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$

5.1.2 Addition

- $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$

5.1.3 Subtraktion

- $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
- $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
- $\tan(\alpha-\beta) = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

5.1.4 Multiplikation

- $\sin(\alpha)\sin(\beta) = -\frac{\cos(\alpha+\beta)-\cos(\alpha-\beta)}{2}$
- $\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$
- $\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$

5.1.5 Potenzen

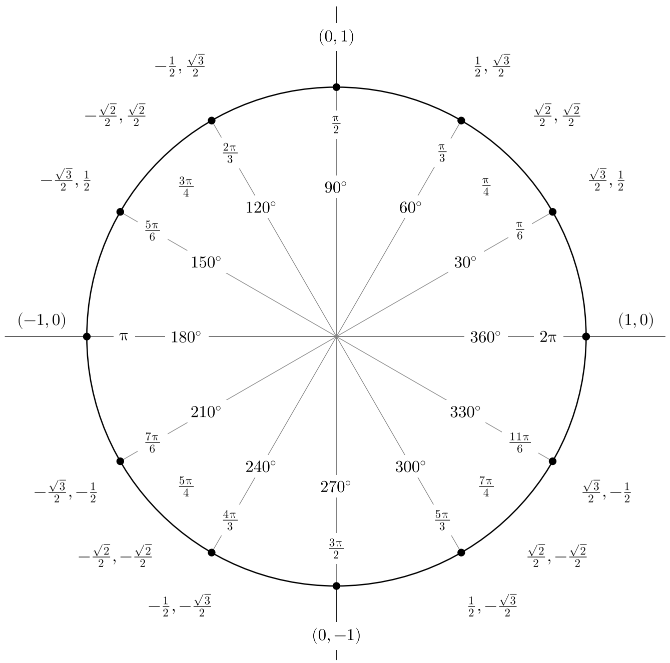
- $\sin^2(\alpha) = \frac{1}{2}(1-\cos(2\alpha))$
- $\cos^2(\alpha) = \frac{1}{2}(1+\cos(2\alpha))$
- $\tan^2(\alpha) = \frac{1-\cos(2\alpha)}{1+\cos(2\alpha)}$

5.1.6 Diverse

- $\sin^2(\alpha) + \cos^2(\alpha) = 1$
- $\cosh^2(\alpha) - \sinh^2(\alpha) = 1$
- $\sin(z) = \frac{e^{iz}-e^{-iz}}{2}$ und $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$

Wichtige Werte

deg	0°	30°	45°	60°	90°	180°
rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$+\infty$	0



6 Tabellen

6.1 Ableitungen

$\mathbf{F(x)}$	$\mathbf{f(x)}$	$\mathbf{f'(x)}$
$\frac{x^{-a+1}}{-a+1}$	$\frac{1}{x^a}$	$\frac{a}{x^{a+1}}$
$\frac{x^{a+1}}{a+1}$	$x^a \ (a \neq 1)$	$a \cdot x^{a-1}$
$\frac{1}{k\ln(a)}a^{kx}$	a^{kx}	$ka^{kx}\ln(a)$
$\ln\lvert x\rvert$	$\frac{1}{x}$	$-\frac{1}{x^2}$
$\frac{2}{3}x^{3/2}$	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$-\cos(x)$	$\sin(x)$	$\cos(x)$
$\sin(x)$	$\cos(x)$	$-\sin(x)$
$\frac{1}{2}(x - \frac{1}{2}\sin(2x))$	$\sin^2(x)$	$2\sin(x)\cos(x)$
$\frac{1}{2}(x + \frac{1}{2}\sin(2x))$	$\cos^2(x)$	$-2\sin(x)\cos(x)$
$-\ln\lvert\cos(x)\rvert$	$\tan(x)$	$\frac{1}{\cos^2(x)}$ $1+\tan^2(x)$
$\cosh(x)$	$\sinh(x)$	$\cosh(x)$
$\log(\cosh(x))$	$\tanh(x)$	$\frac{1}{\cosh^2(x)}$
$\ln\lvert\sin(x)\rvert$	$\cot(x)$	$-\frac{1}{\sin^2(x)}$
$\frac{1}{c} \cdot e^{cx}$	e^{cx}	$c \cdot e^{cx}$
$x(\ln\lvert x\rvert - 1)$	$\ln\lvert x\rvert$	$\frac{1}{x}$
$\frac{1}{2}(\ln(x))^2$	$\frac{\ln(x)}{x}$	$\frac{1-\ln(x)}{x^2}$
$\frac{x}{\ln(a)}(\ln\lvert x\rvert - 1)$	$\log_a\lvert x\rvert$	$\frac{1}{\ln(a)x}$

6.2 Weitere Ableitungen

$\mathbf{F(x)}$	$\mathbf{f(x)}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$x^x \ (x > 0)$	$x^x \cdot (1 + \ln x)$

6.3 Integrale

$\mathbf{f(x)}$	$\mathbf{F(x)}$
$\int f'(x)f(x)\ \mathrm{d}x$	$\frac{1}{2}(f(x))^2$
$\int \frac{f'(x)}{f(x)}\ \mathrm{d}x$	$\ln\lvert f(x)\rvert$
$\int_{-\infty}^{\infty} e^{-x^2}\ \mathrm{d}x$	$\sqrt{\pi}$
$\int (ax+b)^n\ \mathrm{d}x$	$\frac{1}{a(n+1)}(ax+b)^{n+1}$
$\int x(ax+b)^n\ \mathrm{d}x$	$\frac{(ax+b)^{n+2}}{(n+2)a^2} - \frac{b(ax+b)^{n+1}}{(n+1)a^2}$
$\int (ax^p+b)^n x^{p-1}\ \mathrm{d}x$	$\frac{(ax^p+b)^{n+1}}{ap(n+1)}$
$\int (ax^p+b)^{-1} x^{p-1}\ \mathrm{d}x$	$\frac{1}{ap}\ln\lvert ax^p+b\rvert$
$\int \frac{ax+b}{cx+d}\ \mathrm{d}x$	$\frac{ax}{c} - \frac{ad-bc}{c^2}\ln\lvert cx+d\rvert$
$\int \frac{1}{x^2+a^2}\ \mathrm{d}x$	$\frac{1}{a}\arctan\frac{x}{a}$
$\int \frac{1}{x^2-a^2}\ \mathrm{d}x$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$
$\int \sqrt{a^2+x^2}\ \mathrm{d}x$	$\frac{x}{2}f(x) + \frac{a^2}{2}\ln(x+f(x))$

7 Quellen

Ein Grossteil des Cheatsheets wurde stark vom Cheatsheet von Danny Camenisch inspiriert. Ausserdem stammen Teile der Tabellen aus dem Buch “Formeln, Tabellen und Konzepte”. Die Definitionen sind meistens dem Skript “Analysis 1” von Marc Burger und dem Skript “Analysis 2” von E. Kowalski entnommen.