# Cheatsheet InfoTheory

Nicolas Wehrli

June 2024

# 1 Foundations

**Definitions**

Information of an outcome $x$

$$h(x) = -\log(p(x))$$

Cross-Entropy between $p$ and $q$

$$H(p;q) = -\sum_x p(x) \log q(x)$$

Shannon Entropy

$$H(p) = H(p;p)$$

**Notation**
We identify outcomes $x$ with integers $1, ..., m$ and associate probabilities $p(x) \geq 0$.
$H(\frac{1}{m})$ for $H(p)$ with $p(x) = \frac{1}{m}$ (uniform)
$H(X) = H(p) = \mathbb{E}(-\log(p(X)))$ where $p$ is the pdf of $X$

**Jensen's Inequality**
Let $f$ be convex and $g : [m] \to \mathbb{R}$ be an arbitrary function that assigns a value to each outcome.

$$f\left(\sum_x p(x)g(x)\right) \leq \sum_x p(x) f(g(x)), \forall p(x) \geq 0, \sum_x p(x) = 1$$

alternatively

$$f(\mathbb{E}(g(X))) \leq \mathbb{E}(f(g(X)))$$

Applying this inequality to relate **Cross-Entropy** and **Entropy**, we get the following properties.

$$H(p;q) \geq H(p)$$

Definining **KL divergence** or **Relative Entropy** as

$$D(p||q) = \sum_x p(x) \log \frac{p(x)}{q(x)}$$

we get

$$H(p;q) = H(p) + D(p||q)$$

Further investigating KL divergence, we find

$$D(p||q) \geq 0 \tag{1}$$
$$D(p||q) = 0 \iff p = q \tag{2}$$

A further consequence of (1) is that the uniform distribution **maximizes** entropy.

$$H\left(\frac{1}{m}\right) = \max_p H(p)$$

**Definitions - Conditional distributions**

Conditional information

$$h(x|y) = -\log p(x|y)$$

Conditional Entropy

$$H(X|Y=y) = -\sum_x p(x|y) \log p(x|y)$$

$$H(X|Y) = \sum_y p(y) H(X|Y=y)$$

Monotonicity of Conditioning

$$H(X|Y) \leq H(X)$$

**Joint Entropy**

$$H(X,Y) = -\sum_{x,y} p(x,y) \log p(x,y)$$

**Chain Rule**

$$H(X,Y) = H(X|Y) + H(Y)$$

**Subadditivity**

$$H(X,Y) \leq H(X) + H(Y)$$

with equality iff $X \perp Y$.

**Multiple Conditioning**

$$H(X|Y,Y') \leq H(X|Y)$$

Generalized to $X_1, ..., X_n$ we get

$$H(X_1, ..., X_n) = \sum_{i=1}^n H(X_i|X_1, ..., X_{i-1}) \leq \sum_{i=1}^n H(X_i)$$

**Mutual Information**

$$I(X;Y) := H(X) - H(X|Y) = H(Y) - H(Y|X)$$

We further have

$$I(X;Y) = D(P(X,Y)||P(X)P(Y))$$

with $I(X;Y) = 0$ if $X \perp Y$.

**Conditional Mutual Information**

$$I(X;Y|Z) := H(X|Z) - H(X|Y,Z)$$

**Conditional Independence**
If $X \perp Y|Z$

$$I(X;Y|Z) = 0 \text{ and } I(X;Y) \leq I(X;Z)$$

We can deduct that for any function $\phi$ on outcomes of $X$

$$I(\phi(X);Y) \leq I(X;Y)$$

# 2 Compression

**Definition - Code**

A code $C$ is a mapping from outcomes to codewords

$$C : \{1, ..., m\} \to \{0,1\}^*$$

- If there is no codeword that is a prefix of another codeword, the code is a **prefix code**.
- Prefix codes retain injectivity when concatenating codewords.

Sets of codewords fulfilling the prefix property can be uniquely represented by the leaves of a binary tree. Since a leaf node has no children the prefix property is guaranteed.

**Kraft's Inequality**
If $\{c_1, ..., c_m\}$ are codewords of a prefix code, then

$$\sum_x 2^{-l_x} \leq 1, \text{ where } l_x = |c_x| \tag{3}$$

Conversely, given $\{l_1, ..., l_m\} \subset \mathbb{N}$ satisfying (3), there exists a prefix code with those codeword lengths.

- A prefix code is **succinct**, if Kraft's inequality holds with a equality. Else it can be optimized by pruning.
- Succinct codes uniquely define a **dyadic probabilistic model**
  $$q(x) = 2^{-l_x}$$
- Expected codeword length of a prefix code $C$
  $$L(C) = \sum_x p(x) l_x = \sum_x p(x)(-\log q(x)) = H(p;q)$$
- Using $H(p;q) = H(p) + D(p||q)$ we can deduce that the **minimal** $L(C)$ for a binary prefix code $C$ is
  $$L^* = H(p) + \min_{q:\text{dyadic}} D(p||q)$$
- Thus the closer $q$ is to $p$, the more optimal the prefix code is. But since $p$ doesn't have to be dyadic there can be an **inherent suboptimality** based on rounding.

**Weak Law of Large Numbers**

Let $Y_1, ..., Y_n$ be iid. random variables with mean $\mu$. Then

$$\overline{Y}_n := \frac{1}{n}\sum_{i=1}^n Y_i \xrightarrow{\mathbb{P}} \mu \iff \lim_{n\to\infty} \mathbb{P}(|\overline{Y}_n - \mu| < \varepsilon) = 1, \forall \varepsilon > 0$$

**Typicality - Asymptotic Equipartition**
Let $X_1, ..., X_n \overset{\text{iid}}{\sim} p$. The $\varepsilon$-typical outcomes are

$$\mathcal{A}_\varepsilon^n = \left\{ x \in \{1, ..., m\}^n : \left| H(p) + \frac{1}{n}\sum_{i=1}^n \log p(x_i) \right| < \varepsilon \right\}$$

By the law of large numbers for any $p, \varepsilon > 0$ and $\delta > 0$, there exists an $n_0$, s.t. $\forall n \geq n_0$

$$\mathbb{P}(A_\varepsilon^n) > 1 - \delta$$

in particular for $\delta = \varepsilon$.
For all $p, \varepsilon > 0$ and $n \in \mathbb{N}$, let $x \in \mathcal{A}_\varepsilon^n$, then

$$2^{-n(H(p)+\varepsilon)} \leq p(x) \leq 2^{-n(H(p)-\varepsilon)}$$
$$(1-\varepsilon)2^{n(H(p)-\varepsilon)} \leq |\mathcal{A}_\varepsilon^n| \leq 2^{n(H(p)+\varepsilon)}$$
$$\implies |\mathcal{A}_\varepsilon^n| \approx 2^{nH(p)} \text{ and for } x \in \mathcal{A}_\varepsilon^n : p(x) \approx 2^{-nH(p)}$$

We define the AEP Code to encode whole sequences

$$\text{AEP}_\varepsilon^n = \begin{cases} 0B^n(x) & \text{if } x \notin \mathcal{A}_\varepsilon^n \\ 1C^n(x) & \text{otherwise} \end{cases}$$

where we **enumerate** over the typical and atypical sequences.
Then the average codeword length **amortized** over the encoding of the sequence $x$ of $n$ outcomes is

$$\frac{1}{n}|C_\varepsilon^n(x)| \leq \frac{1}{n}(1 + \log|\mathcal{A}_\varepsilon^n|) \leq H(p) + \frac{1}{n} + \varepsilon$$
$$\frac{1}{n}|B^n(x)| \leq \frac{1}{n}(1 + \log m^n) \leq \log m + \frac{1}{n}$$

This result is theoretically optimal but practically not very useful.

**Huffman Codes**
Let $X$ have outcomes $\{1, ..., m\}$ ordered (wlog) st. $p(1) \geq ... \geq p(m)$.
The Huffman contraction $X'$ of $X$ is defined as

$$X' = \min\{m-1, X\}$$

We define the Huffman Code $C$ for $X$ recursively from $C'$ for the H. contraction $X'$

$$C(x) = \begin{cases} x-1 & \text{if } m = 2 \\ C'(x)0 & \text{if } x = m-1 \wedge m > 2 \\ C'(x-1)1 & \text{if } x = m \wedge m > 2 \\ C'(x) & \text{otherwise} \end{cases}$$

Let $C$ be a length-optimal code, then

$$p(x) > p(x') \implies l_x \leq l_{x'}$$
$$\forall c \in \text{Img}(C) \text{ wt. } |c| \text{ maximal} : \exists c' \in \text{Img}(C).\ c' \text{ sibling of } c$$

Assume $p_i$ ordered as above. Then a length-optimal prefix code $C$ with $l_1 \leq ... \leq l_{m-1} = l_m$ and $c_{m-1}, c_m$ only differing in last bit, is called **canonical**.
Huffman codes are length-optimal.

# 3 Prediction

A betting strategy $b$ bets a fraction $b(x)$ on the $x$-th outcome.
The bookmaker provides odds 1-for-$q(x)$ for each outcome $x$.

$$S(X) = \frac{b(X)}{q(X)}$$

is a random variable which describes the wealth growth of a gamble.

- Maximizing $\mathbb{E}(S(X))$ over all possible $b$ results in betting all on the highest probable outcome.
- Let $X_1, ..., X_n \sim p$ iid.

$$S_n := S(X_1, ..., X_n) = \prod_{i=1}^n S(X_i)$$

- Any strategy with $b(x) = 0, p(x) > 0$ for some $x$ will almost surely fail for increasing $n$.
- **Doubling Rate**

$$W(b) = \mathbb{E}(\log S(X)) = \sum_x p(x) \log \frac{b(x)}{q(x)}$$

- Odds 1-for-$q$ are **fair**, if $\sum_x q(x) = 1$
- In general, for fair odds, we have

$$W(b) = D(p||q) - D(p||b)$$

  which is **optimal** for $b = p$, since then $D(p||b) = 0$.
- **Conservation Theorem.** For $q$ uniform, fair and $b = p$.

$$W(b) + H(p) = \log m$$

- With fair odds, withholding part of the budget doesn't gain anything.
- If we have $Q = \sum_x q(x) < 1$, Kelly-betting ($b = p$) remains optimal in expectation. But the **Dutch book**

$$b(x) := \frac{q(x)}{Q} \implies S(X) = \frac{b(X)}{q(X)} = \frac{1}{Q} > 1$$

  has a **guaranteed** doubling rate $W(b) = -\log Q > 0$.

Consider an offered bet, where we can bet $b \in [0,1]$. We receive $\alpha b$ on a win and pay $\beta b$ on a loss. Then

$$W(b) = p\log(1 + \alpha b) + (1-p)\log(1 - \beta b)$$

We can find an optimal strategy using analysis (taking care of border cases). **Kelly Criterion.**

$$b^* = \min\{1, \max\{0, b\}\}, \qquad b = \frac{p}{\beta} - \frac{1-p}{\alpha}$$

**Risky bets.** With $\beta = 1$ and $p \to 0$ while $\alpha p \to \infty$, one gets

$$b = p - \frac{1-p}{\alpha} = \frac{\alpha p - 1 + p}{\alpha} \to p$$

**Log-sum inequality**

$$\sum_{i=1}^n a_i \log \frac{a_i}{b_i} \geq \left(\sum_{i=1}^n a_i\right) \cdot \left(\log \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}\right)$$

# 4 Processes

A semi-infinite sequence of random variables $X_1, X_2, ...$ is a **stochastic process**.

- A process is **stationary** if, for any $n \in \mathbb{N}$ and any $\Delta \geq 0$

$$\mathbb{P}(X_1, ..., X_n) = \mathbb{P}(X_{1+\Delta}, ..., X_{n+\Delta})$$

- **Conditional Entropy Rate.**

$$H(X) = \lim_{t\to\infty} H(X_{t+1}|X_t, ..., X_1)$$

- $X$ stationary $\implies H(X) = H'(X)$ well-defined.
- **Entropy Rate.**

$$H'(X) = \lim_{t\to\infty} \frac{1}{t} H(X_1, ..., X_t)$$

A **Markov Chain** is a stochastic process for which

$$X_{t+1} \perp X_{t-1}, ..., X_1 | X_t$$

Let $X$ be a Markov Chain and $\pi := P(X_1)$.

- then by the independence from past and future

$$\mathbb{P}(X_1, ..., X_t) = \mathbb{P}(X_1)\mathbb{P}(X_2|X_1)\mathbb{P}(X_3|X_2)\cdots\mathbb{P}(X_t|X_{t-1})$$

- $X$ is **time-homogeneous**, if

$$\mathbb{P}(X_{t+1}|X_t) = \mathbb{P}(X_2|X_1), \qquad \forall t \geq 1$$

- A time-homogeneous Markov Chain is fully characterized by its initial distribution and the **transition matrix** $P$ with

$$P_{ij} := \mathbb{P}(X_2 = i | X_1 = j)$$

  then

$$\mathbb{P}(X_{i+r} = b | X_i = a) = (P^r)_{ba}$$

- M.C. $X$ **stationary**
  $\iff \pi$ stationary and $X$ time-homogeneous $\iff P\pi = \pi$
- **Entropy Rate** of a stationary time-homogeneous M.C.

$$H'(X) = H(X) = \sum_a \pi_a \left(-\sum_b P_{ba} \log P_{ba}\right)$$

- A M.C. is **ergodic**, iff. $\exists t \geq 1$ s.t. $(P^t)_{ij} > 0, \forall i, j$.
- An M.C. **ergodic** $\iff$ has a unique stationary distribution
- For stationary M.C. $H(X_t|X_1) \leq H(X_{t+1}|X_1)$.

**Reversible Chains.**
For any finite Markove Chain $X$

$$\mathbb{P}(X_1, ..., X_t) = \mathbb{P}(X_t)\mathbb{P}(X_{t-1}|X_t)\cdots\mathbb{P}(X_1|X_2)$$

For a t.-h. M.C. $X$ with stationary distribution $\pi > 0$ and transition matrix $P$, then the backwards transitions are characterized by

$$U_{ab} = P_{ba}\frac{\pi_a}{\pi_b}$$

and $X$ is **reversible** $\iff P = U \iff P_{ba}\pi_a = P_{ab}\pi_b$.

**Random Walks on Graphs.**
Consider an undirected graph with nodes $\{1, ..., m\}$ and edge weights $w_{ab} = w_{ba} \geq 0$. We define a random walk as a Markov Chain

$$P_{ba} = \frac{w_{ab}}{W_a}, \quad W_a = \sum_b w_{ab}$$

- It has a stationary distribution $\pi_a = \frac{W_a}{W}$ with $W = \sum_a W_a$.
- Graph connected $\implies$ this stationary distribution is unique.
- A random walk on an undirected graph is reversible.
- Every t.-r. M.C. is equivalent to a random walk on a graph.

**Thermodynamics**
Let $(X, Y)$ and $(X', Y')$ be R.V. pairs over the same probability space, then

$$D(\mathbb{P}(X,Y)||\mathbb{P}(X',Y')) = D(\mathbb{P}(X)||\mathbb{P}(X')) + D(\mathbb{P}(Y|X)||\mathbb{P}(Y'|X'))$$

Let $X$ be a t.-h. M.C. with $\mu, v$ different PMF over states, then

$$D(P\mu||Pv) \leq D(\mu||v) \quad \forall \mu, v$$

with $\pi$ stationary

$$D(\mu||\pi) \geq D(P\mu||\pi)$$

if addionally $X$ is reversible

$$D(\mu||\pi) > D(P\mu||\pi), \quad \forall \mu \neq \pi.$$

# 5 Universal Coding

A CDF $F_X$ induces a partition of $[0;1)$ into

$$\{I_x : x \in X(\Omega)\}, \qquad I_x := [F_X(x) - p_X(x); F_X(x))$$

For $I = [a,b) \subseteq [0,1]$ there exists $z \in I, (z)_2 = 0.z_1z_2...z_l$ with $l = \lceil -\log(b-a) \rceil$ (*there's short representation for every Interval* $I$).

**Shannon-Fano-Elias Codes**

- pick midpoint $z_x = \sum_{x'<x} p_X(x') + \frac{1}{2}p_X(x)$
- truncate to $\lceil -\log p_X(x) \rceil + 1$ bits

Prefix-free, but not wasteful. Idea can be translated into

**Arithmetic Coding**

Note that Huffman Codes require knowledge of the distribution and cannot easiy be adapted to changing distributions.

Consider a stochastic process made of $X_t : \Omega \to A, A := \{0, ..., m-1\}$. We define

$$Z = (0.X_1X_2...)_m \in [0,1] \subset \mathbb{R}$$

For any $Z$. If $F_Z$ is a **bijection**, then $U := F_Z(Z) \sim \mathcal{U}([0,1])$.

Then $F_U(u) = u$ and $U = (0.U_1U_2...) \implies U_i \overset{iid}{\sim} \text{Bernoulli}\left(\frac{1}{2}\right)$.

- encoder $x_1, x_2, ... \mapsto z = (0.x_1x_2...)_m \overset{F_Z}{\mapsto} u = (0.u_1u_2...)_2$
- decoder $u \overset{F_Z^{-1}}{\mapsto} z = (0.x_1x_2...)_m \mapsto x_1x_2...$

In general

$$F_Z((0.x_1x_2...)_m) = \sum_{k=1}^{\infty} \sum_{i<x_k} p((0.x_1...x_{k-1}i)_m)$$

Consider $X = (X_1, ..., X_n)$ with $X : \Omega \to A^n$ and thus $m^n$ possible outcomes.

$F_X$ induces a partitioning $\{I_x : x = (x_1, ..., x_n)\}$ as defined above.

The **arithmetic code** is given by the S.-F.-E. code for $x$.

For the arithmetic code $C$, we have

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}(|C(X_1, ..., X_n)|) = H(\{X_t\})$$

**Incremental refinement** of $I_x$ for $x = (x_1, ..., x_n)$ into $m$ subinterval using

$$\mathbb{P}(X_{n+1}|X_1 = x_1, ..., X_n = x_n)$$

**Lempel-Ziv Code**

We denote a string to be compressed as

$$x = x_{-s}x_{-s+1}...x_{-1}x_0x_1...x_t$$

where $x_0, ..., x_t$ still needs to be encoded.

**Matching.**

$$\text{match}(x) := \{(j,l) \mid x_{-j}...x_{-j+l-1} = x_0...x_{l-1} \land j,l \geq 1\}$$

The **Maximal Matching** is $(j^*, l^*) \in \text{match}(x)$ is maximal in $l$ and as tiebreaker minimal in $j$.

**LZ77.**

$$c(x) = \begin{cases} (0, x_0)c(x_1...x_t) & \text{if match}(x) = \emptyset \\ (1, j^*, l^*)c(x_{l^*}...x_t) & \text{otherwise} \end{cases}$$

An integer $x$ can be encoded with $\leq \log x + 2\log\log x + 4$ bits.

$$C'(x) = 00...01(x)_2, \quad |C'(x)| = 2\lceil \log x \rceil + 1$$

and we construct

$$C(x) = C'(\lceil \log x \rceil)1(x)_2, \quad |C(x)| \leq \log x + 2\log\log x + 4$$

LZ77 is optimal (i.e. reaches the shannon limit).

# 6 Channel Coding

A noisy channel is a conditional probability distribution $\mathbb{P}_{Y|X}$, where $X : \Omega \to \mathcal{X}$ is the input and $Y : \Omega \to \mathcal{Y}$ the output. The noise reduces the information from $H(X)$ to $I(X;Y)$.

**Channel Capacity.** Given a channel with $\mathbb{P}_{Y|X}$ its capacity is

$$R^* = \max_{\mathbb{P}_X} I(X;Y)$$

$$w \in [1:m] \overset{\text{enc.}}{\mapsto} x = f(w) \in \mathcal{X} \overset{\text{channel}}{\mapsto} y \in \mathcal{Y} \overset{\text{dec.}}{\mapsto} \hat{w} = g(y) \in [0:m]$$

**Binary Symmetric Channel.**

A $\text{BSC}(n,\eta)$ is a channel with $\mathcal{X} = \mathcal{Y} = \{0,1\}^n$, with

$$\mathbb{P}(Y_{1:n}|X_{1:n}) = \prod_{i=1}^{n} \mathbb{P}(Y_i|X_i), \text{ where } Y_i = X_i \oplus N_i, \; N_i \overset{iid}{\sim} \text{Ber}(\eta)$$

The probability of communication error is $P_e = 1 - (1-\eta)^n$.

A **codebook** $\mathcal{C} = \{x(1), ..., x(m)\} \subseteq \{0,1\}^n$ maps each message $w \in [1:m]$ to a unique binary codeword $x(w)$.

**Bayes-optimal Decoder.** For a codebook $\mathcal{C}$ the min. error probability decoder is given by

$$\hat{w} = g(y) = f^{-1}(\hat{x}), \quad \hat{x} = \arg\max_{x\in\mathcal{C}} \mathbb{P}(x|y)$$

For $X$ uniform, the Bayes-optimal decoder is the **maximum likelihood** decoder, i.e. $\hat{x} = \arg\max_{x\in\mathcal{C}} \mathbb{P}(y|x)$.

**Joint Typicality.** Consider $n$ pairs of R.V. $(X_i, Y_i) \overset{iid}{\sim} \mathbb{P}$. Then for any $\varepsilon > 0$ define the jointly typical sets via

$$\begin{aligned} \mathcal{B}_\varepsilon^n = & \{(x_{1:n}, y_{1:n}) : |H(X,Y) + \frac{1}{n}\log p(x_{1:n}, y_{1:n})| < \varepsilon \land \\ & |H(X) + \frac{1}{n}\log p(x_{1:n})| < \varepsilon \land |H(Y) + \frac{1}{n}\log p(y_{1:n})| < \varepsilon\} \end{aligned}$$

**Joint AEP.** In the same setting as above

(i) $\mathbb{P}(\mathcal{B}_\varepsilon^n) \to 1$, as $n \to \infty$

(ii) $\log|\mathcal{B}_\varepsilon^n| \leq n(H(X,Y) + \varepsilon)$, for all $n$ large enough.

**Decoding by Joint Typicality.** Assume codeb. $\mathcal{C}$ and $y$ received.

1. $\mathcal{C}_\varepsilon(y) := \{x \in \mathcal{C} : (x,y) \in \mathcal{B}_\varepsilon^n\}$
2. If $\mathcal{C}_\varepsilon(y) = \{x\}$, then decode $g(y) = f^{-1}(x)$.
3. otherwise declare an inability to decode by setting, $g(y) = 0$.

**Random Codebooks.** Generate a random codebook from $m \cdot n$ fair coin tosses

$$X_i(w) \overset{iid}{\sim} \text{Ber}\left(\frac{1}{2}\right), \qquad w \in [1:m], i \in [1:n]$$

Define events

$$E_w = (X(w), Y(1)) \in \mathcal{B}_\varepsilon^n$$

Expected probability error of typicality Decoding

$$P_\varepsilon = \mathbb{P}(E_1^c \cup E_2 \cup ... \cup E_m)$$

With Union Bound and typicality ($\mathbb{P}(E_1^c) \leq \varepsilon$ for all $n \geq n_0(\varepsilon)$)

$$P_\varepsilon \leq \mathbb{P}(E_1^c) + \sum_{i=2}^{m} \mathbb{P}(E_i) \leq \varepsilon + m\mathbb{P}(E_2)$$

To further bound the error we consider the following **Lemma:**

Let $(X_i, Y_i) \overset{iid}{\sim} \mathbb{P}(X,Y)$ and $(\overline{X}_i, \overline{Y}_i) \overset{iid}{\sim} \mathbb{P}(X)\mathbb{P}(Y), i \in [1:n]$. For large enough $n$

$$\mathbb{P}((\overline{X}_{1:n}, \overline{Y}_{1:n}) \in \mathcal{B}_\varepsilon^n) \leq 2^{-n(I(X;Y)-3\varepsilon)}$$

With the lemma and noting $m = 2^{nR}$, we get for any $\varepsilon > 0$ and large enough $n$

$$P_\varepsilon \leq \varepsilon + 2^{nR}2^{-n(I(X;Y)-3\varepsilon)} \leq \varepsilon + 2^{-n\kappa}$$

with $\kappa = I(X;Y) - R - 3\varepsilon$. Thus as long as $R < I(X;Y)$ we have $P_\varepsilon \to 0$ for $n \to \infty$. Note then $\kappa > 0$ since $\varepsilon > 0$ can be arbitrary.

Note that $P_\varepsilon$ is the **expected error** over all possible codebooks (uniformly chosen). To find a codebook with an error $\leq P_\varepsilon$ one could do an exhaustive search over all codebooks (**not practical**).

**Channel Coding Theorem.** Consider a $\text{BSC}(n,\eta)$. For any rate $R < R^* = 1 - H(\eta)$ we can communicate $m = 2^{nR}$ distinct messages with an average error $P_\varepsilon \to 0$ for $n \to \infty$.

The **Hamming Distance** between $x, x' \in \{0,1\}^n$ is the number of bits they differ.

$$d_H(x,x') = \sum_{i=1}^{n} x_i(1-x_i') + (1-x_i)x_i' = \sum_{i=1}^{n}(x_i + x_i') - 2x_ix_i'$$

In $\text{BSC}(n,\eta)$ with $\mathbb{P}(X)$ **uniform (!)** and $\eta < \frac{1}{2}$, optimal decoding wrt. to $\mathcal{C}$ is characterized by

$$x^* = \arg\max_{x\in\mathcal{C}} \mathbb{P}(y|x) = \min_{x\in\mathcal{C}} d_H(x,y)$$

**Detecting single-bit corruptions.**

Let $\mathcal{C} = \{0,1\}^n$ be a codebook and $y$ be received with atmost one bit corruption from input $x$. We construct $\mathcal{C}' \subseteq \{0,1\}^{n+1}$ with an appended parity bit for every $x \in \mathcal{C}$.

$$x \mapsto x' = \left(x_1, ..., x_n, \sum_{i=1}^{n} x_i \mod 2\right)$$

Then $\min_{x' \neq z' \in \mathcal{C}'} d_H(x', z') \geq 2$

**Hamming Code.**

Blocklength $n = 2^k - 1$ with $k$ parity bits and $2^k - k - 1$ data bits. The $i$-th parity bit is at position $2^{i-1}$ with $i \in [1:k]$ and covers all positions where the $i$-th least significant bit is set (including itself). The sum of positions of the mismatched parity bits gives the position of the corrupted bit.

The rate achieved is

$$R = \frac{2^k - k - 1}{2^k - 1}$$

**Symmetric Channels**

Consider the transition matrix $P = p(y|x)$ as defined in **this lecture**. The channel is **symmetric** if the rows/columns are permutations of eachother. The channel is **weakly symmetric** if the columns are permutations of eachother and every row sum is equal.

For a **weakly symmetric** channel

$$C = \log|\mathcal{Y}| - H(\textit{column of transition matrix})$$

achieved by uniform distribution over $\mathcal{X}$.

For a channel containing two **parallel channels**: $2^C = 2^{C_1} + 2^{C_2}$.

# 7 Lossy Coding

**Rate-Distortion Theory**
**Distortion measure** $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$, s.t. $d(x, x) = 0 (\forall x)$.
Examples include Hamming Distance and MSE. We consider $\mathbb{E}(d(X, \hat{X}))$, where $X$ is original data and $\hat{X}$ its reconstruction.
**Rate Distortion Theorem.** The maximal rate $R(D)$ at which $X$ ($\mathbb{P}(X)$ given) can be encoded with $\mathbb{E}(d(X, \hat{X})) \leq D$ is given by

$$R(D) = \min_{\mathbb{P}(\hat{X}|X): \mathbb{E}(d(X,\hat{X})) \leq D} I(X, \hat{X})$$

Consider the specific Case $X_t \overset{iid}{\sim} \mathrm{Ber}(p)$ and $d(x^n, \hat{x}^n) = \frac{1}{n} d_H(x^n, \hat{x}^n)$. Then requiring $\mathbb{P}(X_t \neq \hat{X}_t) \leq \eta$ (wlog. $\eta \leq p \leq 1/2$), and minimizing the mutual information $I(X; \hat{X})$ gives us a symmetric backwards channel

$$\mathbb{P}(X|\hat{X}) = \begin{bmatrix} 1-\eta & \eta \\ \eta & 1-\eta \end{bmatrix}, \text{ and thus } \hat{X}_t \overset{iid}{\sim} \mathrm{Ber}(q) \text{ with } q = \frac{p-\eta}{1-2\eta}$$

which gives us a optimal forward channel (it's asymmetric).
This gives a rate of $R(\eta) = H(p) - H(\eta)$ as a specific case of the Rate-Distortion Theorem.
**Distortion-Typicality.** A pair $(x, \hat{x})$ is $(\varepsilon, \delta)$-d-typical, if it is jointly $\varepsilon$-typical and $|\eta - d(x^n, \hat{x}^n)| < \delta$ ($\eta$ is the expected bit error).
**Uniform Quantization** Let $U : \Omega \to R \subseteq \mathbb{R}$ (Scalar Quantization). We partition $R$ into intervals $R_j$ of length $\Delta$, assuming $|R| < \infty$.
We can then approximate the pdf $p(u)$ by a step function $\hat{p}(u)$, which is constant over $R_j$.

$$\hat{p}(u) = \sum_j \mathbb{I}\{u \in R_j\} \frac{\mathbb{P}(u \in R_j)}{\Delta} = \sum_j \mathbb{I}\{u \in R_j\} \frac{\int_{R_j} p(u) du}{\Delta}$$

Let $V$ be an RV characterized by $\hat{p}$. Then $\mathbb{E}((U - V)^2) \approx \frac{\Delta^2}{12}$.
If $U \sim \mathcal{U}([a, b])$ and we want the constant $x$ minimizing MSE. Then the optimal point is $x = \frac{a+b}{2}$ with an MSE of $\frac{(a+b)^2}{12}$.
**Differential Entropy.** Let $U : \Omega \to R \subseteq \mathbb{R}$. The differential entropy is defined as

$$h(U) = -\int_R p(u) \log p(u) du$$

We have $H(V) \approx h(U) - \log \Delta$.
To be more precise we have

$$H(V) + \log \Delta \to h(U), \quad \text{as } \Delta \to 0$$

Let $U \sim \mathcal{N}(0, \sigma^2)$. Then if we accept a MSE of at most $\eta$, $U$ can be quantized at a rate

$$R(\eta) = \begin{cases} \frac{1}{2}(\log \sigma^2 - \log \eta) & \text{if } \eta \leq \sigma^2 \\ 0 & \text{otherwise} \end{cases}$$

**Shannon's Lower Bound.** Let $U : \Omega \to R \subseteq \mathbb{R}$ with $h(U) < \infty$ and $h^*$ the differential entropy $h^*(\sigma^2)$ of a gaussian. Then $U$ can be encoded with MSE at most $\eta$ at a rate $R > R(\eta)$, where

$$R(\eta) \geq h(U) - h^*(\eta)$$

For $U : \Omega \to R \subseteq \mathbb{R}^d, d > 1$ we speak of *vector quantization.* Generally if we are given $\{y_1, ..., y_m\} \subseteq R$, the optimal quantizer $V$ is characterized by

$$u \mapsto \hat{u} = y_k, \quad k \in \arg\min_j ||u - y_j||$$

This induces Voronoi cells around the $y_j$, which individually are convex regions (for $d = 1$ these are intervals).
**Centroid Condition.** Given a partition $\{R_j\}$ of $R \subseteq \mathbb{R}^n$, the optimal code points are

$$y_j = \mathbb{E}(U | U \in R_j), \text{ empirically w/ dataset } \mathcal{X}: \frac{\sum_{x \in \mathcal{X} \cap R_j} x}{|\mathcal{X} \cap R_j|}$$

**Loyld's Algorithm**

1. Generate random code points $y_j$ at random and the induced Voronoi cells $R_j$.
2. For each cell, recompute the centroid. Then update the newly induced Voronoi cells.
3. Repeat Step 2 until convergence.

This algorithm fixes $m$ as the number cells (encoding cost $\log m$ bits) and then optimizes (locally) for distortion.
The relevant quantity for encoding cost should not be the number of cells, but the Entropy of the discretized RV $V$.

$$\ell(V) = \mathbb{E}((U - V)^2) + \lambda H(V), \quad \lambda > 0$$

**Blahut Arimoto Algorithm**

$$\mathbb{P}(y_j|x_i) = \frac{\pi_j \exp(-\lambda ||x_i - y_j||^2)}{\sum_{k=1}^m \pi_k \exp(-\lambda ||x_i - y_k||^2)}, \ \pi_j = \frac{1}{n} \sum_{i=1}^s \mathbb{P}(y_j|x_i)$$

and we generalize the centroid rule with weights $y_j^* = \frac{\sum_{i=1}^s \mathbb{P}(y_j|x_i) x_i}{\sum_{i=1}^s \mathbb{P}(y_j|x_i)}$.

# 8 Estimation

**Laplace Distribution** is characterized by the pdf

$$p(x; \mu, b) = \frac{1}{2b} \exp\left(-\frac{|x - \mu|}{b}\right)$$

with parameters $\theta = (\mu, b) \in \Theta = \mathbb{R} \times \mathbb{R}_+$.
Any function $\phi$ of a RV $X$ is called a **statistic** of $X$.
A statistic $\phi$ of $X$ is **sufficient** for $Y$, if $X \perp Y | \phi(X)$.
It is **minimally sufficient**, if for all other statistics $\phi'$, $X \perp \phi'(X) | \phi(X)$.
**Halmos-Savage Factorization Theorem.** Let $p_\theta$ be family of pdf parameterized by $\theta$. Then $\phi$ is sufficient for $\theta$, iff. there exists non-negative functions $h, g_\theta$ s.t. $p_\theta = h(x) g_\theta(x)$.
**Exponential Family.** Let $\phi$ be a sufficient statistic and $h(x)$ a positive function. The exponential family induced by $\phi, h$ is characterized by the pdf

$$p(x; \theta) = h(x) \exp(\theta \cdot \phi(x) - A(\theta)), \ A(\theta) = \ln \int h(x) \exp(\theta \cdot \phi(x)) dx$$

**Bernoulli** $h \equiv 1, \phi(x) = x, \theta = \ln \frac{p}{1-p}, A(\theta) = \ln(1 + e^\theta)$.
**Binomial** $h(x) = \binom{n}{x}, \phi(x) = x, \theta = \ln \frac{p}{1-p}, A(\theta) = n \ln(1 + e^\theta)$.
**Normal** $h(x) = \frac{1}{\sqrt{2}}, \phi(x) = (x, x^2)^\top, \theta = \left(\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2}\right), A(\theta) = -\frac{\theta_1^2}{4\theta_2} + \ln\left(\frac{1}{\sqrt{-2\theta_2}}\right)$
**Exponential** $h(x) \equiv 1, \phi(x) = -x, \theta = \lambda, A(\theta) = -\ln \lambda, x \geq 0$
Let $X_1, ..., X_n$ be iid. with pdf $p(x; \theta)$ in an exponential family. Then the joint distribution is also an expenonetial family with sufficient statistic $\phi(X_1, ..., X_n) = \sum_{i=1}^n \phi(X_i)$.
When estimating parameters from iid samples, there is a fixed-dimensinoal sufficient statistic iff. the distributions can be represented as an exponential family.
**MLE**
**Binomial** $\hat{p} = \frac{1}{n} \sum_{i=1}^n x_i$
**Normal** $\hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i, \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \hat{\mu})^2$
**Laplace** $\hat{\mu} = \mathrm{median}(x_1, ..., x_n), \hat{b} = \frac{1}{n} \sum_{i=1}^n |x_i - \hat{\mu}|$.
For an exponential family we have

$$\nabla_\theta \sum_{i=1}^n \log p_\theta(x_i) = 0 \iff \mathbb{E}_\theta(\phi(X)) = \frac{1}{n} \sum_{i=1}^n \phi(x_i)$$

**Maximum Entropy Inference**
Statistical inference without parametric families.
Given constraint functions $\phi_i : R \subseteq \mathbb{R} \to \mathbb{R}$ and empirical values $\bar{\phi}_i$. Consider

$$\mathcal{P} = \{p : R \to [0, 1] \mid \mathbb{E}_p(\phi_i(X)) = \bar{\phi}_i (\forall i)\} \neq \emptyset$$

Define the exponential family

$$p_\theta(x) = \exp(\phi(x) \cdot \theta - A(\theta)).$$

Let $p_{\hat{\theta}}$ be the pdf obtained for the MLE $\hat{\theta}$ based on data summaries $\bar{\phi}_i$, then

$$p_{\hat{\theta}} = \arg\max_{p \in \mathcal{P}} h(p), \text{ with } h(p) = -\int p(x) \ln p(x) d$$

**Fisher Information**
Let $p_\theta$ be a parametric family of pdfs.
For an estimator $\hat{\theta}$ and the true parameter $\theta^*$, we define $\mathrm{MSE}(\hat{\theta}) = \mathbb{E}((\hat{\theta}(X_1, ..., X_n) - \theta^*)^2)$.
The **score** is an RV $S(X) = \nabla_\theta \log p_\theta(X)$.
We have $\mathbb{E}(S(X)|\theta) = 0$, and the **Fisher Information** is symmetric positive semi-definite matrix $\mathcal{I}(\theta) = \mathbb{E}(S(X)S(X)^\top | \theta)$.
For $X_1, ..., X_n \overset{iid}{\sim} p_\theta$ we have $S(X_1, ..., X_n) = nS(X_1)$.
For $X, Y \sim p_\theta$ we have $S(X, Y) = S(X|Y) + S(Y)$ and by that $\mathcal{I}_{X,Y}(\theta) = \mathcal{I}_{Y|X}(\theta) + \mathcal{I}_X(\theta)$.
If $p_\theta$ twice-differentiable, then $\mathcal{I}(\theta) = -\mathbb{E}(\nabla_\theta^2 \log p_\theta(X))$.
**Cramér-Rao Bound**
Let $p_\theta(x)$ be a parameterized family of pdfs for a real-valued RV $X$. Let $\hat{\theta} = T(X)$ be an unbiased estimator, i.e. $\mathbb{E}_\theta(T(X) - \theta) = 0$, then

$$\mathrm{MSE}(\hat{\theta}) \geq \mathcal{I}(\theta)^{-1}$$

An unbiased estimator $\hat{\theta}$ is **efficient**, if it attains the Cramér-Rao bound with equality.
**Rao-Blackwell Theorem**
Given an estimator $\hat{\theta}$ for the parameter of a family with sufficient statistics $T$ define an estimator $\bar{\theta} = \mathbb{E}(\hat{\theta}|T)$, then

$$\mathrm{MSE}(\bar{\theta}) \leq \mathrm{MSE}(\hat{\theta})$$